Optimal Pricing with New Models of Consumer Behavior

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Yan Liu

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Dedication

To Liverpool Football Club

Abstract

Revenue management is a commonly used practice in many industries, such as airlines, hotels, fashion, and car rentals. It takes advantage of customers' different valuations for a product or products and charges different prices to different customers to extract customers' surplus.

In revenue management, most literature assumes that customers are myopic and will buy immediately if the price is low and leave otherwise. In recent years there has been much research involving strategic customers who have the ability to predict future prices and thus make a purchase at the price that maximizes their utility. In Chapter 2 and 3, I will study a different type of customer behavior, which we call patient customer behavior. A patient customer will wait up to some fixed number of time periods for the price of the product to fall below his or her valuation at which point the customer will make a purchase. If the price does not fall below a patient customer's valuation at any time during those periods, then that customer will leave without buying. Chapter 4 describes a learning and pricing problem in which the seller does not know the fraction of patient customers.

In practice, customers may wish to search for product information before making purchase decisions. That is, they may wish to research the product or products under consideration. This research behavior will introduce costs to customers, which may include time cost, travel cost, and mental processing cost. Since such research costs could be part of a customer's utility, they may affect

a customer's purchasing behavior and thus the firm's strategy. However, most literature in revenue management does not consider the existence of customers' search cost. In Chapter 5, I consider a pricing problem in which customers face uncertainty about whether they will like certain products. Those customers can incur research costs to learn product information.

In summary, I will focus on deriving optimal pricing decisions for companies that face customer behavior that is more complex than typically assumed in traditional models.

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Chapter 1

Introduction

My research focuses on deriving optimal pricing decisions for companies when customer behavior is more complex than typically assumed in traditional models. In revenue management, most literature assumes that customers' behavior is myopic. That is, a customer will make a purchase immediately if the current price is below his valuation, and leave otherwise. Recent years, customers are becoming more and more sophisticated. They may delay a purchase if the current price is high; they may predict future prices based on their past purchasing experience; they may search for more alternatives before purchasing a specific one. It is important to understand the effect of consumer behavior on a firm's optimal pricing strategy, as much literature has shown that the firm will suffer considerable losses if it does not recognize the existence of strategic customers.

In Chapter 2, I consider what I call patient customers, which can be considered as being "between" myopic customers and strategic customers in terms of sophistication. When faced with a high price for a product, patient customers will wait in the market and possibly purchase the product in the future if the price falls low enough. However, they do not try to predict future prices. We study pricing decisions of a firm facing this type of consumer behavior. In particular, I consider an infinite-horizon single-product pricing problem in which a fraction of customers is patient and the remaining fraction is impatient. A patient customer will wait up to some fixed number of time periods for the price of the product to fall below his or her valuation at which point the customer will make a purchase. If the price does not fall below a patient customer's valuation at any time during those periods, then that customer will leave without buying. In contrast, impatient customers will not wait, and they either buy immediately or leave without buying. I prove that with such customer behavior, there is an optimal dynamic pricing policy comprised of repeating cycles of decreasing prices. I obtain bounds on the length of these cycles, and by exploiting these results we can compute such an optimal policy via an efficient dynamic programming approach. I also consider problems in which customers have variable levels of patience and develop bounds and heuristics.

In Chapter 3, I consider the same problem as Chapter 2 but with a continuous price set. I show that the decreasing cyclic policy is still optimal in this continuous price setting through a discretization procedure. Next, I propose a new approach to derive the optimal decreasing sequence within each cycle for a special case, in which each customer's patience level k is relatively small and customers have uniform valuation distribution. Chapters 2 and 3 include material from Liu and Cooper (2015).

Chapter 4 describes a learning and pricing problem in which the seller does not know the fraction of patient customers. Each period, a stochastic number of customers arrive with heterogeneous patience levels. I propose some algorithms combining learning and pricing to minimize the revenue loss compared to the optimal revenue collected by a clairvoyant who knows the true fraction of patient customers in advance. By assuming a deterministic arrival stream of customers and homogeneous patience level, I derive a regret of $O(\sqrt{T})$, where T is the number of time periods.

Customers often need to search for product information before making purchase decisions. However, in revenue management, most literature assumes that customers have full information about the product, and neglects the effect of customers' search cost on their behavior and then on the firm's strategy. In Chapter 5, I consider a pricing problem in which customers face uncertainty about a product's valuation, but can resolve that uncertainty by incurring a research cost. In the single product setting, customers need to decide whether or not to research a product by comparing the expected utilities of each action. I characterize a customer's optimal policy, and then study the optimal pricing decisions for the seller. It is perhaps surprising that the seller's revenue need not be increasing with an increase in a customer's research cost. I also consider a two-product setting in which customers need not only to decide whether to research, but also to decide in which order to do so. Based on customers' optimal behavior, I study how the seller should manage the research cost of each product to best influence customers' actions and to maximize its own revenue. It is interesting to see that assigning research cost to only one of the two products can better differentiate the customers than assigning research cost to both products.

Chapter 2

Optimal Dynamic Pricing with Patient Customers

2.1 Introduction

Nowadays, as firms make pricing decisions by involving more and more sophisticated techniques, customers also develop their own purchasing strategies to maximize their utilities. It is natural that a customer will wait some amount of time to make a purchase if the current price is too high. When the price falls within his budget, he will make purchase immediately and leave the market then. This is different from strategic customer behaviors in existing literature, wherein customers have the ability to predict the prices in the future and make purchase decisions based on their expectations. In this chapter I study such patient customers who wait and postpone their purchase if the current price is above their valuation. As soon as the price falls below their valuation, they will make the purchase immediately and never come back. I assume there is a deterministic number of arrivals each period and customers are either myopic or patient. Patient customers are homogeneous in their patience levels and there is no inventory consideration.

My main result is that for any customer valuation distribution and any fixed patience level, there is an optimal cyclic policy that is comprised of a decreasing sequence of prices within each cycle. This result is obtained under the assumption that there is a discrete price set. In addition, I derive a lower bound and an upper bound for the cycle length, and exploit it to propose a dynamic programming algorithm to compute the optimal decreasing sequence. I also consider a problem in which customers have heterogeneous patience levels and develop bounds and heuristics.

In my model setting, demand is not only a function of current price, but also depends on the prices in the last k periods. One may wonder if this problem can be analyzed with a dynamic programming approach in which the state is the last k periods' prices and the decision variable is the current price with the aim to maximize the average revenue. However, due to the curse of dimensionality of dynamic programming, it becomes extremely hard to solve this problem especially when k is sufficiently large. In this chapter, I formulate the problem as an infinite-horizon problem involving a monopolist facing myopic customers and patient customers. The monopolist aims to maximize the long-run average revenue. I firstly reduce the original problem to a finite-horizon optimization problem with some constraints, then compare different pricing structures and finally prove that the decreasing structure is optimal for the finite-horizon problem. Thus, repeating this decreasing structure is optimal for the original problem.

Early literature about intertemporal demand models can be found in Stokey (1979). In Stokey's model, the customers stay in the market until the end of the horizon and no new customers enter the market. The firm's objective is to identify a pricing policy that maximizes the present discounted value of its profit, while the customer's objective is to decide whether and when to make the purchase with the knowledge of the pricing policy. For the model without production cost, the result shows that for a large class of customer utility functions the firm would rather sell the product only at the beginning of time horizon instantly than sell to different customers at different dates with different prices.

Coase (1972) studies a model in which a monopolist sells a durable good to customers who stay in the market indefinitely if they did not make the purchase. Each period a fixed amount of customers arrives with two different valuations for the good: high willingness to pay and low willingness to pay. The monopolist has to make a price decision each period to maximize discounted present value. The customers make the purchase when their utility is maximized. The author shows that a decreasing cyclic pricing policy is optimal in which the monopolist sells only to high-value customers except the last period in the cycle and then offer a markdown to the low-value customers at the end of the cycle. Interestingly, their conclusion coincides with a special case of our theoretic results. Sobel (1991) also studies the problem of durable goods monopoly and gets similar results.

Besides the above economics literature, there are a number of papers about dynamic pricing with strategic customers in OR/MS community. We refer readers to these papers and books: Aviv et al. (2009), Popescu and Wu (2007), Shen and Su (2007), Talluri and van Ryzin (2004).

Besanko and Winston (1990) study a game involving intertemporal demand. The customers enter the market when the sale begins and stay there until the time ends. All the customers have rational expectation about the pricing sequence and only make purchase when the utility is maximized. The monopolist has to make a price decision every period given the accumulative sales up to that point. Through backward induction, they figure out that a subgame perfect Nash Equilibrium exists and the optimal pricing sequence decreases monotonically over time.

Borgs et al. (2014) study a multi-period pricing problem with service guarantees. They assume all the customers are strategic with respect to timing of their purchases. The firm announces a vector of prices upfront, and the customer chooses to purchase the service at a time with the lowest price between the time he comes and he leaves if his value is greater than the lowest price. Moreover, the firm has a constraint that all the demand have to be satisfied. Although their optimization problem is non-convex, they provide a polynomial time algorithm that computes the optimal sequence of prices. They also find that when customers are more patient, the firm offers higher prices, which lead to overage capacity, lower revenues and reduced customer welfare.

Su (2007) studies an intertemporal pricing problem with strategic customer behavior. He divides the customers along two dimensions: valuation and patience. Patient-high-types, impatient-high-types, patient-low-types and impatient-low-types. The seller needs to decide the pricing and rationing policies (which will be announced when the time begins) and control policies (cumulative sales processes and departure processes) while the customers aim to maximize their utilities. The author identifies a full spectrum of pricing policies including pure markup policy, pure markdown policy and non-monotone policy as optimal policy depending on the different composition of the four types of customers. More specifically, when high-value customers are less patient than low-value customers, markdown pricing is effective, while markup is optimal when high-value customers are more patient than the low-value customers.

Besbes and Lobel (2007) study a problem where the firm has to offer a sequence of prices at the start of time horizon with the aim to maximize its revenues. The customers are different along two dimensions: their valuation about the product and their willingness to wait. Customers know all the prices and are strategic in timing their purchase in order to maximizing their own utilities. They prove that the cyclic pricing policy is optimal and the cycle length is at most twice the maximum willingness to wait. In addition, by using dynamic programming, they propose a polynomial time algorithm to compute an optimal policy.

Li et al. (2014) conduct an empirical study from the Air-Travel industry to answer whether customers are strategic. They develop a structural model to estimate the fraction of strategic customers. It is notable that in their demand model, they assume that strategic consumers wait for at most one period, thus the demand includes three parts: myopic customers who arrive and purchase, strategic customers who arrive and purchase, and strategic customers who arrive in last period but purchase in this period. Their results show that 4.9 to 44.9 percent of the population are strategic, measured by the 5th and 95th percentiles.

There is one paper very closely related to our work. Ahn et al. (2007) study the pricing and manufacturing decisions when demand is a function of prices in multiple periods. They have myopic customers and patient customers as we assume. Their objective is to choose the optimal pricing and production decisions with the aim to maximize the total profit (or average profit) under some capacity constraint. Their primary result is just a special case with only pricing decision that the customers are willing to wait at most one period and demand function is linear. They establish that the optimal policy in this special case is a 2-period decreasing cyclic policy. They do not have theoretical results for the general k problem (arbitrary patience levels) but come up with some heuristic policies.

My work theoretically identifies the structure of optimal policy for the general k problem with arbitrary customers' valuation distribution. In addition, I provide some good upper and lower bounds for the cycle length of the optimal policy. Then based on the structure and bounds, I propose an efficient algorithm which is able to compute the optimal policy quickly.

The remainder of the section is organized as follows. Section 2.2 introduces the model and states our main result. Section 2.3 contains preliminary results and draws connections with finite-horizon problems. Section 2.4 outlines the proof of the main result through a series of intermediate results. Section 2.5 contains refinements and extensions of our main result as well as a computational algorithm. Section 2.6 describes bounds for problems with variable patience levels. Section 2.7 presents results of numerical experiments. Proofs are contained in appendices in section 2.8.

2.2 Model Setup and Central Result

We study a multi-period single-product pricing problem with deterministic demand and unlimited inventory. We assume that demand is a continuous quantity and that units are scaled so that in a single period a potential new demand of 1 arrives. This demand is comprised of infinitesimal customers. Each such customer has a non-negative valuation drawn from a distribution $G(\cdot)$. For each x, let $F(x) = \lim_{y \uparrow x} G(y)$ be the left limit of $G(\cdot)$ at x. Hence, $F(x)$ is the fraction of customers whose valuation is less than x, and $1 - F(x)$ is the fraction of customers whose valuation is at least x. If $G(\cdot)$ is continuous, then $F(\cdot) = G(\cdot)$. We assume that a fraction $\alpha \in (0, 1]$ of customers are patient and a fraction $1 - \alpha$ are impatient. Each period the firm offers one price. If the price offered is no greater than a particular customer's valuation, then that customer (whether impatient or patient) will make a purchase immediately and then leave. If the price is above that customer's valuation, then the customer's subsequent behavior depends upon whether the customer is patient or impatient. If the customer is impatient, then he will simply leave the market without purchasing. If the customer is patient, then he will wait for up to k more periods. In those k periods, the patient customer will make a purchase as soon as the price falls to or below his valuation. If the price remains above the customer's valuation for the full k periods, then the patient customer will leave without making a purchase. We assume that $k \geq 1$ is fixed.

Let $\mathcal{P} = \{p(1), \ldots, p(m)\}$ denote the set of allowable prices. We assume that $2 \leq m < \infty$; that is, prices are selected from a finite set with cardinality $m \geq 2$. We will be interested in both finite-horizon and infinite-horizon problems. Our analysis of the former will be useful for deriving our main results, which deal with the latter. For an infinite-horizon problem, a sequence $p = (p_1, p_2, \dots) \in \mathcal{P}^\infty$ is called a pricing policy or simply a policy. For finite $L \in \mathbb{N} = \{1, 2, 3, \dots\}$, we will also call $p = (p_1, \dots, p_L) \in \mathcal{P}^L$ a policy for the finite-horizon L-period problem, and sometimes use the notation $L(p)$ to denote the length of p. For $p = (p_1, \dots, p_{L_1}) \in \mathcal{P}^{L_1}$ and $q = (q_1, \dots, q_{L_2}) \in \mathcal{P}^{L_2}$ with $L_1, L_2 \in \mathbb{N}$ we will use (p, q) to denote the policy $(p_1, \dots, p_{L_1}, q_1, \dots, q_{L_2}) \in \mathcal{P}^{L_1+L_2}$ that implements price p_t for $t \in \{1, \dots, L_1\}$ and price q_{t-L_1} for $t \in \{L_1+1, \dots, L_1+L_2\}$. Similarly, for $q \in \mathcal{P}^{L_2}$, we will use $(q, q, q, \dots) \in \mathcal{P}^\infty$ to denote the infinite-horizon policy that charges q_t in periods $t, t+L_2, t+2L_2, t+3L_2, \dots$ for $t = 1, \dots, L_2$.

For $p \in \mathcal{P}^L$ with $L \in \mathbb{N} \cup \{\infty\}$, let

$$\rho_t(p) = p_t\Big[1 - F(p_t) + \alpha \sum_{i=1}^{k} [F(\min\{p_{t-i}, \dots, p_{t-1}\}) - F(p_t)]^+\Big] \quad \text{for } t = 1, 2, \dots \tag{2.1}$$

where we use the convention that $p_t = 0$ for $t \leq 0$. Throughout, $x^+ = \max\{x, 0\}$. The quantity $\rho_t(p)$ represents the revenue accrued in period t under policy p. In (2.1), $1 - F(p_t)$ represents the number of customers that arrive in period t and immediately make a purchase. The expression $[F(\min\{p_{t-i}, \dots, p_{t-1}\}) - F(p_t)]^+$ in (2.1) represents those customers that arrive in period $t - i$ and that have a valuation that is less than all the prices in periods $t-i, \dots, t-1$ but greater than or equal to the price in period t. Keeping in mind that α is the fraction of customers that is patient, we see that $\alpha[F(\min\{p_{t-i}, \dots, p_{t-1}\}) - F(p_t)]^+$ represents the number of customers that initially arrive in period $t - i$ and subsequently make a purchase in period t.

We are now ready to present the objective function of the seller. For $p \in \mathcal{P}^\infty$

let

$$H(p) = \liminf_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T} \rho_t(p)$$

denote the infinite-horizon long-run average revenue from implementing policy $p \in \mathcal{P}^\infty$. The limit inferior above will be a limit for the optimal policy we identify. The seller's goal is to select a pricing policy to maximize the long-run average revenue, that is, it wants to solve

$$\sup_p \{H(p) : p \in \mathcal{P}^\infty\}. \tag{2.2}$$

As in Besbes and Lobel (2007), we say that a policy $p \in \mathcal{P}^\infty$ is *cyclic* if there exists a positive integer L such that $p_{t+L} = p_t$ for all $t \in \mathbb{N}$. The smallest $L > 0$ for which this holds is the *cycle length* of policy p. Note that a cyclic policy with cycle length L is of the form $p = (q, q, q, \dots)$ where $q \in \mathcal{P}^L$. A cyclic policy p is said be a *decreasing cyclic policy* if p is of the form $p = (q, q, q, \dots)$ where $q = (q_1, \dots, q_L)$ is such that $q_1 \geq q_2 \geq \cdots \geq q_L$. (We use decreasing to mean weakly decreasing.)

Our central result is the following.

Theorem 1. *There exists a decreasing cyclic policy with cycle length* $L \in \{1, \dots, m + k - 1\}$ *that is an optimal solution to* (5.7.1).

In the following two sections, we build up to the proof of the preceding theorem. In Section 2.5.2, under some additional mild assumptions, we obtain a lower bound on the cycle length.

2.3 Preliminary Analysis

A key step in the proof of the theorem above involves making a connection between the infinite-horizon problem and a certain finite-horizon problem. To this end, we need to introduce some definitions for finite-horizon problems. For $L \in \mathbb{N}$ and $p \in \mathcal{P}^L$ let $V_L(p)$ and $v_L(p)$ denote, respectively, the total revenue and average revenue accrued over horizon $1, \ldots, L$ under policy p; that is,

$$V_L(p) = \sum_{t=1}^{L} \rho_t(p) \qquad \text{and} \qquad v_L(p) = V_L(p)/L.$$

For $p \in \cup_{L \in \mathbb{N}} \mathcal{P}^L$, let $V(p) = V_{L(p)}(p)$ and $v(p) = v_{L(p)}(p)$. Throughout, we will show the "length subscript" on $V(\cdot)$ or $v(\cdot)$ when we wish to emphasize the length of the finite-horizon policy in question.

Next, we need some terminology. For $p \in \mathcal{P}^\infty$ we say that time $t \geq 2$ is an *I-regeneration point* of p if

$$\min\{p_t, p_{t+1}, \ldots, p_{t+k-1}\} \geq p_{t-1}.$$

For $L \in \mathbb{N}$ and $p \in \mathcal{P}^L$ we say that time $t \in \{2, \ldots, L\}$ is an *F-regeneration point* of p if

$$\min\{p_t, p_{t+1}, \ldots, p_{\min\{t+k-1, L\}}\} \geq p_{t-1}. \tag{2.3}$$

No customers who join the market prior to time t purchase in time t or later, because the prices from time t to time $t + k - 1$ are greater than or equal to the price in time $t - 1$. We have attached the modifiers "I-" and "F-" because it will help later to clearly distinguish between infinite price sequences and finite price sequences. Observe that by definition, time $t = 1$ is not a regeneration point.

For any policy $p \in \mathcal{P}^\infty$, we call the sequence of prices between any two successive I-regeneration points a *component* of p. Let $r(i)$ denote the ith

I-regeneration point, then $p_{r(i)}$ is the price at ith I-regeneration point. We refer to the sequence $C_i(p) := (p_{r(i)}, p_{r(i)+1}, \ldots, p_{r(i+1)-1})$ as the ith component of p for $i \geq 1$. By definition of $r(i)$ and $r(i+1)$, there are no I-regeneration points in time periods $r(i)+1, \ldots, r(i+1)-1$. For simplicity, we also define $r(0) = 1$ and refer to $(p_{r(0)}, \ldots, p_{r(1)-1})$ as $C_0(p)$. Both Ahn et al. (2007) and Besbes and Lobel (2007) use regeneration points and components in their work.

The following lemma provides a link between the infinite-horizon and finite-horizon problems.

Lemma 1. *Fix $L \in \mathbb{N}$, and consider $q \in \mathcal{P}^L$ and $p = (q, q, q, \ldots) \in \mathcal{P}^\infty$.*

1. *$C_i(p) = q$ for $i = 0, 1, 2, \ldots$ if and only if q has no F-regeneration points.*
2. *If q has no F-regeneration points, then $H(p) = v(q)$.*
3. *If q has no F-regeneration points, then p is cyclic with cycle length $L(q)$.*

The lemma is an important ingredient in the proof of the following proposition. Define $\kappa = k(m-1)+1$.

Proposition 1. *There exists a cyclic pricing policy with cycle length at most κ that is an optimal solution to* (5.7.1)*. That is, there exists $q^* = (q_1^*, \ldots, q_L^*)$ with $L \leq \kappa$ such that $p^* = (q^*, q^*, q^*, q^*, \ldots)$ achieves the supremum in* (5.7.1)*. Moreover, the cycles and components of p^* coincide; i.e., p^* is cyclic with cycle length L and $C_i(p^*) = q^*$ for $i = 0, 1, 2, \ldots$.*

By Proposition 1, any optimal solution to problem (2.4) below is also an optimal solution to the original optimization problem (5.7.1).

$$\max_p H(p) \tag{2.4}$$

$$s.t. \quad p = (q, q, q, \dots) \text{ and } C_i(p) = q, \text{ for } i = 0, 1, 2, \dots,$$
$$q = (q_1, q_2, \dots, q_L) \in \mathcal{P}^L \text{ and } L \le \kappa$$

For $L \ge 2$, let

$$\begin{aligned}\Omega(L) &= \{q = (q_1, \dots, q_L) \in \mathcal{P}^L : q_{t-1} > \min\{q_t, \dots, q_{\min\{t+k-1,L\}}\} \text{ for } t = 2, \dots, L\} \\ &= \{q \in \mathcal{P}^L\text{: } q \text{ has no F-regeneration points}\}\end{aligned}$$

For $L = 1$, we define $\Omega(1) = \mathcal{P}$. Let $\Omega = \cup_{L=1}^{\kappa}\Omega(L)$.

We will also be interested in the following finite-horizon optimization problem.

$$\max_q \Big\{ v(q) : q \in \Omega \Big\} \tag{2.5}$$

Proposition 2. *A policy* q *solves* (2.5) *if and only if* $p = (q, q, q, \dots)$ *solves* (2.4).

Combining Propositions 1 and 2, we see that to prove Theorem 1, it will suffice to establish that the optimization problem (2.5) has a decreasing optimal solution of length $L \le m + k - 1$. We take up this task below.

We close this section with a comment about the case of $k = 1$ in which patient customers are willing to wait one period. When $k = 1$ the definition (2.3) says that time $t \ge 2$ is an F-regeneration point of $q \in \mathcal{P}^L$ if $q_t \ge q_{t-1}$. Hence, the (finite) set Ω in (2.5) contains only strictly decreasing price sequences. In addition, $\kappa = m$ when $k = 1$. Therefore, the above propositions imply that a decreasing cyclic policy with cycle length $L \in \{1, \dots, m\}$ is optimal for (5.7.1). So at this point Theorem 1 is proved for $k = 1$ without any more work. It remains only to prove the theorem when $k \ge 2$. Hence, we shall restrict our attention throughout the next section to cases with $k \ge 2$.

2.4 Proof of the Theorem

In this section we establish that there exists an optimal solution to (2.5) that is decreasing, and then use this fact to prove Theorem 1. We begin by exploring the structure of policies in the set Ω.

For a policy $q \in \mathcal{P}^L$ we say that there is a *markup* at time $t \in \{2, \ldots, L\}$ if $q_t > q_{t-1}$.

Given $q \in \Omega(L)$, we let $E_1 = \{t \in \{2, \ldots, L\} : q_t > q_{t-1}\}$. If $E_1 \neq \emptyset$, let $e(1) = \min\{t \in E_1\}$. For $i \geq 2$, let

$$\begin{aligned} E_i = &\{t \in \{e(i-1)+1, \ldots, L\} : q_t > q_{t-1}, q_{t-1} = \min\{q_1, \ldots, q_{t-1}\}, \\ &\text{and } q_{t-1} < q_{e(i-1)-1}\}. \end{aligned}$$

If $E_i \neq \emptyset$, then let $e(i) = \min\{t \in E_i\}$. We say that time $e(i)$ is the time of the ith *strong markup* of q. At time $e(1)$, the price strictly increases from a running minimum in the previous period $e(1) - 1$. For $i > 1$, at time $e(i)$, the price strictly increases from a running minimum price in the previous period $e(i) - 1$ that is *strictly* less than the price in period $e(i-1) - 1$. From the definition, the time of the first strong markup $e(1)$ is also the time of the first markup. Note however that subsequent markups may not be strong markups. A policy has at least one strong markup if and only if it has at least one markup. The left panel of Figure 2.1 shows a policy with two strong markups (at times 3 and 10). In that figure, times 4 and 6 have markups, but not strong markups.

In a number of places throughout the remainder of the paper and in the appendix, we will make comparisons between a policy that we will call q and another policy, say $\widehat{q}$. In the interest of minimizing notational clutter, in those settings we will use $e(i)$ to mean the time of the ith strong markup of q. (This notation does not explicitly show that the time of the ith strong markup — if it exists — depends upon the policy in question.) Hence, $\widehat{q}_{e(i)}$ refers to the price charged by policy $\widehat{q}$ at the time of the ith strong markup of policy q. If we wish to refer to the time of ith strong markup of $\widehat{q}$, we will use the notation $e(i|\widehat{q})$. This convention will apply to other quantities that will be introduced later as well (e.g., $t(i)$ and $M(i)$).

Define $n(q) = \max\{i : E_i \neq \emptyset\}$ to be the number of strong markups of policy q. If $E_1 = \emptyset$, then we define $n(q) = 0$. For $q \in \Omega(L)$, observe that $n(q) = 0$ if and only if $q_1 \geq q_2 \geq \cdots \geq q_L$. Let

$$\begin{aligned}
B^n(L) &= \{q \in \Omega(L) : n(q) = n\} \qquad n = 1, \ldots, L-1 \\
&= \{q \in \Omega(L)\text{: } q \text{ has exactly } n \text{ strong markups}\} \\
B(L) &= \cup_{n=1}^{L-1} B^n(L) \\
&= \{q \in \Omega(L)\text{: } q \text{ has at least one markup}\} \\
D(L) &= \{q \in \Omega(L) : q_1 \geq q_2 \geq \cdots \geq q_L\} \\
&= \{q \in \Omega(L) : n(q) = 0\} \\
&= \{q \in \Omega(L)\text{: } q \text{ is decreasing}\}.
\end{aligned}$$

The left panel of Figure 2.1 shows a policy in a $B^n(L)$.

We note in passing that it is possible to have a decreasing policy $q = (q_1, \ldots, q_L)$ with F-regeneration points (if a price repeats for at least k consecutive periods or if the final two prices are the same); such a policy is not in $D(L)$.

From the proceeding definitions we have

$$\Omega(L) = B(L) \cup D(L). \tag{2.6}$$

Consequently, each policy in the feasible set Ω of problem (2.5) is either decreasing or in some $B^n(L)$.

For $n \geq 1$, define $E^n(L)$ as the set of sequences $q \in \mathcal{P}^L$ with the property that $q = (s_0, s_1, \ldots, s_n)$ where $s_i = (q_{t(i)}, \ldots, q_{t(i+1)-1})$ for some times $\{t(i) : i = 0, \ldots, n+1\}$ such that $1 = t(0) < t(1) < \cdots < t(n+1) = L+1$ and

1. s_i is decreasing ($q_{t(i)} \geq q_{t(i)+1} \geq \cdots \geq q_{t(i+1)-1}$) for $i = 0, \ldots, n$

2. $q_{t(i)-1} < q_{t(i)}$ for $i = 1, \ldots, n$

3. $\{t \in \{t(i)+1, \ldots, t(i+1)-1\} : q_t < q_{t(i)-1}\} \neq \emptyset$ for $i = 1, \ldots, n$

4. $\min\{t \in \{t(i)+1, \ldots, t(i+1)-1\} : q_t < q_{t(i)-1}\} \in \{t(i)+1, \ldots, t(i)+k-1\}$ for $i = 1, \ldots, n$.

Examples of policies in an $E^n(L)$ are shown in the right panel of Figure 2.1 and in the left panel of Figure 2.2. Our interest in $E^n(L)$ stems from the next proposition, which states that for any policy $q \in B^n(L)$ we can find a policy $\bar{q} \in E^n(L)$ that performs at least as well.

Proposition 3. *For any policy $q \in B^n(L)$ with $L \leq \kappa$, there exists a policy $\bar{q} \in E^n(L)$ such that $V_L(q) \leq V_L(\bar{q})$ and $v_L(q) \leq v_L(\bar{q})$.*

The policy $\bar{q}$ above is constructed from q by rearranging the prices $(q_{e(i)}, q_{e(i)+1}, \ldots, q_{e(i+1)-1})$ between each two consecutive strong markups $e(i)$ and $e(i+1)$ of q into a decreasing sequence and also rearranging the prices $(q_{e(n)}, q_{e(n)+1}, \ldots, q_L)$ after the last strong markup into a decreasing sequence. So $\bar{q}$ will consist of $n+1$ decreasing sequences. To get a rough idea why this works, fix $q \in B^n(L)$ and consider the string of prices $(q_{e(1)}, q_{e(1)+1}, \ldots, q_{e(2)-1})$

between $e(1)$ and $e(2)$. At least one of these prices is strictly less than $q_{e(1)-1}$ by the definition of $e(2)$. Let the time of the first such price in the string be called $e(1)+w+1$ (this naming convention is used to match the developments in the proof of the proposition). Also by the definition of $e(2)$, we must have $q_{e(1)+w+1} \geq q_{e(1)+w+2} \geq \cdots \geq q_{e(2)-1}$. Hence, to rearrange $(q_{e(1)}, q_{e(1)+1}, \ldots, q_{e(2)-1})$ into a decreasing sequence, we need only rearrange the prices $(q_{e(1)}, \ldots, q_{e(1)+w})$ into a decreasing sequence and leave the other prices unchanged. See Figure 2.1. After this rearrangement, revenues obtained in periods $1, \ldots, e(1)-1$ remain unchanged, as do revenues in periods $e(1)+w+2, \ldots, L$. However, the total revenue accrued in periods $e(1), \ldots, e(1)+w+1$ increases with the rearrangement. This increase can be attributed to customers who initially arrive in periods $e(1), \ldots, e(1)+w+1$ (but not to customers who initially arrive earlier). Complete details of the argument can be found in the appendix.

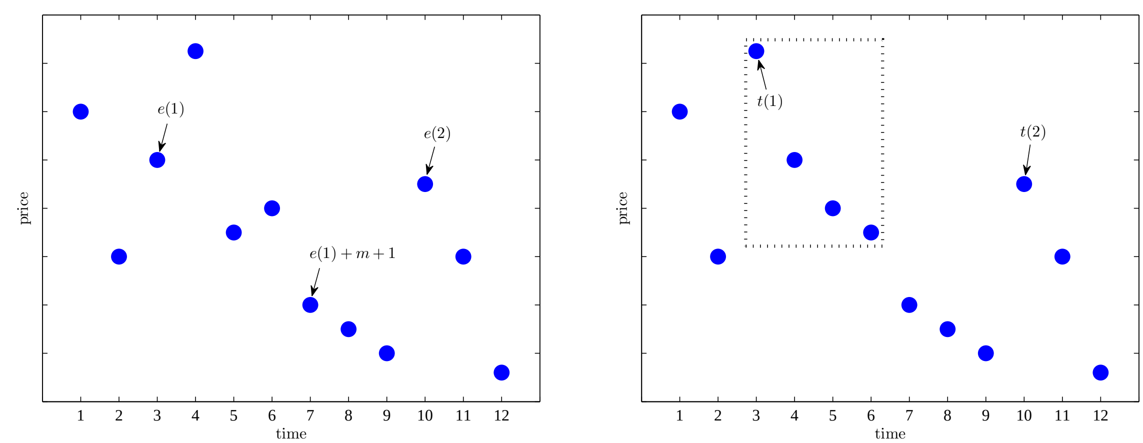


Figure 2.1: Step 2 illustration[1]

[1]Suppose $k \geq 5$. The left panel shows a policy in $B^n(L)$ with $n = 2$ and $L = 12$. (If $k \leq 4$ then

Define

$$B = \cup_{L=1}^{\kappa} B(L)$$

$$D = \cup_{L=1}^{\kappa} D(L)$$

$$E = \cup_{L=1}^{\kappa} \cup_{n=1}^{L} E^n(L)$$

Observe that B, D, and E are finite sets. With these definitions in hand, we are ready for the next ingredient in the proof of our main result.

Proposition 4. *Consider* $d \in \arg\max_{p \in D} v(p)$ *and* $q \in \arg\max_{p \in E} v(p)$. *Then* $v(q) \leq v(d)$.

The proof of the preceding proposition is quite long, and uses an argument by contradiction. This allows for a somewhat more concise presentation than does a direct approach. We do not wish the fact that we argue by contradiction to obscure the constructive underpinnings of the result, which are as follows. We know that $q \in \arg\max_{p \in E} v(p)$ is an element of some $E^n(L)$. Consider the price $q_{t(n)}$ that begins the final piece s_n of q at time $t(n) = t$. If we modify q by adding "enough" copies (say L') of $q_{t(n)}$ starting at time t and we push the prices $q_{t(n)+1}, \ldots, q_L$ later by L' periods, then we obtain a new policy of length $L + L'$ that has an F-regeneration point at time t and that also decreases from time t onward. See Figure 2.2. It turns out — although it is likely not apparent without going through the details of the proof — that the decreasing policy comprised

time $t = 3$ would be an F-regeneration point, and hence the policy would not be in $B^n(L)$.) The right panel shows the rearrangement of prices described above for periods $(e(1), \ldots, e(1)+w) = (3, \ldots, 6)$. Here, the rearrangement yields a policy $\bar{q}$ in $E^n(L)$. In general, we would need to make similar rearrangements between each pair $e(i)$ and $e(i+1)$ and after $e(n)$.

only of the prices of the modified policy from time t onward (p^2 in the figure) is at least as good as the original q.

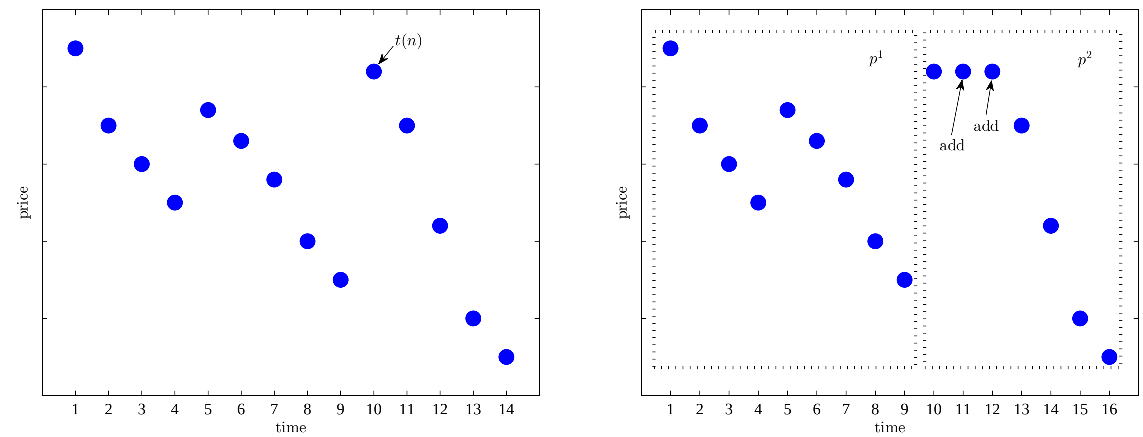


Figure 2.2: Step 3 illustration[2]

We now turn to the problem of maximizing $v(p)$ over $p \in D$. The next proposition states that for this problem, it suffices to consider only those sequences with length less than $k + m$. Of course, the longest possible *strictly* decreasing sequence of prices is m, but recall that "decreasing" should be interpreted as non-increasing, that is, the price can strictly decrease or remain the same as time moves forward (and so the result is not vacuous).

[2]Suppose $k = 5$. The proof of Proposition 4 considers a policy q that maximizes $v(\cdot)$ over E. The left panel shows a case in which this policy q is a element of $E^n(L)$ with $n = 2$ and $L = 14$. Inserting $L' = 2$ additional copies of $q_{t(n)}$ as shown in the right panel yields policy of length $L + L' = 16$ that has an F-regeneration point at time $t = 10$. The average revenue of the policy in the right panel is a convex combination of the average revenues of policies p^1 and p^2 shown there.

Proposition 5. *There exists* $d \in \arg\max\{v(p) : p \in D\}$ *such that* $L(d) \leq m + k - 1$.

The key to the proposition is the fact that no prices need repeat after period k in an optimal policy. Then, since there are m distinct prices in the price set $\mathcal{P}$, there is an optimal decreasing policy with length less than $m + k$. (In this discussion, "optimal" means optimal for the problem $\max\{v(p) : p \in D\}$.) To see why there is no benefit from repeating a price after period k, consider an $L \geq m+k$ and a policy q that maximizes $v(p)$ over $D(L)$. It must be that $q_j = q_{j+1}$ for some $j \geq k$. By removing q_{j+1} and shifting all later prices one period earlier, we arrive at a new policy q' which is in $D(L-1)$. Some careful thought reveals that the total revenue of q' is exactly $q_{j+1}[1 - F(q_{j+1})]$ less than that of q. This amount is no greater than $\max\{x[1 - F(x)] : x \in \mathcal{P}\}$, which is itself no greater than the average revenue of the sequence q because the average revenue of the best policy in $D(L)$ is at least the optimal revenue of a one-period problem (or else the optimal solution to the one-period problem is itself better than q). Therefore, by removing such price repetitions from q, the average revenue gets no worse. Complete details are in the appendix.

We are now ready to present the proof of Theorem 1.

Proof of Theorem 1. By (2.6), the feasible set Ω in problem (2.5) can be expressed as $\Omega = B \cup D$. By Propositions 3, 4, and 5 we have

$$\begin{aligned} \max\{v(p) : p \in B\} &\leq \max\{v(p) : p \in E\} \leq \max\{v(p) : p \in D\} \\ &= \max\{v(p) : p \in \cup_{L=1}^{m+k-1} D(L)\} \,. \end{aligned}$$

Therefore, for $d \in \arg\max\{v(p) : p \in \cup_{L=1}^{m+k-1} D(L)\}$ we have $d \in \arg\max\{v(p) : p \in \Omega\}$. Consequently, $(d, d, d, \dots) \in \mathcal{P}^\infty$ is an optimal solution to (5.7.1) by Propositions 1 and 2. $\square$

2.5 Computation, Refinements, and Extensions

In this section we develop a dynamic programming method for computing an optimal policy. The structure identified in Theorem 1 plays a crucial role in the development of the approach. We also present some additional results, including an extension of Theorem 1 to problems with continuous price sets.

2.5.1 A Computational Approach

As seen in the proof of Theorem 1, to find an optimal policy, it suffices to maximize $v(p)$ over $p \in \cup_{L=1}^{m+k-1} D(L)$. This can be accomplished by maximizing $v(p)$ over $p \in D(L)$ for each individual $L = 1, \dots, m+k-1$ and then selecting the best of those $m+k-1$ maximizers. The search space can be further reduced to include only $L = k+1, \dots, m+k-1$ when the assumption in Proposition 6 below holds.

Next we present a dynamic programming approach with a one-dimensional state variable. The dynamic program computes a decreasing sequence δ^L of length L that maximizes $v(p)$ over all decreasing sequences p of length L. The monotonicity of prices established in our main theorem is what allows us to use a one-dimensional state variable. Without such monotonicity, we would need a k-dimensional state vector, rendering the problem computationally intractable for moderate or large k. To this end, fix L and let $\pi_t(x)$ denote the optimal revenue from time t onward in a L-period problem given that we have used a decreasing policy in periods $i = 1, \ldots, t-1$ and the price in period $t-1$ was $x \in \mathcal{P}$. The algorithm is as follows.

Step 1: Let $\pi_{L+1}(x) = 0$ for each $x \in \mathcal{P}$.

Step 2: For $t = L, \ldots, 2$ recursively compute

$$\pi_t(x) = \max_{z: z \in \mathcal{P}, z \leq x} \left\{ z \Big[1 - F(z) + \alpha \min\{k, t-1\}[F(x) - F(z)] \Big] + \pi_{t+1}(z) \right\}$$

and let $z_t(x)$ denote an optimal solution to $\pi_t(x)$, i.e., $z_t(x) \in \arg\max \pi_t(x)$.

Step 3: Compute

$$\pi_1 = \max_{z: z \in \mathcal{P}} \left\{ z[1 - F(z)] + \pi_2(z) \right\}$$

and let z_1 denote an optimal solution to π_1, i.e., $z_1 \in \arg\max \pi_1$.

Step 4: The policy $\delta^L = (\delta_1^L, \delta_2^L, \ldots, \delta_L^L)$ given by $\delta_1^L = z_1$ and $\delta_t^L = z_t(\delta_{t-1}^L)$ for $t = 2, \ldots, L$ maximizes $V(p)$ — and hence also $v(p)$ — over the set of decreasing policies p of length L. In addition, $V(\delta^L) = \pi_1$ and $v(\delta^L) = \pi_1 / L$.

We note in closing that the decreasing policy δ^L produced by this dynamic programming algorithm may have F-regeneration points if the final two prices in δ^L are identical or if a price is repeated k or more times consecutively in δ^L. If either of these two cases occur, then there is an $L' < L$ and a policy $q \in D(L')$ with $v(q) \geq v(\delta^L)$. Hence, it poses no problem if $\delta^L \notin D(L)$.

2.5.2 A Lower Bound on Optimal Cycle Lengths

The following result provides conditions under which we can further restrict the space in which we search for an optimal policy. Let $q^j \in \arg\max\{V_j(q) : q \in$

$D(j)\} = \arg\max\{v_j(q) : q \in D(j)\}$ for $j = 1, \ldots, k+1$. For clarification, we note that the notation q^j is used differently in the proof of Proposition 4.

Proposition 6. *If $v_2(q^2) \geq v_1(q^1)$, then $v_1(q^1) \leq v_2(q^2) \leq \cdots \leq v_{k+1}(q^{k+1})$ and therefore, there exists an optimal solution $q \in D(L)$ to (2.5) with $L \geq k+1$. If $v_2(q^2) > v_1(q^1)$, then $v_1(q^1) < v_2(q^2) < \cdots < v_{k+1}(q^{k+1})$.*

Each of the following conditions (i) and (ii) is individually sufficient for $v_2(q^2) > v_1(q^1)$ to hold. For $v_2(q^2) \geq v_1(q^1)$ to hold, we can simply replace the strict inequalities by weak inequalities in the following.

(i) There exists some price $p \in \mathcal{P}$ such that $p > q^1$ and $v_2(p, q^1) > v_1(q^1)$. (Here, (p, q^1) is the policy that charges p in period 1 and q^1 in period 2.)

(ii) There exists some price $\check{p} \in \mathcal{P}$ such that $\check{p} < q^1$ and $v_2(q^1, \check{p}) > v_1(q^1)$.

Note also that it is not difficult to compute $v_2(q^2)$ and $v_1(q^1)$, and therefore it is easy to check directly whether it is true that $v_2(q^2) \geq v_1(q^1)$.

2.5.3 Two-Point Valuation Distributions

The papers by Conlisk et al. (1984) and Sobel (1991) consider intertemporal pricing problems in which each customer has one of two possible valuations. In this section we see what our results say about problems in which customers have just two possible valuations.

Suppose that the valuation distribution G is discrete and assigns mass at just two points, a and b with $0 < a < b$. Let θ be the fraction of customers that have the higher valuation b, and $1-\theta$ be the fraction of customers with lower valuation a. Then $G(x) = (1-\theta)1\{x \geq a\} + \theta 1\{x \geq b\}$ and $F(x) = (1-\theta)1\{x > a\} + \theta 1\{x > b\}$. It is apparent that the seller has no incentive to set a price other than a or b.

Therefore, with no loss of optimality we take the price set to be $\mathcal{P} = \{a, b\}$ so that $m = |\mathcal{P}| = 2$.

For $m = 2$, Theorem 1 establishes that there is a decreasing optimal solution to (2.5) with length $L \in \{1, \ldots, m + k - 1\} = \{1, \ldots, k + 1\}$. The proof of that theorem shows that it suffices to solve $\max\{v(d) : d \in \cup_{L=1}^{k+1} D(L)\}$. Proposition 6 shows that if $v_2(q^2) \geq v_1(q^1)$ then $q^{k+1} \in \arg\max\{v(d) : d \in D(k + 1)\}$ also maximizes $v(d)$ over $\cup_{L=1}^{k+1} D(L)$. That is, if $v_2(q^2) \geq v_1(q^1)$, then it suffices to find the best decreasing policy of length $k + 1$ with no F-regeneration points. When there are just two prices a, b, it turns out that there is only one policy in $D(k+1)$. That policy sets the high price b for k consecutive periods and then sets the low price a for one. Any other decreasing policy of length $k+1$ has an F-regeneration point and hence is not in $D(k + 1)$.

In view of the preceding observations, we immediately have the following result, where the assumptions on a, b, and θ in (2.7) state that $v_2(q^2) \geq v_1(q^1)$ as in Proposition 6.

Proposition 7. *Suppose a and b satisfy*

$$\frac{1}{2} [b\theta + a + \alpha(1 - \theta)a] \geq \max\{b\theta, a\} \tag{2.7}$$

Then an optimal solution to (5.7.1) *is $p^* = (q, q, q, \ldots)$ where $q = (q_1, \ldots, q_{k+1}) = (b, b, \ldots, b, b, a)$.*

Even if (2.7) does not hold, it is still particularly easy to solve $\max\{v(d) : d \in \cup_{L=1}^{k+1} D(L)\}$ because $D(1) = \mathcal{P} = \{a, b\}$ has only two elements and $D(j) = \{(b, \ldots, b, a)\}$ has only one element for $j = 2, \ldots, k+1$. Hence, just $k+2$ policies need to be evaluated.

The intuition behind Proposition 7 is as follows. In the first few periods of a cycle, the seller charges a high price and only high-valuation customers make purchases immediately. The seller continues charging a high price because new customers arrive each period. At the same time, the market is accumulating low-valuation customers who do not purchase at the high price. When enough low-valuation customers accumulate, the seller then charges the low price and all these customers purchase and leave as do the high-valuation customers that arrive that period.

The preceding proposition has some similarities to results of Conlisk et al. (1984) and Sobel (1991), who establish under some different assumptions that a cyclic decreasing policy is optimal for a setting with a two-point valuation distribution. (They do not consider other valuation distributions.) In contrast to our proposition, prices decline steadily during a cycle in these papers. However, as in our setting, only in the final period of a cycle do low-valuation customers buy. The steadily declining prices in their results can be attributed in part to their models of customer behavior wherein customers, all which are strategic, discount the value of future purchases and time their purchases to maximize their surplus.

2.6 Variable Patience Levels: Bounds and Heuristics

In this section we consider a problem involving customers with different patience levels. The model is the same as that described in Section 2.2, except we now assume that customers' patience levels range from 0 to $K < \infty$, and each level k

accounts for a fraction of α_k of customers where $\sum_{k=0}^{K} \alpha_k = 1$. A customer with a patience level k will wait up to k periods to make a purchase. So, a customer with patience level $k = 0$ is impatient in the sense described earlier. For $p \in \mathcal{P}^\infty$ let

$$\widetilde{\rho}_t(p) = p_t\Big[1 - F(p_t) + \sum_{k=1}^{K}\sum_{i=k}^{K} \alpha_i [F(\min\{p_{t-k}, \ldots, p_{t-1}\}) - F(p_t)]^+\Big] \qquad \text{for } t = 1, 2, \ldots \tag{2.8}$$

For $p \in \mathcal{P}^\infty$ let $\widetilde{H}(p) = \liminf_{T\to\infty} T^{-1} \sum_{t=1}^{T} \widetilde{\rho}_t(p)$. The seller now wants to solve

$$\widetilde{H} = \sup_p \{\widetilde{H}(p) : p \in \mathcal{P}^\infty\}. \tag{2.9}$$

Proposition 8. *There exists a cyclic pricing policy with cycle length at most* $K(m-1)+1$ *that is an optimal solution to* (2.9).

The proof of this result is identical to that of Proposition 1, and is omitted. It is seemingly a difficult task to identify additional structure of an optimal policy. It is also a difficult task even to compute an optimal policy. Note that the computational approach described in Section 2.5.1 relies on the fact that optimal cycles are decreasing, which need not be the case here. That approach can readily be modified to cover the variable patience levels now under consideration, but doing so yields a dynamic program with K-dimensional state variable. The main idea is to recursively compute $\widetilde{\pi}_t(x_{t-K}, \ldots, x_{t-1})$, the maximum revenue from time t onward in an L-period problem given that the prices in periods $t-K, \ldots, t-1$ are $x_{t-K}, \ldots, x_{t-1}$. This can be done via the relation

$$\widetilde{\pi}_t(x_{t-K}, \ldots, x_{t-1}) = \max_{z: z\in\mathcal{P}} \Big\{ z\Big[1 - F(z) + \sum_{k=1}^{K}\sum_{i=k}^{K} \alpha_i [F(\min\{x_{t-k}, \ldots, x_{t-1}\})$$

$$- F(z)]^+ \Big] + \widetilde{\pi}_{t+1}(x_{t-K+1}, \ldots, x_{t-1}, z) \Big\} \qquad (2.10)$$

with the convention that $x_j = 0$ for $j \leq 0$. Such dynamic programs are intractable except when K is small. Hence, we will confine ourselves to developing bounds and heuristics.

To do so, let $\rho_t^k(p)$ denote the revenue function defined in (2.1) with $\alpha = 1 - \alpha_0$. Likewise, let $H^k(p) = \liminf_{T\to\infty} T^{-1} \sum_{t=1}^T \rho_t^k(p)$ be the value of a policy p and let $H^k = \sup\{H^k(p) : p \in \mathcal{P}^\infty\}$ denote the optimal value in (5.7.1). A little algebra shows that

$$\widetilde{\rho}_t(p) = \sum_{k=1}^K \frac{\alpha_k}{1-\alpha_0} \rho_t^k(p).$$

Therefore, for the optimal cyclic policy $\widetilde{p}$ whose existence was established in Proposition 8, we have

$$\begin{aligned} \widetilde{H} = \widetilde{H}(\widetilde{p}) &= \liminf_{T\to\infty} \frac{1}{T} \sum_{t=1}^T \widetilde{\rho}_t(\widetilde{p}) = \lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^T \widetilde{\rho}_t(\widetilde{p}) = \lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^T \sum_{k=1}^K \frac{\alpha_k}{1-\alpha_0} \rho_t^k(\widetilde{p}) \\ &= \sum_{k=1}^K \frac{\alpha_k}{1-\alpha_0} \liminf_{T\to\infty} \frac{1}{T} \sum_{t=1}^T \rho_t^k(\widetilde{p}) \leq \sum_{k=1}^K \frac{\alpha_k}{1-\alpha_0} \sup_p \liminf_{T\to\infty} \frac{1}{T} \sum_{t=1}^T \rho_t^k(p). \end{aligned}$$

Hence, we have established the following result, which states that the optimal value in (2.9) for the problem with variable patience levels is bounded above by a convex combination of optimal objective values of problems with fixed patience levels.

Proposition 9. *The optimal value in* (2.9) *satisfies*

$$\widetilde{H} \leq \bar{H} := \sum_{k=1}^K \frac{\alpha_k}{1-\alpha_0} H^k.$$

We close this section by noting that it is straightforward to maximize $\widetilde{H}(\cdot)$ over decreasing cyclic policies using a slight variation of the algorithm from

Section 2.5.1. Specifically, in (2.10) we maximize over $\{z \in \mathcal{P} : z \leq x_{t-1}\}$ and we can drop all arguments except x_{t-1} from the function $\widetilde{\pi}(\cdot)$ because $\min\{x_{t-k}, \ldots, x_{t-1}\} = x_{t-1}$ for a decreasing policy. Such an optimization is tractable because the state variable is one-dimensional. The policy (say $\widetilde{q} = (\widetilde{d}, \widetilde{d}, \ldots)$ where $\widetilde{d}$ is decreasing) produced by this procedure will generally not be an optimal policy — even though it is the best decreasing cyclic policy — because in this setting of variable patience levels, it may be the case that the optimal policy does not have decreasing cycles (i.e., $\widetilde{H}(\widetilde{q}) < \widetilde{H}$). Nevertheless, this does provide a heuristic approach. We will consider this approach in some of the examples provided in the next section.

2.7 Numerical Experiments

In this section we present the results of some numerical experiments. In the first set of experiments, we consider settings with fixed patience levels and we suppose that valuations are drawn from the beta distribution with parameters (a, b). The density of the beta distribution is given by $f(x) = F'(x) = x^{a-1}(1-x)^{b-1}/B(a, b)$ for $x \in (0, 1)$ where $B(a, b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt$. Taking $a = b = 1$ yields the uniform distribution on $(0, 1)$. The mean and variance of the beta distribution with parameters (a, b) are given by $\mu = a/(a+b)$ and $\sigma^2 = (ab)/[(a+b)^2(a+b+1)]$.

In Table 2.1, we consider examples with $a = b$ so that the mean valuation is fixed at $\mu = 1/2$, the variance is $\sigma^2 = 1/(8a + 4)$, the coefficient of variation is $\text{cv} = \sigma/\mu = 1/\sqrt{2a + 1}$, and the valuation density is symmetric about $1/2$. We set $m = 10$ with prices evenly spaced on $(0, 1]$; that is, the price set is $\mathcal{P} = \{1/10, 2/10, \ldots, 9/10, 1\}$. The pairs of numbers in the table denote the optimal cycle length and the optimal average revenue for particular values of α, k, and a. For example, the optimal cycle length is 4 and optimal average revenue is 0.2736 for $\alpha = 0.5$, $k = 2$, and $a = b = 2$. The variance of the valuations decreases as a (and therefore b) increases. Hence, as we move to the right in the table, we get examples with lower variances and lower coefficients of variation. When $a = b > 1$ the mode (the maximum value of the density) occurs at 1/2, and as a gets bigger the distribution becomes more concentrated around 1/2. For $a = b < 1$ the density has a "U shape" and is minimized at 1/2 and grows without bound as $x \downarrow 0$ or $x \uparrow 1$. As a becomes smaller, the distribution becomes more concentrated just above 0 and just below 1.

As expected, the revenue increases in k and in α (it is simple to prove this). It is interesting to note that for fixed k and α the revenue is lowest when $a = 1$, which is the case of the uniform distribution. Revenue increases as a function of the variance of the valuation distribution for $a < 1$ and decreases in the variance when $a > 1$. This is perhaps counterintuitive at first because one might expect revenue to decrease as a function of the variance, given a fixed mean. This phenomenon can be explained as follows. Although the variance increases as a goes down to 0, the distribution becomes more concentrated in a sense as noted in the previous paragraph, and therefore one may view the "variability" to be low even though the variance is itself high (of course, variance is only a summary measure of variability). In the limit as $a \downarrow 0$, one may view the distribution as essentially placing mass 1/2 at both 0 and 1, and indeed it turns out that the revenue converges to 1/2 as $a \downarrow 0$. (Note that if the valuation distribution places mass 1/2 at both 0 and 1, then it is optimal to always price at 1, yielding a revenue of 1/2.)

As $a \uparrow \infty$, the valuation distribution becomes essentially a unit mass at 1/2. In that case the revenue also approaches 1/2. (Note that if the valuation distribution places mass 1 at 1/2, then it is optimal to always price at 1/2, yielding a revenue of 1/2.)

It is also interesting to note that the optimal cycle length is either 1 (fixed price), $k+1$, or $k+2$ for all the examples shown on Table 2.1. We see the same thing in Table 2.2 below as well. In fact, we have not been able to find any examples in which the optimal cycle length exceeds $k+2$. We conjecture that there are broad conditions under which there is an optimal policy with cycle length of either $k+1$ or $k+2$. Recall that for uniform valuations and continuous price set, Ahn et al. (2007) have previously established that there is an optimal policy with cycles of length 2 for problems with $k=1$. As further support for the conjecture, we have also proved through a different argument (not presented here) that for uniform valuations and continuous price set, there is an optimal policy with cycles of length 3 for problems with $k=2$. However, neither of these arguments appears to readily extend to more general settings. Therefore, it is an open problem to prove the conjecture.

In Table 2.2 we consider the same setup as in Table 2.1, but suppose that valuations follow the gamma distribution. The gamma distribution with parameters (n, λ) has density function $f(x) = F'(x) = \exp(-\lambda x)\lambda^n x^{n-1}/\Gamma(n)$ for $x > 0$ where $\Gamma(n)$ is the gamma function. We take $m = 20$ with prices evenly spaced on $(0, 5]$. The parameters (n, λ) for the table are selected so that the mean valuation is $\mu = n/\lambda = 1/2$ for all examples. The variance $\sigma^2 = n/\lambda^2$

---

[3] Optimal Cycle Lengths and Average Revenues for Beta-Distributed Valuations.

Table 2.1: For Beta-Distributed Valuations [3]

α	$a = b \longrightarrow$ $k \downarrow$	1/8 cv = 0.89	1/2 cv = 0.71	1 cv = 0.58	2 cv = 0.45	8 cv = 0.24
0.2	1	(1, 0.3484)	(2, 0.2639)	(1, 0.2500)	(2, 0.2605)	(1, 0.3148)
	2	(3, 0.3504)	(3, 0.2647)	(3, 0.2520)	(3, 0.2610)	(1, 0.3148)
	5	(6, 0.3526)	(6, 0.2682)	(6, 0.2550)	(6, 0.2638)	(6, 0.3180)
	10	(11, 0.3543)	(11, 0.2715)	(12, 0.2575)	(11, 0.2666)	(11, 0.3209)
0.5	1	(2, 0.3529)	(2, 0.2705)	(2, 0.2600)	(2, 0.2694)	(1, 0.3148)
	2	(3, 0.3567)	(3, 0.2786)	(3, 0.2667)	(4, 0.2736)	(3, 0.3211)
	5	(6, 0.3632)	(6, 0.2900)	(6, 0.2783)	(6, 0.2858)	(6, 0.3302)
	10	(11, 0.3692)	(11, 0.2995)	(11, 0.2864)	(11, 0.2923)	(11, 0.3343)
0.8	1	(2, 0.3577)	(2, 0.2840)	(2, 0.2730)	(2, 0.2826)	(2, 0.3283)
	2	(3, 0.3637)	(3, 0.2974)	(3, 0.2887)	(3, 0.2973)	(4, 0.3404)
	5	(6, 0.3750)	(6, 0.3209)	(6, 0.3143)	(6, 0.3206)	(6, 0.3604)
	10	(11, 0.3863)	(11, 0.3405)	(12, 0.3330)	(12, 0.3380)	(11, 0.3698)
1	1	(2, 0.3609)	(2, 0.2943)	(2, 0.2850)	(2, 0.2944)	(2, 0.3398)
	2	(3, 0.3700)	(3, 0.3159)	(3, 0.3067)	(3, 0.3163)	(4, 0.3621)
	5	(6, 0.3839)	(6, 0.3479)	(6, 0.3467)	(6, 0.3567)	(6, 0.3963)
	10	(11, 0.3981)	(11, 0.3751)	(12, 0.3750)	(11, 0.3842)	(11, 0.4164)

and coefficient of variation cv $= 1/\sqrt{n}$ of the valuations again decrease as we move rightward on the table. The column with $(n, \lambda) = (1, 2)$ corresponds to the exponential distribution with parameter $\lambda = 2$. In this case, we see that revenues increase monotonically as we move rightward. This differs from what we saw in Table 2.1, and can perhaps be explained by the fact that the gamma distributions are decreasing in the convex order as we move rightward on the table; i.e., $\int_x h(x)f(x)dx$ decreases as we move rightward for any convex $h(\cdot)$. This is a markedly stronger notion of decreasing variability than merely having decreasing variance with fixed mean; see, for example, Müller and Stoyan (2002). Hence, the results in Table 2.2 are consistent with intuition.

The next set of examples is shown in Table 2.3, where we consider variable

[4]Optimal Cycle Lengths and Average Revenues for Gamma-Distributed Valuations.

Table 2.2: For Gamma-Distributed Valuations [4]

α	$(n,\lambda) \longrightarrow$ $k \downarrow$	$(1/10, 1/5)$ cv = 3.16	$(1/4, 1/2)$ cv = 2.00	$(1/2, 1)$ cv = 1.41	$(1, 2)$ cv = 1.00	$(2, 4)$ cv = 0.71
0.2	1	(2, 0.1495)	(2, 0.1571)	(2, 0.1669)	(1, 0.1839)	(1, 0.2030)
	2	(3, 0.1517)	(3, 0.1593)	(3, 0.1701)	(1, 0.1839)	(3, 0.2076)
	5	(6, 0.1561)	(6, 0.1631)	(6, 0.1733)	(6, 0.1885)	(6, 0.2136)
	10	(11, 0.1602)	(11, 0.1673)	(11, 0.1770)	(11, 0.1919)	(11, 0.2163)
0.5	1	(2, 0.1592)	(2, 0.1665)	(2, 0.1780)	(2, 0.1937)	(2, 0.2141)
	2	(3, 0.1682)	(3, 0.1759)	(3, 0.1867)	(4, 0.2008)	(3, 0.2241)
	5	(7, 0.1848)	(6, 0.1923)	(7, 0.2020)	(6, 0.2169)	(6, 0.2342)
	10	(12, 0.1992)	(12, 0.2052)	(11, 0.2155)	(11, 0.2248)	(11, 0.2387)
0.8	1	(2, 0.1719)	(2, 0.1793)	(2, 0.1912)	(2, 0.2061)	(3, 0.2283)
	2	(3, 0.1889)	(3, 0.1981)	(3, 0.2094)	(4, 0.2255)	(3, 0.2503)
	5	(7, 0.2192)	(7, 0.2334)	(7, 0.2428)	(7, 0.2604)	(6, 0.2776)
	10	(12, 0.2436)	(12, 0.2622)	(12, 0.2727)	(11, 0.2828)	(11, 0.2900)
1	1	(2, 0.1813)	(3, 0.1894)	(2, 0.2013)	(2, 0.2178)	(3, 0.2407)
	2	(3, 0.2031)	(4, 0.2168)	(3, 0.2270)	(4, 0.2458)	(4, 0.2701)
	5	(7, 0.2423)	(7, 0.2654)	(7, 0.2768)	(7, 0.2950)	(6, 0.3173)
	10	(12, 0.2735)	(12, 0.3047)	(12, 0.3173)	(11, 0.3316)	(11, 0.3436)

patience levels as described in Section 2.6. We take $K = 2$ so that patient customers are willing to wait either 1 or 2 periods (and impatient customers will not wait). In the examples we assume that valuations are uniformly distributed on $(0, 1)$ and we assume that the fraction of customers that is impatient is $\alpha_0 = 0.2$. We take $m = 50$ prices evenly spaced on $(0, 1]$. For $K = 2$, we can use the recursion (2.10) to compute an optimal policy and the associated optimal revenue. Hence, these examples allow us to make some assessment of the quality of the upper bound $\bar{H}$ from Proposition 9 and also to evaluate the performance $\widetilde{H}(\widetilde{q})$ of the best decreasing cyclic policy $\widetilde{q}$ for problems with variable patience levels. The columns in the table correspond to different combinations of (α_1, α_2), which are the fractions of customers with patience levels 1 and 2. As we move rightward in the table, customers become less patient. In three of the seven examples, a decreasing cyclic policy turns out to be optimal. In the other two, the best decreasing policy achieves 99.9% of the optimal revenue. The upper bound on the optimal revenue is also quite tight. Overall, Table 2.3 suggests that the best decreasing policy works well for $K = 2$ when customer valuations are uniform. Is this still the case with larger K and different valuation distribution? We take this up next.

Table 2.3: Variable patience levels with $K = 2$

$(\alpha_1, \alpha_2) \longrightarrow$	(0.1,0.7)	(0.3,0.5)	(0.5,0.3)	(0.6,0.2)	(0.7,0.1)
optimal revenue $\widetilde{H}$	0.2869	0.2816	0.2769	0.2749	0.2745
upper bound $\bar{H}$	0.2878	0.2840	0.2801	0.2782	0.2763
best decreasing $\widetilde{H}(\widetilde{q})$	0.2869	0.2816	0.2769	0.2747	0.2744
ratio $\widetilde{H}(\widetilde{q})/\widetilde{H}$	1.00	1.00	1.00	0.9993	0.9996
ratio $\widetilde{H}(\widetilde{q})/\bar{H}$	0.9969	0.9915	0.9886	0.9874	0.9931

In Table 2.4, we consider variable patience levels but with larger values of K, namely $K = 4$ and $K = 10$. For these values, it is not practical to compute an optimal policy with (2.10), so we instead obtain the best decreasing cyclic policy and compare it against the upper bound $\bar{H}$ from Proposition 9. In these examples we suppose that valuations are drawn from the gamma distribution with parameters $n = \lambda = 1/2$ so that the mean valuation is $\mu = 1$ and the variance is $\sigma^2 = 2$. The table shows various different values of α_0. For each, we considered three different problem instances. In instance i, we take $\alpha_i = (1 - \alpha_0)/K$ for $i = 1, \ldots, K$, which means the fraction of customers with each patience level is the same. In instance ii, we take $\alpha_1 = \alpha_K = (1 - \alpha_0)/2$, which means half of the patient customers have patience level 1 and the other half have patience level K. In instance iii, we take $\alpha_{K/2} = \alpha_{K/2+1} = (1 - \alpha_0)/2$, which means half of the patient customers have patience level $K/2$ and the other half have patience level

$K/2+1$. Instances *i* and *ii* represent situations where the problem in question is dissimilar to a problem with fixed patience levels, whereas instance *iii* represents a case where the problem is "almost" a problem with a fixed patience level. Not surprisingly, the best decreasing policy does quite well in instance *iii*, attaining at least 99% of the optimal revenue in each case. However, in instances *i* and *ii* the best decreasing policy may not perform as well. For example, when $\alpha_0 = 0$ and $K = 10$, the best decreasing policy attains roughly 89% of the upper bound. Note that the comparison is made against the upper bound, so it is not possible to tell if the gap arises from a shortcoming of decreasing cyclic policies or because the upper bound is loose. The table suggests that the ratio $\widetilde{H}(\widetilde{q})/\bar{H}$ decreases as K gets bigger, which is not surprising. Even for $K = 10$, the best decreasing policy performs reasonably well in all instances.

Table 2.4: Variable patience levels with $K = 4$ and $K = 10$

		$K=4$			$K=10$		
α_0		i	ii	iii	i	ii	iii
0	upper bound $\bar{H}$	0.4718	0.4651	0.4785	0.5455	0.5155	0.5626
	best decreasing $\widetilde{H}(\widetilde{q})$	0.4559	0.4429	0.4753	0.5094	0.4591	0.5601
	ratio $\widetilde{H}(\widetilde{q})/\bar{H}$	0.9663	0.9523	0.9933	0.9338	0.8906	0.9956
0.2	upper bound $\bar{H}$	0.43	0.4257	0.4344	0.4836	0.4635	0.496
	best decreasing $\widetilde{H}(\widetilde{q})$	0.4184	0.4091	0.4327	0.4570	0.4210	0.494
	ratio $\widetilde{H}(\widetilde{q})/\bar{H}$	0.9730	0.9610	0.9961	0.9450	0.9083	0.9960
0.5	upper bound $\bar{H}$	0.3785	0.3766	0.3803	0.4035	0.3942	0.4087
	best decreasing $\widetilde{H}(\widetilde{q})$	0.3717	0.3674	0.3790	0.3894	0.3741	0.4077
	ratio $\widetilde{H}(\widetilde{q})/\bar{H}$	0.9820	0.9756	0.9966	0.9651	0.9490	0.9976
0.8	upper bound $\bar{H}$	0.3423	0.3418	0.3427	0.3487	0.3466	0.3499
	best decreasing $\widetilde{H}(\widetilde{q})$	0.3402	0.3389	0.3420	0.3447	0.3416	0.3497
	ratio $\widetilde{H}(\widetilde{q})/\bar{H}$	0.9939	0.9915	0.9980	0.9885	0.9856	0.9994

2.8 Appendix

2.8.1 Proof of Lemma 1 and Propositions 1 and 2

Lemma 1. *For any policy $p \in \mathcal{P}^\infty$, the length of any component of p is at most κ; that is $r(i+1) - r(i) \leq \kappa$ for all $i = 0, 1, 2, \ldots$.*

Proof. The proof is essentially identical to that of Lemma 1 of Besbes and Lobel (2007). Consider a component $(p_{r(i)}, \ldots, p_{r(i+1)-1})$ of a policy p, and suppose for a contradiction that the length of the component is greater than κ; i.e, $r(i+1) - r(i) > \kappa$.

Let $t_0 = r(i)$. There exists $t_1 \in \{t_0 + 1, \ldots, t_0 + k\}$ such that $p_{t_1} < p_{t_0}$. Otherwise $t_0 + 1 = r(i) + 1$ would be an I-regeneration point of p in which case $r(i+1) = t_0 + 1 = r(i) + 1$, which would contradict our supposition that $r(i+1) - r(i) > \kappa$. Note that $t_1 \leq t_0 + k$.

Likewise, there exists $t_2 \in \{t_1 + 1, \ldots, t_1 + k\}$ such that $p_{t_2} < p_{t_1}$. Otherwise $t_1 + 1$ would be an I-regeneration point of p in which case $r(i+1) = t_1 + 1 \leq t_0 + k + 1 = r(i) + k + 1$, which would contradict our supposition that $r(i+1) - r(i) > \kappa$. So there must exist a t_2 as claimed. Note that $t_2 \leq t_1 + k \leq t_0 + 2k$.

Continuing in this fashion for $j = 1, \ldots, m$, we see that there exist times $\{t_j\}$ with $t_j \in \{t_{j-1} + 1, \ldots, t_{j-1} + k\}$ such that $p_{t_j} < p_{t_{j-1}}$ and $t_j \leq t_{j-1} + k \leq t_0 + jk$.

We have now established that $p_{t_0} > p_{t_1} > \cdots > p_{t_m}$, which is not possible because the cardinality of the price set $\mathcal{P}$ is m. Hence, the supposition that $r(i+1) - r(i) > \kappa$ cannot hold, and thus $r(i+1) - r(i) \leq \kappa$. $\square$

Lemma 2. *Consider $p \in \mathcal{P}^\infty$ and suppose $C_i(p) = q \in \mathcal{P}^L$ for some $i \geq 0$. Then q*

has no F-regeneration points. That is, if $q \in \mathcal{P}^L$ is a component of policy $p \in \mathcal{P}^\infty$, then q has no F-regeneration points.

Proof. Suppose $q = C_i(p) = (p_{r(i)}, \ldots, p_{r(i+1)-1})$ for some i so that $L(q) = r(i+1) - r(i)$ and $q_j = p_{r(i)+j-1}$ for $j = 1, \ldots, r(i+1) - r(i)$. If $L(q) = 1$ the lemma is true. So we need only consider the case with $L(q) \geq 2$. Suppose for a contradiction that q has an F-regeneration point at some time $t \in \{2, \ldots, r(i+1) - r(i)\}$. Then $q_{t-1} \leq \min\{q_t, \ldots, q_{\min\{t+k-1, r(i+1)-r(i)\}}\}$, and so

$$p_{r(i)+t-2} \leq \min\{p_{r(i)+t-1}, \ldots, p_{\min\{r(i)+t+k-2, r(i+1)-1\}}\}$$

If $r(i) + t + k - 2 \leq r(i+1) - 1$, then $p_{r(i)+t-2} \leq \min\{p_{r(i)+t-1}, \ldots, p_{r(i)+t+k-2}\}$, hence $r(i) + t - 1 \in \{r(i) + 1, \ldots, r(i+1) - 1\}$ is an I-regeneration point of p, which contradicts the fact that q is a component of p. If $r(i) + t + k - 2 > r(i+1) - 1$, then $p_{r(i)+t-2} \leq \min\{p_{r(i)+t-1}, \ldots, p_{r(i+1)-1}\}$. Moreover, $p_{r(i+1)-1} \leq \min\{p_{r(i+1)}, \ldots, p_{r(i+1)+k-1}\}$ because $r(i+1)$ is an I-regeneration point of p. The preceding two inequalities imply that $r(i) + t - 1$ is an I-regeneration point of p, which is again a contradiction. □

Proof of Lemma 1. Fix $L \in \mathbb{N}$, $q = (q_1, \ldots, q_L) \in \mathcal{P}^L$, and $p = (q, q, q, \ldots) \in \mathcal{P}^\infty$.

Part 1a: We will show if q has no F-regeneration points, then $C_i(p) = q$ for $i = 0, 1, 2, \ldots$.

Suppose q has no F-regeneration points. Then for each $t = 2, 3, \ldots, L$, there exists some $s \in \{t, \ldots, \min\{t + k - 1, L\}\}$ with $q_{t-1} > q_s$. Now consider $p = (q, q, q, \ldots)$. For each $t = 2, \ldots, L$ and $i = 0, 1, 2, \ldots$, the time $u = t + iL$ is such that $p_{u-1} = p_{t+iL-1} = q_{t-1} > q_s = p_{s+iL}$ and

$$s + iL \in \{t + iL, \ldots, \min\{L, t + k - 1\} + iL\} \subset \{t + iL, \ldots, t + iL + k - 1\}$$
$$= \{u, \ldots, u + k - 1\}$$

So $p_{u-1} > p_{s+iL}$ and $s + iL \in \{u, \ldots, u + k - 1\}$. Hence, u is not an I-regeneration point of p, and consequently the only possible I-regeneration points of p are $\{1 + L, 1 + 2L, 1 + 3L, \ldots\}$.

Next observe that $q_L = \min\{q_1, \ldots, q_L\}$. (Otherwise, there would exist $n \in \{1, 2, \ldots, L - 1\}$ such that $q_n = \min\{q_1, \ldots, q_L\}$ in which case time $n + 1$ would be an F-regeneration point of q, contradicting the original assumption that q has no F-regeneration points.) Therefore, for each $i = 0, 1, 2, \ldots,$ we have $p_{L+iL} = q_L \leq \min\{p_{1+L+iL}, \ldots, p_{k+L+iL}\}$. Consequently $1 + L + iL$ for $i = 0, 1, 2, \ldots$ are I-regeneration points of p. This establishes that $C_i(p) = q$ for $i = 0, 1, 2, \ldots$.

Part 1b: If $C_i(p) = q$ for $i = 0, 1, 2, \ldots,$ then q has no F-regeneration points by Lemma 2. This completes the proof of part 1 of the lemma.

Part 2: Suppose q has no F-regeneration points. By part 1, we have $C_i(p) = q$. Let $n_T = \lfloor T/L \rfloor$. Thus

$$\begin{aligned} H(p) &= \liminf_{T\to\infty} \frac{1}{T}\Big\{ \sum_{j=1}^{n_T} \sum_{t=(j-1)L+1}^{jL} \rho_t(p) + \sum_{t=n_T L+1}^{T} \rho_t(p) \Big\} \\ &= \liminf_{T\to\infty} \frac{1}{T}\Big\{ n_T L v(q) + \sum_{t=n_T L+1}^{T} \rho_t(p) \Big\} \\ &= v(q) \end{aligned}$$

This completes the proof of part 2 of the lemma.

Part 3: Suppose q has no F-regeneration points. It is clear that $p = (q, q, q, \ldots)$ is cyclic. To show the cycle length is $L = L(q)$, it suffices to show there does not exist a sequence $\widehat{q} = (\widehat{q}_1, \ldots, \widehat{q}_l) \in \mathcal{P}^l$ and finite integer $n \geq 2$ such that $q = (\widehat{q}, \widehat{q}, \ldots, \widehat{q})$ and $L = nl$.

Suppose for a contradiction that there exists such a sequence $\widehat{q}$. Suppose $\widehat{q}_i = \min\{\widehat{q}_1, \ldots, \widehat{q}_l\}$. Thus $q_i = \widehat{q}_i = \min\{q_1, \ldots, q_L\}$. Therefore, we have $q_i \leq \min\{q_{i+1}, q_{i+2}, \ldots, q_L\}$, which implies that period $i+1$ is an F-regeneration point, contradicting the fact that q has no F-regeneration points. This completes the proof of part 3 of the lemma. □

Proof of Proposition 1. For policy $p \in \mathcal{P}^\infty$, define $N_T = \max\{i : r(i) \leq T\}$. Then

$$
\begin{aligned}
H(p) &= \liminf_{T\to\infty} \frac{1}{T}\Big\{ \sum_{j=1}^{N_T} \sum_{t=r(j-1)}^{r(j)-1} \rho_t(p) + \sum_{t=r(N_T)}^{T} \rho_t(p) \Big\} \\
&= \liminf_{T\to\infty} \Big\{ \sum_{j=1}^{N_T} \frac{r(j) - r(j-1)}{T} v(C_{j-1}(p)) + \frac{\sum_{t=r(N_T)}^{T} \rho_t(p)}{T} \Big\} \\
&\leq \limsup_{T\to\infty} \max\{v(C_0(p)), \ldots, v(C_{N_T-1}(p))\} + \limsup_{T\to\infty} \frac{\sum_{t=r(N_T)}^{T} \rho_t(p)}{T} \\
&= \limsup_{T\to\infty} \max\{v(C_0(p)), \ldots, v(C_{N_T-1}(p))\} \qquad (2.11) \\
&= \sup_i v(C_i(p)).
\end{aligned}
$$

The equality (2.11) is justified as follows. By Lemma 1, we know the length of each component of p is at most κ, thus $\sum_{t=r(N_T)}^{T} \rho_t(p)$ is the sum of at most κ terms, each of which is bounded. Consequently, $\limsup_{T\to\infty} \frac{1}{T} \sum_{t=r(N_T)}^{T} \rho_t(p) = 0$ and hence the equality holds.

Since the length of each component of p is at most κ and $\mathcal{P}$ is finite, there are finitely many distinct components. Consider $q = \arg\max_i v(C_i(p))$. By Lemma 2,

q has no F-regeneration points. By Lemma 1, the policy $p' = (q, q, q, \dots)$ built by repeating the component q in p with the largest average revenue is such that

$$H(p') = v(q) = \sup_i v(C_i(p)) \tag{2.12}$$

and $C_j(p') = q$ for $j = 0, 1, 2, \dots$.

Combining (2.11) and (2.12), we have that for any $p \in \mathcal{P}^\infty$, there exists p' for which $H(p) \le H(p')$, where p' is a cyclic policy with cycle length at most κ, and $C_j(p') = q$ for $j = 0, 1, 2, \dots$. Therefore, we can rewrite the problem (5.7.1) as

$$\begin{aligned} &\sup_p H(p) && (2.13)\\ s.t.\quad & p = (q, q, q, \dots) \text{ and } C_i(p) = q, \text{ for } i = 0, 1, 2, \dots, \\ & \text{where } q = (q_1, q_2, \dots, q_L) \text{ and } L \le \kappa \end{aligned}$$

The length of q is at most κ, so it follows that the set of q above is finite. Therefore, there exists a cyclic policy $p^* = (q^*, q^*, q^*, \dots)$ with $L(q^*) \le \kappa$ that achieves the supremum in (5.7.1). Moreover, the cycle length of p^* is $L(q^*)$ by part 3 of Lemma 1. We have also established that $C_i(p^*) = q^*$ for $i = 0, 1, 2, \dots$. $\square$

Proof of Proposition 2. Let $Z = \max_p H(p)$ in (2.4) and $Z' = \max_q v(q)$ in (2.5).

Consider $q = (q_1, \dots, q_L)$ with $L \le \kappa$ and $p = (q, q, q, \dots)$ with $C_i(p) = q$ for every i. Since $C_i(p) = q$, by Lemma 1, q has no F-regeneration points and $H(p) = v(q)$. Thus, $Z \le Z'$. Next consider $q = (q_1, \dots, q_L)$ with $L \le \kappa$ such that q has no F-regeneration points. Then for $p = (q, q, q, \dots)$, by Lemma 1, we have $C_i(p) = q$, and $H(p) = v(q)$. Thus, $Z' \le Z$. Therefore, $Z = Z'$, which establishes the equivalence of problems (2.4) and (2.5). $\square$

2.8.2 Proof of Proposition 3

We begin introducing a new definition. For $n \geq 1$, define $C^n(L)$ as the set of sequences $q \in \mathcal{P}^L$ with the property that $q = (s_0, s_1, \ldots, s_n)$ where $s_i = (q_{\tau(i)}, \ldots, q_{\tau(i+1)-1})$ for some times $\{\tau(i) : i = 0, \ldots, n+1\}$ such that $1 = \tau(0) < \tau(1) < \cdots < \tau(n+1) = L+1$ and

1. s_0 is decreasing ($q_1 \geq q_2 \geq \cdots \geq q_{\tau(1)-1}$)
2. $q_{\tau(i)-1} < q_{\tau(i)}$ for $i = 1, \ldots, n$
3. $\{t \in \{\tau(i)+1, \ldots, \tau(i+1)-1\} : q_t < q_{\tau(i)-1}\} \neq \emptyset$ for $i = 1, \ldots, n$
4. $\min\{t \in \{\tau(i)+1, \ldots, \tau(i+1)-1\} : q_t < q_{\tau(i)-1}\} \in \{\tau(i)+1, \ldots, \tau(i)+k-1\}$ for $i = 1, \ldots, n$.
5. $q_{\tau(i)+m(i)+1} \geq q_{\tau(i)+m(i)+2} \geq \cdots \geq q_{\tau(i+1)-1}$ where $m(i) = \min\big\{t \in \{\tau(i)+1, \ldots, \tau(i+1)-1\} : q_t < q_{\tau(i)-1}\big\} - \tau(i) - 1$ for $i = 1, \ldots, n$.

It is apparent that if $q \in E^n(L)$, then $q \in C^n(L)$ with $\{\tau(i)\} = \{t(i)\}$.

For $q \in C^n(L)$, time $\tau(i)+m(i)+1$ is the first time in $\{\tau(i)+1, \ldots, \tau(i+1)-1\}$ that the price goes below $q_{\tau(i)-1}$. Immediately after time $\tau(i)-1$, there are $m(i)+1$ consecutive prices greater than or equal to $q_{\tau(i)-1}$. The length of sequence s_i is $\tau(i+1)-\tau(i)$. There is at least one price lower than $q_{\tau(i)-1}$ in s_i by condition 3; therefore, we have $\tau(i+1)-\tau(i)-[m(i)+1] \geq 1$, thus $m(i) \leq \tau(i+1)-\tau(i)-2$. By condition 4, $\tau(i)+m(i)+1 \leq \tau(i)+k-1$, thus $m(i) \leq k-2$. Hence we have $m(i) \in \{0, \ldots, \min\{k-2, \tau(i+1)-\tau(i)-2\}\}$.

Observe also that for $q \in C^n(L)$, we have

$$\min\{q_{\tau(i)}, q_{\tau(i)+1}, \ldots, q_{\tau(i)+m(i)}\} \geq q_{\tau(i)-1} > q_{\tau(i)+m(i)+1} \geq q_{\tau(i)+m(i)+2}$$

$$\geq \cdots \geq q_{\tau(i+1)-1}. \tag{2.14}$$

Lemma 3. *Consider* $q = (q_1, q_2, \ldots, q_L)$*. If* $q \in B^n(L)$*, then* $q \in C^n(L)$*.*

Proof. Consider $q \in B^n(L)$. By the definition of $B^n(L)$, we know that q has no F-regeneration points and that q has exactly n strong markups: $2 \leq e(1) < e(2) < \cdots < e(n) \leq L$ such that $q_{e(i)} > q_{e(i)-1} = \min\{q_1, \ldots, q_{e(i)-1}\} < q_{e(i-1)-1}$ for $i = 2, \ldots, n$. Let $e(0) = 1$, $e(n+1) = L+1$, then $1 = e(0) < e(1) < \cdots < e(n+1) = L+1$. Thus sequence q can be written in the form of $q = (s_0, s_1, \ldots, s_n)$ where $s_i = (q_{e(i)}, \ldots, q_{e(i+1)-1})$ for $i = 0, \ldots, n$.

Next we will show that conditions 1–5 in the definition of $C^n(L)$ are satisfied by taking $\tau(i) = e(i)$ for $i = 0, \ldots, n+1$.

Consider sequence s_0. We have $e(1) \geq 2$. Also we have $q_1 \geq q_2 \geq \cdots \geq q_{e(1)-1}$, otherwise there would be a strong markup at some time $t < e(1)$, which would contradict the definition of $e(1)$ as the time of the first strong markup of q. Therefore, condition 1 holds.

Next consider sequence $s_i = (q_{e(i)}, q_{e(i)+1}, \ldots, q_{e(i+1)-1})$ for $i = 1, \ldots, n$. Observe that $q_{e(i)-1} < q_{e(i)}$, so condition 2 holds. In sequence s_i, there must exist at least one price strictly lower than $q_{e(i)-1}$. (For $i = 1, \ldots, n-1$, otherwise $e(i+1)$ would not be a strong markup of q. For $i = n$, otherwise $e(i)$ would be an F-regeneration point.) This establishes condition 3. Suppose $e(i) + w(i) + 1$ is the first time that the price is strictly lower than $q_{e(i)-1}$, then we have

$$\min\{q_{e(i)}, \ldots, q_{e(i)+w(i)}\} \geq q_{e(i)-1},$$

which means that beginning at time $e(i)$, there are $w(i) + 1$ consecutive prices greater than or equal to $q_{e(i)-1}$. We must have that $w(i) + 1 \leq k - 1$, because otherwise $e(i)$ would be an F-regeneration point (and we know q has no F-regeneration points). Hence, condition 4 holds.

Finally, if there are multiple prices strictly lower than $q_{e(i)-1}$ in s_i, we must have $q_{e(i)+w(i)+1} \geq q_{e(i)+w(i)+2} \geq \cdots \geq q_{e(i+1)-1}$, otherwise there would be a strong markup at some time $t \in \{e(i)+w(i)+2, \ldots, e(i+1)-1\}$, which would contradict the definition of $e(i+1)$ as the time of the $(i+1)$th strong markup of q. Thus condition 5 holds.

By taking $\tau(i) = e(i)$ and $m(i) = w(i)$, we see that $q \in C^n(L)$. □

Lemma 4. *For any policy $y \in \mathcal{P}^L$ with $L \leq k+1$, consider the policy $\widetilde{y}$ constructed by rearranging the prices in y from largest to smallest. Then $V(y) \leq V(\widetilde{y})$ and $v(y) \leq v(\widetilde{y})$. Therefore, for an L-period problem with $L \leq k+1$, there exists an optimal policy that is decreasing.*

Proof. Consider $\widetilde{y} = (\widetilde{y}_1, \ldots, \widetilde{y}_L) = (y_{i(1)}, \ldots, y_{i(L)})$ obtained by re-arranging the prices in $y = (y_1, \ldots, y_L)$ from the largest to smallest. Hence, $i(t)$ denotes the time at which price $\widetilde{y}_t$ (the tth largest price in y) appears in y. If a price (say p) appears more than once in y (say at times $i_1 < i_2 < \cdots < i_n$) then for some t we have $\widetilde{y}_t = \widetilde{y}_{t+1} = \cdots = \widetilde{y}_{t+n-1} = p$ and we take $i(t) = i_1, i(t+1) = i_2, \ldots, i(t+n-1) = i_n$.

For each $i = 1, \ldots, L$, define

$$\varphi_i^L(y) = y_i[1 - F(y_i)] + \alpha \sum_{j=i+1}^{\min\{L,i+k\}} y_j[F(\min\{y_i, \ldots, y_{j-1}\}) - F(y_j)]^+ . \quad (2.15)$$

The quantity $\varphi_i^L(y)$ is the revenue obtained from customers who initially arrive in period i under policy y. Hereafter, we drop the superscript. When $L \leq k+1$, the formula (2.15) reduces to

$$\varphi_i(y) = y_i[1 - F(y_i)] + \alpha \sum_{j=i+1}^{L} y_j[F(\min\{y_i, \ldots, y_{j-1}\}) - F(y_j)]^+ . \tag{2.16}$$

Therefore,

$$\begin{aligned}\sum_{i=1}^{L} \varphi_i(y) &= \sum_{i=1}^{L} \Big\{ y_i[1 - F(y_i)] + \alpha \sum_{j=i+1}^{L} y_j[F(\min\{y_i, \ldots, y_{j-1}\}) - F(y_j)]^+ \Big\} \\ &= \sum_{j=1}^{L} y_j[1 - F(y_j)] + \alpha \sum_{j=1}^{L} \sum_{i=1}^{j-1} y_j[F(\min\{y_i, \ldots, y_{j-1}\}) - F(y_j)]^+ \\ &= \sum_{j=1}^{L} \rho_j(y) = V(y) .\end{aligned}$$

Consequently, $V(y) = \sum_{i=1}^{L} \varphi_i(y) = \sum_{t=1}^{L} \varphi_{i(t)}(y)$. We also have $V(\widetilde{y}) = \sum_{t=1}^{L} \varphi_t(\widetilde{y})$ where

$$\begin{aligned}\varphi_t(\widetilde{y}) &= \widetilde{y}_t[1 - F(\widetilde{y}_t)] + \alpha \sum_{s=t+1}^{L} \widetilde{y}_s[F(\min\{\widetilde{y}_t, \ldots, \widetilde{y}_{s-1}\}) - F(\widetilde{y}_s)]^+ \\ &= \widetilde{y}_t[1 - F(\widetilde{y}_t)] + \alpha \sum_{s=t+1}^{L} \widetilde{y}_s[F(\widetilde{y}_{s-1}) - F(\widetilde{y}_s)] .\end{aligned} \tag{2.17}$$

The final equality above holds because $\widetilde{y}$ is decreasing.

Hence, to complete the proof, it suffices to show that $\varphi_{i(t)}(y) \le \varphi_t(\widetilde{y})$ for $t = 1, \ldots, L$. To this end, fix $t \in \{1, \ldots, L\}$ and consider (2.16) with $i = i(t)$:

$$\varphi_{i(t)}(y) = y_{i(t)}[1 - F(y_{i(t)})] + \alpha \sum_{j=i(t)+1}^{L} y_j[F(\min\{y_{i(t)}, \ldots, y_{j-1}\}) - F(y_j)]^+ . \tag{2.18}$$

Notice that the jth term in the summation in (2.18) is non-zero only if $y_j < \min\{y_{i(t)}, \ldots, y_{j-1}\}$, that is only if y_j is the strict minimum of $\{y_{i(t)}, \ldots, y_j\}$.

(Figure 2.3 depicts an example with the following bookkeeping.) Let $j(1) < j(2) < \cdots < j(N)$ denote those times $j \in \{i(t)+1, \ldots, L\}$ at which $y_j < \min\{y_{i(t)}, \ldots, y_{j-1}\}$. By (2.18) we have

$$\varphi_{i(t)}(y) = y_{i(t)}[1 - F(y_{i(t)})] + \alpha \sum_{\ell=1}^{N} y_{j(\ell)}[F(y_{j(\ell-1)}) - F(y_{j(\ell)})] \tag{2.19}$$

where we take $j(0) = i(t)$ so $y_{j(0)} = y_{i(t)} = \widetilde{y}_t$.

Recall that $\widetilde{y}$ contains the prices of y rearranged in decreasing order. Therefore, $\{y_{j(1)}, \ldots, y_{j(N)}\} \subset \{\widetilde{y}_{t+1}, \ldots, \widetilde{y}_L\}$. That is, $y_{j(\ell)} = \widetilde{y}_{z(\ell)}$ for some $\{z(\ell)\}$ with $t+1 \le z(1) < z(2) < \cdots < z(N) \le L$. Let $z(0) = t$. Hence, for the ℓth term in the sum in (2.19), we have

$$\begin{aligned} y_{j(\ell)}[F(y_{j(\ell-1)}) - F(y_{j(\ell)})] &= \widetilde{y}_{z(\ell)}[F(\widetilde{y}_{z(\ell-1)}) - F(\widetilde{y}_{z(\ell)})] \\ &= \widetilde{y}_{z(\ell)} \sum_{s=z(\ell-1)+1}^{z(\ell)} [F(\widetilde{y}_{s-1}) - F(\widetilde{y}_s)] \\ &\le \sum_{s=z(\ell-1)+1}^{z(\ell)} \widetilde{y}_s[F(\widetilde{y}_{s-1}) - F(\widetilde{y}_s)] \end{aligned}$$

where the inequality holds because $\widetilde{y}$ is decreasing.

Plugging back into (2.19) and then using (2.17) yields

$$\varphi_{i(t)}(y) \le y_{i(t)}[1 - F(y_{i(t)})] + \alpha \sum_{\ell=1}^{N} \sum_{s=z(\ell-1)+1}^{z(\ell)} \widetilde{y}_s[F(\widetilde{y}_{s-1}) - F(\widetilde{y}_s)]$$

[5]The left panel shows a policy y and the right panel shows policy $\widetilde{y}$ obtained by rearranging the elements of y from largest to smallest. In this example, for $t = 1$, we have $i(1) = 2$ and $N = 2$ with $j(1) = 3$, $j(2) = 5$, $z(1) = 4$, $z(2) = 5$. For $t = 2$, we have $i(2) = 1$ and $N = 2$ with $j(1) = 3$, $j(2) = 5$, $z(1) = 4$, $z(2) = 5$. For $t = 3$, we have $i(3) = 4$ and $N = 1$ with $j(1) = 5$ and $z(1) = 5$. For $t = 4$, we have $i(4) = 3$ and $N = 1$ with $j(1) = 5$ and $z(1) = 5$. For $t = 5$, we have $i(5) = 5$ and the summation in (2.18) is empty.

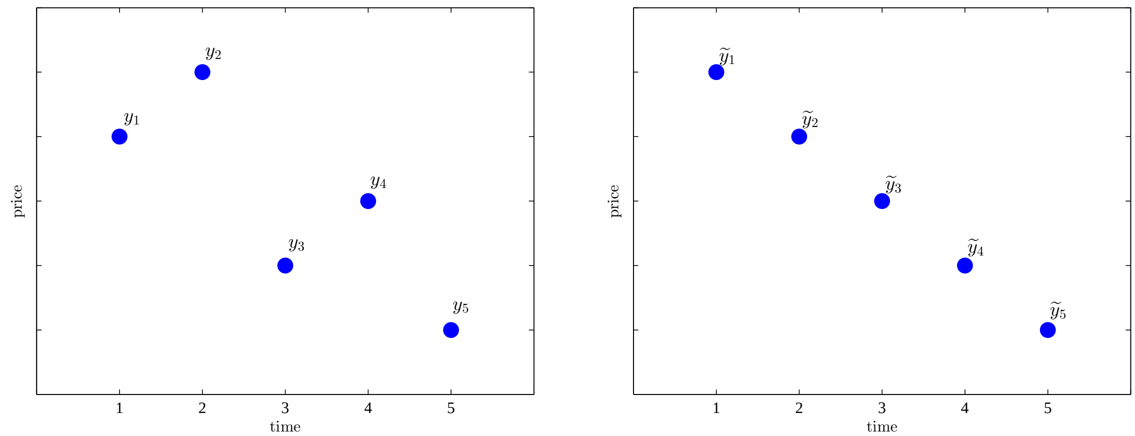


Figure 2.3: Rearrange the prices [5]

$$
\begin{aligned}
&= \widetilde{y}_t[1 - F(\widetilde{y}_t)] + \alpha \sum_{s=t+1}^{z(N)} \widetilde{y}_s[F(\widetilde{y}_{s-1}) - F(\widetilde{y}_s)] \\
&\le \widetilde{y}_t[1 - F(\widetilde{y}_t)] + \alpha \sum_{s=t+1}^{L} \widetilde{y}_s[F(\widetilde{y}_{s-1}) - F(\widetilde{y}_s)] = \varphi_t(\widetilde{y})\,.
\end{aligned}
$$

This completes the proof. □

Proof of Proposition 3. Consider a sequence $q \in B^n(L)$. By Lemma 3, $q \in C^n(L)$. Then $q = (s_0, s_1, \ldots, s_n)$ where $s_i = (q_{\tau(i)}, q_{\tau(i)+1}, \ldots, q_{\tau(i+1)-1})$ for $i = 0, \ldots, n$.

In subsequence s_1, we know by (2.14) that there exists $m(1) \in \{0, \ldots, \min\{k-2, \tau(2)-\tau(1)-2\}\}$ such that

$$
\min\{q_{\tau(1)}, q_{\tau(1)+1}, \ldots, q_{\tau(1)+m(1)}\} \ge q_{\tau(1)-1} > q_{\tau(1)+m(1)+1} \ge \cdots \ge q_{\tau(2)-1}.
$$

Let $(q_{i(1)}, q_{i(2)}, \ldots, q_{i(m(1)+1)})$ be the re-arrangement of $q_{\tau(1)}, \ldots, q_{\tau(1)+m(1)}$ from largest to smallest, and consider another sequence $\widetilde{q}$ such that

$$\widetilde{q}_t = \begin{cases} q_t & \text{if} \quad t \leq \tau(1) - 1 \\ q_{i(1)} & \text{if} \quad t = \tau(1) \\ \vdots & \\ q_{i(m(1)+1)} & \text{if} \quad t = \tau(1) + m(1) \\ q_t & \text{if} \quad t \geq \tau(1) + m(1) + 1 \end{cases}$$

Observe that $\widetilde{q}_{\tau(1)} \geq \widetilde{q}_{\tau(1)+1} \geq \cdots \geq \widetilde{q}_{\tau(1)+m(1)} \geq \widetilde{q}_{\tau(1)-1} > \widetilde{q}_{\tau(1)+m(1)+1} \geq \cdots \geq \widetilde{q}_{\tau(2)-1}$. The difference between sequences q and $\widetilde{q}$ is that subsequence s_1 in q is replaced by a reordered decreasing sequence in $\widetilde{q}$. Let $\bar{s}_1 = (\widetilde{q}_{\tau(1)}, \ldots, \widetilde{q}_{\tau(1)+m(1)}, \ldots, \widetilde{q}_{\tau(2)-1})$. Then $\widetilde{q} = (s_0, \bar{s}_1, s_2, \ldots, s_n)$.

Next we will show $V_L(q) \leq V_L(\widetilde{q})$.

Since $\widetilde{q}_t = q_t$ for $t \leq \tau(1) - 1$, we have

$$\rho_t(\widetilde{q}) = \rho_t(q) \quad \text{for} \quad t \leq \tau(1) - 1 \tag{2.20}$$

Since s_0 is decreasing, we have

$$\begin{aligned} q_{\tau(1)+m(1)+1} &= \min\{q_1, \ldots, q_{\tau(1)+m(1)+1}\} \\ &= \widetilde{q}_{\tau(1)+m(1)+1} = \min\{\widetilde{q}_1, \ldots, \widetilde{q}_{\tau(1)+m(1)+1}\} \end{aligned} \tag{2.21}$$

By (2.21) and the fact that $\widetilde{q}_t = q_t$ for $t \geq \tau(1) + m(1) + 1$, we have

$$\rho_t(\widetilde{q}) = \rho_t(q) \quad \text{for} \quad t \geq \tau(1) + m(1) + 2 \tag{2.22}$$

Let $y = (q_{\tau(1)}, q_{\tau(1)+1}, \ldots, q_{\tau(1)+m(1)}, q_{\tau(1)+m(1)+1})$, $\widetilde{y} = (\widetilde{q}_{\tau(1)}, \widetilde{q}_{\tau(1)+1}, \ldots, \widetilde{q}_{\tau(1)+m(1)}, \widetilde{q}_{\tau(1)+m(1)+1})$, and recall that $\widetilde{q}_{\tau(1)+m(1)+1} = q_{\tau(1)+m(1)+1}$.

Consider policy q. Since $\min\{q_{\tau(1)}, q_{\tau(1)+1}, \ldots, q_{\tau(1)+m(1)}\} \geq q_{\tau(1)-1}$, and $q_{\tau(1)+m(1)+1} < q_{\tau(1)-1}$, the customers who initially arrive before period $\tau(1)$ will not buy anything in periods $\tau(1), \tau(1)+1, \ldots, \tau(1)+m(1)$ and then some of them will purchase in period $\tau(1)+m(1)+1$. We use X to denote the revenue accrued in periods $\tau(1), \ldots, \tau(1)+m(1)+1$ from sales to customers who arrived in time $\tau(1)-1$ or earlier. By the preceding comment, such revenues are received only in period $\tau(1)+m(1)+1$. Hence

$$\sum_{t=\tau(1)}^{\tau(1)+m(1)+1} \rho_t(q) = V_{L(y)}(y) + X$$

where $L(y) = m(1)+2$ and

$$X = \alpha q_{\tau(1)+m(1)+1} \sum_{i=m(1)+2}^{k} \Big[F(\min\{q_{\tau(1)+m(1)+1-i}, \ldots, q_{\tau(1)-2}, q_{\tau(1)-1}\}) - F(q_{\tau(1)+m(1)+1})\Big]^{+}$$

Similarly,

$$\sum_{t=\tau(1)}^{\tau(1)+m(1)+1} \rho_t(\widetilde{q}) = V_{L(\widetilde{y})}(\widetilde{y}) + \widetilde{X}$$

where $L(\widetilde{y}) = m(1)+2$ and

$$\widetilde{X} = \alpha \widetilde{q}_{\tau(1)+m(1)+1} \sum_{i=m(1)+2}^{k} \Big[F(\min\{\widetilde{q}_{\tau(1)+m(1)+1-i}, \ldots, \widetilde{q}_{\tau(1)-2}, \widetilde{q}_{\tau(1)-1}\}) - F(\widetilde{q}_{\tau(1)+m(1)+1})\Big]^{+}$$

By the definition of $\widetilde{q}$, it is easy to see $X = \widetilde{X}$. Lemma 4 implies $V_{L(y)}(y) \leq V_{L(\widetilde{y})}(\widetilde{y})$ because $L(y) = L(\widetilde{y}) = m(1)+2 \leq k$. Thus

$$\sum_{t=\tau(1)}^{\tau(1)+m(1)+1} \rho_t(q) \leq \sum_{t=\tau(1)}^{\tau(1)+m(1)+1} \rho_t(\widetilde{q}) \tag{2.23}$$

Therefore, by (2.20), (2.22), and (2.23) it follows that $V_L(q) \leq V_L(\widetilde{q})$.

Continuing in this fashion for subsequences $s_2, s_3, \ldots, s_n$, we obtain a sequence $\bar{q} = (s_0, \bar{s}_1, \ldots, \bar{s}_n) \in E^n(L)$ such that $V_L(q) \leq V_L(\bar{q})$. □

2.8.3 Proof of Proposition 4

Lemma 5. *For any policy $y = (y_1, \ldots, y_{M+1}) \in \mathcal{P}^{M+1}$ with $y_1 \geq y_2 \geq \cdots \geq y_{M+1}$ and $M \leq k$, consider the policy $\check{y}$ defined by $\check{y}_t = y_{t+1}$ for $t = 1, \ldots, M$. Then*

$$V_{M+1}(y) - V_M(\check{y}) \;=\; y_1[1 - F(y_1)] + \alpha \sum_{t=1}^{M} y_{t+1}[F(y_t) - F(y_{t+1})] \;=\; \varphi_1^{M+1}(y),$$

where $\varphi_1^{M+1}(y)$ is defined in (2.15).

Proof. Both y and $\check{y}$ are decreasing, so by the definition of $\rho_t(\cdot)$,

$$\begin{aligned} \rho_{t+1}(y) &= y_{t+1}\big\{1 - F(y_{t+1}) + \alpha t[F(y_t) - F(y_{t+1})]\big\} \\ \rho_t(\check{y}) &= \check{y}_t\big\{1 - F(\check{y}_t) + \alpha(t-1)[F(\check{y}_{t-1}) - F(\check{y}_t)]\big\} \\ &= y_{t+1}\big\{1 - F(y_{t+1}) + \alpha(t-1)[F(y_t) - F(y_{t+1})]\big\} \end{aligned}$$

for $t = 1, \ldots, M$. Thus $\rho_{t+1}(y) - \rho_t(\check{y}) = \alpha y_{t+1}[F(y_t) - F(y_{t+1})]$ for $t = 1, \ldots, M$. Therefore,

$$\begin{aligned} V_{M+1}(y) - V_M(\check{y}) &= \rho_1(y) + \sum_{t=1}^{M} \big[\rho_{t+1}(y) - \rho_t(\check{y})\big] \\ &= y_1[1 - F(y_1)] + \alpha \sum_{t=1}^{M} y_{t+1}[F(y_t) - F(y_{t+1})] \end{aligned}$$

□

Preparation for the Proof of Proposition 4. Consider an arbitrary $q \in E^n(L)$ and define

$$M(i) = \min\{t \in \{t(i)+1, \ldots, t(i+1)-1\} : q_t < q_{t(i)-1}\} - t(i) - 1 \text{ for } i = 1, \ldots, n.$$

The time $t(i) + M(i) + 1$ is the first time in $\{t(i) + 1, \ldots, t(i + 1) - 1\}$ that the price drops below $q_{t(i)-1}$. Immediately after time $t(i) - 1$, there are $M(i) + 1$ consecutive prices greater than or equal to $q_{t(i)-1}$. The length of sequence s_i is $t(i+1) - t(i)$, and there is at least one price lower than $q_{t(i)-1}$ in s_i by condition 3 in the definition of $E^n(L)$; therefore, we have $t(i+1) - t(i) - [M(i)+1] \geq 1$. Thus $M(i) \leq t(i + 1) - t(i) - 2$. By condition 4 in the definition, $t(i) + M(i) + 1 \leq t(i) + k - 1$, and thus $M(i) \leq k - 2$. Hence,

$$M(i) \in \{0, \ldots, \min\{k - 2, t(i + 1) - t(i) - 2\}\}. \tag{2.24}$$

Moreover,

$$\begin{aligned} q_{t(i)} \geq q_{t(i)+1} \geq \cdots \geq q_{t(i)+M(i)} \geq q_{t(i)-1} > q_{t(i)+M(i)+1} \geq q_{t(i)+M(i)+2} \\ \geq \cdots \geq q_{t(i+1)-1}. \end{aligned}$$

For policy $q \in E^n(L)$, consider another policy q' with length $L - 1$ as follows.

$$q'_j = \begin{cases} q_j & \text{for} \quad 1 \leq j \leq t(n) - 1 \\ q_{j+1} & \text{for} \quad t(n) \leq j \leq L - 1 \end{cases} \tag{2.25}$$

Intuitively, q' is constructed from q by removing price $q_{t(n)}$ from q and shifting all prices originally in periods $t(n) + 1, \ldots, L$ one period earlier. We also consider another policy $q'' = \psi(q)$ with length $L + 1$ as follows.

$$q''_j = \begin{cases} q_j & \text{for} \quad 1 \leq j \leq t(n) \\ q_{j-1} & \text{for} \quad t(n) + 1 \leq j \leq L + 1 \end{cases} \tag{2.26}$$

If we want to emphasize the dependence of q'' on q, we will use the notation $\psi(q)$ to denote the sequence defined in (2.26). Intuitively, q'' is constructed from q by inserting a copy of price $q_{t(n)}$ in period $t(n)+1$ and shifting those prices originally in periods $t(n)+1, \ldots, L$ one period later. Define

$$\Delta_1(q) = V_L(q) - V_{L-1}(q')$$

$$\Delta_2(q) = V_{L+1}(q'') - V_L(q).$$

It may be helpful to refer to Figure 2.4 to visualize the arguments in the proof of the following lemma.

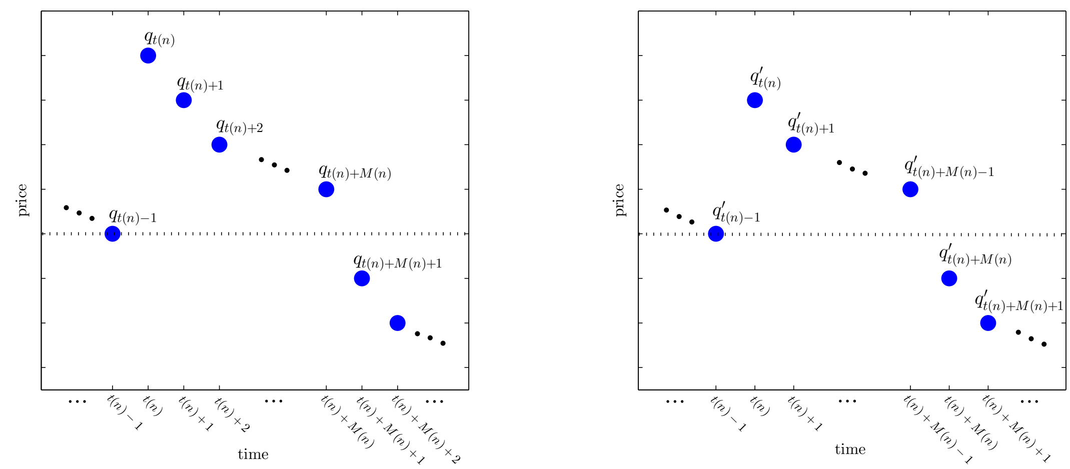


Figure 2.4: Lemma 6 illustration [6]

Lemma 6. *For any $q \in E^n(L)$ with $t(n)+M(n) \geq k+1$, we have $\Delta_1(q) = \Delta_2(q)$.*

[6]The left panel shows a portion of an element q of some $E^n(L)$ and the right panel shows a portion of q' as defined in (2.25). The policy q' is constructed from q by removing price $q_{t(n)}$ from q and shifting all prices originally in periods $t(n)+1, \ldots, L$ one period earlier. The dashed line through $q_{t(n)-1}$ and $q'_{t(n)-1}$ may be helpful for understanding the bookkeeping in the proof of Lemma 6.

Proof. Fix $q \in E^n(L)$. Consider q' as defined in (2.25). We begin by obtaining an expression for $\Delta_1(q)$ in terms of the entries of q.

We have $\rho_t(q') = \rho_t(q)$ for $t \leq t(n) - 1$, because $q'_j = q_j$ for $j \leq t(n) - 1$. Also

$$\begin{aligned} q_{t(n)+M(n)+1} &= \min\{q_1, \ldots, q_{t(n)+M(n)+1}\} \\ &= \min\{q'_1, \ldots, q'_{t(n)+M(n)}\} \\ &= q'_{t(n)+M(n)}, \end{aligned}$$

$q'_j = q_{j+1}$ for $j \geq t(n) + M(n) + 1$, and $t(n) + M(n) \geq k + 1$ so we have $\rho_t(q') = \rho_{t+1}(q)$ for $t \geq t(n) + M(n) + 1$.

Therefore,

$$\Delta_1(q) = V_L(q) - V_{L-1}(q') = \sum_{t=t(n)}^{t(n)+M(n)+1} \rho_t(q) - \sum_{t=t(n)}^{t(n)+M(n)} \rho_t(q') \tag{2.27}$$

We will now evaluate the terms on the right side of (2.27).

No customers who initially arrive prior to period $t(n)$ purchase in periods $t(n), \ldots, t(n) + M(n)$ when prices are set according to q because $q_{t(n)-1} \leq \min\{q_{t(n)}, \ldots, q_{t(n)+M(n)}\}$. Therefore,

$$\sum_{t=t(n)}^{t(n)+M(n)} \rho_t(q) = \sum_{t=1}^{M(n)+1} \rho_t(y) = V_{M(n)+1}(y)$$

where $y = (y_1, \ldots, y_{M(n)+1}) = (q_{t(n)}, \ldots, q_{t(n)+M(n)})$. Likewise, no customers who arrive prior to period $t(n)$ will purchase in $t(n), \ldots, t(n) + M(n) - 1$ when prices are set according to q'. Hence,

$$\sum_{t=t(n)}^{t(n)+M(n)-1} \rho_t(q') = \sum_{t=1}^{M(n)} \rho_t(\check{y}) = V_{M(n)}(\check{y})$$

where $\check{y} = (\check{y}_1, \ldots, \check{y}_{M(n)}) = (q_{t(n)+1}, \ldots, q_{t(n)+M(n)})$.

In addition, $q_{t(n)} \geq q_{t(n)+1} \geq \cdots \geq q_{t(n)+M(n)}$ and $M(n)+1 \leq k-1$. Therefore by Lemma 5, we have

$$\begin{aligned}
&\sum_{t=t(n)}^{t(n)+M(n)} \rho_t(q) - \sum_{t=t(n)}^{t(n)+M(n)-1} \rho_t(q') \\
=&q_{t(n)}[1-F(q_{t(n)})] + \alpha \sum_{i=1}^{M(n)} q_{t(n)+i}[F(q_{t(n)+i-1}) - F(q_{t(n)+i})]
\end{aligned} \tag{2.28}$$

Next we consider $\rho_{t(n)+M(n)+1}(q)$ and $\rho_{t(n)+M(n)}(q')$ in (2.27). We have

$$\begin{aligned}
&\rho_{t(n)+M(n)+1}(q) \\
&= q_{t(n)+M(n)+1}\Big\{1 - F(q_{t(n)+M(n)+1}) \\
&\qquad + \alpha \sum_{j=1}^{k} [F(\min\{q_{t(n)+M(n)+1-j}, \ldots, q_{t(n)+M(n)}\}) - F(q_{t(n)+M(n)+1})]^+\Big\} \\
&= q_{t(n)+M(n)+1}\Big\{1 - F(q_{t(n)+M(n)+1}) \\
&\qquad + \alpha(M(n)+1)[F(q_{t(n)+M(n)}) - F(q_{t(n)+M(n)+1})] \\
&\qquad + \alpha(k-1-M(n))[F(q_{t(n)-1}) - F(q_{t(n)+M(n)+1})]\Big\}
\end{aligned}$$

The last equality above holds because $\min\{q_{t(n)+M(n)+1-j}, \ldots, q_{t(n)+M(n)}\} = q_{t(n)+M(n)} > q_{t(n)+M(n)+1}$ for $j \in \{1, \ldots, M(n)+1\}$, and $\min\{q_{t(n)+M(n)+1-j}, \ldots, q_{t(n)+M(n)}\} = q_{t(n)-1} > q_{t(n)+M(n)+1}$ for $j \in \{M(n)+2, \ldots, k\}$. Notice that the indices in the summation above satisfy $t(n)+M(n)+1-j \geq 2$ for $j = 1, \ldots, k$ because $t(n)+M(n) \geq k+1$ by assumption.

We also have

$$\begin{aligned}
\rho_{t(n)+M(n)}(q') =& q'_{t(n)+M(n)}\Big\{1 - F(q'_{t(n)+M(n)}) \\
&+ \alpha \sum_{j=1}^{k} [F(\min\{q'_{t(n)+M(n)-j}, \ldots, q'_{t(n)+M(n)-1}\}) - F(q'_{t(n)+M(n)})]^+\Big\}
\end{aligned}$$

$$
\begin{aligned}
=& q_{t(n)+M(n)+1}\Big\{1 - F(q_{t(n)+M(n)+1}) \\
&+ \alpha M(n)[F(q_{t(n)+M(n)}) - F(q_{t(n)+M(n)+1})] \\
&+ \alpha(k - M(n))[F(q_{t(n)-1}) - F(q_{t(n)+M(n)+1})]\Big\}
\end{aligned}
$$

Notice that here the indices in the summation satisfy $t(n)+M(n)-j \geq 1$ because $t(n) + M(n) \geq k+1$ again by assumption.

From the preceding expressions for $\rho_{t(n)+M(n)+1}(q)$ and $\rho_{t(n)+M(n)}(q')$ it follows that

$$
\rho_{t(n)+M(n)+1}(q) - \rho_{t(n)+M(n)}(q') = \alpha q_{t(n)+M(n)+1}[F(q_{t(n)+M(n)}) - F(q_{t(n)-1})] \tag{2.29}
$$

Substituting (2.28) and (2.29) into (2.27) yields

$$
\begin{aligned}
\Delta_1(q) = q_{t(n)}[1 - F(q_{t(n)})] + \Big\{\alpha \sum_{i=1}^{M(n)} q_{t(n)+i}[F(q_{t(n)+i-1}) - F(q_{t(n)+i})]\Big\} \\
+ \alpha q_{t(n)+M(n)+1}[F(q_{t(n)+M(n)}) - F(q_{t(n)-1})]
\end{aligned}
$$

By essentially duplicating the preceding analysis we obtain

$$
\begin{aligned}
\Delta_2(q) =& \sum_{t=t(n)}^{t(n)+M(n)+2} \rho_t(q'') - \sum_{t=t(n)}^{t(n)+M(n)+1} \rho_t(q) \\
=& q_{t(n)}[1 - F(q_{t(n)})] + \Big\{\alpha \sum_{i=1}^{M(n)} q_{t(n)+i}[F(q_{t(n)+i-1}) - F(q_{t(n)+i})]\Big\} \\
&+ \alpha q_{t(n)+M(n)+1}[F(q_{t(n)+M(n)}) - F(q_{t(n)-1})] \\
=& \Delta_1(q).
\end{aligned}
$$

□

Proof of Proposition 4. Let $D_0 = \cup_{L=1}^{\kappa+k} D_0(L)$ where $D_0(L) = \{q \in \mathcal{P}^L : q_1 \geq q_2 \geq \cdots \geq q_L\}$. Observe that $D \subset D_0$. The set D_0 differs from D in two ways: elements of D_0 can have regeneration points and elements of D_0 can be slightly longer than those of D. Any element $x \in D_0$ with say ℓ regeneration points can be expressed as $(x^1, x^2, \ldots, x^\ell, x^{\ell+1})$ where $x^1, \ldots, x^{\ell+1} \in D$ and the average revenue of x is a convex combination of the average revenues of $x^1, \ldots, x^{\ell+1}$. In particular, $v(x) = \sum_{i=1}^{\ell+1} \lambda_i v(x^i)$ where $\lambda_i = L(x^i)/L(x)$. Hence, $d \in \arg\max_{p \in D} v(p)$ satisfies

$$v(d) \geq v(x) \quad \text{for all } x \in D_0. \tag{2.30}$$

Consider $q \in \arg\max_{p \in E} v(p)$ and suppose for a contradiction that

$$v(q) > v(d). \tag{2.31}$$

Let $L = L(q)$. We have $q \in E$ and therefore $q \in E^n(L)$ for some n. We consider two cases.

Case 1: $t(n) + M(n) \geq k + 1$.

Consider sequence q' with length $L - 1$ as defined in (2.25). By the definition of $E^n(L)$, we have $q_{t(n)-1} < q_{t(n)}$, $q_{t(n)} \geq \cdots \geq q_{t(n)+M(n)} \geq q_{t(n)-1}$, and $q_{t(n)-1} > q_{t(n)+M(n)+1} \geq \cdots \geq q_{t(n+1)-1}$.

If $M(n) = 0$, then $q_{t(n)-1} < q_{t(n)}$ and $q_{t(n)-1} > q_{t(n)+1} \geq \cdots \geq q_{t(n+1)-1}$. Thus

$$q' \in \begin{cases} E^{n-1}(L-1) & \text{if} \quad n \geq 2 \\ D_0(L-1) & \text{if} \quad n = 1 \end{cases}$$

If $M(n) \neq 0$ and $q_{t(n)+1} = \cdots = q_{t(n)+M(n)} = q_{t(n)-1}$, then

$$q' \in \begin{cases} E^{n-1}(L-1) & \text{if} \quad n \geq 2 \\ D_0(L-1) & \text{if} \quad n = 1 \end{cases}$$

If $M(n) \neq 0$ and $\max\{q_{t(n)+1}, \ldots, q_{t(n)+M(n)}\} > q_{t(n)-1}$, then $q' \in E^n(L-1)$.

Taking the above three situations into consideration, $q' \in E$ or $q' \in D_0$. Thus

$$v(q') \leq \max\{v(q), v(d)\} = v(q) \tag{2.32}$$

where the last equality holds because of our supposition (2.31) that $v(q) > v(d)$. Thus,

$$\begin{aligned}
\Delta_1(q) &= V_L(q) - V_{L-1}(q') && \text{[by the definition of } \Delta_1(q)] \\
&= L \cdot v_L(q) - (L-1)v_{L-1}(q') \\
&= (L-1)[v_L(q) - v_{L-1}(q')] + v_L(q) \\
&\geq v_L(q) && \text{[by (2.32)]}
\end{aligned} \tag{2.33}$$

Below we will use the notation $M(n|q)$ and $t(n|q)$ to emphasize the dependence upon q. We will consider two subcases, (i) and (ii). Recall (2.24).

(i) Suppose that $M(n|q) = k-2$. (Here $M(n|q) = k-2-l$ with $l = 0$.) In this case $t(n|q)$ is an F-regeneration point of policy $q^1 = \psi(q) = q''$ as defined in (2.26). To see why this is so, note that $t(n|q) + M(n|q) + 1 = t(n|q) + k - 1$ because $M(n|q) = k-2$. In addition, $q_{t(n|q)+i} \geq q_{t(n|q)-1}$ for $i = 0, \ldots, M(n|q)$, because $q \in E^n(L)$. It follows by construction of q^1 that

$$q^1_{t(n|q)-1} \leq \min\{q^1_{t(n|q)}, \ldots, q^1_{t(n|q)+k-1}\}$$

and hence $t(n|q)$ is an F-regeneration point of q^1.

(ii) Suppose that $M(n|q) < k-2$. Then $M(n|q) = k-2-l$ for some $l \in \{1, \ldots, k-2\}$. Consider $q^1 = \psi(q)$. Then $q^1 \in E^n(L+1)$ with $t(n|q^1) = t(n|q)$ and $M(n|q^1) = M(n|q) + 1$. Consider policy $q^2 = \psi(q^1) = \psi(\psi(q))$. If $l = 1$, then q^2 has an F-regeneration point at $t(n|q)$.

For general $l \in \{1, \ldots, k-2\}$, consider $q^{l+1} \in \mathcal{P}^{L+l+1}$ defined as follows: let $q^0 = q$, and $q^i = \psi(q^{i-1}) \in \mathcal{P}^{L+i}$ for $i = 1, \ldots, l+1$. Intuitively, q^i is obtained from q by inserting i new copies of price $q_{t(n|q)}$ into q at time periods $t(n|q)+1, \ldots, t(n|q)+i$ and shifting all prices in q that appeared after $t(n|q)$ "to the right" by i time periods. By construction $t(n|q)$ is an F-regeneration point of q^{l+1}. (It now may be helpful to refer back to Figure 2.2 in Section 2.4. In the example depicted there with $k = 5$, the left panel shows q and the right panel shows q^{l+1} where $l + 1 = L' = 2$.) Note also that $q^i \in E^n(L+i)$ for $i = 0, \ldots, l$. Moreover, $t(n|q^i) = t(n|q^{i-1}) = t(n|q)$ and $M(n|q^i) = M(n|q^{i-1}) + 1 = M(n|q) + i$ for $i = 1, \ldots, l$.

We can now combine subcases (i) and (ii) to see that if $M(n|q) = k - 2 - l$ for some $l \in \{0, \ldots, k-2\}$ (which exhausts all possibilities because $q \in E^n(L)$), then q^{l+1} has an F-regeneration point at time $t(n|q)$. In addition, in either subcase, by construction, $q^i_{t(n|q^i)} = q_{t(n|q)}$ for $i = 0, \ldots, l$.

By Lemma 6, it follows that

$$\Delta_2(q^i) = \Delta_1(q^i) \qquad \text{for } i = 0, \ldots, l \tag{2.34}$$

By the definition of Δ_1 and Δ_2, we have

$$\begin{aligned} \Delta_2(q^i) &= V_{L+i+1}(q^{i+1}) - V_{L+i}(q^i) && \text{for } i = 0, \ldots, l \\ \Delta_1(q^{i+1}) &= V_{L+i+1}(q^{i+1}) - V_{L+i}(q^i) && \text{for } i = 0, \ldots, l-1 \end{aligned}$$

and hence

$$\Delta_2(q^i) = \Delta_1(q^{i+1}) \qquad \text{for } i = 0, \ldots, l-1 \tag{2.35}$$

From (2.34) and (2.35), it follows that

$$\Delta_2(q^i) = \Delta_1(q^0) = \Delta_1(q) \qquad \text{for } i = 0, \ldots, l \tag{2.36}$$

Consequently,

$$
\begin{aligned}
v_{L+l+1}(q^{l+1}) &= \frac{1}{L+l+1} V_{L+l+1}(q^{l+1}) \\
&= \frac{1}{L+l+1}\Big[V_L(q^0) + \sum_{i=0}^{l} \Delta_2(q^i)\Big] \\
&= \frac{1}{L+l+1}\Big[V_L(q) + (l+1)\Delta_1(q)\Big] && \text{[by (2.36)]} \\
&\geq \frac{1}{L+l+1}\Big[V_L(q) + (l+1)v_L(q)\Big] && \text{[by (2.33)]} \\
&= v_L(q) && (2.37)
\end{aligned}
$$

Since $t(n|q)$ is an F-regeneration point of q^{l+1}, we can decompose q^{l+1} into two independent subsequences $p^1 = (q_1^{l+1}, \ldots, q_{t(n|q)-1}^{l+1})$ and $p^2 = (q_{t(n|q)}^{l+1}, \ldots, q_{L+l+1}^{l+1})$ with $L(p^1) = t(n|q) - 1$ and $L(p^2) = L + l - t(n|q) + 2$ so that $q^{l+1} = (p^1, p^2)$ and

$$
\begin{aligned}
v_{L+l+1}(q^{l+1}) &= \frac{1}{L+l+1}\Big[\sum_{i=1}^{t(n|q)-1} \rho_t(q^{l+1}) + \sum_{i=t(n|q)}^{L+l+1} \rho_t(q^{l+1})\Big] \\
&= \frac{L(p^1)}{L+l+1} v(p^1) + \frac{L(p^2)}{L+l+1} v(p^2). && (2.38)
\end{aligned}
$$

Observe that

$$
p^1 \in \begin{cases} E^{n-1}(t(n|q)-1) & \text{if} \quad n \geq 2 \\ D_0(t(n|q)-1) & \text{if} \quad n = 1 \end{cases}
$$

and $p^2 \in D_0$. If $n \geq 2$, then $v(q) \geq v(p^1)$ and if $n = 1$, then $v(q) > v(d) \geq v(p^1)$. In either case, $v(q) > v(d) \geq v(p^2)$. Here, we have used (2.30). The strict inequalities follow from supposition (2.31). Hence $v(q) > v_{L+l+1}(q^{l+1})$ by (2.38), which contradicts (2.37). Therefore (2.31) cannot hold.

Case 2: $t(n) + M(n) \leq k$.

Write $q = (p^1, p^2)$, where $p^1 = (q_1, \ldots, q_{t(n)+M(n)})$ and $p^2 = (q_{t(n)+M(n)+1}, \ldots, q_L)$. Reorder p^1 into a decreasing sequence $p^3 =$

$(q_{i(1)}, \ldots, q_{i(t(n)+M(n))})$. Then consider another sequence p^0 defined by

$$p^0_j = \begin{cases} q_{i(j)} & \text{if} \quad j \leq t(n) + M(n) \\ q_j & \text{if} \quad j \geq t(n) + M(n) + 1 \end{cases}$$

Then $p^0 = (p^3, p^2) \in D_0$.

Since $t(n) + M(n) \leq k$, it follows that $t(n) + M(n) + 1 \leq k + 1$. Observe that $(p^0_1, \ldots, p^0_{t(n)+M(n)+1})$ is $(q_1, \ldots, q_{t(n)+M(n)+1})$ rearranged into a decreasing sequence. Hence, by Lemma 4, we have

$$\sum_{t=1}^{t(n)+M(n)+1} \rho_t(p^0) \geq \sum_{t=1}^{t(n)+M(n)+1} \rho_t(q) \tag{2.39}$$

Since $q_{t(n)+M(n)+1} = \min\{q_1, \ldots, q_{t(n)+M(n)+1}\} = p^0_{t(n)+M(n)+1} = \min\{p^0_1, \ldots, p^0_{t(n)+M(n)+1}\}$, and $q_j = p^0_j$ for $j \geq t(n) + M(n) + 1$, we have

$$\rho_t(p^0) = \rho_t(q) \quad \text{for } t \geq t(n) + M(n) + 2 \tag{2.40}$$

Thus by (2.39) and (2.40), we have $V(p^0) \geq V(q)$, hence $v(p^0) \geq v(q)$. Moreover, $v(d) \geq v(p^0)$ by (2.30) because $p^0 \in D_0$. Hence $v(d) \geq v(q)$, which contradicts our supposition (2.31). So (2.31) cannot hold. This completes the proof. □

2.8.4 Proof of Proposition 5

Proof. Consider $L \geq k + m$ and $q \in \arg\max\{v_L(p) : p \in D(L)\}$. We will establish that there exists a policy $q^\circ \in D(k+m-1) \cup D(1)$ such that $v(q^\circ) \geq v(q)$, from which the proposition follows. To this end, observe that it must be that $q_j = q_{j+1}$ for some $j \in \{k, \ldots, L-1\}$ because there are m prices in $\mathcal{P}$ and q is decreasing. (For a policy of length at least $k + m$, at least one price must appear multiple times in period k or later. The policy q is decreasing, so such a price must appear in consecutive periods.)

Let $x^* \in \arg\max\{v_1(x) : x \in D(1)\} = \arg\max\{x[1 - F(x)] : x \in \mathcal{P}\}$. If $v_L(q) < v_1(x^*)$, then we are done. Therefore we just need to consider the case that

$$v_L(q) \geq v_1(x^*). \tag{2.41}$$

Consider the sequence $q^\dagger \in D(L-1) \subset D$ as follows:

$$q_t^\dagger = \begin{cases} q_t & \text{for} \quad t = 1, \ldots, j \\ q_{t+1} & \text{for} \quad t = j+1, \ldots, L-1 \end{cases}$$

From the definition of $q^\dagger$ we have

$$\sum_{t=1}^{j} \rho_t(q^\dagger) = \sum_{t=1}^{j} \rho_t(q). \tag{2.42}$$

Moreover,

$$\rho_t(q) = q_t\big\{1 - F(q_t) + \alpha k[F(q_{t-1}) - F(q_t)]\big\} \qquad \text{for } t = j+1, \ldots, L \tag{2.43}$$

$$\rho_t(q^\dagger) = q_{t+1}\big\{1 - F(q_{t+1}) + \alpha k[F(q_t) - F(q_{t+1})]\big\} \quad \text{for } t = j+1, \ldots, L-1 \tag{2.44}$$

because $j \geq k$ and both q and $q^\dagger$ are decreasing. Thus by (2.43) and (2.44) it is easy to see that $\rho_t(q^\dagger) = \rho_{t+1}(q)$ for $t = j+1, \ldots, L-1$, and hence

$$\sum_{t=j+1}^{L-1} \rho_t(q^\dagger) = \sum_{t=j+2}^{L} \rho_t(q)\,. \tag{2.45}$$

Therefore, by (2.42) and (2.45)

$$V_L(q) - V_{L-1}(q^\dagger) = \rho_{j+1}(q) = q_{j+1}[1 - F(q_{j+1})] \leq v_1(x^*). \tag{2.46}$$

The last inequality holds by the definition of x^*. Thus,

$$\begin{aligned}
v_{L-1}(q^\dagger) - v_L(q) &= \frac{V_{L-1}(q^\dagger)}{L-1} - \frac{V_L(q)}{L} \\
&= \frac{V_L(q) - L[V_L(q) - V_{L-1}(q^\dagger)]}{L(L-1)} \\
&\geq \frac{V_L(q) - L \cdot v_1(x^*)}{L(L-1)} && \text{[by (2.46)]} \\
&= \frac{v_L(q) - v_1(x^*)}{L-1} \\
&\geq 0 && \text{[by (2.41)]}
\end{aligned}$$

If $L-1 = k+m-1$ we are done with $q^\circ = q^\dagger$. Otherwise we can remove a repeated price from $q^\dagger$ in some period later than k while improving (or keeping the same) the average revenue as above. We continue in this fashion until we arrive at q° with length $k+m-1$ as desired. □

2.8.5 Proof of Proposition 6

Proof. We will prove by induction that $v_j(q^j) \leq v_{j+1}(q^{j+1})$ for $j = 1, \ldots, k$ when $v_1(q^1) \leq v_2(q^2)$.

When $j = 1$, we have $v_1(q^1) \leq v_2(q^2)$.

For any $j \in \{2, \ldots, k\}$, suppose for the inductive hypothesis that

$$v_{j-1}(q^{j-1}) \leq v_j(q^j). \tag{2.47}$$

To complete the proof, we will show that $v_j(q^j) \leq v_{j+1}(q^{j+1})$. By (2.15), we have

$$\varphi_1^j(q^j) = q_1^j[1 - F(q_1^j)] + \alpha q_2^j[F(q_1^j) - F(q_2^j)] + \cdots + \alpha q_j^j[F(q_{j-1}^j) - F(q_j^j)]$$

Let $\widehat{q}^{j-1} = (q_2^j, \ldots, q_j^j)$. Then $\widehat{q}^{j-1} \in D(j-1)$ and

$$v_{j-1}(\widehat{q}^{j-1}) - \varphi_1^j(q^j) \leq v_{j-1}(q^{j-1}) - \varphi_1^j(q^j)$$

$$
\begin{aligned}
&= v_{j-1}(q^{j-1}) - [V_j(q^j) - V_{j-1}(\widehat{q}^{j-1})] \qquad \text{[by Lemma 5]} \\
&\le v_{j-1}(q^{j-1}) + V_{j-1}(q^{j-1}) - V_j(q^j) \\
&= j v_{j-1}(q^{j-1}) - j v_j(q^j) \\
&\le 0 \qquad \text{[by (2.47)]} \qquad (2.48)
\end{aligned}
$$

Thus

$$
\begin{aligned}
v_j(q^j) = \frac{1}{j} V_j(q^j) &= \frac{1}{j}\Big[\varphi_1^j(q^j) + V_{j-1}(\widehat{q}^{j-1})\Big] \\
&= \frac{1}{j}\Big[\varphi_1^j(q^j) + (j-1) v_{j-1}(\widehat{q}^{j-1})\Big] \\
&\le \varphi_1^j(q^j) \qquad \text{[by (2.48)]} \qquad (2.49)
\end{aligned}
$$

Consider $\widetilde{q}^{j+1} \in D(j+1)$ as follows.

$$
\widetilde{q}_t^{j+1} = \begin{cases} q_1^j & \text{for} \quad t = 1 \\ q_{t-1}^j & \text{for} \quad t = 2, \ldots, j+1 \end{cases}
$$

Then

$$
\begin{aligned}
\varphi_1^{j+1}(\widetilde{q}^{j+1}) &= \widetilde{q}_1^{j+1}[1 - F(\widetilde{q}_1^{j+1})] + \alpha\Big\{\widetilde{q}_2^{j+1}[F(\widetilde{q}_1^{j+1}) - F(\widetilde{q}_2^{j+1})] \\
&\qquad + \sum_{t=3}^{j+1} \widetilde{q}_t^{j+1}[F(\widetilde{q}_{t-1}^{j+1}) - F(\widetilde{q}_t^{j+1})]\Big\} \\
&= q_1^j[1 - F(q_1^j)] + \alpha \sum_{t=2}^{j} q_t^j [F(q_{t-1}^j) - F(q_t^j)] \\
&= \varphi_1^j(q^j) \ge v_j(q^j) \qquad \text{[by (2.49)]} \qquad (2.50)
\end{aligned}
$$

Hence

$$
v_{j+1}(\widetilde{q}^{j+1}) = \frac{1}{j+1} V_{j+1}(\widetilde{q}^{j+1}) = \frac{1}{j+1}\Big[\varphi_1^{j+1}(\widetilde{q}^{j+1}) + V_j(q^j)\Big]
$$

$$= \frac{1}{j+1}\Big[\varphi_1^{j+1}(\widetilde{q}^{j+1}) + jv_j(q^j)\Big] \geq v_j(q^j)$$

by (2.50). Therefore, we have $v_{j+1}(q^{j+1}) \geq v_{j+1}(\widetilde{q}^{j+1}) \geq v_j(q^j)$, which completes the inductive step.

We have now proved that $v_1(q^1) \leq v_2(q^2) \leq \cdots \leq v_{k+1}(q^{k+1})$. Consequently, $v_{k+1}(q^{k+1}) \geq \max\{v(q) : q \in \cup_{L=1}^{k} D(L)\}$ where $q^{k+1} \in D(k+1)$, from which the second statement in the proposition follows. If $v_1(q^1) < v_2(q^2)$, then the above argument is easily modified to show that $v_1(q^1) < v_2(q^2) < \cdots < v_{k+1}(q^{k+1})$. $\square$

Chapter 3

Pricing with Continuous Price Set

3.1 Introduction

In Chapter 2, we demonstrated that a decreasing cyclic pricing policy achieves optimality in the presence of patient customers. However, there are still two questions that remain. First, the main result is derived with the assumption of a discrete price set. It is natural that people may be interested in the performance of the decreasing cyclic policy with a continuous price set. Second, although a dynamic programming algorithm is proposed to compute the optimal decreasing sequence, we could not derive the decreasing sequence explicitly.

In order to address the above two issues, we study the same pricing problem in this chapter, but with a continuous price set. Section 3.2 establishes the optimization problem and introduces some notation. Section 3.3 shows the optimality of decreasing cyclic policy through a discretization approach. In section 3.4, we solve the optimal decreasing sequence in a special case that customers have uniform valuation distribution and their patience level k is small.

3.2 Problem Description

We assume prices are selected from a continuous set $[0, \bar{P}]$ where $\bar{P} < \infty$. Thus our objective is to solve the following optimization problem.

$$\sup_{p}\{H(p) : p \in [0, \bar{P}]^{\infty}\}. \tag{3.1}$$

First, we will introduce some notations. Let

$$D_0(L) = \{q \in [0, \bar{P}]^L : q_1 \geq q_2 \geq \cdots \geq q_L\}$$
$$D(L) = \{q \in D_0(L) : q \text{ has no F-regeneration points}\}.$$

Throughout this section, we use $D(L)$ to denote the set of decreasing price sequences with no F-regeneration points where the individual prices are selected from the continuous price set $[0, \bar{P}]$. Note that this is slightly different from $D(L)$ as used Section 2.4, where prices are selected from a finite price set.

For $L \geq 1$, let $d^L \in \arg\max\{v_L(q) : q \in D_0(L)\}$. Consider the problem $\sup\{v_{k+1}(q) : q \in [0, \bar{P}]^{k+1}\}$. Since $v_{k+1}(\cdot)$ is continuous and $[0, \bar{P}]^{k+1}$ is compact, the supremum is a maximum. By Lemma 4 (which holds when $\mathcal{P} = [0, \bar{P}]$) there is a decreasing sequence that is optimal. Therefore $d^{k+1} = (d_1^{k+1}, d_2^{k+1}, \ldots, d_{k+1}^{k+1}) \in \arg\max\{v_{k+1}(q) : q \in [0, \bar{P}]^{k+1}\}$, where $d_1^{k+1} \geq d_2^{k+1} \geq \cdots \geq d_{k+1}^{k+1}$.

Let $\widehat{\mathcal{P}}(n) = \{d_1^{k+1}, d_2^{k+1}, \ldots, d_{k+1}^{k+1}\} \cup \{0, \frac{1}{n}, \frac{2}{n}, \ldots, V^n - \frac{2}{n}, V^n - \frac{1}{n}, V^n\}$ where $V^n = \lfloor n\bar{P} \rfloor / n$. Observe that $\widehat{\mathcal{P}}(n)$ is a finite set.

Define

$$\widehat{D}_0(n,L) = \{q \in \widehat{\mathcal{P}}(n)^L : q_1 \geq q_2 \geq \cdots \geq q_L\}$$
$$\widehat{D}(n,L) = \{q \in \widehat{D}_0(n,L) : q \text{ has no F-regeneration points}\}$$

Observe that d^{k+1} is an element of $\widehat{D}_0(n,k+1)$ for all n by construction of $\widehat{\mathcal{P}}(n)$. Note also that $\widehat{D}_0(n,L)$ and $\widehat{D}(n,L)$ are finite sets, and therefore the supremum of any real-valued function taken over those sets is in fact a maximum.

3.3 Main Result (Discretization)

First, I will introduce an upper bound, which is independent of the number of prices in $\widehat{\mathcal{P}}(n)$, for the cycle length of the optimal decreasing cyclic policy given price set $\widehat{\mathcal{P}}(n)$.

Lemma 7. *There exists a finite integer c, which does not depend upon n, with $c \geq k+1$ such that for all $L > c$, the following inequality holds:*

$$\max_q \{v(q) : q \in \widehat{D}_0(n,L)\} \leq v_{k+1}(d^{k+1}) \quad \textit{for all } n \geq 1.$$

Proof. It suffices to show $\sup_q\{v_L(q) : q \in D_0(L)\} \leq v_{k+1}(d^{k+1})$, because $\widehat{D}_0(n,L) \subset D_0(L)$.

Consider $L \geq 1$ and $d = d^L \in \arg\max_{q \in D_0(L)} v_L(q)$. Then

$$\begin{aligned} V_L(d) &= \sum_{i=1}^{L} \rho_i(d) \\ &= d_1[1 - F(d_1)] + \sum_{i=2}^{k} d_i\Big\{1 - F(d_i) + \alpha(i-1)[F(d_{i-1}) - F(d_i)]\Big\} \end{aligned}$$

$$+ \sum_{i=k+1}^{L} d_i \Big\{ 1 - F(d_i) + \alpha k [F(d_{i-1}) - F(d_i)] \Big\}$$

$$= \sum_{i=1}^{k} \Big\{ d_i[1 - F(d_i)] + \alpha \sum_{j=i+1}^{L} d_j [F(d_{j-1}) - F(d_j)] \Big\} + \sum_{i=k+1}^{L} d_i [1 - F(d_i)]$$

$$\leq k\bar{P} + (L-k) v_1(d^1)$$

The last inequality holds because each item inside the large curly braces is bounded above by $\bar{P}$ and because d^1 maximizes $v_1(x) = x[1 - F(x)]$ over $x \in [0, \bar{P}]$. It follows that

$$v_L(d^L) \leq v_1(d^1) + \frac{k(\bar{P} - v_1(d^1))}{L}. \tag{3.2}$$

Note that $F(\bar{P}) = 1$, so $v_1(\bar{P}) = 0$. Thus, $d^1 < \bar{P}$. For $x \in [d^1, \bar{P}]$, let

$$R(x) = v_2(x, d^1) - v_1(d^1) = \frac{1}{2} \Big\{ x[1 - F(x)] + \alpha d^1 [F(x) - F(d^1)] - d^1 [1 - F(d^1)] \Big\}$$

Observe that $R(d^1) = 0$, and $R'(x) = \frac{1}{2}\{1 - F(x) - xf(x) + \alpha d^1 f(x)\}$ where $f(x) = F'(x)$. The price d^1 maximizes $x[1 - F(x)]$, and hence by the first order condition, we have $1 - F(x) - xf(x)|_{x=d^1} = 0$. Therefore, $R'(d^1) = \frac{1}{2}\alpha d^1 f(d^1) > 0$ where the inequality holds because $F(\cdot)$ is strictly increasing. It follows that there must exist some $p \in (d^1, \bar{P}]$ such that $R(p) > 0$. Hence the sufficient condition (i) following Proposition 6 is satisfied. Then by the same proof as Proposition 6, we have

$$v_1(d^1) < v_2(d^2) < \cdots < v_{k+1}(d^{k+1}). \tag{3.3}$$

Therefore, $v_L(d^L) \leq v_{k+1}(d^{k+1})$ for all L sufficiently large by (3.2) and (3.3). Hence, there exists some $c < \infty$ such that $v_L(d^L) = \sup_q \{v_L(q) : q \in D_0(L)\} \leq v_{k+1}(d^{k+1})$ when $L > c$. □

Next, I will show the lipschitz continuity of the long-run average revenue function $H(\cdot)$.

Lemma 8. *Suppose that $|F(x) - F(y)| \leq A|x - y|$ for all $x, y \in [0, \bar{P}]$ for some finite A; that is, $F(\cdot)$ is Lipschitz continuous on $[0, \bar{P}]$. Then $|H(p) - H(q)| \leq C \sup_t |p_t - q_t|$ for any $p, q \in [0, \bar{P}]^\infty$ where C is a finite constant.*

Proof. Consider $p = (p_1, p_2, \dots) \in \mathcal{P}^\infty$ and $q = (q_1, q_2, \dots) \in \mathcal{P}^\infty$. We have

$$\begin{aligned}
|H(p) - H(q)| &= \Big| \liminf_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} \rho_t(p) - \liminf_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} \rho_t(q) \Big| \\
&\leq \limsup_{T\to\infty} \Big| \frac{1}{T} \sum_{t=1}^{T} [\rho_t(p) - \rho_t(q)] \Big| \\
&\leq \sup_t \Big| \rho_t(p) - \rho_t(q) \Big| \\
&\leq \sup_t \Big| p_t[1 - F(p_t)] - q_t[1 - F(q_t)] \Big| \\
&\quad + \alpha \sum_{i=1}^{k} \sup_t \Big| p_t[F(\min\{p_{t-i}, \dots, p_{t-1}\}) - F(p_t)]^+ \\
&\qquad - q_t[F(\min\{q_{t-i}, \dots, q_{t-1}\}) - F(q_t)]^+ \Big|
\end{aligned}$$

Now we will evaluate the terms in the final expression above.

$$\begin{aligned}
\Big| p_t[1 - F(p_t)] - q_t[1 - F(q_t)] \Big| &= \Big| p_t[F(q_t) - F(p_t)] + [1 - F(q_t)](p_t - q_t) \Big| \\
&\leq \Big| p_t[F(q_t) - F(p_t)] \Big| + \Big| [1 - F(q_t)](p_t - q_t) \Big| \\
&\leq p_t A \Big| p_t - q_t \Big| + [1 - F(q_t)] \Big| p_t - q_t \Big| \\
&\leq (\bar{P}A + 1) \Big| p_t - q_t \Big|
\end{aligned}$$

Similarly,

$$\Big| q_t[1 - F(\min\{q_{t-i}, \dots, q_{t-1}\})] - p_t[1 - F(\min\{p_{t-i}, \dots, p_{t-1}\})] \Big|$$

$$
\begin{aligned}
&\leq \Big|q_t[F(\min\{p_{t-i},\ldots,p_{t-1}\}) - F(\min\{q_{t-i},\ldots,q_{t-1}\})]\Big| \\
&\quad + \Big|[1 - F(\min\{p_{t-i},\ldots,p_{t-1}\})](q_t - p_t)\Big| \\
&\leq \bar{P}A \sup_j \Big|p_j - q_j\Big| + \Big|q_t - p_t\Big|
\end{aligned}
$$

So

$$
\begin{aligned}
&\Big|p_t[F(\min\{p_{t-i},\ldots,p_{t-1}\}) - F(p_t)]^+ - q_t[F(\min\{q_{t-i},\ldots,q_{t-1}\}) - F(q_t)]^+\Big| \\
&\leq \Big|p_t[F(\min\{p_{t-i},\ldots,p_{t-1}\}) - F(p_t)] - q_t[F(\min\{q_{t-i},\ldots,q_{t-1}\}) - F(q_t)]\Big| \\
&= \Big|p_t[1 - F(p_t)] - p_t[1 - F(\min\{p_{t-i},\ldots,p_{t-1}\})] - q_t[1 - F(q_t)] \\
&\qquad + q_t[1 - F(\min\{q_{t-i},\ldots,q_{t-1}\})]\Big| \\
&\leq \Big|p_t[1 - F(p_t)] - q_t[1 - F(q_t)]\Big| + \Big|q_t[1 - F(\min\{q_{t-i},\ldots,q_{t-1}\})] \\
&\qquad - p_t[1 - F(\min\{p_{t-i},\ldots,p_{t-1}\})]\Big| \\
&\leq 2(\bar{P}A + 1) \sup_j \Big|p_j - q_j\Big|
\end{aligned}
$$

Therefore, taking $C = (1 + 2k\alpha)(\bar{P}A + 1)$ we have

$$
\begin{aligned}
\Big|H(p) - H(q)\Big| &\leq \sup_t \Big\{(\bar{P}A + 1)\big|p_t - q_t\big|\Big\} + \alpha k \sup_t \Big\{2(\bar{P}A + 1) \sup_j \big|p_j - q_j\big|\Big\} \\
&\leq C \sup_t \big|p_t - q_t\big|.
\end{aligned}
$$

Theorem 2. *Suppose $\mathcal{P} = [0, \bar{P}]$ and $F(\cdot)$ is Lipschitz continuous and strictly increasing on $[0, \bar{P}]$. Then there exists a decreasing cyclic policy that is an optimal solution to* (3.1)*.*

The uniform distribution function is Lipschitz continuous, and therefore Theorem 2 applies to settings where customers' valuations follow the uniform distribution.

Proof. Consider a sequence of pricing policies $\{p^n\} \in \mathcal{P}^\infty = [0, \bar{P}]^\infty$ such that

$$H(p^n) \geq H^* - \frac{1}{n} \tag{3.4}$$

where $H^* = \sup_p\{H(p) : p \in [0, \bar{P}]^\infty\}$.

Let $\widehat{p}^n = (\widehat{p}_1^n, \widehat{p}_2^n, \dots)$ where $\widehat{p}_t^n$ is equal to p_t^n rounded to the closest element in the set $\widehat{\mathcal{P}}(n)$. By Lemma 8,

$$|H(p^n) - H(\widehat{p}^n)| \leq C \sup_t |p_t^n - \widehat{p}_t^n|, \tag{3.5}$$

where C is a finite constant. By (3.4) and (3.5), we have $H(\widehat{p}^n) \geq H^* - \frac{C+1}{n}$.

Define $H_n = \sup_p\{H(p) : p \in \widehat{\mathcal{P}}(n)^\infty\}$. By definition, $H_n \geq H(\widehat{p}^n)$ because $\widehat{p}^n \in \widehat{\mathcal{P}}(n)^\infty$. So $H_n \geq H^* - \frac{C+1}{n}$. Given any $\varepsilon > 0$, it follows that $H_n \geq H^* - \varepsilon$ for $n > n(\varepsilon) = \frac{C+1}{\varepsilon}$. Therefore, $\liminf_n H_n \geq H^* - \varepsilon$. This holds for any $\varepsilon > 0$, and so

$$\liminf_n H_n \geq H^* \,. \tag{3.6}$$

Moreover,

$$H_n = \max_q\{H(p) : p = (q, q, q, \dots), q \in \cup_{L=1}^{|\widehat{\mathcal{P}}(n)|+k-1} \widehat{D}(n, L)\} \tag{3.7}$$

by Theorem 1 (and its proof, which shows that the decreasing cycles in the theorem have no F-regeneration points when viewed in isolation).

We now take a brief detour and observe that $\cup_{L=1}^M D(L) \subset \cup_{L=1}^M D_0(L)$ for any M. Any element $x \in \cup_{L=1}^M D_0(L)$ with say ℓ regeneration points can be expressed as $(x^1, x^2, \dots, x^\ell, x^{\ell+1})$ where $x^1, \dots, x^{\ell+1} \in \cup_{L=1}^M D(L)$ and the average revenue of x is $v(x) = \sum_{i=1}^{\ell+1} \lambda_i v(x^i)$ for some $\{\lambda_i\}$. Hence, for any $d \in \arg\max_{p \in \cup_{L=1}^M D_0(L)} v(p)$ we have

$$v(d) \leq v(y) \quad \text{ for some } y \in \cup_{L=1}^M D(L). \tag{3.8}$$

By (3.7), we now have

$$
\begin{aligned}
H_n &= \max_q \{v(q) : q \in \cup_{L=1}^{|\widehat{\mathcal{P}}(n)|+k-1} \widehat{D}(n, L)\} && \text{[by Lemma 1]} \\
&\leq \max_q \{v(q) : q \in \cup_{L=1}^{|\widehat{\mathcal{P}}(n)|+k-1} \widehat{D}_0(n, L)\} && [\widehat{D}(n, L) \subset \widehat{D}_0(n, L)] \\
&\leq \max_q \{v(q) : q \in \cup_{L=1}^{c} \widehat{D}_0(n, L)\} && [\text{by Lemma 7 and } d^{k+1} \in \widehat{D}_0(n, k+1)] \\
&\leq \sup_q \{v(q) : q \in \cup_{L=1}^{c} D_0(L)\} && [\widehat{D}_0(n, L) \subset D_0(L)] \\
&= \max_q \{v(q) : q \in \cup_{L=1}^{c} D_0(L)\} && [\cup_{L=1}^{c} D_0(L) \text{ is compact} \\
&= \max_q \{v(q) : q \in \cup_{L=1}^{c} D(L)\} && \text{[by (3.8)].}
\end{aligned} \tag{3.9}
$$

As an aside, we note that the second inequality above becomes an equality for n so large that $|\widehat{\mathcal{P}}(n)| + k - 1 \geq c$.

By (3.9) and Lemma 1, we see that

$$H_n \leq \max_q \{H(p) : p = (q, q, q, \dots), q \in \cup_{L=1}^{c} D(L)\}\,. \tag{3.10}$$

By (3.6) and (3.10) it follows that

$$\max_q \{H(p) : p = (q, q, q, \dots), q \in \cup_{L=1}^{c} D(L)\} \geq H^*$$

and hence

$$\max_q \{H(p) : p = (q, q, q, \dots), q \in \cup_{L=1}^{c} D(L)\} = H^*. \tag{3.11}$$

Therefore there exists a decreasing cyclic policy that is an optimal solution to (3.1) when $\mathcal{P} = [0, \bar{P}]$. In particular, any policy that attains the maximum in the optimization problem on the left side of (3.11) solves (3.1) for $\mathcal{P} = [0, \bar{P}]$.

□

3.4 Optimal Policy for a Special Case

Ahn et al. (2007) derived the optimal policy in a special case that customers are willing to wait just one period and have uniform valuation distribution. Their method apparently does not readily extend to cases with $k > 1$. In this section, we propose another method to analyze the optimal policy when k is relatively small. Our approach works for $k = 2$ (and also $k = 1$) and can possibly adapted to work for other "small" values of k.

Through this section, we assume that all the customers have patience level $k = 2$ and their valuation has uniform distribution on the support of [0,1]. (Readers will find it is easy to apply the same method to another values of k.) Our main result here is that a decreasing cyclic policy of length $3 = k + 1$ is optimal.

By Theorem 2, it suffices to solve the following optimization problem.

$$\max_{q}\{v(q) : q \in \cup_{L\geq 1} D(L)\} \tag{3.12}$$

Recall that $q^j \in \arg\max_{q\in D(j)} V(q)$ denotes the optimal decreasing sequence among all the decreasing sequences with length j. Hence, for any $t \geq 3$,

$$\rho_t(q^j) = q_t^j\{1 - q_t^j + 2(q_{t-1}^j - q_t^j)\}.$$

The idea is as follows. For any n-period decreasing sequence A, we construct a $n-1$-period decreasing sequence B and argue that the total revenue of sequence A is less than the sum of that of B and the average revenue of the optimal three-period decreasing sequence. Thus $V(q^n) \leq V(q^{n-1}) + v(q^3)$. Then by induction, we are able to show $v(q^3) \geq v(q^n)$, $\forall n \geq 4$. And it is easy to show $v(q^3) > v(q^2) > v(q^1)$. Therefore, the optimality of three-period decreasing sequence is proved.

Lemma 9. *For $n \geq 3$, the optimal decreasing price policy $\{q^n \in \arg\max_{q \in D(n)} V(q)\}$ is unique.*

Proof.

$$V(q^n) = \sum_{t=1}^{n} \rho_t(q^n) = q_1^n(1 - q_1^n) + q_2^n[1 - q_2^n + q_1^n - q_2^n]$$
$$+ q_3^n[1 - q_3^n + 2(q_2^n - q_3^n)] + \cdots + q_n^n[1 - q_n^n + 2(q_{n-1}^n - q_n^n)]$$

By taking derivatives, we get the hessian matrix of $V(q^n)$ is as follows:

$$\begin{pmatrix} -2 & 1 & & & & \\ 1 & -4 & 2 & & & \\ & 2 & -6 & 2 & & \\ & & & \ddots & & \\ & & & 2 & -6 & 2 \\ & & & & 2 & -6 \end{pmatrix}$$

It is not hard to verify this matrix is negative definite, which implies the strict concavity of $V(q^n)$. Hence the optimal solution q^n is unique. □

The above lemma establishes the uniqueness of optimal decreasing sequence, therefore, $q^n \in \arg\max_{q \in D(n)} V(q)$ can be written as $q^n = \arg\max_{q \in D(n)} V(q)$.

Proposition 10. *Consider the optimal decreasing price policies $\{q^n = \arg\max_{q \in D(n)} V(q), n = 3, 4, \ldots\}$. Then $q_3^n \leq q_3^{n+1}$ for all $n \geq 3$.*

Proof. We shall first show that if $q_3^{n+1} < q_3^n$, then it must be that $q_4^{n+1} < q_4^n$.

Suppose $q_3^{n+1} < q_3^n$. For a contradiction, suppose also that $q_4^{n+1} \geq q_4^n$. Note that $q_3^n > q_3^{n+1} \geq q_4^{n+1} \geq q_4^n$.

Consider the policy $\widetilde{q} = (\widetilde{q}_1, \ldots, \widetilde{q}_{n+1}) \in D(n+1)$ given by

$$\widetilde{q}_t = \begin{cases} q_t^n & \text{for} \quad t = 1, \ldots, 3 \\ q_t^{n+1} & \text{for} \quad t = 4, \ldots, n+1 \end{cases}$$

From the definition of $\widetilde{q}$ and q^{n+1} that,

$$0 < V(q^{n+1}) - V(\widetilde{q}) = \sum_{t=1}^{4} \rho_t(q^{n+1}) - \sum_{t=1}^{4} \rho_t(\widetilde{q}) \tag{3.13}$$

Consider now the policy $\widehat{q} = (\widehat{q}_1, \ldots, \widehat{q}_n) \in D(n)$ given by

$$\widehat{q}_t = \begin{cases} q_t^{n+1} & \text{for} \quad t = 1, 2, 3 \\ q_t^n & \text{for} \quad t = 4, \ldots, n \end{cases}$$

Observe next that

$$\begin{aligned} &\rho_4(q^n) - \rho_4(\widehat{q}) - [\rho_4(\widetilde{q}^n) - \rho_4(q^{n+1})] \\ &= 2q_4^n(\widetilde{q}_3 - q_3^{n+1}) + 2q_4^{n+1}(q_3^{n+1} - \widetilde{q}_3) \\ &= 2(q_3^n - q_3^{n+1})(q_4^n - q_4^{n+1}) \\ &\leq 0 \end{aligned}$$

Hence,

$$\rho_4(q^n) - \rho_4(\widehat{q}) \leq \rho_4(\widetilde{q}^n) - \rho_4(q^{n+1}) \tag{3.14}$$

Then

$$\begin{aligned} 0 &< V(q^n) - V(\widehat{q}) \\ &= \sum_{t=1}^{4} \rho_t(q^n) - \sum_{t=1}^{4} \rho_t(\widehat{q}) \\ &= \Big[\sum_{t=1}^{3} \rho_t(q^n) + \rho_4(q^n) \Big] - \Big[\sum_{t=1}^{3} \rho_t(\widehat{q}) + \rho_4(\widehat{q}) \Big] \end{aligned}$$

$$= \Big[\sum_{t=1}^{3} \rho_t(q^n) - \sum_{t=1}^{3} \rho_t(q^{n+1}) \Big] + \Big[\rho_4(q^n) - \rho_4(\widehat{q}) \Big] \qquad \text{[by the definition of } \widehat{q}\text{]}$$

$$\leq \Big[\sum_{t=1}^{3} \rho_t(q^n) - \sum_{t=1}^{3} \rho_t(q^{n+1}) \Big] + \Big[\rho_4(\widetilde{q}) - \rho_4(q^{n+1}) \Big] \quad \text{[by (3.14)]}$$

$$= \sum_{t=1}^{4} \rho_t(\widetilde{q}) - \sum_{t=1}^{4} \rho_t(q^{n+1}) \qquad \text{[by the definition of } \widetilde{q}\text{]}$$

That is,

$$\sum_{t=1}^{4} \rho_t(\widetilde{q}) - \sum_{t=1}^{4} \rho_t(q^{n+1}) > 0,$$

which contradicts with (3.13).

By similar analysis, we have if $q_k^{n+1} < q_k^n$, then $q_{k+1}^{n+1} < q_{k+1}^n$ for $k = 3, \ldots, n-1$. Especially,

$$q_n^{n+1} < q_n^n. \tag{3.15}$$

Consider now the policy $\bar{q} = (\bar{q}_1, \bar{q}_2, \ldots, \bar{q}_{n+1}) \in D(n+1)$ given by

$$\bar{q}_t = \begin{cases} q_t^n & \text{for} \quad t = 1, \ldots, n \\ q_t^{n+1} & \text{for} \quad t = n+1 \end{cases}$$

Therefore,

$$V(\bar{q}) - V(q^{n+1})$$

$$= \sum_{t=1}^{n} \rho_t(\bar{q}) + \rho_{n+1}(\bar{q}) - [\sum_{t=1}^{n} \rho_t(q^{n+1}) + \rho_{n+1}(q^{n+1})]$$

$$= \sum_{t=1}^{n} \rho_t(q^n) + \rho_{n+1}(\bar{q}) - [\sum_{t=1}^{n} \rho_t(q^{n+1}) + \rho_{n+1}(q^{n+1})] \quad \text{[By the definition of } \bar{q}\text{]}$$

$$> \rho_{n+1}(\bar{q}) - \rho_{n+1}(q^{n+1}) \qquad \text{[By the optimality of } q^n\text{]}$$

$$= 2q_{n+1}^{n+1}(q_n^n - q_n^{n+1})$$

> 0 [By (3.15)]

which contradicts with the definition of q^{n+1}.

□

Lemma 10. *1.* $v(q^1) < v(q^2) < v(q^4) < v(q^3)$ *;*

2. $q_3^5 = 66/115$.

Proof. By Lemma 9, we know $V(q^n)$ is jointly concave. By taking derivatives, we get

$$v(q^1) = \frac{1}{4}, v(q^2) = \frac{2}{7}, v(q^3) = \frac{89.25}{289}, v(q^4) = \frac{297}{968}.$$

When $n = 5$, we solved that $q_3^5 = 66/115$. □

Lemma 11. *For* $n \geq 5$*, consider* $q^n = \arg\max_{q \in D(n)} V(q)$*. Define* $\widetilde{q}=(\widetilde{q}_1,\widetilde{q}_2,\ldots,\widetilde{q}_{n-1})$ *as following*

$$\widetilde{q}_t = \begin{cases} \frac{1+q_3^n}{2} & \text{for} \quad t = 1 \\ q_{t+1}^n & \text{for} \quad t = 2, \ldots, n-1 \end{cases}$$

Then

$$V(q^n) - V(\widetilde{q}) < v(q^3). \tag{3.16}$$

Proof. By the definition of $\widetilde{q}$, it is easy to see that $\rho_t(q^n) = \rho_{t-1}(\widetilde{q})$ for $t = 4, \ldots, n$. Hence,

$$\begin{aligned} & V(q^n) - V(\widetilde{q}) \\ & = \rho_1(q^n) + \rho_2(q^n) + \rho_3(q^n) - [\rho_1(\widetilde{q}) + \rho_2(\widetilde{q})] \\ & = q_1^n(1 - q_1^n) + q_2^n\big[1 - q_2^n + (q_1^n - q_2^n)\big] + q_3^n\big[1 - q_3^n + 2(q_2^n - q_3^n)\big] \\ & \quad - \frac{1 + q_3^n}{2}\Big(1 - \frac{1 + q_3^n}{2}\Big) - q_3^n\Big[1 - q_3^n + \frac{1 + q_3^n}{2} - q_3^n\Big] \end{aligned}$$

$$= q_1^n(1-q_1^n) + q_2^n\big[1 - q_2^n + (q_1^n - q_2^n)\big] + q_3^n\Big(-\frac{5}{4}q_3^n + 2q_2^n - \frac{1}{2}\Big) - \frac{1}{4}$$
$$=: G(q_1^n, q_2^n, q_3^n)$$

By taking derivatives, we get the hessian matrix of the function G is:

$$\begin{pmatrix} -2 & 1 & 0 \\ 1 & -4 & 2 \\ 0 & 2 & -5/2 \end{pmatrix}$$

It is not hard to verify this matrix is negative definite, hence $G(q_1^n, q_2^n, q_3^n)$ is jointly concave. By Lemma 10, we know $q_1^5 > q_2^5 > q_3^5 = \frac{66}{115}$. Thus, by Proposition 10, we have that $q_1^n \geq q_2^n \geq q_3^n \geq q_3^5 = \frac{66}{115}$ for $n \geq 5$. To complete the proof, we will next show that

$$G^* := \max\{G(q_1, q_2, q_3) : q_1 \geq q_2 \geq q_3 \geq \frac{66}{115}\} < v(q^3).$$

From the KKT conditions, we get $G^* = \frac{358288}{648025} - \frac{1}{4} < \frac{89.25}{289} = v(q^3)$. This completes the proof. □

Theorem 3. $v(q^3) > v(q^n)$ *for* $n \neq 3$. *That is, the best three-period decreasing sequence is optimal to problem* (3.12).

Proof. By Lemma 10, it suffices to show $v(q^3) > v(q^n)$, for $n \geq 4$. We will prove this by induction. For the base case $n = 4$, we have $v(q^3) > v(q^4)$ by Lemma 10. For arbitrary $n \geq 5$, suppose $v(q^3) > v(q^{n-1})$ and consider $\widetilde{q}$ defined in Lemma 11, then

$$\begin{aligned} v(q^n) &= \frac{V(q^n)}{n} \\ &< \frac{V(\widetilde{q}) + v(q^3)}{n} && \text{[by Lemma 11]} \end{aligned}$$

$$\leq \frac{(n-1)v(q^{n-1}) + v(q^3)}{n} \qquad \text{[by the definition of } q^{n-1}\text{]}$$

$$< v(q^3) \qquad \text{[by } v(q^3) > v(q^{n-1})\text{]}$$

This completes the proof. □

Chapter 4

Learning and Pricing with Patient Customers

4.1 Introduction

In dynamic pricing and revenue management areas, much research assumes that the seller has full information about customers' demand functions (such as function parameters, parametric families, etc) or population-level characteristics (such as the fraction of strategic customers). However, in reality, this is impossible even with the help of current technology. Hence, in most cases, the decision maker has to make pricing or allocation decisions without knowing such information. Thus, lack of information raises several fundamental research questions.

1. How to estimate revenue loss due to imperfect information? How to quantify the value of full information?

2. Is there any way to make revenue loss as small as possible by combining

learning and pricing strategies?

In this chapter, we study a finite horizon learning and pricing problem without knowing the fraction of patient customers. We assume customers are divided into two groups, myopic customers and patient customers. The myopic customers make purchase only when the price offered in the period they arrived is lower than their valuation, while the patient customers will wait up to some amount of periods if the current period's price is larger than their valuation. Each period a stochastic amount of customers arrives and they have heterogeneous patience levels. We will propose some algorithms combining learning and pricing to minimize the revenue loss compared to the optimal revenue collected by a clairvoyant who knows the fraction of patient customers in advance.

4.2 Literature Review

In recent years there is a growing stream of research in learning and pricing area. This work focuses on balancing learning parameter values and gaining revenue.

Besbes and Zeevi (2009) study a single-product pricing problem with finite initial inventory. They assume the seller does not know the underlying functional relationship between price and mean demand rate. For the case of non-parametric learning, they develop a policy achieving a bound of $O(n^{-1/4})$ for the revenue loss. For the parametric case, they obtain a bound of $O(n^{-1/3})$. In addition, they derive a lower bound of $O(n^{-1/2})$ for both cases. Wang et al. (2014) study the same problem and propose a learning-while-doing algorithm, which can achieve a regret of $O(n^{-1/2})$, thus closing the gap between upper bound and lower bound of this problem. This algorithm only involves function value estimation to achieve a near-optimal performance.

Besbes and Zeevi (2012) extend their method to a general class of network revenue management problems, where the mean demand is determined by a vector of prices. In a discrete feasible price setting, they derived a regret of $O(n^{-1/3})$. In a continuous price setting, a regret of $O(n^{-1/(d+3)})$ is obtained, where d denotes the number of products. They show the asymptotic optimality of their algorithm as the volume of sales increases.

Border and Rusmevichientong (2012) consider a dynamic pricing problem in which the seller faces a sequence of T customers. The parameters of a general parametric choice model are unknown to the seller who wants to minimize the regret. They show that the regret of the optimal pricing strategy is $O(\sqrt{T})$ for the general case and reduce the regret to $O(log(T))$ when the demand curves satisfy a "well-separated" condition. Keskin and Zeevi (2014) show regret of $O(\sqrt{T})$ if the seller knows nothing about the parameters of the demand curve, and regret of $O(log(T))$ if he knows the expected demand under an incumbent price.

Den Boer and Zwart (2014) propose a controlled variance pricing strategy which enhances the certainty equivalent policy by constructing a lower bound on the sample variance of the chosen prices. Thus, they can control the speed at which the prices converge and collect enough new information before it converges. They demonstrate that the value of the optimal price will be learned and the regret is $O(T^{-1/2+\delta})$, where δ is positive and can be arbitrarily small.

Harrison et al. (2012) study a problem in which the seller needs to offer prices sequentially to a stream of customers. Different from the above models, the authors impose a binary assumption on the demand model with an initial probability. They argue the myopic Bayesian policy can lead to incomplete learning and thus incur large revenue loss. To solve this problem, a constrained variant of the myopic Bayesian policy is proposed and they demonstrated that the revenue loss is bounded by a constant as the number of sales attempts becomes large.

There is one common assumption that customers are myopic in all the above papers. In this chapter, we incorporate patient customer behavior into the learning and pricing problem, trying to see what kind of regret result we can obtain and what effect does the customer behavior have on the learning process. In section 4.3, I establish our model and introduce some notation. In section 4.4 and 4.5, by assuming deterministic customer arrivals and homogeneous patience levels, I propose an algorithm and demonstrate a result of regret $O(\sqrt{T})$. Section 4.6 describes my future research directions.

4.3 Model Description

I study a finite horizon discrete time learning and pricing problem. In each period t, N customers arrive, where N is a random variable with some distribution. If the price offered in period t is larger than customer's valuation, he will wait in the market up to some number of periods and make a purchase as soon as the price falls below his valuation. We assume $V_t(i)$ to be the value of ith customer arrival in period t and say a customer has patience level j $(0 \leq j \leq k)$ if this customer is willing to wait up to j periods in the market. Let $Z_t(i)$ denote the ith customer's patience level in period t and $P(Z_t(i) = j)$ denote the probability that customer's patience level is j. To simplify notation, we let $\alpha_j = P(Z_t(i) = j)$ for each j. Hence $\sum_{j=0}^{k} \alpha_j = \sum_{j=0}^{k} P(Z_t(i) = j) = 1$.

The firm has no inventory considerations. I use S_t to denote the demand function in period t. Thus, for a pricing policy $p = (p_1, p_2, \dots)$, I have

$$S_t = \sum_{i=1}^{N} 1_{\{V_t(i) \geq p_t\}} + \sum_{j=1}^{k} \sum_{i=1}^{N} 1_{\{p_t \leq V_{t-j}(i) < \min\{p_{t-1}, \dots, p_{t-j}\}, Z_{t-j}(i) \geq j\}} \tag{4.1}$$

I use π to denote the pricing policy which implies a price process $(p_t, 1 \leq t \leq T)$. Let $J(\pi^*, T|\alpha)$ denote the optimal revenue given α is known, $J(\widehat{\pi}, T; \alpha)$ be the revenue accrued by implementing policy $\widehat{\pi}$ given the estimated parameter $\widehat{\alpha}$ while the underlying true value is α. Therefore,

$$J(\pi^*, T|\alpha) = \sup_{\pi \in \mathcal{P}^T} J(\pi, T|\alpha) = E[\sum_{t=1}^{T} S_t \pi_t^*].$$

$$J(\widehat{\pi}, T; \alpha) = E[\sum_{t=1}^{T} S_t \widehat{\pi}_t].$$

Assuming the true value of parameter is α^*, my objective is

$$\inf_{\widehat{\pi}} \sup_{\alpha^*} J(\pi^*, T|\alpha^*) - J(\widehat{\pi}, T; \alpha^*) \tag{4.2}$$

I assume that a fixed number of customers arrive each period and all the patient customers have the same fixed patience level k. That is, $N = n$ and $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_k) = (\alpha_0, 0, \dots, 0, \alpha_k)$. To simplify notation, I let $\alpha_k = \alpha$, and thus $\alpha_0 = 1 - \alpha$. We also assume valuation $V_t(i)$ and patience level $Z_t(i)$ are independent. With the assumption of fixed patience level and independence, I have the expected demand each period $\lambda_t(p, \alpha)$ as follows

$$\lambda_t(p, \alpha) = n[1 - F(p_t)] + n\alpha \sum_{i=1}^{k} \Big[F(\min\{p_{t-i}, \dots, p_{t-1}\}) - F(p_t)\Big]^+$$

4.4 Algorithm

Algorithm $\widehat{\pi}^l$:

Step 1:

(a) Set the number of steps to be l and define $\Delta^{(i)}$, $i = 1, \ldots, l$ such that $\sum_{i=1}^{l} \Delta^{(i)} = T - (l-1)$.

(b) Choose a two-period decreasing sequence $\widehat{p}^1 = (p_1, p_2)$.

Step 2:

Set $t_1 = 0$.

For $i = 1, \ldots, l$.

(a) Let $L^i = L(\widehat{p}^i)$, $\eta_i = \lfloor \frac{\Delta^{(i)}}{L^i} \rfloor$, and $\delta_i = \Delta^{(i)} - \eta_i L^i$. Apply $\widehat{p}^i$ on the interval $[t_i + 1, t_i + \eta_i L^i]$. Define $Y_j^i = \sum_{m=1}^{L^i} S_{t_i+m+(j-1)L^i}$ where $j = 1, \ldots, \eta_i$.

(b) Compute

$$\widehat{\alpha}^i = \frac{\frac{1}{\eta_i}\sum_{j=1}^{\eta_i} Y_j^i - \sum_{t=1}^{L^i} n[1 - F(\widehat{p}_t^i)]}{\sum_{t=1}^{L^i} \min\{t-1, k\} n[F(\widehat{p}_{t-1}^i) - F(\widehat{p}_t^i)]}$$

and the optimal decreasing sequence $\widehat{p}^{i+1}$ given $\widehat{\alpha}^i$.

(c) Apply a δ_i-period decreasing sequence to the interval $[t_i + \eta_i L^i + 1, \ldots, t_i + \Delta^{(i)}]$, and apply $\underline{p}$ to the period $t_i + \Delta^{(i)} + 1$. Set $t_{i+1} = t_i + \Delta^{(i)} + 1$.

End for.

On interval $[t_i + 1, t_i + \Delta^{(i)}]$, we apply the decreasing price sequence $\widehat{p}^i$ for the number of η_i times on interval $[t_i + 1, t_i + \eta_i L^i]$ (Y_j^i is the total demand within each cycle) and apply another decreasing sequence to $[t_i + \eta_i L^i + 1, t_i + \Delta^{(i)}]$. To avoid the customers who arrived in phase $\Delta^{(i)}$ to buy product in phase $\Delta^{(i+1)}$, we apply a lowest price $\underline{p}$ at the end of phase $\Delta^{(i)}$ for each i. Hence $\sum_{i=1}^{l} \Delta^{(i)} = T - (l-1)$.

4.5 Revenue Loss Estimation

Now I will evaluate $J(\pi^*, T|\alpha^*) - J(\widehat{\pi}^l, T; \alpha^*)$.

First I introduce some notations. We use $v(\lambda(p,\alpha);\alpha)$ to denote the average-revenue accrued by implementing price sequence p when the true parameter is α, $r(\lambda(p,\alpha);\alpha)$ to denote the total revenue accrued by implementing the price sequence p when the true parameter is α. That is,

$$r(\lambda(p,\alpha);\alpha) = \sum_{t=1}^{L(p)} \lambda_t(p,\alpha) p_t$$

and

$$v(\lambda(p,\alpha);\alpha) = r(\lambda(p,\alpha);\alpha)/L(p).$$

We let

$$D(L) = \{p \in \mathcal{P}^L : p_1 \geq p_2 \geq \cdots \geq p_L\}$$

denote the set of decreasing price sequences with length L.

Theorem 2 in Chapter 3 says that there exists a decreasing cyclic policy which is optimal for an infinite horizon problem ($T = \infty$). Assume $q = (p^*, p^*, p^*, \dots)$ where p^* is decreasing is an optimal policy when $T = \infty$.

If $T/L(p^*)$ is an integer, it's easy to see $\pi^* = (p^*, p^*, \dots, p^*)$, and

$$\begin{aligned} J(\pi^*, T|\alpha^*) &= E[\sum_{t=1}^{T} S_t \pi_t^*] \\ &= Tv(\lambda(p^*, \alpha^*); \alpha^*). \end{aligned}$$

Otherwise, it is not hard to see $J(\pi^*, T|\alpha^*) < Tv(\lambda(p^*,\alpha^*);\alpha^*)$ due to the finite-horizon end effect. Therefore, letting $L = L(p^*)$,

$$J(\pi^*, T|\alpha^*) \leq Tv(\lambda(p^*, \alpha^*); \alpha^*)$$

On the other hand,

$$J(\widehat{\pi}^l, T; \alpha^*) \geq E\Big[\sum_{i=2}^{l} \eta_i L^i v(\lambda(\widehat{p}^i, \alpha^*); \alpha^*)\Big].$$

Therefore,

$$\begin{aligned}
&J(\pi^*, T|\alpha^*) - J(\widehat{\pi}^l, T; \alpha^*) \\
&\leq Tv(\lambda(p^*, \alpha^*); \alpha^*) - E\Big[\sum_{i=2}^{l} \eta_i L^i v(\lambda(\widehat{p}^i, \alpha^*); \alpha^*)\Big] \\
&= \eta_1 L^1 v(\lambda(p^*, \alpha^*); \alpha^*) + E\Big[\sum_{i=2}^{l} \eta_i L^i \big(v(\lambda(p^*, \alpha^*); \alpha^*) - v(\lambda(\widehat{p}^i, \alpha^*); \alpha^*)\big)\Big] \\
&\quad + (\sum_{i=1}^{l} \delta_i + l - 1) v(\lambda(p^*, \alpha^*); \alpha^*) \\
&\leq \eta_1 L^1 n\bar{p} + \sum_{i=2}^{l} \eta_i L^i E[v(\lambda(p^*, \alpha^*); \alpha^*) - v(\lambda(\widehat{p}^i, \alpha^*); \alpha^*)] + (\sum_{i=1}^{l} \delta_i + l - 1) n\bar{p}
\end{aligned} \tag{4.3}$$

I will take the task of evaluating $E[v(\lambda(p^*, \alpha^*); \alpha^*) - v(\lambda(\widehat{p}, \alpha^*); \alpha^*)]$ next.

Lemma 12.

$$v(\lambda(p^*, \alpha^*); \alpha^*) - v(\lambda(\widehat{p}^i, \alpha^*); \alpha^*) \leq A_i |\widehat{\alpha}^i - \alpha^*| \tag{4.4}$$

where

$$\begin{aligned}
A_i = &\frac{n\bar{p}}{L(p^*)} \sum_{t=1}^{L(p^*)} \min\{t-1, k\}[F(p^*_{t-1}) - F(p^*_t)] \\
&+ \frac{n\bar{p}}{L(\widehat{p}^i)} \sum_{t=1}^{L(\widehat{p}^i)} \min\{t-1, k\}[F(\widehat{p}^i_{t-1}) - F(\widehat{p}^i_t)]
\end{aligned}$$

Proof. For any $p \in D(L)$ with $L = k+1$ or $k+2$,

$$\lambda_t(p, \alpha) - \lambda_t(p, \alpha')$$

$$= n(\alpha - \alpha') \sum_{i=1}^{k} \Big[F(\min\{p_{t-i}, \ldots, p_{t-1}\}) - F(p_t)\Big]^+$$

$$= n(\alpha - \alpha') \min\{k, t-1\}[F(p_{t-1}) - F(p_t)] \qquad \text{[by } p \text{ is decreasing]} \tag{4.5}$$

Therefore,

$$\begin{aligned}
&r(\lambda(p, \alpha); \alpha) - r(\lambda(p, \alpha'); \alpha') \\
&= \sum_{t=1}^{L} \lambda_t(p, \alpha) p_t - \sum_{t=1}^{L} \lambda_t(p, \alpha') p_t = \sum_{t=1}^{L} p_t[\lambda_t(p, \alpha) - \lambda_t(p, \alpha')] \\
&\leq \bar{p} n(\alpha - \alpha') \sum_{t=1}^{L} \min\{k, t-1\}[F(p_{t-1}) - F(p_t)] \qquad \text{[by (4.5)]} \\
&= C_1(\alpha - \alpha')
\end{aligned} \tag{4.6}$$

where $C_1 = n\bar{p} \sum_{t=1}^{L} \min\{k, t-1\}[F(p_{t-1}) - F(p_t)]$.

Hence, by letting $L = L(p)$, it follows

$$\begin{aligned}
v(\lambda(p, \alpha); \alpha) - v(\lambda(p, \alpha'); \alpha') &= \frac{1}{L}[r(\lambda(p, \alpha); \alpha) - r(\lambda(p, \alpha'); \alpha')] \\
&\leq \frac{C_1}{L}(\alpha - \alpha') = C_2(\alpha - \alpha')
\end{aligned} \tag{4.7}$$

where $C_2 = C_1/L$.

Thus,

$$\begin{aligned}
&v(\lambda(p^*, \alpha^*); \alpha^*) - v(\lambda(\widehat{p}, \alpha^*); \alpha^*) \\
&= v(\lambda(p^*, \alpha^*); \alpha^*) - v(\lambda(p^*, \widehat{\alpha}); \widehat{\alpha}) + v(\lambda(p^*, \widehat{\alpha}); \widehat{\alpha}) - v(\lambda(\widehat{p}, \widehat{\alpha}); \widehat{\alpha}) \\
&\quad + v(\lambda(\widehat{p}, \widehat{\alpha}); \widehat{\alpha}) - v(\lambda(\widehat{p}, \alpha^*); \alpha^*) \\
&\leq v(\lambda(p^*, \alpha^*); \alpha^*) - v(\lambda(p^*, \widehat{\alpha}); \widehat{\alpha}) + v(\lambda(\widehat{p}, \widehat{\alpha}); \widehat{\alpha}) - v(\lambda(\widehat{p}, \alpha^*); \alpha^*) \\
&\leq C_3|\widehat{\alpha} - \alpha^*| + C_4|\widehat{\alpha} - \alpha^*| \qquad \text{[by (4.7)]} \\
&= C_5|\widehat{\alpha} - \alpha^*|
\end{aligned} \tag{4.8}$$

where

$$C_3 = \frac{n\bar{p}}{L}\sum_{t=1}^{L}\min\{k,t-1\}[F(p^*_{t-1}) - F(p^*_t)]$$

$$C_4 = \frac{n\bar{p}}{L(\widehat{p})}\sum_{t=1}^{L(\widehat{p})}\min\{k,t-1\}[F(\widehat{p}_{t-1}) - F(\widehat{p}_t)]$$

$$C_5 = C_3 + C_4$$

and the first inequality above holds because $v(\lambda(p^*,\widehat{\alpha});\widehat{\alpha}) \leq v(\lambda(\widehat{p},\widehat{\alpha});\widehat{\alpha})$.

□

Next I will evaluate $|\widehat{\alpha}^i - \alpha^*|$.

Lemma 13.

$$E[|\widehat{\alpha}^i - \alpha^*|] \leq B_i\eta_i^{-1/2}$$

for some suitable B_i.

Proof. By the definition of Y^i_j, I have

$$\begin{aligned} E[Y^i_j] &= E[Y^i_1] = E[S_{t_i+1} + S_{t_i+2} + \cdots + S_{t_i+L^i}] \\ &= \sum_{t=1}^{L^i} n[1 - F(\widehat{p}^i_t)] + \sum_{t=1}^{L^i}\min\{t-1,k\}n\alpha^*[F(\widehat{p}^i_{t-1}) - F(\widehat{p}^i_t)] \end{aligned}$$

Since $Y^i_1, Y^i_2, \ldots, Y^i_{\eta_i}$ are iid observations such that $E[Y^i_j]$ equals to the above formula and $0 \leq Y^i_j \leq L^i n$, $j = 1, \ldots, \eta_i$, then for any $\epsilon > 0$, we have

$$P\Big(\big|\frac{1}{\eta_i}\sum_{1}^{\eta_i} Y^i_j - E[Y^i_j]\big| > \epsilon\Big) \leq 2e^{-2\eta_i\epsilon^2/n^2(L^i)^2}.$$

Through some algebra, we have

$$P\Big(\big|\widehat{\alpha}^i - \alpha^*\big|\sum_{t=1}^{L^i}\min\{t-1,k\}n[F(\widehat{p}^i_{t-1}) - F(\widehat{p}^i_t)] > \epsilon\Big) \leq 2e^{-2\eta_i\epsilon^2/n^2(L^i)^2}.$$

Hence,

$$
\begin{aligned}
E[|\widehat{\alpha}^i - \alpha^*|] &= \int_0^\infty P(|\widehat{\alpha}^i - \alpha^*| > s)ds \\
&\le a + \int_0^\infty 2e^{-2\eta_i s^2(\mu^i)^2/(L^i)^2} ds \\
&= a + \frac{e^{-2\eta_i a^2(\mu^i)^2/(L^i)^2}}{2a\eta_i(\mu^i)^2/(L^i)^2} \\
&= B_i \eta_i^{-1/2} \qquad [\text{taking } a = \eta_i^{-1/2}]
\end{aligned} \tag{4.9}
$$

□

Theorem 4. *The revenue loss are bounded by below*

$$
J(\pi^*, T|\alpha^*) - J(\widehat{\pi}^l, T; \alpha^*) \le O(T^{1/2}) \tag{4.10}
$$

Proof. Plugging (4.4) and (4.9) into (4.3), we have

$$
\begin{aligned}
&J(\pi^*, T|\alpha^*) - J(\widehat{\pi}^l, T; \alpha^*) \\
&\le \eta_1 L^1 n\bar{p} + \sum_{i=2}^{l} \eta_i L^i A_i B_i \eta_i^{-1/2} + (l - 1 + \sum_{i=1}^{l} \delta_i) n\bar{p} \\
&\le \Delta^{(1)} n\bar{p} + \sum_{i=2}^{l} D_i (\Delta^{(i)})^{1/2} + (l - 1 + \sum_{i=1}^{l} \delta_i) n\bar{p} \\
&= O(T^{1/2})
\end{aligned}
$$

by taking $\Delta^{(1)} = T^X$ where $X \le 1/2$, and $\Delta^{(i)} = \beta_i(T - l + 1 - \Delta^{(1)})$ in which $\sum_{i=2}^{l} \beta_i = 1$. □

Chapter 5

Revenue Management with Consumer Research Costs

5.1 Introduction

It is common for customers to search for information about products before making purchase decisions. Besides testing or examining a product, customers may read online descriptions, or reviews, or even request expert opinions. We refer to such activities as researching a product. A 2010 survey from Zillow Mortgage Marketplace (Zillow.com 2010) reveals that on average customers in America spend 40 hours for a new home, 10 hours for a major home improvement, 10 hours for a car, 5 hours for a vacation or a mortgage, 4 hours for a computer, and 2 hours for a television set.

The research process may come with a cost, which may include a time cost and information processing cost. When customers have to incur such research costs prior to making a purchase, they may refrain from information search, and purchase immediately without conducting research if their expected utility is high. That is, customers may weigh the benefits and costs of researching to maximize their utility. It is easy to understand a customer would rather not search anymore if she is already sure that a particular product can satisfy most of her needs and the cost of further research is high. For example, a scientific researcher might decide to buy a laptop if her main need, the computing ability, is satisfied, and thus she would be reluctant to spend time and energy researching.

In practice, retailers can control customer research costs to some extent, especially online retailers. They can decrease or increase the research cost by disclosing or hiding more product information. Traditionally, they may describe product attributes through advertisements, phones, posters, and product brochures. Recent developments in information technology have greatly reduced customers' cost of acquiring information. The sellers can post online instructions, customer reviews, or even make videos or trial versions of products enabling customers to have a free test. For example, the movie seller can post a trailer of that movie, the video game seller can offer a 10-minutes free-playing demo, and software retailers can provide a 3-days trial version, through which customers can gain a direct feeling about whether they like a movie, video game, or software, and the research cost is much less than making a phone call, reading instructions and listening to a salesperson's explanation.

The cost (or lack thereof) to a customer in time, money, or effort to research a product may affect that customer's purchase decisions. Firms may consider it as a good way to differentiate its products (even homogeneous products) by assigning different search costs to different products. For example, on Best Buy's website, even for the video games under the same genre and developed by the same company, some have free trailers but others do not. Dell uploads videos for the most popular PC models but not for others. While there may be some reasons for these differences, but a possible reason behind this phenomenon is that the firm expects to differentiate the products.

Given that customers may behave differently under the existence of research cost, how should retailers manage the research cost to affect customers' behavior to achieve their revenue-maximizing objective? What are the optimal pricing decisions under different research cost scenarios? And how does the optimal revenue change with feature uncertainty and customer research cost? This chapter will take up the task of answering these questions.

We consider a revenue management problem in which customers face uncertainty about whether they will like products under consideration. Customers can either make a purchase decision based on the expected utility, or resolve the uncertainty by incurring a research cost. Customers need to decide whether to engage in search by comparing the expected utilities of each action. We characterize a customer's optimal policy, based on which we study the optimal pricing decisions for the seller. We show that, somewhat surprisingly, the firm's revenue may increase with an increase in customer's research cost. The reason is as follows. When customers' research cost is large enough, customers would refrain from search and prefer to purchase directly. Thus, without incurring a research cost, the customers are willing to pay a relatively higher price than they would with a payment of the research cost. Therefore, the seller's revenue may increase with customers' research cost, which suggests the retailers that reducing research cost is not always beneficial.

We also consider a problem with two substitutable products in which customers not only need to decide whether to research, but also need to decide in which order to do so. We study how the seller should manage the research cost of each product to influence customers' action to maximize its revenue. In the setting with two homogeneous products, we study three scenarios: (1) both products have research cost (it can be considered as two uncertain products, hereafter denoted by (U, U)); (2) only one product has research cost (it can be considered as one uncertain and one certain product, hereafter denoted by (U, C)); (3) both products have no research cost (it can be considered as two certain products, hereafter denoted by (C, C)). In each scenario, based on customers' optimal behavior, we derive the optimal pricing decisions with the objective to maximize the seller's total revenue. Then we compare the optimal revenue accrued in these three different scenarios to see which one is the best. We find that when the features of the two products are independent, the revenue accrued by offering (U, U) is always smaller than that by offering (U, C). One reason is that by assigning a different research cost (i.e. zero and a positive value) to the otherwise homogeneous products, the seller can better differentiate the customers. The homogeneous products with different search costs become "non-homogeneous" to customers in some sense. The second reason is the value of disclosing information. When customers need to resolve the uncertainty of both products, the customers just need to pay one unit of research cost in the scenario (U, C), and thus they are willing to pay a higher price than the scenario (U, U) in which customers need to pay two units of research cost.

The insight derived from this argument is that even for homogeneous products, assigning different search costs enables the seller to better differentiate customers and obtain a higher revenue.

Moreover, we find that with feature independence, the seller can accrue a larger revenue by offering (C, C) than by offering (U, C) when the feature uncertainty is high, and a lower revenue otherwise. The underlying reason is that when the uncertainty is large, customers would always choose to search before making a purchase. Hence, disclosing all the information saves customers one unit of research cost, and thus customers are willing to pay more. However, when the uncertainty is small, search is not necessary, thus there is no information advantage in (C, C) anymore. On the other hand, offering (U, C) enables the seller to better differentiate the customers due to the different research cost assigned to each product. This result indicates that disclosing all the information is not always optimal for sellers.

In addition, we study the effect of feature correlation. With perfectly positive correlation, we find that offering (U, C) is better than offering (U, U) when uncertainty is large, and is worse otherwise. This is consistent with our intuition that disclosing one product's information (which is equal to disclosing both products' information in the presence of perfect correlation) is better because customers would always search before making a purchase when uncertainty is large. With perfectly negative correlation, offering (U, C) (no research cost) is always better than offering (U, U). Because in scenario (U, C) customers like either product 1 or product 2, charging the highest possible price for both products guarantees the seller the largest possible revenue. We also find that with general negative correlation, the revenue by offering (U, C) is always larger than that by offering (U, U), which can be explained by the same reasoning as the independence case.

The reminder of this chapter is organized as follows. Section 5.2 reviews related literature. Section 5.3 introduces the single product model and discusses the main result in this setting. Section 5.4 describes the two-product model under different research cost scenarios and contains results and discussion for cases in which the features of the two products are independent. Section 5.5 studies the effect of correlation. Section 5.6 presents some future research directions. Section 5.7 contains generalizations and proofs. Section 5.8 contains additional details of computations used in the proofs.

5.2 Literature Review

There are two papers that are closely related to our research. In the paper by Branco et al. (2012), the utility U of consuming a single product consists of two parts,

$$U = v + \sum_{i=1}^{T} x_i$$

where v is the ex ante expected utility prior to learning, and x_i is the value of the ith random attribute of the product. (Imagine a car has thousands of attributes, such as color, pattern, safety, and engine.) It is assumed all $x_i, i = 1, 2, \ldots,$ are binary taking value z and $-z$ with equal probability. In each step, the customer can incur search cost c to learn the realization of a single attribute x_i. There are infinite number of attributes to learn (T is ∞). For simplicity, the authors work with a continuous process (Imagine there are so many small-valued attributes that the change in utility from each attribute gets infinitely small as the number of attributes goes to infinity.). Then the expected utility follows a Brownian motion as customer keeps searching. The authors characterize the optimal stopping rule for customers' search process. They derive an upper bound and a lower bound such that when the expected utility hits the upper (lower) bound the customer would stop the search and purchase (not purchase) the product. In addition, the authors study the firm's optimal pricing decisions.

The other closely related paper is by Ke et al. (2014) who extend the above model to a multi-product setting. They consider a continuous setting where information about the product being researched changes according to a Brownian motion. At the beginning the consumer chooses which product to start researching. Without having complete information on the first product, she might decide to switch, and search for information on the other product. At some point, the consumer may decide to stop searching and purchase one of the products, or stop searching and leave the market without making any purchase. They investigate a consumer's optimal search, switch, and purchase or exit strategy.

There are two main differences between our research and the above two papers. The first one is that there are infinite number of attributes in the product in their models and consumers learn one attribute each time by incurring a search cost c. So the consumers gradually learn the information about the products. However, in our model, research is a one step action. By incurring a research cost c, customers learn all the unobservable information of the product. This leads to a simpler model in our case that avoids the study of diffusions. The second major difference is that operational decisions are the focus of our paper, but not theirs. They focus on characterizing the customers' optimal behavior. Although Branco et al. (2012) study the pricing decision, they do not consider how to manage the research cost. Ke et al. (2014) do not involve any pricing decision and research cost decision, whereas our model is trying to see how the seller should manage research cost and make pricing decisions under different search cost scenarios.

There is some literature in the operations management area that incorporates consumer's search cost for product quality information. Boyaci and Akcay (2015) assume customers have limited attention and capability to process the product information. They employ a new choice model called the Generalized MNL model, and study pricing decisions under a monopolist setting (one seller, one customer, one product) and a competitive setting (two sellers, one customer, two products). In the competitive setting, they also study when the quality of the two products are perfectly positively correlated and perfectly negative correlated. There is a big difference between their central assumptions and ours. They assume the firm knows the quality level of the product, either low or high, and all customers consider the product as either low quality or high quality. We do not have a true quality level, but rather assume a fraction q of customers like the product and the rest dislike the product. Another difference is that they study when customers have limited attention and do some imperfect search for information, while we assume customers can do a perfect search for information.

Cachon et al. (2006) study a model in which consumers search among multiple competing firms for products that match their preferences at a reasonable price. They focus on how easier search influences equilibrium prices, assortments, firm profits, and consumer welfare. They demonstrate that easier search exhibits a market-expansion effect that encourages firms to expand their assortment. Thus customers are more likely to find products that better match their ideal preferences, improving the efficiency of the market, which may manifest itself in higher prices, more profits, and increased welfare. In the conclusion, they argue that search costs may be partially endogenous and reducing search costs allows firms to expand their assortments.

Sahin and Wang (2014) investigate dynamic search policies given customers are facing multiple products and need to incur a search cost to resolve the uncertainty of each product. They study several heuristics and propose a consider-then-search policy. Based on the tractability of this policy, they further study the assortment problem. They argue that revenue ordered assortments fail to be optimal, but that an optimal assortment can be obtained efficiently by dynamic programming. Finally, they also study price competition and identify how the equilibrium changes as search cost increases. They find that fewer firms would serve the market when search cost is sufficiently high. Moreover, if the product quality levels are not significantly different from each other, the firms should make their product information easy to access and reduce consumer search cost. In their future research directions, they mention that each firm may have control over the search cost to some extent in practice. For example, the seller may make the information about certain products easier or harder to access, which is not uncommon in many firms, especially online sellers. Thus it would be interesting to see what the optimal policy looks like in terms of pricing and controlling search cost.

In marketing, there is much research involving consumers' search cost for product's quality information. For example, Lal and Sarvary (1999) argue that the internet reduces consumer search costs for digital attributes of a product (such as its price and dimension) but not for nondigital attributes (such as how well it fits, in the case of clothing). They study when and how the internet is likely to decrease the level of price competition between firms. In their model, customers need to gather information on two types of product attributes, digital attributes and nondigital attributes. Consumers choose between two brands but are familiar with the nondigital attributes of only the brand purchased on the last purchase occasion. Firms use traditional stores and internet to inform consumers about their products' attributes and to sell their products. They show that the impact of the internet on competition will be different depending on the relative importance of parameters describing the relevant shopping and distribution context. They also show the use of internet can not only lead to higher prices but can also discourage consumers from engaging in search.

Lynch and Ariely (2000) test conditions under which lowered search costs should increase or decrease price sensitivity. They conduct an experiment in which they varied search cost via electronic shopping (search cost for quality information within a given store). They show that for differentiated products like wines, lowering the cost of search for quality information reduces price sensitivity.

In addition, there are many discussions related to consumers' search costs effects on product prices. Brynjolfsson et al. (2011) argue that even if the same product availability and prices are offered, the sales are different between an Internet channel and traditional channel due to the lower search costs (on prices) on the Internet. Kuksov (2004) discusses the effects of changing search costs on prices when product differentiation is fixed and when it is endogenously determined in equilibrium. Bakos (1997) argues that electronic marketplaces which lower the buyer's cost to acquire information about seller prices and product offerings can reduce the ability of sellers to extract monopolistic profits and increase the ability of market to optimally allocate productive resources.

There is also research related to costly product information acquisition in economics. Chan and Leland (1982) assume that sellers need to decide the prices and quality levels of their products and customers can acquire price/quality information about individual sellers at a cost. They study the equilibrium existence under four different scenarios: (i) the price is costlessly observable, while the quality is costly observable, (ii) the price is costly observable, while the quality is costlessly observable, (iii) both price and quality are costly observable, (iv) both price and quality are costlessly observable. They show price advertising that makes price information costlessly observable can improve the social welfare.

5.3 One Product Model

We consider a monopolist selling a single product to a fixed population of infinitesimal customers. Each customer has a valuation $v + \epsilon$ for the product, where v is based on openly observable characteristics of the product and ϵ is based on features that are not openly observable to customers ("hidden"). The quantity v is the same for all customers, whereas ϵ is not. We assume that a fraction q of customers like the hidden features. For such customers, ϵ is a positive number δ. The remaining fraction $1 - q$ of customers dislike the hidden features, and for such customers, ϵ equals $-\delta$. A priori, each individual customer does not know his/her value of ϵ, but does know what fraction of customers like hidden feature. Hence, each individual customer knows a priori that his/her ϵ has distribution as follows:

$$\epsilon = \begin{cases} \delta & \text{with probability} \quad q \\ -\delta & \text{with probability} \quad 1-q. \end{cases}$$

To further illustrate the meaning of q, consider a scenario in which a customer is considering whether to buy a popular book on Amazon.com. In addition to some observable features (such as author, abstract, etc), customer reviews show that a fraction q of customers like the book and the rest dislike the book. This forms this customer's prior knowledge about the hidden features of the book. At the same time, since there are abundant past customers who participated in customer reviews, it is reasonable to assume that customer composition in the market is truly reflected by the customer reviews. That is, the probability a given customer likes the hidden features is q.

In our basic problem, the seller's decision is the price p of the product. Given p, a customer's utility u of purchasing this product is $u = v - p + \epsilon$. In the following, we assume $v \geq \delta$, which means that getting the product for free always generates a positive utility. We assume that the customers know the fixed valuation v, the price p, and the distribution of ϵ.

Before deciding whether to purchase the product, an individual customer may wish to research the product to determine if s/he likes the product's hidden features. Conducting such research comes at a cost (possibly in time or effort) to the customer. To model this, we assume that by incurring a cost c, a customer can find out his/her realization of ϵ. The cost c is not collected by the seller.

Given a price p, a customer has the following options: (i) choose to research and then buy if $\epsilon = \delta$ and not buy if $\epsilon = -\delta$, (ii) choose to buy immediately without research, (iii) leave without purchasing (or researching). We refer to these options as s, b, and 0 respectively. Customers are a priori identical, so all customers make the same choice among b, s, and 0 in our model. However, if the customers select s, then some will buy the product and some will not.

Let $\mathcal{A} = \{b, s, 0\}$ be the set of the above options and $w_a(p)$ be the expected utility of a customer choosing action $a \in \mathcal{A}$ given price p. We have

$$w_a(p) = \begin{cases} q(v-p+\delta) + (1-q)(v-p-\delta) & \text{if} \quad a = b \\ -c + q(v-p+\delta) & \text{if} \quad a = s \\ 0 & \text{if} \quad a = 0. \end{cases} \tag{5.1}$$

We assume each customer maximizes his/her expected utility. Some readers may notice that customers could consider actions other than those in $\mathcal{A}$. For example, customers could choose to search and then buy the product no matter what the search result is. However, the utility of this action is $-c + q(v-p+\delta) + (1-q)(v-p-\delta)$, which is always less than $w_b(p)$. Customers might also choose to search and then buy the product if they dislike the hidden information, and leave otherwise. The utility of this action is $-c+(1-q)(v-p-\delta)$, which is always less than $\max\{w_b(p), 0\}$. Therefore, there is no loss of optimality in assuming each customer's action set is $\mathcal{A}$ as defined above, and the utilities are (5.1).

Let $a(p)$ be the optimal action of the customer given price p. Then

$$a(p) = \begin{cases} b & \text{if} \quad w_b(p) \geq w_s(p) \quad \text{and} \quad w_b(p) \geq 0 \\ s & \text{if} \quad w_s(p) > w_b(p) \quad \text{and} \quad w_s(p) \geq 0 \\ 0 & \text{if} \quad 0 > \max\{w_b(p), w_s(p)\}. \end{cases}$$

In case of "ties" we suppose for simplicity customers choose b over s and s over 0. The seller's revenue from charging price p, denoted by $r(p)$, can be written as

$$r(p) = r(p|a(p)) = \begin{cases} p & \text{if} \quad a(p) = b \\ q \cdot p & \text{if} \quad a(p) = s \\ 0 & \text{if} \quad a(p) = 0. \end{cases}$$

The seller's problem is:

$$\max_{p} \quad r(p) \tag{5.2}$$

Hereafter, we assume $q = 1/2$ to simplify the presentation. Most of the results below hold with slight modification for arbitrary $q \in (0, 1)$. These more general results are stated in section 5.7. In the following, we derive the optimal solution p^* to problem (5.2) and study the comparative statics of the optimal solution. We have the following result.

Theorem 5.

1. *If $\delta \geq \frac{1}{3}v + 2c$, then $p^* = v + \delta - 2c$. In this case, the customers' optimal action is to research the product (action s). The seller's optimal revenue $r(p^*) = \frac{1}{2}(v + \delta) - c$.*

2. *If $2c \leq \delta < \frac{1}{3}v + 2c$, then $p^* = v - \delta + 2c$. In this case, the customers' optimal action is to purchase directly without research (action b). The seller's optimal revenue $r(p^*) = v - \delta + 2c$.*

3. *If $\delta < 2c$, then $p^* = v$. In this case, the customers' optimal action is to purchase directly without research (action b). The seller's optimal revenue $r(p^*) = v$.*

Now we give some explanation of the results in Theorem 1. When δ is large (as in part 1), customers research the product before making a purchase because the value of knowing the realization of ϵ is large. In this case, the seller's optimal strategy is to choose the largest price that can keep customers' utility $w_s(p)$ positive and the consumer's surplus is fully extracted by the seller. When δ is small (as in part 3), customers opt not to research because the value of information derived from searching is not worth the cost. In this case, the seller's optimal strategy is to choose the largest price that can keep customer's utility $w_b(p)$ positive and the customer's surplus is again fully extracted by the seller. When δ is in the middle (as in part 2), both actions b and s may be possible for the customers. (Obviously the seller will not set the price that makes the customers leave without purchase, otherwise the seller will get 0.) However, by calculation, we find that inducing customers to purchase directly can generate higher revenue than inducing customers to search in this case and the optimal price for the seller is to choose the largest price that can keep $w_b(p) \geq w_s(p)$ (inducing customers to buy directly). Note that in this case customers' surplus is positive, because decreasing the price makes option b more attractive, since lowering the price will increase $w_b(p)$ by 1 while $w_s(p)$ by only 1/2.

Next we provide some comparative statics of the optimal price and the optimal revenue. We have the following. Proofs, which will follow immediately from Theorem 5, are omitted.

Proposition 11. *Both p^* and $r(p^*)$ increase with v.*

Proposition 11 is quite intuitive. It says that both the optimal price and the optimal revenue increase with the known portion of the valuation of the product.

Next we consider how the optimal revenue changes with the magnitude of the uncertainty parameter δ.

Proposition 12.

1. *If* $\delta \geq \frac{1}{3}v + 2c$, *then* $r(p^*)$ *increases in* δ.

2. *If* $2c \leq \delta < \frac{1}{3}v + 2c$, *then* $r(p^*)$ *decreases in* δ.

3. *If* $\delta < 2c$, *then* $r(p^*)$ *does not change with* δ.

Next we illustrate the results in Proposition 12. As we have argued earlier, when δ is large enough (as in part 1), customers will always prefer research to immediate purchase, and the seller's optimal strategy is to set the highest price that can keep $w_s(p)$ positive. However, in this case $w_s(p)$ depends on the value of the favorable outcome, which increases in δ. Therefore, so does the optimal price and the optimal revenue. This leads to the part 1 of Proposition 12. When δ is small, customers will never choose to search, and decide to whether to purchase based on the expected utility which equals to v. Hence the revenue does not change with δ. This leads to the third part of Proposition 12. When δ is in the middle (as in part 2), customers can choose either to search or to purchase directly. According to our earlier discussions, the seller needs to induce the customers to purchase directly. Note that as δ increases, customers' incentive to research also increases, thus the seller has to decrease the price to keep $w_b(p) \geq w_s(p)$ (note that as the price decreases, although both $w_b(p)$ and $w_s(p)$ increase, $w_b(p)$ increases faster than $w_s(p)$). Therefore, both price and revenue decrease in δ in this case.

To help visualize the results of Proposition 12, we plot $r(p^*)$ against δ in Figure 5.1 for an example. As we can see, the seller's revenue doesn't always

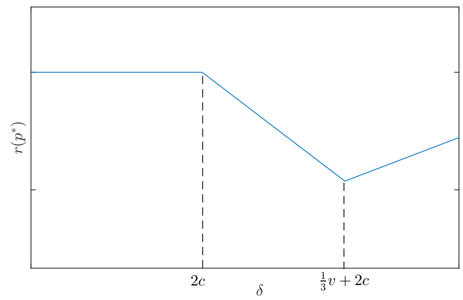


Figure 5.1: Optimal revenue as a function of δ

decrease with the uncertainty parameter δ. To the contrary, it may increase in it in certain range.

Next we study how the optimal revenue changes with the research cost c. We have the following results:

Proposition 13.

1. *If $\delta \geq \frac{1}{3}v + 2c$, then $r(p^*)$ decreases in c.*

2. *If $2c \leq \delta < \frac{1}{3}v + 2c$, then $r(p^*)$ increases in c.*

3. *If $\delta < 2c$, then $r(p^*)$ does not change with c.*

In addition, if the seller can choose the research cost, then it is optimal to set $c \geq \frac{1}{2}\delta$, i.e., to set the research cost high.

Now we explain the results in Proposition 13. When c is small (as in part 1), customers will always choose to research the product and hence their utility is decreased by the amount of the research cost. To keep the utility $w_s(p)$ of action s positive, the seller needs to lower its price to make up customers' utility. Therefore, both the price and the revenue decrease in c in this case. When c

is large enough (as in part 3), customers would never choose to research the product. Hence the revenue is independent of c in this case. When c is in the middle (as in part 2), either to research or to purchase directly may be optimal for the customers. By calculation, we find that if $\delta < \frac{1}{3}v + 2c$, then the revenue when customers research is smaller than that when customers purchase directly. Hence, in this case, the seller should induce customers to purchase directly. As c decreases in this range, customers' incentive to search increases, thus to keep customers purchasing directly, the seller has to lower the price. As a result, the optimal price and revenue increase in c in this case.

To help visualize the results of Proposition 13, we plot $r(p^*)$ against c in Figure 5.2. As we can see, the seller's revenue first decreases with research cost c then increases and reaches a steady maximum. This suggests that to obtain better revenue, a seller should either eliminate the research cost of the product, or set the research cost high enough. In our setting, the latter one leads to higher revenue since it is equivalent to a perfect price discrimination policy which effectively makes half of the customers to buy at price $v + \delta$ and the rest to buy at price $v - \delta$.

5.4 Two Product Model

In this section, we study a problem in which the monopolist sells two substitutable products (indexed by $i = 1, 2$) to customers. A customer's utility from consuming product i is denoted by

$$u_i = v_i - p_i + \epsilon_i \quad \text{for } i = 1, 2.$$

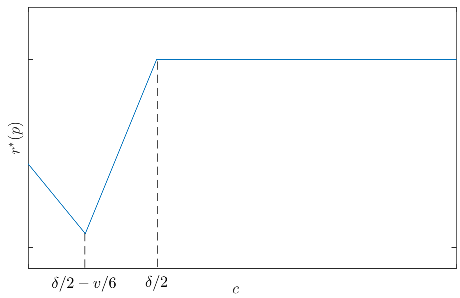


Figure 5.2: Optimal revenue as a function of c

Similar to the previous section, for each i, we have $\epsilon_i = \delta$ for a fraction q of the population and $\epsilon_i = -\delta$ for a fraction $1-q$ of the population. To simplify our analysis, we assume $v_1 = v_2 = v$ and $v \geq \delta$. We allow the customers' values of these two products to be correlated, and the fraction of the population with particular values for (ϵ_1, ϵ_2) pairs is given in Table 5.1.

Table 5.1: Fraction of customers with different values of (ϵ_1, ϵ_2)

		ϵ_2	
		δ	$-\delta$
ϵ_1	δ	β	$q-\beta$
	$-\delta$	$q-\beta$	$1-2q+\beta$

We assume that $\max\{0, 2q-1\} \leq \beta \leq q$. As before, each customer knows the information above and consequently has a joint distribution on his/her values of (ϵ_1, ϵ_2) as given by Table 5.1. From Table 5.1, it is easy to see that for $i \neq j$ we have

$$P(\epsilon_i = \delta | \epsilon_j = \delta) = \frac{\beta}{q},$$

$$P(\epsilon_i = \delta | \epsilon_j = -\delta) = \frac{q-\beta}{1-q}.$$

That is, if the customer learns that he likes the hidden features of one product, then the probability that he likes the hidden features of the other product is $\frac{\beta}{q}$. If the customer learns that he dislikes the hidden features of one product, then the probability that he likes the hidden features of the other product is $\frac{q-\beta}{1-q}$. Note that $\beta = \max\{0, 2q-1\}$ denotes the maximal possible negative correlation between ϵ_1 and ϵ_2 and $\beta = q$ corresponds a perfectly positive correlation.

In this section, we assume there are two decisions for the seller. First, the seller can choose whether to impose search costs for the uncertain (to the customers) feature of each product. If the seller does not impose the cost for a product i, then all customers will know their values of ϵ_i. To simplify our analysis, we assume the research cost is either c which corresponds to the situation that customers need to spend some amount of time and effort to research the product, or 0 which corresponds to the situation that all the information of the hidden features has been disclosed to the customers and the research cost is so small that it can be neglected. Second, he can choose the prices for each product. We are interested in how the seller should manage the research cost of these two products to achieve the largest revenue. Therefore, we consider three scenarios: (U, U) in which both products have research costs, (U, C) in which one product has research cost, and (C, C) in which neither product has research cost. Here, "U" stands for uncertain and "C" stands for certain. Our objective is to find which scenario is optimal (with respect to revenue) to the seller.

5.4.1 Setup of the Three Scenarios

Scenario (U, U): Two Uncertain Products

In this scenario, we assume the seller hides the feature information of both products, keeping customers uncertain about the product absent research effort on their part. Each customer does not know his/her values of (ϵ_1, ϵ_2), but knows the joint distribution as given in Table 5.1. That is, they do not know whether they like or dislike the hidden features of each product, but have a prior distribution. Each customer may learn his individual values of ϵ_1, and/or ϵ_2 at a cost of c for each. Each customer buys at most one of the two products.

Without loss of generality, we assume the seller sets prices p_1 and p_2 such that $p_1 \leq p_2$. Faced with prices (p_1, p_2) a customer must select an action, i.e., a contingency plan for researching/buying products. By symmetry, there are only six possible actions that expected utility-maximizing customers might choose. Those six actions are as follows.

- Purchase product 1 immediately without research. We use "b" to denote this option.

- Research product 1 and buy it if $\epsilon_1 = \delta$. Otherwise, research product 2 and buy it if $\epsilon_2 = \delta$. Otherwise, leave without purchase. We use "ss" to denote this option.

- Research product 1 and buy it if $\epsilon_1 = \delta$. Otherwise, leave without purchase. We use "$s0$" to denote this option.

- Research product 1 and buy it if $\epsilon_1 = \delta$. Otherwise, buy product 2 immediately without research. We use "sb" to denote this option.

- Research product 2 and buy it if $\epsilon_2 = \delta$. Otherwise, buy product 1 immediately without research. We use "$-sb$" to denote this option.

- Do not research or buy either product. We denote this by 0.

Let $w_a(\boldsymbol{p})$ denote the expected utility of a customer choosing action a given price vector $\boldsymbol{p} = (p_1, p_2)$. Thus,

$$w_a(\boldsymbol{p}) = \begin{cases} v - p_1 + (2q-1)\delta & \text{if } \ a = b \\ -c + q(v - p_1 + \delta) + (1-q)[-c + \frac{q-\beta}{1-q}(v - p_2 + \delta)] & \text{if } \ a = ss \\ -c + q(v - p_1 + \delta) & \text{if } \ a = s0 \\ -c + q(v - p_1 + \delta) + (1-q)[v - p_2 + (2\frac{q-\beta}{1-q} - 1)\delta] & \text{if } \ a = sb \\ -c + q(v - p_2 + \delta) + (1-q)[v - p_1 + (2\frac{q-\beta}{1-q} - 1)\delta] & \text{if } \ a = -sb \\ 0 & \text{if } \ a = 0 \end{cases}$$

The expression $(2\frac{q-\beta}{1-q} - 1)\delta$ that appears for $a = sb, -sb$ is simply $E[\epsilon_i | \epsilon_j = -\delta]$ for $i \neq j$.

Recall that we assume $p_1 \leq p_2$. From the above expression we see that $w_{sb}(\boldsymbol{p}) \geq w_{-sb}(\boldsymbol{p})$ if $q \geq 1/2$. Likewise, $w_{sb}(\boldsymbol{p}) < w_{-sb}(\boldsymbol{p})$ if $q < 1/2$. Thus, we shall henceforth assume without loss of optimality that

$$\mathcal{A} = \begin{cases} \{b, ss, s0, sb, 0\} & \text{if } \ q \geq 1/2 \\ \{b, ss, s0, -sb, 0\} & \text{if } \ q < 1/2 \end{cases}$$

Let $a(\boldsymbol{p})$ be the optimal action of the customer given price vector $\boldsymbol{p}$, i.e., $a(\boldsymbol{p}) = \arg\max_{a \in \mathcal{A}} w_a(\boldsymbol{p})$. Note that in case of ties, we assume the customer would choose the action which maximizes the seller's revenue.

The seller's revenue from charging a price vector $\boldsymbol{p}$, denoted by $R(\boldsymbol{p})$, can be written as

$$R(\boldsymbol{p}) = R(\boldsymbol{p}|a(\boldsymbol{p})) = \begin{cases} p_1 & \text{if} \quad a(\boldsymbol{p}) = b \\ qp_1 + (q-\beta)p_2 & \text{if} \quad a(\boldsymbol{p}) = ss \\ qp_1 & \text{if} \quad a(\boldsymbol{p}) = s0 \\ qp_1 + (1-q)p_2 & \text{if} \quad a(\boldsymbol{p}) = sb \\ qp_2 + (1-q)p_1 & \text{if} \quad a(\boldsymbol{p}) = -sb \\ 0 & \text{if} \quad a(\boldsymbol{p}) = 0 \end{cases}$$

The seller's pricing problem is as follows:

$$R^*_{UU} := \max_{\boldsymbol{p}} R(\boldsymbol{p}) \tag{5.3}$$

<u>Scenario (U, C): One Uncertain Product, One Certain Product</u>

In this case, we assume the seller discloses the information about the hidden features of product 2 to the public but keeps that of product 1 secret. This means that each individual customer knows his/her value of ϵ_2, and has posterior information about the distribution of ϵ_1 (conditional upon the customer's value of ϵ_2). Each customer can find out his/her realization of ϵ_1 by incurring a cost c. A fraction q of customers have realization δ for ϵ_2, and consuming product 2 generates an utility of $v - p_2 + \delta$ to those customers. We call these customers group 1. Let $q_1 = \beta/q$. Group 1 customers have $\epsilon_2 = \delta$. A fraction q_1 of group 1 customers have $\epsilon_1 = \delta$, and a fraction $1 - q_1$ of group 1 customers have $\epsilon_1 = -\delta$. Hence for a group 1 customer the conditional distribution of his ϵ_1 is as follows:

$$\epsilon_1 = \begin{cases} \delta & \text{with conditional probability} \quad q_1 \\ -\delta & \text{with conditional probability} \quad 1 - q_1. \end{cases}$$

Similarly, a fraction $1 - q$ of customers dislike the hidden features of product 2. For them, consuming it generates an utility of $v - p_2 - \delta$. We call these customers group 2. Let $q_2 = \frac{q-\beta}{1-q}$. Group 2 customers have $\epsilon_2 = -\delta$. A fraction q_2 of group 2 customers have $\epsilon_1 = \delta$, and a fraction $1 - q_2$ of group 2 customers have $\epsilon_1 = -\delta$. Hence for a group 2 customer the conditional distribution of his ϵ_1 is as follows:

$$\epsilon_1 = \begin{cases} \delta & \text{with conditional probability} \quad q_2 \\ -\delta & \text{with conditional probability} \quad 1 - q_2. \end{cases}$$

Given a price vector $\boldsymbol{p}$, we can show both groups of customers have five possible actions that expected utility-maximizing customers might choose. Those five actions are as follows.

- Purchase product 2. We use "$b2$" to denote this option.
- Purchase product 1 immediately without research. We use "$b1$" to denote this option.
- Research product 1 and buy it if he likes its hidden features, and buy product 2 otherwise. We use "sb" to denote this option.
- Research product 1 and buy it if he likes its hidden features, and leave otherwise. We use "$s0$" to denote this option.
- Do not research or buy either product. We denote this by 0.

The set of actions for the customers is $\mathcal{A} = \{b2, b1, sb, s0, 0\}$. Let $w_a^j(\boldsymbol{p})$ denote the expected utility of group j ($j = 1, 2$) customers if they choose action $a \in \mathcal{A}$

given prices $\boldsymbol{p}$. We have

$$w_a^j(\boldsymbol{p}) = \begin{cases} v - p_2 + \delta & \text{if } \ a = b2, j = 1 \\ v - p_2 - \delta & \text{if } \ a = b2, j = 2 \\ v - p_1 + (2q_j - 1)\delta & \text{if } \ a = b1 \\ -c + q_j(v - p_1 + \delta) + (1 - q_j)(v - p_2 + \delta) & \text{if } \ a = sb, j = 1 \\ -c + q_j(v - p_1 + \delta) + (1 - q_j)(v - p_2 - \delta) & \text{if } \ a = sb, j = 2 \\ -c + q_j(v - p_1 + \delta) & \text{if } \ a = s0 \\ 0 & \text{if } \ a = 0 \end{cases}$$

Let $a^j(\boldsymbol{p})$ be the optimal actions of customers of group j given price $\boldsymbol{p}$, i.e., $a^j(\boldsymbol{p}) = \arg\max_{a \in \mathcal{A}} w_a^j(\boldsymbol{p})$. As in scenario (U, U), we assume customers would choose the action which maximizes the seller's revenue in case of ties.

The seller's revenue collected from group j customers by charging a price vector $\boldsymbol{p}$, denoted by $R^j(\boldsymbol{p})$, can be written as

$$R^j(\boldsymbol{p}) = R^j(\boldsymbol{p}|a^j(\boldsymbol{p})) = \begin{cases} p_2 & \text{if } \ a^j(\boldsymbol{p}) = b2 \\ p_1 & \text{if } \ a^j(\boldsymbol{p}) = b1 \\ q_j p_1 + (1 - q_j) p_2 & \text{if } \ a^j(\boldsymbol{p}) = sb \\ q_j p_1 & \text{if } \ a^j(\boldsymbol{p}) = s0 \\ 0 & \text{if } \ a^j(\boldsymbol{p}) = 0 \end{cases}$$

The seller's pricing problem is as follows:

$$R_{UC}^* := \max_{\boldsymbol{p}} R_{UC}(\boldsymbol{p}) = qR^1(\boldsymbol{p}) + (1 - q)R^2(\boldsymbol{p}) \tag{5.4}$$

<u>Scenario (C, C): Two Certain Products</u>

In this case, we assume the seller discloses the information about the (previously) hidden features of both products, and each customer knows his/her values of

(ϵ_1, ϵ_2). That is, each customer knows whether s/he likes or dislikes the hidden features of each product. We have the following four segments of customers.

- Segment 1: a fraction β of them like the features of both product 1 and 2, and the utilities of consuming product 1 and 2 are $v - p_1 + \delta$ and $v - p_2 + \delta$ respectively.

- Segment 2: a fraction $q - \beta$ like the features of product 1 but dislike those of product 2, and the utilities of consuming product 1 and 2 are $v - p_1 + \delta$ and $v - p_2 - \delta$ respectively.

- Segment 3: a fraction $q - \beta$ like the features of product 2 but dislike those of product 1, and the utilities of consuming product 1 and 2 are $v - p_1 - \delta$ and $v - p_2 + \delta$ respectively.

- Segment 4: a fraction $1-2q+\beta$ dislike the features of both product 1 and 2, and the utilities of consuming product 1 and 2 are $v - p_1 - \delta$ and $v - p_2 - \delta$ respectively.

It is easy to see that in this case, the action set for each customer is $\mathcal{A} = \{b1, b2, 0\}$ where $b1$ denotes buying product 1 and $b2$ denotes buying product 2. Let $w_a^j(\boldsymbol{p})$ denote the expected utility of segment j ($j = 1, 2, 3, 4$) customers if they choose action $a \in \mathcal{A}$ given prices $\boldsymbol{p}$. We have

$$w_a^j(\boldsymbol{p}) = \begin{cases} v - p_1 + \delta & \text{if } \; a = b1, j = 1, 2 \\ v - p_1 - \delta & \text{if } \; a = b1, j = 3, 4 \\ v - p_2 + \delta & \text{if } \; a = b2, j = 1, 3 \\ v - p_2 - \delta & \text{if } \; a = b2, j = 2, 4 \\ 0 & \text{if } \; a = 0. \end{cases}$$

Let $a^j(\boldsymbol{p})$ be the optimal actions of customers of segment j given price $\boldsymbol{p}$, i.e., $a^j(\boldsymbol{p}) = \arg\max_{a\in\mathcal{A}} w_a^j(\boldsymbol{p})$. As in scenario (U, U), we assume customers would choose the action which maximizes the seller's revenue in case of ties.

Therefore, the revenue collected from segment j customers, denoted by $R^j(\boldsymbol{p})$, can be written as follows:

$$R^j(\boldsymbol{p}) = R^j(\boldsymbol{p}|a^j(\boldsymbol{p})) = \begin{cases} p_1 & \text{if} \quad a^j(\boldsymbol{p}) = b1 \\ p_2 & \text{if} \quad a^j(\boldsymbol{p}) = b2 \\ 0 & \text{if} \quad a^j(\boldsymbol{p}) = 0. \end{cases}$$

The seller's pricing problem is

$$\begin{aligned} R^*_{CC} :&= \max_{\boldsymbol{p}\in\mathcal{P}^2} R_{CC}(\boldsymbol{p}) \\ &= \beta R^1(\boldsymbol{p}) + (q-\beta)R^2(\boldsymbol{p}) + (q-\beta)R^3(\boldsymbol{p}) + (1-2q+\beta)R^4(\boldsymbol{p}). \end{aligned} \tag{5.5}$$

5.4.2 Independent Features

Again, to simplify the presentation, we assume $q = 1/2$. In addition, we assume that $\beta = q^2 = 1/4$ in this subsection, so that ϵ_1 and ϵ_2 are independent (see Table 5.1). In the following, we compare the three scenarios described in the previous section under these assumptions. Most of the results below hold with slight modification for arbitrary $q \in (0, 1)$. These more general results are stated in Section 5.7.

First, we have the comparison between scenario (U, U) and (U, C).

Proposition 14. *Suppose $\beta = q^2 = \frac{1}{4}$. Then $R^*_{UU} \leq R^*_{UC}$. That is, if the features are independent, the optimal revenue from offering (U, U) is always smaller than the optimal revenue from offering (U, C).*

In case (U, C) the seller can differentiate the customers better than in case (U, U), because the products in (U, C) become "non-homogeneous" in some sense by assigning a different research cost (c or 0) to each product. For any price vector under which customers choose not research in case (U, U), the seller can accrue a higher revenue in case (U, C) by setting the same prices. In other words, as long as customers choose not research in case (U, U), the seller can induce the customers to choose the same or better option in case (U, C) which gives the same or greater revenue. For example, if customers choose to not search and buy directly in case (U, U), the seller can set the same price to induce all customers to not search and buy directly the uncertain product in case (U, C) which gives the seller the exact amount of revenue as in case (U, U). However, the seller can induce a fraction $1 - q$ of customers who dislike the certain product (group 2 customers) to buy the uncertain product directly without research, and induce a fraction q of customers who like the certain product (group 1 customers) to purchase at the largest possible price $v + \delta$. In this way, the seller can collect more revenue from the fraction q of customers than in case (U, U).

Another reason is the value of disclosing information. Whenever customers in case (U, U) choose to research at least one product, they need to pay the research cost at least once, which can be saved by the customer in case (U, C). Thus, in case (U, C) customers are willing to pay more and the revenue is larger.

Hence, no matter whether the customers research or not in case (U, U), offering (U, C) can give the seller a higher revenue. Next we describe which of (U, C) and (C, C) is better for the seller. By Proposition 4, the better of these two is also the best among the three (U, U), (U, C) and (C, C).

Proposition 15. *Suppose* $\beta = q^2 = \frac{1}{4}$.

1. *If* $\delta \geq 2c$ *and* $\delta \geq \max\{\frac{1}{5}v + \frac{6}{5}c, \frac{1}{7}v + \frac{12}{7}c\}$, *then offering* (C, C) *is optimal, the optimal prices are* $p_1^* = p_2^* = v + \delta$, *and* $R_{CC}^* = \frac{3}{4}(v + \delta)$.

2. *If* $\delta \geq 2c$ *and* $\delta < \max\{\frac{1}{5}v + \frac{6}{5}c, \frac{1}{7}v + \frac{12}{7}c\}$, *then offering* (U, C) *is optimal, either* $p_1^* = v - \delta + 2c$, $p_2^* = v - \delta + 4c$ *or* $p_1^* = v - \delta + 2c$, $p_2^* = v + \delta$. *In addition,* $R_{UC}^* = \max\{v - \frac{1}{2}\delta + \frac{3}{2}c, v - \delta + 3c\}$.

3. *If* $\delta < 2c$, *then offering* (U, C) *is optimal,* $p_1^* = v$, $p_2^* = v + \delta$, *and* $R_{UC}^* = v + \frac{1}{2}\delta$.

When the feature uncertainty is large, customers always choose to research. Thus offering (C, C) is better than offering (U, C), because customers save one unit of research cost and thus they are willing to pay more. When the feature uncertainty is small, customers may not research and they are more willing to buy based on their expectation. Hence there is no value of providing extra information. Moreover, offering (U, C) allows the seller to better differentiate the customers because of the "non-homogeneous" products.

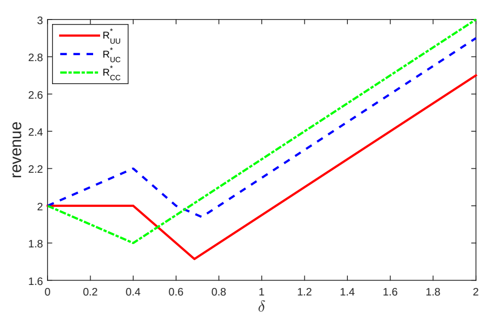


Figure 5.3: Comparison of the three scenarios $(v = 2, q = 0.5, c = 0.2)$

Figure 5.3 illustrates the proceeding two propositions. The figure shows that the optimal revenue from offering (U, U) is always lower than that from offering (U, C), which is consistent with Proposition 14. Moreover, the optimal revenue from offering (U, C) is larger than that from offering (C, C) when δ is small, and is smaller than from offering (C, C) when δ is large. This is consistent with Proposition 15.

Proposition 16. *In case (U, C), we have $p_1^* < p_2^*$. That is, the price of the uncertain product is always smaller than that of the certain product.*

There are two main reasons for the preceding result. The first one is that for the certain product, the seller has already disclosed the feature information and thus customers do not need to incur a research cost to learn the information. Therefore, the seller can charge a higher price than the uncertain product. The second reason is charging different prices for the two products allows the seller to differentiate customers better.

Let $R_{all}^* = \max\{R_{UU}^*, R_{UC}^*, R_{CC}^*\}$. Next we study some comparative statics of R_{all}^*. We have the following.

Proposition 17. *For any fixed parameter set (v, c),*

1. *If $\delta < 2c$, then R_{all}^* is increasing in δ.*

2. *If $\delta \geq 2c$ and $\delta < \max\{\frac{1}{5}v + \frac{6}{5}c, \frac{1}{7}v + \frac{12}{7}c\}$, then R_{all}^* is decreasing in δ.*

3. *If $\delta \geq 2c$ and $\delta \geq \max\{\frac{1}{5}v + \frac{6}{5}c, \frac{1}{7}v + \frac{12}{7}c\}$, then R_{all}^* is increasing in δ.*

Now we explain the results in Proposition 17. When δ is large (as in part 3), customers always choose to research. Thus offering (C, C) is the best choice.

Moreover, the best strategy is to seize the high valuation customers who like at least one of the products and let them buy at the largest possible price $v + \delta$. Hence, it is easy to see that R^*_{all} is increasing in δ.

When δ is small (as in part 1), customers choose to not search because it is not worth the research cost. As we argued before, offering (U, C) is the best. Since group 2 customers dislike the certain product and group 1 customers like the certain product, the best strategy is to let group 2 customers buy the uncertain product based on their expected utility v (which is equivalent to a perfect price discrimination to group 2 customers) and let group 1 customers buy the certain product at the largest possible price $v + \delta$. Therefore, R^*_{all} is increasing in δ.

When δ is in the middle (as in part 2), by computation, offering (U, C) is better than offering (C, C). For group 2 customers who dislike the certain product, the optimal strategy is to induce them to buy the uncertain product based on their expected utility. As we argued in part 2 of Proposition 12, as δ increases, customers' incentive to research also increases, thus the seller has to decrease the price of the uncertain product ($p^*_2 = v - \delta + 2c$) to let them choose to not research. For group 1 customers, it is easy to see that their optimal decision is either to buy the certain product or to research the uncertain product first and buy if they like it, otherwise buy the certain product. If the optimal action is to buy the certain product, then the price of the certain product has to be lower than some bound which is connect to the price of the uncertain product (otherwise, they will choose to search the feature information of the uncertain product). As δ increases, since the price of the uncertain product decreases, the price of the certain product also decreases. Hence the revenue is decreasing in δ in this case. If the optimal action is to research the uncertain product first, then the price of the certain product can be as high as $v + \delta$. Since there are more customers who purchases the uncertain product in this case, the revenue is decreasing in δ.

To summarize the results of Proposition 17, we plot R^*_{all} against δ in Figure 5.4. As we can see, the seller's revenue does not always decrease with the uncertainty parameter δ. To the contrary, it may increase in it in certain range. This is consistent with the finding in the single product setting.

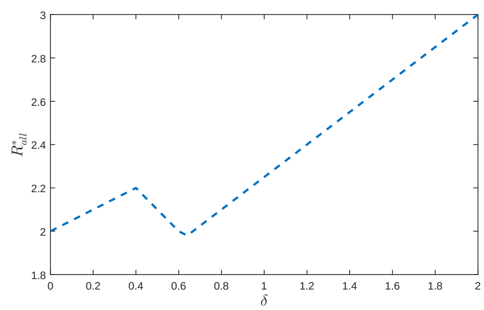


Figure 5.4: R^*_{all} as a function of δ

Next, we study how the optimal revenue R^*_{all} changes with the research cost c. We have the following results:

Proposition 18. *For any fixed parameter set* (v, δ)*,*

1. *If* $\delta < 2c$*, then* R^*_{all} *is constant.*

2. *If* $\delta \geq 2c$*,*

   (a) *If* $\delta < \max\{\frac{1}{5}v + \frac{6}{5}c, \frac{1}{7}v + \frac{12}{7}c\}$*, then* R^*_{all} *is increasing in* c*.*

   (b) *If* $\delta \geq \max\{\frac{1}{5}v + \frac{6}{5}c, \frac{1}{7}v + \frac{12}{7}c\}$*, then* R^*_{all} *is constant.*

Next, we illustrate the results in Proposition 18. When δ is large (as in part 3), customers always choose to search, and offering (C, C) is the best choice. Thus, R^*_{all} is a constant in c. When δ is small (as in part 1), as we argued before, offering (U, C) is optimal and the optimal strategy is to let group 2 customers purchase the uncertain product directly based on their expected utility and let group 1 customers purchase the certain product. No customers choose to research in this case. Thus, R^*_{all} is a constant to c. When δ is in the middle, for group 2 customers, as c decreases, customers' incentive to research also increases, thus the seller has to decrease the price of the uncertain product ($p^*_2 = v - \delta + 2c$) to let them choose to not research. Hence, the price of the uncertain product is increasing in c. For group 1 customers, if the optimal action is to buy the certain product, then the price of the certain product has to be lower than some bound which is connect to the price of the uncertain product (otherwise, they will choose to search the feature information of the uncertain product). As c decreases, since the price of the uncertain product decreases, the price of the certain product also decreases. Hence the revenue is increasing in c in this case. If the optimal action is to research the uncertain product first, then the price of the certain product can be as high as $v + \delta$. Thus, the revenue is increasing in c.

To help visualize the results of Proposition 18, we plot R^*_{all} against c in Figure 5.5. As we can see, the seller's revenue first does not change with research cost c then increases and reaches a steady maximum. This confirms the insight in the single product setting that in order to obtain better revenue, a seller should set the research cost high enough.

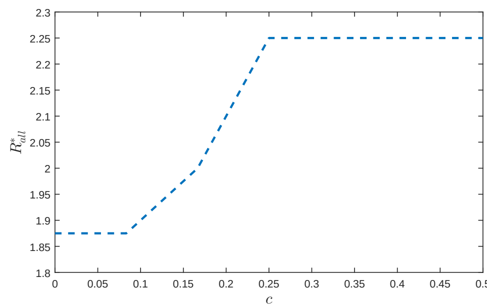


Figure 5.5: R^*_{all} as a function of c

5.5 Correlated Features

5.5.1 Perfectly Positive Correlation

In this subsection, we study the scenario that the features of the two products are perfectly positively correlated. This is a special case with $\beta = q$ of the model considered in section 5.4. Specifically, we assume the customer composition is as given in Table 5.2.

Table 5.2: Fraction of customers with different values of (ϵ_1, ϵ_2)

		ϵ_2	
		δ	$-\delta$
ϵ_1	δ	q	0
	$-\delta$	0	$1-q$

With perfectly positive correlation, it is easy to see that case (U, U) is reduced to the one uncertain product case. Hence the optimal solution is the same as in Theorem 5.7.1. That is,

$$R^*_{UU} = \begin{cases} \max\{q(v+\delta) - c, v - \delta + \frac{c}{1-q}\} & \text{if} \quad \delta \geq \frac{c}{2q(1-q)} \\ v - \delta + 2q\delta & \text{if} \quad \delta < \frac{c}{2q(1-q)} \end{cases}$$

Moreover, both case (U, C) and case (C, C) are reduced to the one certain product case in which a fraction q of customers like the product and the rest dislike the product. Thus, the optimal revenue is $R^*_{UC} = \max\{q(v+\delta), v-\delta\}$. Then we have the following proposition.

Proposition 19. *Suppose the features are perfectly positively correlated.*

1. *Suppose* $\delta \geq 2c$

   (a) *If* $\delta \geq \frac{1}{3}v + \frac{4}{3}c$, *then* $R^*_{UU} \leq R^*_{UC}$.

   (b) *If* $\delta < \frac{1}{3}v + \frac{4}{3}c$, *then* $R^*_{UU} > R^*_{UC}$.

2. *Suppose* $\delta < 2c$. *Then* $R^*_{UU} > R^*_{UC}$.

If δ is large, customers would always research before purchasing the uncertain product. In case (U, C), customers save one unit of research cost and thus are more willing to pay more. Hence, the seller can collect more revenue. Therefore, disclosing the feature information is better than hiding it. If δ is small, in case (U, U) customers are not willing to research and would make a purchase directly, as long as their expected utility is positive. And it is equivalent that a fraction q of customers buy at high price $v + \delta$ and the rest buy at the low price $v - \delta$, which is a perfect price discrimination. Hence, the seller can collect more revenue from selling the uncertain product. Therefore, hiding the feature information is better than disclosing. We can see that disclosing all information is not necessarily always good to firm.

5.5.2 Perfectly Negative Correlation

In this subsection, we study the scenario that the features of the two products are perfectly negatively correlated. We assume the customer composition is as given in Table 5.3. Note that this is not a special case of the model in section 5.4, unless we take $q = 1/2$ above (which corresponds to $q = 1/2, \beta = 0$ in Table 5.1 in Section 5.4).

Table 5.3: Fraction of customers with different values of (ϵ_1, ϵ_2)

		ϵ_2	
		δ	$-\delta$
ϵ_1	δ	0	q
	$-\delta$	$1-q$	0

Case (U, U): two uncertain products

Note that since the features are perfectly correlated, it suffices to research the feature of only one product if customers want to search. Then customers have the following possible optimal options to choose.

- Purchase product 1 directly without research. We use "$b1$" to identify this option.
- Purchase product 2 directly without research. We use "$b2$" to identify this option.
- Research product 1 and buy if $\epsilon_1 = \delta$. Otherwise, buy product 2. We use "sb" to identify this option.

Thus, $\mathcal{A} = \{b1, b2, sb, 0\}$. And customer's expected utility can be written as

$$w_a(\boldsymbol{p}) = \begin{cases} v - p_1 + (2q-1)\delta & \text{if} \quad a = b1 \\ v - p_2 + (1-2q)\delta & \text{if} \quad a = b2 \\ -c + q(v - p_1 + \delta) + (1-q)(v - p_2 + \delta) & \text{if} \quad a = sb \\ 0 & \text{if} \quad a = 0 \end{cases}$$

Notice that customers may have other actions. For example, a customer may search product 2 , buy if $\epsilon_2 = \delta$, and buy product 1 otherwise. This action gives customers expected utility $-c+(1-q)(v-p_2+\delta)+q(v-p_1+\delta)$ which is equal to $w_{sb}(\boldsymbol{p})$. And customers may research product 1, buy if $\epsilon_1 = \delta$, and leave otherwise. This action gives customers $-c + q(v - p_1 + \delta)$ which is less than $w_{sb}(\boldsymbol{p})$. Hence, the optimal action set is $\mathcal{A} = \{b1, b2, sb, 0\}$.

Let $a(\boldsymbol{p})$ be the optimal action of the customer given price $\boldsymbol{p}$. Then

$$a(\boldsymbol{p}) = \begin{cases} b1 & \text{if} \quad w_{b1}(\boldsymbol{p}) \geq \max\{w_a(\boldsymbol{p}), a \in \mathcal{A}\} \\ b2 & \text{if} \quad w_{b2}(\boldsymbol{p}) \geq \max\{w_a(\boldsymbol{p}), a \in \mathcal{A}\} \\ sb & \text{if} \quad w_{sb}(\boldsymbol{p}) \geq \max\{w_a(\boldsymbol{p}), a \in \mathcal{A}\} \\ 0 & \text{if} \quad w_0(\boldsymbol{p}) \geq \max\{w_a(\boldsymbol{p}), a \in \mathcal{A}\} \end{cases}$$

The revenue collected by charging a price vector $\boldsymbol{p}$, denoted by $R(\boldsymbol{p})$, can be written as

$$R(\boldsymbol{p}) = R(\boldsymbol{p}|a(\boldsymbol{p})) = \begin{cases} p_1 & \text{if} \quad a(\boldsymbol{p}) = b1 \\ p_2 & \text{if} \quad a(\boldsymbol{p}) = b2 \\ qp_1 + (1-q)p_2 & \text{if} \quad a(\boldsymbol{p}) = sb \\ 0 & \text{if} \quad a(\boldsymbol{p}) = 0 \end{cases}$$

Therefore, the seller's problem is as follows:

$$R^*_{UU} = \max_{\boldsymbol{p}} R(\boldsymbol{p}) \tag{5.6}$$

Lemma 14. *The optimal solution is*

1. *If* $\delta \geq c$, *then* $R^*_{UU} = v + \delta - c$.

2. *If* $\delta < c$, *then* $R^*_{UU} = v$.

Case (U, C) and Case (C, C):

Since the features are perfectly correlated, then case (U, C) is exactly the same as case (C, C). That is, in both cases, there is q fraction of customers who like product 1 but dislike product 2 (call them group 1), and $1 - q$ fraction who dislike product 1 but like product 2 (call them group 2). The revenue function of each group, denoted by $R_1(\boldsymbol{p})$ and $R_2(\boldsymbol{p})$, can be written as

$$R_1(\boldsymbol{p}) = \begin{cases} p_1 & \text{if } v - p_1 + \delta \geq v - p_2 - \delta \text{ and } v - p_1 + \delta \geq 0 \\ p_2 & \text{if } v - p_2 - \delta > v - p_1 + \delta \text{ and } v - p_2 - \delta \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

and

$$R_2(\boldsymbol{p}) = \begin{cases} p_1 & \text{if } v - p_1 - \delta \geq v - p_2 + \delta \text{ and } v - p_1 - \delta \geq 0 \\ p_2 & \text{if } v - p_2 + \delta > v - p_1 - \delta \text{ and } v - p_2 + \delta \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Therefore, the seller's problem is as follows:

$$R^*_{UC} = \max_{\boldsymbol{p} \in \mathcal{P}^2} qR_1(\boldsymbol{p}) + (1 - q)R_2(\boldsymbol{p}) \tag{5.7}$$

It is easy to figure out that the optimal solution is $p_1^* = p_2^* = v + \delta$, and $R^*_{UC} = R^*_{CC} = v + \delta$.

Proposition 20. *Suppose the features are perfectly negatively correlated. Then* $R^*_{UU} < R^*_{UC} = R^*_{CC}$.

This result is quite intuitive. With perfectly negative correlation, in case (U, C) the seller can set the highest price $v+\delta$ for both products and all customers make a purchase because all customers like at least one of the products. Actually, $v+\delta$ is the largest possible revenue that the seller can collect in any scenarios. Hence, the revenue in case (U, C) is larger.

5.6 Future Directions

1. In the single product setting, I assume that customers' valuation v, which is based on openly observable characteristics of the product, is the same across all customers. I am interested in the situation that v has a continuous distribution.

2. In the single product setting, I assume that customers' valuation ϵ, which is based on the hidden features of the product, has a binary distribution. What if it has a continuous distribution, such as a normal distribution?

3. In the two product setting, I assume that the two products are homogeneous since ϵ_1 and ϵ_2 have the same distribution. What if the two products are heterogeneous? For example, I could assume one product has a larger uncertainty than the other. Mathematically, we let

$$\epsilon_i = \begin{cases} \delta_i & \text{with probability} \quad q \\ -\delta_i & \text{with probability} \quad 1-q, \end{cases}$$

and $\delta_1 > \delta_2$.

I hope to answer the following questions:

- When the seller offers two uncertain products, which one should have a lower price? The one with high uncertainty or the one with low uncertainty?
- Does the result still hold that offering two uncertain products is worse than offering one uncertain product and one certain product?

5.7 Appendix

In Section 5.7.1 below we state the general versions (general q) of our results of this chapter. In Sections 5.7.2, 5.7.3, and 5.7.4, we provide the solutions to problems (U, U), (U, C), and (C, C) respectively. The reader is directed to Section 5.8 for the lengthy derivations of the solutions. Section 5.7.5 contains proofs of the results stated in Section 5.7.1. Those proofs use the solutions presented in Sections 5.7.2-5.7.4.

5.7.1 Results with General q

In this section, we state all the results with a general q ($0 < q < 1$). All the theorems, lemmas, and propositions in the previous sections in this chapter can be obtained by putting $q = 1/2$ into the results in this section. We append a G below to indicate the general version of theorems and propositions from Sections 5.2-5.4. For instance, Theorem 5G is the general version of Theorem 5. Theorem 5 can be obtained as a special case of Theorem 5G by taking $q = 1/2$.

Theorem 5G.

1. *If* $\delta \geq \frac{1-q}{1+q}v + \frac{2-q}{1-q^2}c$*, then* $p^* = v + \delta - \frac{c}{q}$*. In this case, customers' optimal action is to search the product (action s). The seller's optimal revenue*

$r(p^*) = q(v + \delta - \frac{c}{q})$.

2. *If* $\frac{c}{2q(1-q)} \leq \delta < \frac{1-q}{1+q}v + \frac{2-q}{1-q^2}c$*, then* $p^* = v - \delta + \frac{c}{1-q}$*. In this case, customers' optimal action is to purchase directly without search (action b). The seller's optimal revenue* $r(p^*) = v - \delta + \frac{c}{1-q}$.

3. *If* $\delta < \frac{c}{2q(1-q)}$*, then* $p^* = v + (2q-1)\delta$*. In this case, customers' optimal action is to purchase directly without search (action b). The seller's optimal revenue* $r(p^*) = v + (2q-1)\delta$.

Proposition 11G. *Both* p^* *and* $r(p^*)$ *increase with* v *and* q*.*

Proposition 12G.

1. *If* $\delta \geq \frac{1-q}{1+q}v + \frac{2-q}{1-q^2}c$*, then* $r(p^*)$ *increases in* δ*.*

2. *If* $\frac{c}{2q(1-q)} \leq \delta < \frac{1-q}{1+q}v + \frac{2-q}{1-q^2}c$*, then* $r(p^*)$ *decreases in* δ*.*

3. *If* $\delta < \frac{c}{2q(1-q)}$*, then* $r(p^*)$ *increases in* δ *when* $q > \frac{1}{2}$ *and decreases in* δ *when* $q < \frac{1}{2}$.

Proposition 13G.

1. *If* $\delta \geq \frac{1-q}{1+q}v + \frac{2-q}{1-q^2}c$*, then* $r(p^*)$ *decreases in* c*.*

2. *If* $\frac{c}{2q(1-q)} \leq \delta < \frac{1-q}{1+q}v + \frac{2-q}{1-q^2}c$*, then* $r(p^*)$ *increases in* c*.*

3. *If* $\delta < \frac{c}{2q(1-q)}$*, then* $r(p^*)$ *does not change with* c*.*

In addition, if the seller can choose the research cost, then it is optimal to set $c \geq 2\delta q(1-q)$*, i.e., to set the research cost high.*

Proposition 14G. *If* $\beta = q^2$*, then* $R^*_{UU} \leq R^*_{UC}$*. That is, if the features are independent, the optimal revenue by offering* (U, U) *is always smaller than by offering* (U, C)*.*

Let $\theta = \max\{\frac{(1-q)^2}{1+q^2}v + \frac{q^2+1-q}{(1-q)(1+q^2)}c, \frac{(1-q)^2}{1+2q-q^2}v + \frac{2-q}{(1-q)(1+2q-q^2)}c\}$.

Proposition 15G. *Suppose* $\beta = q^2$*.*

1. *If* $\delta \geq \frac{c}{2q(1-q)}$*.*
   (a) *If* $\delta \geq \theta$*, then offering* (C, C) *is optimal. And* $R^*_{CC} = q(2-q)(v+\delta)$*.*
   (b) *Otherwise, offering* (U, C) *is optimal. And* $R^*_{UC} = \max\{[q^2 + (1-q)](v - \delta + \frac{c}{1-q}) + q(1-q)(v+\delta), v - \delta + \frac{2-q}{1-q}c\}$*.*
2. *If* $\delta < \frac{c}{2q(1-q)}$*, then offering* (U, C) *is optimal. And* $R^*_{UC} = v - \delta + 2q\delta + 2q(1-q)\delta$*.*

Proposition 16G. *In case* (U, C)*,* $p^*_1 < p^*_2$*. That is, the price of the stochastic product is always smaller than that of the deterministic product.*

Proposition 17G. *For any fixed parameter set* (v, c, q)*,*

1. *If* $\delta < \frac{c}{2q(1-q)}$*, then* R^*_{all} *is increasing in* δ *if* $q \geq \frac{2-\sqrt{2}}{2}$*, and decreasing in* δ *otherwise.*
2. *If* $\delta \geq \frac{c}{2q(1-q)}$ *and* $\delta < \theta$*, then* R^*_{all} *is decreasing in* δ*.*
3. *If* $\delta \geq \frac{c}{2q(1-q)}$ *and* $\delta \geq \theta$*, then* R^*_{all} *is increasing in* δ*.*

Let $\gamma = \min\{\frac{(1+q^2)(1-q)}{q^2+1-q}\delta - \frac{(1-q)^2(1-q)}{q^2+1-q}v, \frac{(1+2q-q^2)(1-q)}{2-q}\delta - \frac{(1-q)^2(1-q)}{2-q}v\}$.

Proposition 18G. *For any fixed parameter set* (v, δ, q)*,*

1. *If* $c \geq 2q(1-q)\delta$*, then* R^*_{all} *is constant.*

2. *If* $c < 2q(1-q)\delta$*,*

   (a) *If* $c \geq \gamma$*, then* R^*_{all} *is increasing in c.*

   (b) *If* $c < \gamma$*, then* R^*_{all} *is constant.*

Proposition 19G. *Suppose the features are perfectly positively correlated.*

1. *Suppose* $\delta \geq \frac{c}{2q(1-q)}$

   (a) *If* $\delta \geq \frac{1-q}{1+q}v + \frac{1}{1-q^2}c$*, then* $R^*_{UU} \leq R^*_{UC}$*.*

   (b) *If* $\delta < \frac{1-q}{1+q}v + \frac{1}{1-q^2}c$*, then* $R^*_{UU} \geq R^*_{UC}$*.*

2. *Suppose* $\delta < \frac{c}{2q(1-q)}$*. Then* $R^*_{UU} \geq R^*_{UC}$*.*

Lemma 14G. *The optimal solution is*

1. *Suppose* $q \geq \frac{1}{2}$*.*

   (a) *If* $\delta \geq \frac{c}{2(1-q)}$*, then* $R^*_{UU} = v + \delta - c$*.*

   (b) *If* $\delta < \frac{c}{2(1-q)}$*, then* $R^*_{UU} = v - \delta + 2q\delta$*.*

2. *Suppose* $q < \frac{1}{2}$*.*

   (a) *If* $\delta \geq \frac{c}{2q}$*, then* $R^*_{UU} = v + \delta - c$*.*

   (b) *If* $\delta < \frac{c}{2q}$*, then* $R^*_{UU} = v + \delta - 2q\delta$*.*

Proposition 20G. *Suppose the features are perfectly negatively correlated. Then* $R^*_{UU} < R^*_{UC} = R^*_{CC}$*.*

5.7.2 Optimal Solution of Case (U, U)

Part 1: Positive correlation and $q \geq 1/2$ The optimal solution is as follows:

- If $\delta \geq \frac{1-\beta-q+q^2}{2(1-q)(q-\beta)}c$, then $R^*_{UU} = \max\{v-\delta+\frac{c}{1-q}, (2q-\beta)(v+\delta) - \frac{(1-q)(2q-\beta)}{q-\beta}c, q(v+\delta)-c\}$.
- If $\frac{c}{2q(1-q)} \leq \delta < \frac{1-\beta-q+q^2}{2(1-q)(q-\beta)}c$, then $R^*_{UU} = \max\{v-\delta+\frac{c}{1-q}, q(v+\delta)-c\}$.
- If $\delta < \frac{c}{2q(1-q)}$, then $R^*_{UU} = v-\delta+2q\delta$.

Part 2: Positive correlation and $q < 1/2$ The optimal solution is as follows:

- If $\delta \geq \frac{c}{2(q-\beta)}$ and $\beta \geq \frac{q(1-2q)}{1-q}$, then $R^*_{UU} = \max\{v-\delta+\frac{c}{1-q}, (2q-\beta)(v+\delta) - \frac{(1-q)(2q-\beta)}{q-\beta}c, q(v+\delta)-c\}$.
- If $\delta \geq \frac{c}{2(q-\beta)}$ and $\beta < \frac{q(1-2q)}{1-q}$, then $R^*_{UU} = \max\{v-\delta+\frac{c}{1-q}, (2q-\beta)(v+\delta) - \frac{(1-q)(2q-\beta)}{q-\beta}c, q(v+\delta)-c, v-\delta+\frac{1-q}{1-2q+\beta}c-\frac{2\beta(q-\beta)}{q(1-2q+\beta)}\delta+\frac{\beta}{q(1-2q+\beta)}c+2\delta(q-\beta)-c\}$.
- If $\frac{1-\beta-q+q^2}{2(1-q)(q-\beta)}c \leq \delta < \frac{c}{2(q-\beta)}$, then $R^*_{UU} = \max\{v-\delta+\frac{c}{1-q}, (2q-\beta)(v+\delta) - \frac{(1-q)(2q-\beta)}{q-\beta}c, q(v+\delta)-c\}$.
- If $\frac{c}{2q(1-q)} \leq \delta < \frac{1-\beta-q+q^2}{2(1-q)(q-\beta)}c$, then $R^*_{UU} = \max\{v-\delta+\frac{c}{1-q}, q(v+\delta)-c\}$.
- If $\delta < \frac{c}{2q(1-q)}$, then $R^*_{UU} = v-\delta+2q\delta$.

Part 3: Negative correlation and $q \geq 1/2$ The optimal solution is as follows:

- If $\delta \geq \frac{2-3q+\beta}{2(2q-\beta)(1-2q+\beta)}c$, then $R^*_{UU} = \max\{v-\delta+\frac{1-q}{1-2q+\beta}c, (2q-\beta)(v+\delta) - (2-q)c\}$.
- If $\frac{c}{2(q-\beta)}\frac{(1-q)^2}{1-2q+\beta} \leq \delta < \frac{2-3q+\beta}{2(2q-\beta)(1-2q+\beta)}c$, then $R^*_{UU} = \max\{(2q-\beta)(v+\delta) - (2-q)c, v+(4q-2\beta-1)\delta-c\}$.

- If $\frac{c}{2q(1-q)} \leq \delta < \frac{c}{2(q-\beta)}\frac{(1-q)^2}{1-2q+\beta}$, then $R^*_{UU} = v + (4q - 2\beta - 1)\delta - c$.
- If $\frac{c}{2(q-\beta)} \leq \delta < \frac{c}{2q(1-q)}$, then $R^*_{UU} = v + (4q - 2\beta - 1)\delta - c$.
- If $\delta \leq \frac{c}{2(q-\beta)}$, then $R^*_{UU} = v - \delta + 2q\delta$.

Part 4: Negative correlation and $q < 1/2$

- Case 1:
  - If $\delta \geq \frac{c}{2q(1-q)}$, then $R^*_{UU} = v - \delta + \frac{c}{1-q}$.
  - If $\delta < \frac{c}{2q(1-q)}$, then $R^*_{UU} = v - \delta + 2q\delta$.
- Case 2: Upper bound: If $\delta \geq \frac{2-q}{2(1-q)(2q-\beta)}c$, then $R^*_{UU} = (2q - \beta)(v + \delta) - (2 - q)c$.
- Case 3:
  - If $\delta \geq \frac{c}{2q(1-q)}$, then $R^*_{UU} = q(v + \delta) - c$.
  - Otherwise, infeasible.
- Case 5: Upper bound
  - if $\delta \leq \frac{2-3q+\beta}{2(2q-\beta)(1-2q+\beta)}c$, then $R^*_{UU} \leq v + (4q - 2\beta - 1)\delta - c$.
  - if $\delta \geq \frac{2-3q+\beta}{2(2q-\beta)(1-2q+\beta)}c$, then $R^*_{UU} \leq R^C$ or $R^*_{UU} \leq \min\{R^A, R^B\}$, where $R^A = v - \delta - \frac{2\beta(q-\beta)}{q(1-2q+\beta)}\delta + \frac{1-q}{1-2q+\beta}c + \frac{\beta}{q(1-2q+\beta)}c + 2\delta(q - \beta) - c$, $R^B = v + (3q - 2\beta - 1)\delta + \frac{q(2q-2\beta-1)}{1-q}\delta + \frac{q^2(1-q)}{(q-\beta)(1-2q)}c$, and $R^C = v - \delta + \frac{1-q}{1-2q+\beta}c$
  - We also proved that R^*_{UU} is decreasing in β.

5.7.3 Optimal Solution of Case (U, C)

Part 1: Positive Correlation

- If $\delta \geq \max\{\frac{c}{2(q-\beta)}\frac{q^2}{\beta}, \frac{c}{2(q-\beta)}\frac{(1-q)^2}{1-2q+\beta}\}$, then $R^*_{UC} = \max\{v-\delta+\frac{1-q}{1-2q+\beta}c+\frac{q^2}{\beta}c, (2q-\beta)(v+\delta)-\frac{(1-q)(2q-\beta)}{q-\beta}c+\frac{q^2}{\beta}c, q(v+\delta), v-\delta+\frac{1-q}{1-2q+\beta}c+2\delta(q-\beta)-qc, \min\{(2q-\beta)(v+\delta)-\frac{q(1-q)}{q-\beta}c, (2q-\beta)(v+\delta)-\frac{(2q-\beta)(1-q)}{q-\beta}c+2(q-\beta)\delta-qc\}\}$.

- If $\frac{c}{2(q-\beta)}\frac{(1-q)^2}{1-2q+\beta} \leq \delta < \frac{c}{2(q-\beta)}\frac{q^2}{\beta}$ (hold only when $q \geq 1/2$), then $R^*_{UC} = \max\{v-\delta+\frac{1-q}{1-2q+\beta}c+2(q-\beta)\delta, (2q-\beta)(v+\delta)-\frac{(1-q)(2q-\beta)}{q-\beta}c+2(q-\beta)\delta, q(v+\delta)\}$.

- If $\frac{c}{2(q-\beta)}\frac{q^2}{\beta} \leq \delta < \frac{c}{2(q-\beta)}\frac{(1-q)^2}{1-2q+\beta}$ (hold only when $q < 1/2$), then $R^*_{UC} = \max\{v-\delta+\frac{2(q-\beta)}{1-q}\delta+\frac{q^2}{\beta}c, q(v+\delta), v-\delta+2\frac{q-\beta}{1-q}\delta+2\delta(q-\beta)-qc\}$.

- If $\delta < \min\{\frac{c}{2(q-\beta)}\frac{(1-q)^2}{1-2q+\beta}, \frac{c}{2(q-\beta)}\frac{q^2}{\beta}\}$, then $R^*_{UC} = \max\{v-\delta+\frac{2(q-\beta)}{1-q}\delta+2(q-\beta)\delta, q(v+\delta)\}$.

Part 2: Negative Correlation

- If $\delta \geq \frac{q-2q^2+\beta}{2\beta(1-2q+\beta)}c$, then $R^*_{UC} = \max\{v-\delta+\frac{1-q}{1-2q+\beta}c+\frac{q^2}{\beta}c, (2q-\beta)(v+\delta)-(1-q)c, v+\delta+\frac{(1-q)(1+\beta-q)}{1-2q+\beta}c-2\delta(1+\beta-q)\}$.

- If $\max\{\frac{c}{2(q-\beta)}\frac{q^2}{\beta}, \frac{c}{2(q-\beta)}\frac{(1-q)^2}{1-2q+\beta}\} \leq \delta \leq \frac{q-2q^2+\beta}{2\beta(1-2q+\beta)}c$, then $R^*_{UC} = \max\{v+(2q-1)\delta+\frac{(1-q)^2}{1-2q+\beta}c, (2q-\beta)(v+\delta)-(1-q)c\}$.

- If $\frac{c}{2(q-\beta)}\frac{(1-q)^2}{1-2q+\beta} \leq \delta < \frac{c}{2(q-\beta)}\frac{q^2}{\beta}$, then $R^*_{UC} = \max\{v+(2q-1)\delta+\frac{(1-q)^2}{1-2q+\beta}c, (2q-\beta)(v+\delta)-(1-q)c\}$.

- If $\frac{c}{2(q-\beta)}\frac{q^2}{\beta} \leq \delta < \frac{c}{2(q-\beta)}\frac{(1-q)^2}{1-2q+\beta}$, then $R^*_{UC} = v+(2q-1)\delta+2(q-\beta)\delta$.

- If $\delta \leq \min\{\frac{c}{2(q-\beta)}\frac{(1-q)^2}{1-2q+\beta}, \frac{c}{2(q-\beta)}\frac{q^2}{\beta}\}$, then $R^*_{UC} = v+(2q-1)\delta+2(q-\beta)\delta$.

5.7.4 Optimal Solution of Case (C, C)

$R^*_{CC} = \max\{(q-\beta)(v+\delta) + (1-q+\beta)(v-\delta), (2q-\beta)(v+\delta)\}$.

5.7.5 Proofs of Propositions and Lemmas in Section 5.7.1

In this section, we will prove all the results stated in Subsection 5.7.1.

Proof of Theorem 5G. We first consider the case when $\delta \geq \frac{c}{2q(1-q)}$. In this case, we have the following.

- If $p \leq v - \delta + \frac{c}{1-q}$, then $w_b(p) \geq \max\{w_s(p), 0\}$. Thus $a(p) = b$. That is, the customers' optimal action is to purchase directly without search.
- If $v - \delta + \frac{c}{1-q} < p \leq v + \delta - \frac{c}{q}$, then $w_s(p) > w_b(p)$ and $w_s(p) \geq 0$. Thus $a(p) = s$. That is, the customers' optimal action is to search the product.
- If $p > v + \delta - \frac{c}{q}$, then $0 > \max\{w_b(p), w_s(p)\}$. Thus $a(p) = 0$. That is, the customers' optimal action is to leave without purchase.

Therefore, when $\delta \geq \frac{c}{2q(1-q)}$, we have

$$r(p) = \begin{cases} p & \text{if} \quad p \leq v - \delta + \frac{c}{1-q} \\ q \cdot p & \text{if} \quad v - \delta + \frac{c}{1-q} < p \leq v + \delta - \frac{c}{q} \\ 0 & \text{if} \quad p > v + \delta - \frac{c}{q}. \end{cases}$$

Maximizing over p, we obtain

$$p^* = \begin{cases} v - \delta + \frac{c}{1-q} & \text{if} \quad v - \delta + \frac{c}{1-q} > q(v + \delta - \frac{c}{q}) \\ v + \delta - \frac{c}{q} & \text{if} \quad v - \delta + \frac{c}{1-q} \leq q(v + \delta - \frac{c}{q}) \end{cases}$$

and

$$r(p^*) = \begin{cases} v - \delta + \frac{c}{1-q} & \text{if} \quad v - \delta + \frac{c}{1-q} > q(v + \delta - \frac{c}{q}) \\ q(v + \delta - \frac{c}{q}) & \text{if} \quad v - \delta + \frac{c}{1-q} \leq q(v + \delta - \frac{c}{q}) \end{cases}$$

Further, we note that $\delta < \frac{1-q}{1+q}v + \frac{2-q}{1-q^2}c$ is equivalent to $v - \delta + \frac{c}{1-q} > q(v + \delta - \frac{c}{q})$. Therefore, parts 1 and 2 of the theorem are proved.

Next we consider the case where $\delta < \frac{c}{2q(1-q)}$. In this case, we have:

- If $p \leq v + (2q-1)\delta$, then $w_b(p) \geq \max\{w_s(p), 0\}$. Thus $a(p) = b$. That is, the customers' optimal action is to purchase directly without search.

- If $p > v + (2q-1)\delta$, then $0 > \max\{w_b(p), w_s(p)\}$. Thus $a(p) = 0$. That is, the customers' optimal action is to leave without purchase.

Hence, $p^* = v + (2q-1)\delta$ and $r(p^*) = v + (2q-1)\delta$. This completes the proof. □

Next I will introduce Lemma 15, Lemma 16, and Lemma 17. They will be used to prove Proposition 14G and Proposition 15G.

Lemma 15. *If $\beta = q^2$, then the optimal solution in case (U, U) is as follows:*

1. *If $\delta \geq \frac{c}{2q(1-q)}$ and $q \geq \frac{3-\sqrt{5}}{2}$, then $R^*_{UU} = \max\{v - \delta + \frac{c}{1-q}, q(2-q)(v+\delta) - (2-q)c\}$.*

2. *If $\delta \geq \frac{c}{2q(1-q)}$ and $q < \frac{3-\sqrt{5}}{2}$, then $R^*_{UU} = \max\{v - \delta + \frac{c}{(1-q)^2} - \frac{2q^2}{1-q}\delta + 2q(1-q)\delta - c, q(2-q)(v+\delta) - (2-q)c\}$.*

3. *If $\delta < \frac{c}{2q(1-q)}$, then $R^*_{UU} = v - \delta + 2q\delta$.*

Proof. We put $\beta = q^2$ into the optimal solution in Part 1 in Section 5.7.2, we get that if $q \geq \frac{1}{2}$,

- If $\delta \geq \frac{c}{2q(1-q)}$, then $R^*_{UU} = \max\{v - \delta + \frac{c}{1-q}, q(2-q)(v+\delta) - (2-q)c, q(v+\delta) - c\}$.

- If $\delta < \frac{c}{2q(1-q)}$, then $R^*_{UU} = v - \delta + 2q\delta$.

We put $\beta = q^2$ into the optimal solution in Part 2 in Section 5.7.2, we get that if $q < \frac{1}{2}$,

- If $\delta \geq \frac{c}{2q(1-q)}$ and $q \geq \frac{3-\sqrt{5}}{2}$, then $R^*_{UU} = \max\{v - \delta + \frac{c}{1-q}, q(2-q)(v+\delta) - (2-q)c, q(v+\delta) - c\}$.

- If $\delta \geq \frac{c}{2q(1-q)}$ and $q < \frac{3-\sqrt{5}}{2}$, then $R^*_{UU} = \max\{v - \delta + \frac{c}{1-q}, v - \delta + \frac{c}{(1-q)^2} - \frac{2q^2}{1-q}\delta + 2q(1-q)\delta - c, q(2-q)(v+\delta) - (2-q)c, q(v+\delta) - c\}$.

- If $\delta < \frac{c}{2q(1-q)}$, then $R^*_{UU} = v - \delta + 2q\delta$.

Next, we will combine the solutions when $q \geq \frac{1}{2}$ and when $q < \frac{1}{2}$. Note that

$$\begin{aligned}
&q(2-q)(v+\delta) - (2-q)c - [q(v+\delta) - c] \\
&= (q - q^2)(v+\delta) - (1-q)c \\
&\geq q(1-q)2\delta - (1-q)c && \text{(by } v \geq \delta\text{)} \\
&= (1-q)(2q\delta - c) \\
&\geq 0 && \text{(by } \delta \geq \frac{c}{2q(1-q)}\text{).}
\end{aligned}$$

Therefore, $q(v+\delta) - c$ is kicked out of the optimal solution when $\delta \geq \frac{c}{2q(1-q)}$. Hence, part 1 is proved.

To prove part 2, it suffices to show

$$v - \delta + \frac{c}{(1-q)^2} - \frac{2q^2}{1-q}\delta + 2q(1-q)\delta - c - [v - \delta + \frac{c}{1-q}]$$

$$= \frac{c}{(1-q)^2} - \frac{c}{1-q} - \frac{2q^2}{1-q}\delta + 2q(1-q)\delta - c$$
$$= \frac{-1+3q-q^2}{(1-q)^2}c + \frac{2q(1-3q+q^2)}{1-q}\delta$$
$$= \frac{1-3q+q^2}{1-q}(2q\delta - \frac{c}{1-q})$$
$$\geq 0 \qquad \text{(by } \delta \geq \frac{c}{2q(1-q)}\text{)}.$$

It is apparent that part 3 holds. Then proved. □

Lemma 16. *If $\beta = q^2$, then the optimal solution in case (U, C) is as follows:*

1. *If $\delta \geq \frac{c}{2q(1-q)}$, then $R^*_{UC} = \max\{q(2-q)(v+\delta) - (1-q)c, (q^2+1-q)(v - \delta + \frac{c}{1-q}) + q(1-q)(v+\delta), v - \delta + \frac{2-q}{1-q}c\}$.*

2. *Otherwise, $R^*_{UC} = v - \delta + 2q\delta + 2q(1-q)\delta$.*

Lemma 17. *If $\beta = q^2$, then the optimal solution in case (C, C) is $R^*_{CC} = \max\{q(2-q)(v+\delta), (q^2+1-q)(v-\delta) + q(1-q)(v+\delta)\}$.*

Proof of Proposition 14G. It is easy to see

$$v - \delta + \frac{c}{1-q} < v - \delta + \frac{2-q}{1-q}c,$$
$$q(2-q)(v+\delta) - (2-q)c < q(2-q)(v+\delta) - (1-q)c.$$

Therefore, $R^*_{UU} \leq R^*_{UC}$ if $\delta \geq \frac{c}{2q(1-q)}$ and $q \geq \frac{3-\sqrt{5}}{2}$.

Note that

$$v - \delta + \frac{c}{(1-q)^2} - \frac{2q^2}{1-q}\delta + 2q(1-q)\delta - c$$
$$- [(q^2+1-q)(v - \delta + \frac{c}{1-q}) + q(1-q)(v+\delta)]$$

$$
\begin{aligned}
&= v - \delta + \frac{c}{(1-q)^2} - \frac{2q^2}{1-q}\delta + 2q(1-q)\delta - c \\
&\quad - [v - \delta + 2q\delta - 2q^2\delta + \frac{q^2+1-q}{1-q}c] \\
&= \frac{4q - 3q^2 - 1 + q^3}{(1-q)^2}c - \frac{2q^2}{1-q}\delta \\
&\leq \frac{4q - 3q^2 - 1 + q^3}{(1-q)^2}c - \frac{q}{(1-q)^2}c \qquad\qquad (\text{by } \delta \geq \frac{c}{2q(1-q)}) \\
&= \frac{(q-1)^3}{(1-q)^2}c \\
&\leq 0.
\end{aligned}
$$

Hence $R^*_{UU} \leq R^*_{UC}$ if $\delta \geq \frac{c}{2q(1-q)}$ and $q < \frac{3-\sqrt{5}}{2}$.

Clearly, $R^*_{UU} \leq R^*_{UC}$ if $\delta < \frac{c}{2q(1-q)}$. Then proved. □

Proof of Proposition 15G. We first consider when $\delta \geq \frac{c}{2q(1-q)}$. It is easy to see

$$
\begin{aligned}
&q(2-q)(v+\delta) - (1-q)c \leq q(2-q)(v+\delta), \\
&(q^2+1-q)(v - \delta + \frac{c}{1-q}) + q(1-q)(v+\delta) \\
&\geq (q^2+1-q)(v-\delta) + q(1-q)(v+\delta).
\end{aligned}
$$

Hence, $R^*_{CC} \geq R^*_{UC}$ if and only if $q(2-q)(v+\delta) \geq \max\{(q^2+1-q)(v-\delta+\frac{c}{1-q}) + q(1-q)(v+\delta), v-\delta+\frac{2-q}{1-q}c\}$, from which we derive $\delta \geq \theta$. Thus, part 1 is proved.

Next, we study when $\delta < \frac{c}{2q(1-q)}$. We have

$$
\begin{aligned}
&v - \delta + 2q\delta + 2q(1-q)\delta - [q(2-q)(v+\delta)] \\
&= (1-2q+q^2)v - (1-2q+q^2)\delta \\
&= (1-q)^2(v-\delta) \geq 0
\end{aligned}
$$

$$v - \delta + 2q\delta + 2q(1-q)\delta - [(q^2 + 1 - q)(v - \delta) + q(1-q)(v + \delta)]$$
$$= v - \delta + 2q\delta + 2q(1-q)\delta - [v + (2q - 2q^2 - 1)\delta]$$
$$= 2q\delta \geq 0.$$

Therefore, $R^*_{CC} < R^*_{UC}$ if $\delta < \frac{c}{2q(1-q)}$. Then proved. □

Proof of Proposition 16G. If $p^*_1 = p^*_2$, then group 1 customers' optimal action is to buy product 2, and group 2 customers' optimal action is to buy product 1. And it is easy to figure out $p^*_1 = p^*_2 = v + (2q_2 - 1)\delta$, and $R^*_{UC} = qp_2 + (1-q)p_1 = v + (2q_2 - 1)\delta$.

Next I will show there exists a price vector (p'_1, p'_2) in which $p'_1 = v + (2q_2 - 1)\delta$ and $p'_2 > p'_1$ such that customers behavior is the same as above. Then we have $r(p'_1, p'_2) > R^*_{UC}$. Through some analysis, it suffices to show $w^1_{b2}(p^*_1, p^*_2) > \max\{w^1_a(p^*_1, p^*_2), a \in \{b1, sb, s0, 0\}\}$ and $w^2_{b1}(p^*_1, p^*_2) > \max\{w^2_a(p^*_1, p^*_2), a \in \{b2, sb, s0, 0\}\}$. It is easy to see

$$v - p^*_2 + \delta - [v - p^*_1 + (2q_1 - 1)\delta] > 0,$$
$$v - p^*_2 + \delta - [-c + q_1(v - p^*_1 + \delta) + (1 - q_1)(v - p^*_2 + \delta)] > 0,$$
$$v - p^*_2 + \delta - [-c + q_1(v - p^*_1 + \delta)] > 0,$$
$$v - p^*_2 + \delta - 0 > 0.$$

Thus, increasing p^*_2 can still guarantee group 1's customer buy product 2. That is, $w^1_{b2}(p^*_1, p^*_2) > \max\{w^1_a(p^*_1, p^*_2), a \in \{b1, sb, s0, 0\}\}$.

Also, we have

$$v - p^*_1 + (2q_2 - 1)\delta - [v - p_2 - \delta] > 0,$$

$$v - p_1^* + (2q_2 - 1)\delta - [-c + q_2(v - p_1^* + \delta) + (1 - q_2)(v - p_2^* - \delta)] > 0,$$

$$v - p_1^* + (2q_2 - 1)\delta - [-c + q_2(v - p_1^* + \delta)] > 0,$$

$$v - p_1^* + (2q_2 - 1)\delta - 0 > 0.$$

Thus, increasing p_2^* can still guarantee group 2's customer buy product 1. That is, $w_{b1}^2(p_1^*, p_2^*) > \max\{w_a^2(p_1^*, p_2^*), a \in \{b2, sb, s0, 0\}\}$. This completes the proof.

□

Proof of Proposition 17G and Proposition 18G. They immediately follows Proposition 15G. □

Proof of Proposition 19G. It is easy to see that if $\delta \geq \frac{c}{2q(1-q)}$, $R_{UU}^* \geq R_{UC}^*$ if and only if $v - \delta + \frac{c}{1-q} \geq q(v + \delta)$, from which we derive $\delta < \frac{1-q}{1+q}v + \frac{1}{1-q^2}c$. Thus part 1 is proved. And part 2 holds clearly. Then proved. □

Proof of Lemma 14G. Problem 5.6 can be written into several linear optimization problems. We first solve

$$\begin{aligned} &\max\{p_1\} \\ s.t.\quad & p_2 - p_1 \geq 2(1 - 2q)\delta, \\ & p_2 - p_1 \geq 2\delta - \frac{c}{1 - q}, \\ & p_1 \leq v + (2q - 1)\delta, \end{aligned}$$

and get $R(\boldsymbol{p}^*) = v + (2q - 1)\delta$.

We then solve

$$\max\{p_2\}$$

$$
\begin{aligned}
s.t.\quad & p_1 - p_2 \geq 2(2q-1)\delta, \\
& p_1 - p_2 \geq 2\delta - \frac{c}{q}, \\
& p_2 \leq v + (1-2q)\delta
\end{aligned}
$$

and get $R(\boldsymbol{p}^*) = v + (1-2q)\delta$.

We next solve

$$
\begin{aligned}
& \max\{qp_1 + (1-q)p_2\} \\
s.t.\quad & p_2 - p_1 \leq 2\delta - \frac{c}{1-q}, \\
& p_2 - p_1 \geq -2\delta + \frac{c}{q}, \\
& qp_1 + (1-q)p_2 \leq v + \delta - c,
\end{aligned}
$$

and get $R(\boldsymbol{p}^*) = v + \delta - c$ if $\delta \geq \frac{c}{4q(1-q)}$ and infeasible otherwise.

By comparing these three possible optimal solutions, we arrive the conclusion.

□

Proof of Proposition 20G. Compare the optimal solution of each case, and it is easy to see this proposition holds. □

5.8 Supplemental Material

5.8.1 Optimal Solution of Case (U, U)

Solving UU

Here we solve the optimization problem in scenario (U, U). That is, we wish to solve for $R^*_{UU} = \max_{\boldsymbol{p} \in \mathcal{P}^2} R(\boldsymbol{p}) = \max_{p_1 \leq p_2} R(\boldsymbol{p})$. To do so, we solve for $R^a_{UU} \equiv \max_{\boldsymbol{p} \in \mathcal{P}^2} \{R(\boldsymbol{p}|a(\boldsymbol{p})) : p_1 \leq p_2, a(\boldsymbol{p}) = a\}$ for each $a \in \mathcal{A}$. Then $R^*_{UU} = \max_{a \in \mathcal{A}} R^a_{UU}$. Note that the constraint $a(\boldsymbol{p}) = a$ can be written as $w_a(\boldsymbol{p}) \geq w_{a'}(\boldsymbol{p})$ for all $a' \in \mathcal{A} \setminus \{a\}$.

By assuming $p^*_1 \leq p^*_2$, the customers' action set is $\mathcal{A} = \{b, ss, s0, 0, sb\}$ if $q \geq 1/2$, and $\mathcal{A} = \{b, ss, s0, 0, -sb\}$ if $q < 1/2$. Next, we obtain R^a_{UU} for each action $a \in \mathcal{A}$. In a few cases, we will not be able to compute R^a_{UU} exactly. However, in those cases we will find bounds on R^a_{UU} which will allow us to reach our ultimate goal of solving for R^*_{UU}.

<u>Case 1:</u> $a(\boldsymbol{p}) = b$. Then,

$$\max_{\boldsymbol{p}} p_1$$

$$s.t.\quad \begin{cases} w_b(\boldsymbol{p}) \geq w_{ss}(\boldsymbol{p}) \Rightarrow (1-q)p_1 - (q-\beta)p_2 \leq (1-2q+\beta)v - (1-\beta)\delta \\ \qquad\qquad + (2-q)c \\ w_b(\boldsymbol{p}) \geq w_{s0}(\boldsymbol{p}) \Rightarrow p_1 \leq v - \delta + \dfrac{c}{1-q} \\ w_b(\boldsymbol{p}) \geq w_{sb}(\boldsymbol{p}) \Rightarrow p_2 - p_1 \geq 2\dfrac{q-\beta}{1-q}\delta - \dfrac{c}{1-q} \quad (\text{if} \;\; q \geq \dfrac{1}{2}) \\ w_b(\boldsymbol{p}) \geq w_{-sb}(\boldsymbol{p}) \Rightarrow p_1 - p_2 \leq \dfrac{-2\delta(q-\beta)+c}{q} \quad (\text{if} \;\; q < \dfrac{1}{2}) \\ w_b(\boldsymbol{p}) \geq w_0(\boldsymbol{p}) \Rightarrow p_1 \leq v + (2q-1)\delta \\ p_1 \leq p_2 \end{cases}$$

It is easy to see the optimal solution in Case 1 is

- if $\delta \geq \frac{c}{2q(1-q)}$, then $p_1^* = v - \delta + \frac{c}{1-q}$, and $R_{UU}^b = v - \delta + \frac{c}{1-q}$.
- if $\delta < \frac{c}{2q(1-q)}$, then $p_1^* = v - \delta + 2q\delta$, and $R_{UU}^b = v - \delta + 2q\delta$.

Case 2: $a(\boldsymbol{p}) = ss$. Then,

$$\max_{\boldsymbol{p}} qp_1 + (q-\beta)p_2$$

Subcase 2.1: $q \geq \frac{1}{2}$

$$s.t.\quad \begin{cases} w_{ss}(\boldsymbol{p}) \geq w_b(\boldsymbol{p}) \Rightarrow (1-q)p_1 - (q-\beta)p_2 \geq (1-2q+\beta)v - (1-\beta)\delta \\ \qquad\qquad + (2-q)c \\ w_{ss}(\boldsymbol{p}) \geq w_{s0}(\boldsymbol{p}) \Rightarrow p_2 \leq v + \delta - \dfrac{1-q}{q-\beta}c \\ w_{ss}(\boldsymbol{p}) \geq w_{sb}(\boldsymbol{p}) \Rightarrow p_2 \geq v - \delta + \dfrac{1-q}{1-2q+\beta}c \\ w_{ss}(\boldsymbol{p}) \geq w_0(\boldsymbol{p}) \Rightarrow qp_1 + (q-\beta)p_2 \leq (2q-\beta)(v+\delta) - (2-q)c \\ p_1 \leq p_2 \end{cases}$$

Sub-subcase 2.1.1: $\beta \geq q^2$ We first study the situation when the features are positively correlated, that is, $\beta \geq q^2$. We draw a graph of the feasible region in Figure 5.6. In this figure, we define the following price vectors:

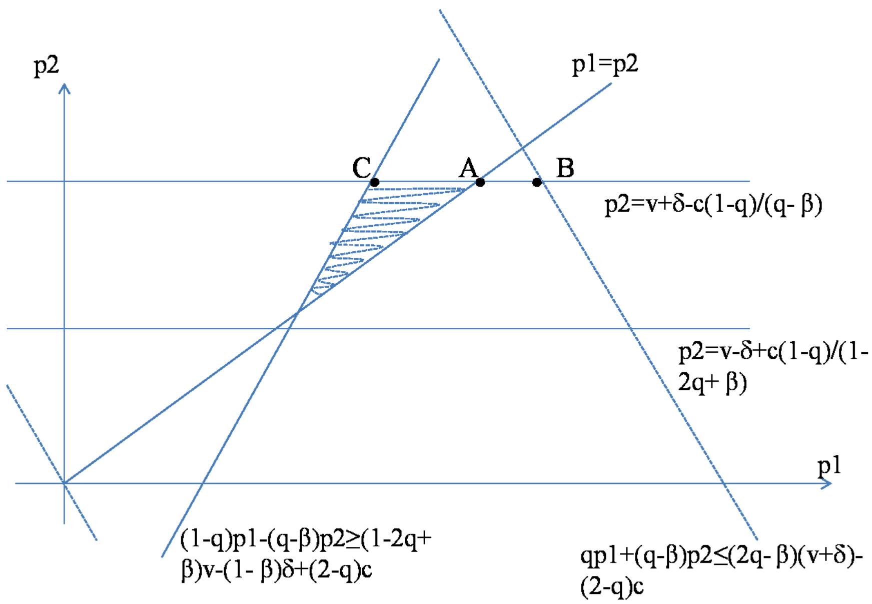


Figure 5.6: $q \geq \frac{1}{2}$ and positive correlation

$$A : p_1^A = p_2^A = v + \delta - \frac{1-q}{q-\beta}c$$
$$B : p_1^B = v + \delta - \frac{c}{q}, p_2^B = v + \delta - \frac{1-q}{q-\beta}c$$
$$C : p_1^C = v - \delta + \frac{c}{1-q}, p_2^C = v + \delta - \frac{1-q}{q-\beta}c$$

We can easily see that $\beta \geq q^2$ implies $p_1^A \leq p_1^B$ (so point B lies to the right of point A in Figure 5.6). Moreover, only if $p_1^C \leq p_1^A$, then there is a feasible option. Thus, a necessary condition is $p_1^C \leq p_1^A$ which can be written as $\delta \geq \frac{1-\beta-q+q^2}{2(1-q)(q-\beta)}c$. (Note that when this condition holds, $v-\delta+\frac{1-q}{1-2q+\beta}c \leq v+\delta-\frac{1-q}{q-\beta}c$ automatically holds.) Hence, the optimal solution in Sub-subcase 2.1.1 is

- If $\delta \geq \frac{1-\beta-q+q^2}{2(1-q)(q-\beta)}c$, then $p_1^* = p_2^* = v + \delta - \frac{1-q}{q-\beta}c$, and $R_{UU}^{ss} = (2q-\beta)(v + \delta) - \frac{(1-q)(2q-\beta)}{q-\beta}c$.

- Otherwise, infeasible.

Sub-subcase 2.1.2: $\beta < q^2$ Next, we study the negative correlation problem. That is, $\beta \leq q^2$. We draw a graph of the feasible region in Figure 5.7. Note that point A is to the right of point B. In this figure, we define the following price vectors:

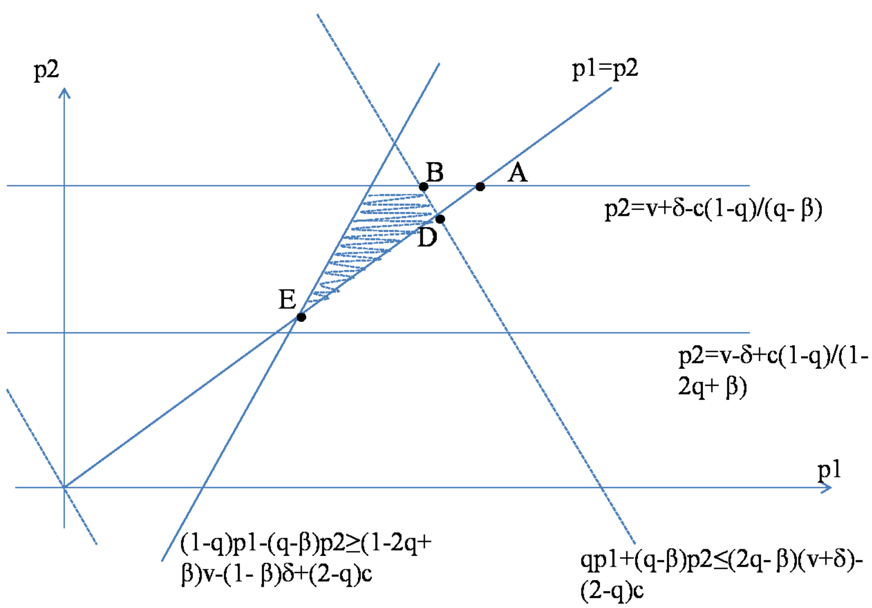


Figure 5.7: $q \geq \frac{1}{2}$ and negative correlation

$$D : p_1^D = p_2^D = v + \delta - \frac{2-q}{2q-\beta}c$$
$$E : p_1^E = p_2^E = v - \frac{1-\beta}{1-2q+\beta}\delta + \frac{2-q}{1-2q+\beta}c$$

It is easy to see that only if $p_1^E \le p_1^D$, then there is a feasible solution. Note also that $p_1^E \le p_1^D \Leftrightarrow \delta \ge \frac{2-q}{2(1-q)(2q-\beta)}c$. Moreover, $v+\delta-\frac{1-q}{q-\beta}c \ge v-\delta+\frac{1-q}{1-2q+\beta}c \Leftrightarrow \delta \ge \frac{c}{2(q-\beta)}\frac{(1-q)^2}{1-2q+\beta}$. Thus, the optimal solution in Sub-subcase 2.1.2 is

- If $\delta \ge \max\{\frac{2-q}{2(1-q)(2q-\beta)}c, \frac{c}{2(q-\beta)}\frac{(1-q)^2}{1-2q+\beta}\}$, then $p_1^* = p_2^* = v+\delta-\frac{2-q}{2q-\beta}c$ and $R_{UU}^{ss} = (2q-\beta)(v+\delta)-(2-q)c$.
- Otherwise, infeasible.

Subcase 2.2: $q < 1/2$

$$
s.t. \begin{cases}
w_{ss}(\boldsymbol{p}) \ge w_b(\boldsymbol{p}) \Rightarrow (1-q)p_1 - (q-\beta)p_2 \ge (1-2q+\beta)v - (1-\beta)\delta \\
\qquad\qquad + (2-q)c \\
w_{ss}(\boldsymbol{p}) \ge w_{s0}(\boldsymbol{p}) \Rightarrow p_2 \le v+\delta-\dfrac{1-q}{q-\beta}c \\
w_{ss}(\boldsymbol{p}) \ge w_{-sb}(\boldsymbol{p}) \Rightarrow (1-2q)p_1 + \beta p_2 \ge (1-2q+\beta)(v-\delta) + (1-q)c \\
\qquad\qquad ....(*) \\
w_{ss}(\boldsymbol{p}) \ge w_0(\boldsymbol{p}) \Rightarrow qp_1 + (q-\beta)p_2 \le (2q-\beta)(v+\delta) - (2-q)c \\
p_1 \le p_2
\end{cases}
$$

Sub-subcase 2.2.1: $\beta \ge q^2$ We first study the situation when the features are positively correlated, that is, $\beta \ge q^2$. Compared with the constraints when $q \ge \frac{1}{2}$ in Sub-subcase 2.1.1, we find that the only difference is $p_2 \ge v-\delta+\frac{1-q}{1-2q+\beta}c$ is replaced by $(*)$. Note that $p_2 \ge v-\delta+\frac{1-q}{1-2q+\beta}c$ is not tight for the optimal solution in Sub-subcase 2.1.1. We put the optimal solution for Sub-subcase 2.1.1 into $(*)$, and find that if $\delta \ge \frac{1-\beta-q+q^2}{2(1-q)(q-\beta)}c$ then $(*)$ holds. Hence, the optimal solution when $q < \frac{1}{2}$ is the same as that when $q \ge \frac{1}{2}$. Therefore, the optimal solution in Sub-subcase 2.2.1 is

- If $\delta \geq \frac{1-\beta-q+q^2}{2(1-q)(q-\beta)}c$, then $p_1^* = p_2^* = v + \delta - \frac{1-q}{q-\beta}c$, and $R_{UU}^{ss} = (2q-\beta)(v+\delta) - \frac{(1-q)(2q-\beta)}{q-\beta}c$.

- Otherwise, infeasible.

<u>Sub-subcase 2.2.2: $\beta < q^2$</u> Now, we study the negative correlation problem. That is, $\beta \leq q^2$. This subproblem becomes more complicated than that when $q \geq \frac{1}{2}$. Here, we will establish some upper bound on R_{UU}^{ss} instead of the exact solution. By the analysis of Sub-subcase 2.1.2, we can get the following upper bound in <u>Sub-subcase 2.2.2</u>.

- If $\delta \geq \frac{2-q}{2(1-q)(2q-\beta)}c$, then either $R_{UU}^{ss} \leq (2q-\beta)(v+\delta) - (2-q)c$, or infeasible.

<u>Case 3:</u> $a(\boldsymbol{p}) = s0$. Then,

$$\max_{\boldsymbol{p}} qp_1$$

$$s.t. \quad \begin{cases} w_{s0}(\boldsymbol{p}) \geq w_b(\boldsymbol{p}) \Rightarrow p_1 \geq v - \delta + \dfrac{c}{1-q} \\ w_{s0}(\boldsymbol{p}) \geq w_{ss}(\boldsymbol{p}) \Rightarrow p_2 \geq v + \delta - \dfrac{1-q}{q-\beta}c \\ w_{s0}(\boldsymbol{p}) \geq w_{sb}(\boldsymbol{p}) \Rightarrow p_2 \geq v + (2\dfrac{q-\beta}{1-q} - 1)\delta \quad (\text{if} \quad q \geq \dfrac{1}{2}) \\ w_{s0}(\boldsymbol{p}) \geq w_{-sb}(\boldsymbol{p}) \Rightarrow (1-2q)p_1 + qp_2 \geq (1-q)v + (3q-2\beta-1)\delta \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\text{if} \quad q < \dfrac{1}{2}) \\ w_{s0}(\boldsymbol{p}) \geq w_0(\boldsymbol{p}) \Rightarrow p_1 \leq v + \delta - \dfrac{c}{q} \\ p_1 \leq p_2 \end{cases}$$

It is easy to see the optimal solution in <u>Case 3</u> is

- if $\delta \geq \frac{c}{2q(1-q)}$, then $p_1^* = v + \delta - \frac{c}{q}$, and $R_{UU}^{s0} = q(v+\delta) - c$.

- otherwise, infeasible.

Case 4: $q \geq \frac{1}{2}$ and $a(\boldsymbol{p}) = sb$. Recall that $sb \in \mathcal{A}$ if and only if $q \geq 1/2$. Then,

$$\max_{\boldsymbol{p}} qp_1 + (1-q)p_2$$

$$s.t. \quad \begin{cases} w_{sb}(\boldsymbol{p}) \geq w_b(\boldsymbol{p}) \Rightarrow p_2 - p_1 \leq \dfrac{2(q-\beta)}{1-q}\delta - \dfrac{c}{1-q} \quad ......(*) \\ w_{sb}(\boldsymbol{p}) \geq w_{ss}(\boldsymbol{p}) \Rightarrow p_2 \leq v - \delta + \dfrac{1-q}{1-2q+\beta}c \\ w_{sb}(\boldsymbol{p}) \geq w_{s0}(\boldsymbol{p}) \Rightarrow p_2 \leq v + (2\dfrac{q-\beta}{1-q} - 1)\delta \\ w_{sb}(\boldsymbol{p}) \geq w_0(\boldsymbol{p}) \Rightarrow qp_1 + (1-q)p_2 \leq v + (4q - 2\beta - 1)\delta - c......(+) \\ p_1 \leq p_2 \quad ......(**) \end{cases}$$

By $(*)$ and $(**)$, we have that only if $\delta \geq \frac{c}{2(q-\beta)}$, there is a feasible solution.

Subcase 4.1: $\beta \geq q^2$ We first study the positive correlation problem. That is, $\beta \geq q^2$. With $\delta \geq \frac{c}{2(q-\beta)}$, we have $v - \delta + \frac{1-q}{1-2q+\beta}c \leq v + (2\frac{q-\beta}{1-q} - 1)\delta$. We draw a graph of the feasible region in Figure 5.8.

$$A : p_1^A = p_2^A = v - \delta + \frac{1-q}{1-2q+\beta}c$$
$$B : p_1^B = v - \delta + \frac{4q-2\beta}{q}\delta - \frac{c}{q} - \frac{(1-q)^2}{q(1-2q+\beta)}c, p_2^B = v - \delta + \frac{1-q}{1-2q+\beta}c.$$

We can show $p_1^A \leq p_1^B$ if $\beta \geq q^2$ and $\delta \geq \frac{c}{2(q-\beta)}$. Therefore, the optimal solution in Subcase 4.1 is

- if $\delta \geq \frac{c}{2(q-\beta)}$, then $p_1^* = p_2^* = v - \delta + \frac{1-q}{1-2q+\beta}c$, and $R_{UU}^{sb} = v - \delta + \frac{1-q}{1-2q+\beta}c$.

- otherwise, infeasible.

Subcase 4.2: $\beta < q^2$ Next, we study the negative correlation problem. That is, $\beta < q^2$. Again, by $(*)$ and $(**)$, we must have $\delta \geq \frac{c}{2(q-\beta)}$ for feasibility. When

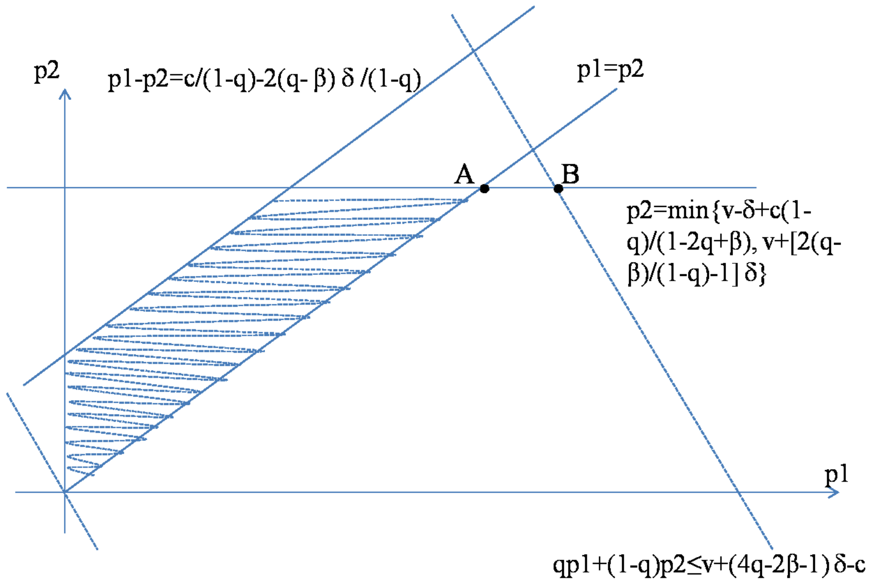


Figure 5.8: Case 4: $q \geq \frac{1}{2}$

$\delta \geq \frac{(1-q)^2}{2(q-\beta)(1-2q+\beta)}c$, we have $v - \delta + \frac{1-q}{1-2q+\beta}c \leq v + (2\frac{q-\beta}{1-q} - 1)\delta$. And we find that $p_1^A \leq p_1^B$ if $\delta \geq \frac{2-3q+\beta}{2(2q-\beta)(1-2q+\beta)}c$, and $p_1^A > p_1^B$ otherwise.

Since $\frac{2-3q+\beta}{2(2q-\beta)(1-2q+\beta)}c \geq \frac{(1-q)^2}{2(q-\beta)(1-2q+\beta)}c$, the optimal solution when $\delta \geq \frac{(1-q)^2}{2(q-\beta)(1-2q+\beta)}c$ in Subcase 4.2 is

- if $\delta \geq \frac{2-3q+\beta}{2(2q-\beta)(1-2q+\beta)}c$, then $p_1^* = p_2^* = R_{UU}^{sb} = v - \delta + \frac{1-q}{1-2q+\beta}c$.

- if $\frac{(1-q)^2}{2(q-\beta)(1-2q+\beta)}c \leq \delta < \frac{2-3q+\beta}{2(2q-\beta)(1-2q+\beta)}c$, then line $qp_1 + (1-q)p_2 = v + (4q - 2\beta - 1)\delta - c$ cuts the shaded region in Figure 5.8, and the optimal pricing solution lies on the intersection of the line and the region, and $R_{UU}^{sb} = v + (4q - 2\beta - 1)\delta - c$. To understand this, note that the left-side of constraint (+) is the same as the objective function.

When $\frac{c}{2(q-\beta)} \leq \delta < \frac{(1-q)^2}{2(q-\beta)(1-2q+\beta)}c$, then $v - \delta + \frac{1-q}{1-2q+\beta}c \geq v + (2\frac{q-\beta}{1-q} - 1)\delta$.

$$A : p_1^A = p_2^A = v + (2\frac{q-\beta}{1-q} - 1)\delta$$
$$B : p_1^B = v + \delta - \frac{c}{q}, p_2^B = v + (2\frac{q-\beta}{1-q} - 1)\delta.$$

We can show that $p_1^B \leq p_1^A$ if $\beta \leq q^2$ and $\delta < \frac{(1-q)^2}{2(q-\beta)(1-2q+\beta)}c$. Therefore, the optimal solution when $\delta < \frac{(1-q)^2}{2(q-\beta)(1-2q+\beta)}c$ in Subcase 4.2 is

- if $\frac{c}{2(q-\beta)} \leq \delta < \frac{(1-q)^2}{2(q-\beta)(1-2q+\beta)}c$, then the optimal pricing solution is an interval on the line $qp_1 + (1-q)p_2 = v + (4q - 2\beta - 1)\delta - c$, and $R_{UU}^{sb} = v + (4q - 2\beta - 1)\delta - c$.

- if $\delta < \frac{c}{2(q-\beta)}$, then infeasible.

Putting the four bullets above together, we get the optimal solution for negative correlation in Subcase 4.2.

Case 5: $q < 1/2$ and $a(\boldsymbol{p}) = -sb$.

$$\max_{\boldsymbol{p}} qp_2 + (1-q)p_1$$
$$s.t. \begin{cases} w_{-sb}(\boldsymbol{p}) \geq w_b(\boldsymbol{p}) \Rightarrow p_1 - p_2 \geq \frac{c}{q} - \frac{2\delta(q-\beta)}{q} \\ w_{-sb}(\boldsymbol{p}) \geq w_{ss}(\boldsymbol{p}) \Rightarrow (1-2q)p_1 + \beta p_2 \leq (1-2q+\beta)(v-\delta) + (1-q)c \\ \qquad\qquad ......(*) \\ w_{-sb}(\boldsymbol{p}) \geq w_{s0}(\boldsymbol{p}) \Rightarrow (1-2q)p_1 + qp_2 \leq (1-q)v + (3q-2\beta-1)\delta \\ w_{-sb}(\boldsymbol{p}) \geq w_0(\boldsymbol{p}) \Rightarrow (1-q)p_1 + qp_2 \leq v + (4q-2\beta-1)\delta - c \quad ......(**) \\ p_1 \leq p_2 \end{cases}$$

By $w_{-sb}(\boldsymbol{p}) \geq w_b(\boldsymbol{p})$ and $p_1 \leq p_2$, we have that $\delta \geq \frac{c}{2(q-\beta)}$ is necessary condition for a feasible solution.

Subcase 5.1: $\beta \geq q^2$ We first study the positive correlation case. That is, $\beta \geq q^2$. We draw a graph of the feasible region in Figure 5.9.

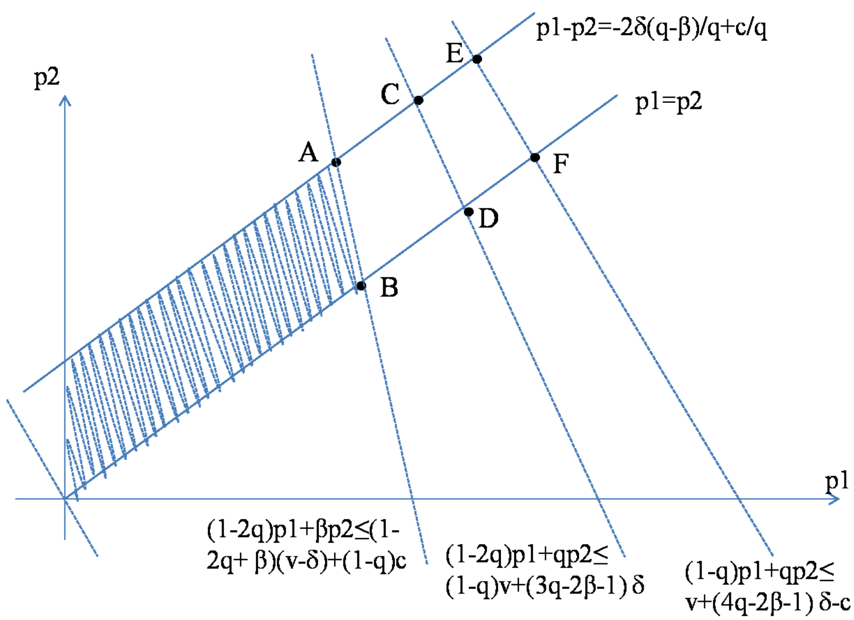


Figure 5.9: Case 5 with positive correlation (Subcase 5.1)

$$A : p_1^A = v - \delta - \frac{2\beta(q-\beta)}{q(1-2q+\beta)}\delta + \frac{1-q}{1-2q+\beta}c + \frac{\beta}{q(1-2q+\beta)}c$$
$$C : p_1^C = v - \delta + \frac{c}{1-q}$$
$$E : p_1^E = v - \delta + 2q\delta$$

By $\delta \geq \frac{c}{2(q-\beta)}$ and $\beta \geq q^2$, we have $p_1^A \leq p_1^C \leq p_1^E$.

$$B : p_1^B = v - \delta + \frac{1-q}{1-2q+\beta}c$$

$$D: p_1^D = v + \frac{3q - 2\beta - 1}{1-q}\delta$$

$$F: p_1^F = v + (4q - 2\beta - 1)\delta - c$$

Again, we have $p_1^B \leq p_1^D \leq p_1^F$. Hence, the feasible region is as shown in Figure 5.9, and the optimal solution is either point B or point A, depending on the slopes of constraint $(*)$ and $(**)$. The optimal solution in Subcase 5.1 is as follows,

- If $\delta \geq \frac{c}{2(q-\beta)}$.
  - If $\frac{1-2q}{\beta} \leq \frac{1-q}{q}$, then point B is optimal and $p_1^* = p_2^* = R_{UU}^{-sb} = v - \delta + \frac{1-q}{1-2q+\beta}c$.
  - If $\frac{1-2q}{\beta} > \frac{1-q}{q}$, then point A is optimal and $p_1^* = v - \delta - \frac{2\beta(q-\beta)}{q(1-2q+\beta)}\delta + \frac{1-q}{1-2q+\beta}c + \frac{\beta}{q(1-2q+\beta)}c$, $p_2^* = p_1^* + \frac{2\delta(q-\beta)}{q} - \frac{c}{q}$, $R_{UU}^{-sb} = v - \delta - \frac{2\beta(q-\beta)}{q(1-2q+\beta)}\delta + \frac{1-q}{1-2q+\beta}c + \frac{\beta}{q(1-2q+\beta)}c + 2\delta(q-\beta) - c$.
- Otherwise, infeasible.

Subcase 5.2: $\beta < q^2$ Next, we consider the negative correlation case. It is too complex to characterize the optimal solution completely. However, we are able to (i) derive bounds which can be used to compare with R_{UC}^*, and (ii) show that R_{UU}^{-sb} is decreasing with β if $p_2 \geq v - \delta$.

We do part (i) first. It is easy to see $R_{UU}^{-sb} \leq v + (4q - 2\beta - 1)\delta - c$ by $(**)$. However, this upper bound is not good enough for our purposes. We derive some other tighter bounds. We draw a graph of the feasible region in Figure 5.10.

$$A: p_1^A = v - \delta - \frac{2\beta(q-\beta)}{q(1-2q+\beta)}\delta + \frac{1-q}{1-2q+\beta}c + \frac{\beta}{q(1-2q+\beta)}c,$$

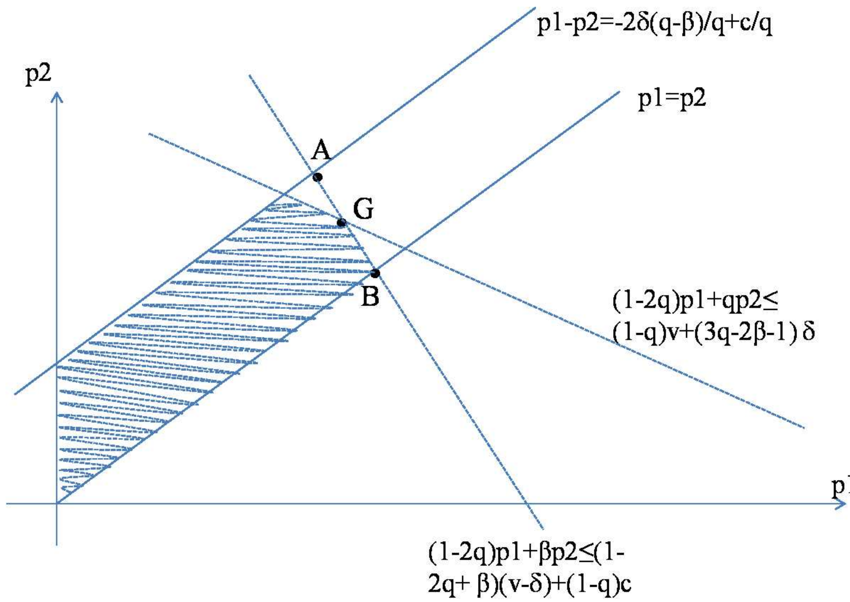


Figure 5.10: Case 5 with negative correlation (Subcase 5.2)

$$p_2^A = p_1^A + \frac{2\delta(q-\beta)}{q} - \frac{c}{q},$$

$$R(\boldsymbol{p}^A| - sb) = v - \delta - \frac{2\beta(q-\beta)}{q(1-2q+\beta)}\delta + \frac{1-q}{1-2q+\beta}c + \frac{\beta}{q(1-2q+\beta)}c$$
$$+ 2\delta(q-\beta) - c.$$

$$G : p_1^G = v + \frac{2q-2\beta-1}{1-2q}\delta + \frac{q(1-q)}{(q-\beta)(1-2q)}c, p_2^G = v + \delta - \frac{1-q}{q-\beta}c,$$

$$R(\boldsymbol{p}^G| - sb) = v + (3q-2\beta-1)\delta + \frac{q(2q-2\beta-1)}{1-2q}\delta + \frac{q^2(1-q)}{(q-\beta)(1-2q)}c.$$

$$B : p_1^B = p_2^B = R(\boldsymbol{p}^B| - sb) = v - \delta + \frac{1-q}{1-2q+\beta}c.$$

Through some analysis, we can show either $R_{UU}^{-sb} \leq R(\boldsymbol{p}^B| - sb)$ or $R_{UU}^{-sb} \leq \min\{R(\boldsymbol{p}^A| - sb), R(\boldsymbol{p}^G| - sb)\}$. Now we have the following upper bounds for Subcase 5.2:

- $R_{UU}^{-sb} \leq v + (4q - 2\beta - 1)\delta - c$.
- Either $R_{UU}^{-sb} \leq R(\boldsymbol{p}^B | -sb)$ or $R_{UU}^{-sb} \leq \min\{R(\boldsymbol{p}^A | -sb), R(\boldsymbol{p}^G | -sb)\}$.

In the future, we will use those upper bounds to compare R_{UU}^* with R_{UC}^*. We will use the first bullet above if $\delta \leq \frac{2-3q+\beta}{2(2q-\beta)(1-2q+\beta)}c$ and we use the second bullet otherwise.

Next, we do part (ii). Suppose $\beta_1 \leq \beta_2$ and $p_2 \geq v - \delta$. We use $(p_1^*(\beta_i), p_2^*(\beta_i), R_{UU}^{-sb}(\beta_i))$ to denote the optimal revenue and pricing solution when $\beta = \beta_i$. Then, we want to show $R_{UU}^{-sb}(\beta_1) \geq R_{UU}^{-sb}(\beta_2)$.

The idea is to show $(p_1^*(\beta_2), p_2^*(\beta_2))$ is a feasible solution when $\beta = \beta_1$. We need to verify $(p_1^*(\beta_2), p_2^*(\beta_2))$ satisfies all the constraints when $\beta = \beta_1$.

For the first constraint,

$$p_1^*(\beta_2) - p_2^*(\beta_2) \geq \frac{c}{q} - \frac{2\delta(q - \beta_2)}{q} \geq \frac{c}{q} - \frac{2\delta(q - \beta_1)}{q}.$$

Let's rewrite the second constraint as $(1-2q)p_1 \leq (1-2q)(v-\delta) + \beta(v - \delta - p_2) + (1-q)c$. Since $p_2 \geq v - \delta$, we have $v - \delta - p_2 \leq 0$. Thus,

$$\begin{aligned}(1-2q)p_1^*(\beta_2) &\leq (1-2q)(v-\delta) + \beta_2(v - \delta - p_2^*(\beta_2)) + (1-q)c \\ &\leq (1-2q)(v-\delta) + \beta_1(v - \delta - p_2^*(\beta_2)) + (1-q)c\end{aligned}$$

For the third constraint,

$$(1-2q)p_1^*(\beta_2) + qp_2^*(\beta_2) \leq (1-q)v + (3q - 2\beta_2 - 1)\delta \leq (1-q)v + (3q - 2\beta_1 - 1)\delta.$$

For the fourth constraint,

$$(1-q)p_1^*(\beta_2) + qp_2^*(\beta_2) \leq v + (4q - 2\beta_2 - 1)\delta - c \leq v + (4q - 2\beta_1 - 1)\delta - c.$$

Therefore, $R_{UU}^{-sb}(\beta_1) \geq R_{UU}^{-sb}(\beta_2)$.

Case 6: $a(\boldsymbol{p}) = 0$. Then the optimal solution for Case 6 is

- $R^0_{UU} = 0$.

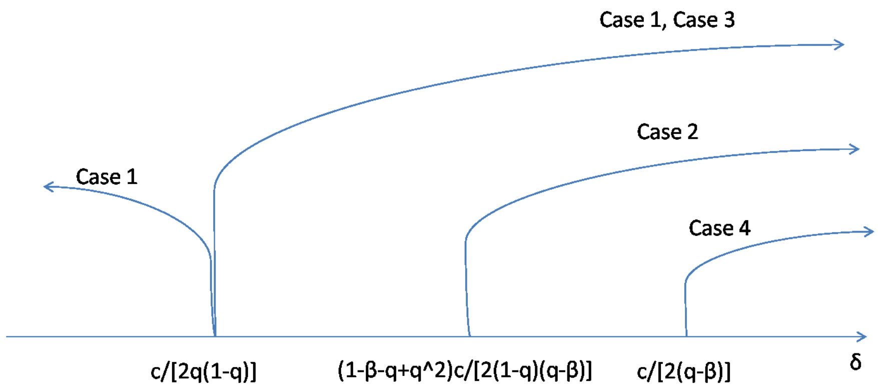
Summary

Part 1: Positive correlation ($\beta \geq q^2$) and $q \geq 1/2$

- Case 1:
  - If $\delta \geq \frac{c}{2q(1-q)}$, then $R^b_{UU} = v - \delta + \frac{c}{1-q}$.
  - If $\delta < \frac{c}{2q(1-q)}$, then $R^b_{UU} = v - \delta + 2q\delta$.
- Case 2:
  - If $\delta \geq \frac{1-\beta-q+q^2}{2(1-q)(q-\beta)}c$, then $R^{ss}_{UU} = (2q-\beta)(v+\delta) - \frac{(1-q)(2q-\beta)}{q-\beta}c$.
  - Otherwise, infeasible.
- Case 3:
  - If $\delta \geq \frac{c}{2q(1-q)}$, then $R^{s0}_{UU} = q(v+\delta) - c$.
  - Otherwise, infeasible.
- Case 4:
  - If $\delta \geq \frac{c}{2(q-\beta)}$, then $R^{sb}_{UU} = v - \delta + \frac{1-q}{1-2q+\beta}c$.
  - Otherwise, infeasible.

We have $v-\delta+\frac{1-q}{1-2q+\beta}c \leq v-\delta+\frac{c}{1-q}$ (by $\beta \geq q^2$), hence Case 4 is not optimal. Therefore, the optimal solution for $\beta \geq q^2$ and $q \geq 1/2$ is

- If $\delta \geq \frac{1-\beta-q+q^2}{2(1-q)(q-\beta)}c$, then $R^*_{UU} = \max\{v - \delta + \frac{c}{1-q}, (2q-\beta)(v+\delta) - \frac{(1-q)(2q-\beta)}{q-\beta}c, q(v+\delta) - c\}$.



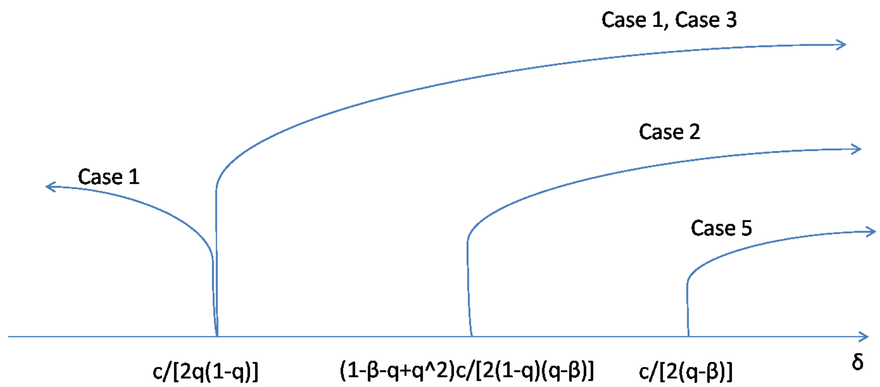
- If $\frac{c}{2q(1-q)} \leq \delta < \frac{1-\beta-q+q^2}{2(1-q)(q-\beta)}c$, then $R^*_{UU} = \max\{v - \delta + \frac{c}{1-q}, q(v+\delta) - c\}$.
- If $\delta < \frac{c}{2q(1-q)}$, then $R^*_{UU} = v - \delta + 2q\delta$.

Part 2: Positive correlation ($\beta \geq q^2$) and $q < 1/2$

- Case 1, 2, and 3 is the same as Part 1.
- Case 5:
  - If $\delta \geq \frac{c}{2(q-\beta)}$ and $q(1-2q) \leq \beta(1-q)$, then $R^{-sb}_{UU} = v - \delta + \frac{1-q}{1-2q+\beta}c$.
  - If $\delta \geq \frac{c}{2(q-\beta)}$ and $q(1-2q) > \beta(1-q)$, then $R^{-sb}_{UU} = v - \delta + \frac{1-q}{1-2q+\beta}c - \frac{2\beta(q-\beta)}{q(1-2q+\beta)}\delta + \frac{\beta}{q(1-2q+\beta)}c + 2\delta(q-\beta) - c$.
  - Otherwise, infeasible.

Note that $v - \delta + \frac{1-q}{1-2q+\beta}c \leq v - \delta + \frac{c}{1-q}$. So Case 5 is not optimal if $q(1-2q) \leq \beta(1-q)$. We now have that the optimal solution for $\beta \geq q^2$ and $q < 1/2$ is as follows:

- If $\delta \geq \frac{c}{2(q-\beta)}$ and $\beta \geq \frac{q(1-2q)}{1-q}$, then $R^*_{UU} = \max\{v - \delta + \frac{c}{1-q}, (2q-\beta)(v+\delta) - \frac{(1-q)(2q-\beta)}{q-\beta}c, q(v+\delta) - c\}$.



- If $\delta \geq \frac{c}{2(q-\beta)}$ and $\beta < \frac{q(1-2q)}{1-q}$, then $R^*_{UU} = \max\{v-\delta+\frac{c}{1-q}, (2q-\beta)(v+\delta) - \frac{(1-q)(2q-\beta)}{q-\beta}c, q(v+\delta)-c, v-\delta+\frac{1-q}{1-2q+\beta}c - \frac{2\beta(q-\beta)}{q(1-2q+\beta)}\delta + \frac{\beta}{q(1-2q+\beta)}c + 2\delta(q-\beta) - c\}$.

- If $\frac{1-\beta-q+q^2}{2(1-q)(q-\beta)}c \leq \delta < \frac{c}{2(q-\beta)}$, then $R^*_{UU} = \max\{v-\delta+\frac{c}{1-q}, (2q-\beta)(v+\delta) - \frac{(1-q)(2q-\beta)}{q-\beta}c, q(v+\delta)-c\}$.

- If $\frac{c}{2q(1-q)} \leq \delta < \frac{1-\beta-q+q^2}{2(1-q)(q-\beta)}c$, then $R^*_{UU} = \max\{v-\delta+\frac{c}{1-q}, q(v+\delta)-c\}$.

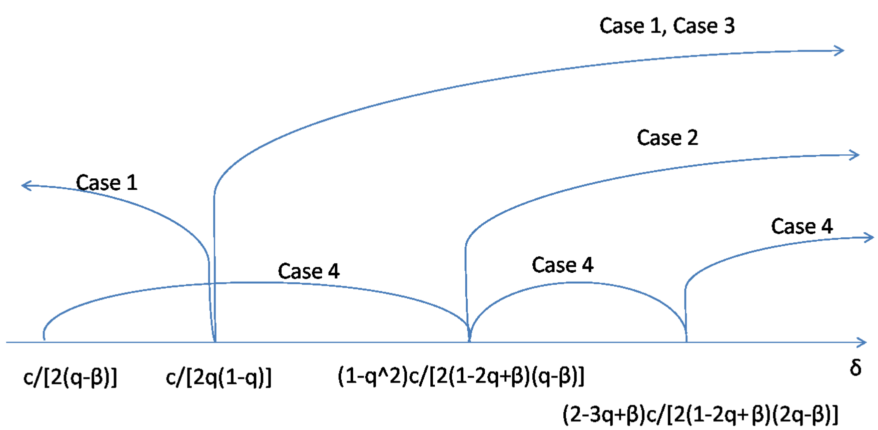
- If $\delta < \frac{c}{2q(1-q)}$, then $R^*_{UU} = v-\delta+2q\delta$.

Part 3: Negative correlation ($\beta < q^2$) and $q \geq 1/2$

- Case 1:
  - If $\delta \geq \frac{c}{2q(1-q)}$, then $R^b_{UU} = v-\delta+\frac{c}{1-q}$.
  - If $\delta < \frac{c}{2q(1-q)}$, then $R^b_{UU} = v-\delta+2q\delta$.

- Case 2:
  - If $\delta \geq \frac{c}{2(q-\beta)}\frac{(1-q)^2}{1-2q+\beta}$, then $R^{ss}_{UU} = (2q-\beta)(v+\delta)-(2-q)c$.
  - Otherwise, infeasible.

- Case 3:
  - If $\delta \geq \frac{c}{2q(1-q)}$, then $R^{s0}_{UU} = q(v+\delta) - c$.
  - Otherwise, infeasible.
- Case 4:
  - if $\delta \geq \frac{2-3q+\beta}{2(2q-\beta)(1-2q+\beta)}c$, then $R^{sb}_{UU} = v - \delta + \frac{1-q}{1-2q+\beta}c$.
  - if $\frac{(1-q)^2}{2(q-\beta)(1-2q+\beta)}c \leq \delta < \frac{2-3q+\beta}{2(2q-\beta)(1-2q+\beta)}c$, then $R^{sb}_{UU} = v + (4q - 2\beta - 1)\delta - c$.
  - if $\frac{c}{2(q-\beta)} \leq \delta < \frac{(1-q)^2}{2(q-\beta)(1-2q+\beta)}c$, then $R^{sb}_{UU} = v + (4q - 2\beta - 1)\delta - c$.



Before we introduce the optimal solution, let's make several comparisons. (Note that $\beta \leq q^2$ and $v \geq \delta$)

$$v - \delta + \frac{c}{1-q} \leq v - \delta + \frac{1-q}{1-2q+\beta}c \tag{5.8}$$

$$q(v+\delta) - c \leq (2q-\beta)(v+\delta) - (2-q)c \quad \text{if} \quad \delta \geq \frac{c}{2(q-\beta)} \frac{(1-q)^2}{1-2q+\beta} \tag{5.9}$$

$$v - \delta + \frac{c}{1-q} \leq v + (4q - 2\beta - 1)\delta - c \quad \text{if} \quad \delta \geq \frac{2-q}{2(1-q)(2q-\beta)}c \tag{5.10}$$

$$q(v + \delta) - c \leq v + (4q - 2\beta - 1)\delta - c \tag{5.11}$$

$$v - \delta + 2q\delta \leq v + (4q - 2\beta - 1)\delta - c \quad \text{if} \quad \delta \geq \frac{c}{2(q-\beta)} \tag{5.12}$$

The optimal solution for $\beta < q^2$ and $q \geq 1/2$ is as follows:

- If $\delta \geq \frac{2-3q+\beta}{2(2q-\beta)(1-2q+\beta)}c$, then $R^*_{UU} = \max\{v - \delta + \frac{1-q}{1-2q+\beta}c, (2q-\beta)(v+\delta) - (2-q)c\}$ (By (5.8) and (5.9)).

- If $\frac{c}{2(q-\beta)}\frac{(1-q)^2}{1-2q+\beta} \leq \delta < \frac{2-3q+\beta}{2(2q-\beta)(1-2q+\beta)}c$, then $R^*_{UU} = \max\{(2q-\beta)(v+\delta) - (2-q)c, v + (4q-2\beta-1)\delta - c\}$ (By (5.9) and (5.10)).

- If $\frac{c}{2q(1-q)} \leq \delta < \frac{c}{2(q-\beta)}\frac{(1-q)^2}{1-2q+\beta}$, then $R^*_{UU} = v + (4q-2\beta-1)\delta - c$ (by (5.10) and (5.11)).

- If $\frac{c}{2(q-\beta)} \leq \delta < \frac{c}{2q(1-q)}$, then $R^*_{UU} = v + (4q-2\beta-1)\delta - c$ (by (5.12)).

- If $\delta \leq \frac{c}{2(q-\beta)}$, then $R^*_{UU} = v - \delta + 2q\delta$.

Part 4: Negative correlation ($\beta < q^2$) and $q < 1/2$

- Case 1:
  - If $\delta \geq \frac{c}{2q(1-q)}$, then $R^b_{UU} = v - \delta + \frac{c}{1-q}$.
  - If $\delta < \frac{c}{2q(1-q)}$, then $R^b_{UU} = v - \delta + 2q\delta$.

- Case 2: Upper bound: If $\delta \geq \frac{2-q}{2(1-q)(2q-\beta)}c$, then $R^{ss}_{UU} = (2q-\beta)(v+\delta) - (2-q)c$.

- Case 3:

- If $\delta \geq \frac{c}{2q(1-q)}$, then $R_{UU}^{s0} = q(v+\delta) - c$.
- Otherwise, infeasible.

- Case 5: Upper bound
  - if $\delta \leq \frac{2-3q+\beta}{2(2q-\beta)(1-2q+\beta)}c$, then $R_{UU}^{-sb} \leq v + (4q-2\beta-1)\delta - c$.
  - if $\delta \geq \frac{2-3q+\beta}{2(2q-\beta)(1-2q+\beta)}c$, then $R_{UU}^{-sb} \leq R^B$ or $R_{UU}^{-sb} \leq \min\{R^A, R^G\}$, where $R^A = v - \delta - \frac{2\beta(q-\beta)}{q(1-2q+\beta)}\delta + \frac{1-q}{1-2q+\beta}c + \frac{\beta}{q(1-2q+\beta)}c + 2\delta(q-\beta) - c$, $R^G = v + (3q-2\beta-1)\delta + \frac{q(2q-2\beta-1)}{1-q}\delta + \frac{q^2(1-q)}{(q-\beta)(1-2q)}c$, and $R^B = v - \delta + \frac{1-q}{1-2q+\beta}c$.
  - We also proved that R_{UU}^{-sb} is decreasing in β in this case.

5.8.2 Optimal Solution of Case (U, C)

Solving (U, C)

Customers' action set $\mathcal{A} = \{b2, b1, sb, s0, 0\}$. For the convenience of presentation, we will refer to $b2$ as action 1, $b1$ as action 2, sb as action 3, $s0$ as action 4, and 0 as action 5. We use case $i + j$ to denote the case that group 1 customers' optimal action is i and group 2 customers' optimal action is j. It is easy to see there are 25 cases. Next, we will go over them one by one.

Case 1+1 ($b2 + b2$):

$$\max_{\boldsymbol{p}} qp_2 + (1-q)p_2 = p_2$$

$$s.t. \quad \begin{cases} v - p_2 + \delta \geq v - p_1 + (2q_1 - 1)\delta \\ v - p_2 + \delta \geq -c + q_1(v - p_1 + \delta) + (1-q_1)(v - p_2 + \delta) \\ v - p_2 + \delta \geq -c + q_1(v - p_1 + \delta) \\ v - p_2 + \delta \geq 0 \end{cases}$$

$$\begin{cases} v - p_2 - \delta \geq v - p_1 + (2q_2 - 1)\delta \\ v - p_2 - \delta \geq -c + q_2(v - p_1 + \delta) + (1 - q_2)(v - p_2 - \delta) \\ v - p_2 - \delta \geq -c + q_2(v - p_1 + \delta) \\ v - p_2 - \delta \geq 0 \end{cases}$$

$\Rightarrow$

$$\max_{\boldsymbol{p}} qp_2 + (1-q)p_2 = p_2$$

$$s.t. \quad \begin{cases} p_1 - p_2 \geq 2(q_1 - 1)\delta \\ p_1 - p_2 \geq -\dfrac{c}{q_1} \\ q_1 p_1 - p_2 \geq -c - (1 - q_1)(v + \delta) \\ p_2 \leq v + \delta \end{cases}$$

$$\begin{cases} p_1 - p_2 \geq 2q_2\delta \\ p_1 - p_2 \geq 2\delta - \dfrac{c}{q_2} \\ q_2 p_1 - p_2 \geq (1 + q_2)\delta - (1 - q_2)v - c \\ p_2 \leq v - \delta \end{cases}$$

Note that we write these constraints separately because we can copy either part when we study the other cases. The first group of bracketed constraints deal with group 1 customers. The second deal with group 2 customers. The optimal solution has to satisfy all eight constraints.

It is easy to see the optimal solution is

$$\begin{cases} p_2^* = v - \delta \\ r_{1,1}^* = v - \delta \end{cases}$$

Case 1+2 ($b2 + b1$):

$$\max_{\boldsymbol{p}} qp_2 + (1-q)p_1$$

$$s.t. \quad \begin{cases} p_1 - p_2 \geq 2(q_1 - 1)\delta \\ p_1 - p_2 \geq -\dfrac{c}{q_1} \\ q_1 p_1 - p_2 \geq -c - (1-q_1)(v+\delta) \\ p_2 \leq v + \delta \end{cases}$$

$$\begin{cases} p_1 - p_2 \leq 2q_2\delta \\ p_1 - p_2 \leq \dfrac{c}{1-q_2} \\ p_1 \leq v - \delta + \dfrac{c}{1-q_2} \\ p_1 \leq v - \delta + 2q_2\delta \end{cases}$$

$\Rightarrow$ s.t.

$$\begin{cases} \max\{-2(1-q_1)\delta, -\dfrac{c}{q_1}\} \leq p_1 - p_2 \leq \min\{2q_2\delta, \dfrac{c}{1-q_2}\} \\ p_1 \leq \min\{v - \delta + \dfrac{c}{1-q_2}, v - \delta + 2q_2\delta\} \\ p_2 \leq v + \delta \\ q_1 p_1 - p_2 \geq -c - (1-q_1)(v+\delta) \end{cases}$$

(i) Suppose positive correlation. That is, $\beta \geq q^2 \Rightarrow q_1 \geq q_2$. The optimal solution is as follows:

If $\delta \geq \frac{c}{2q_2(1-q_2)}$,

- If $\delta \geq \frac{c}{2q_1(1-q_1)}$, then $p_1^* = v - \delta + \frac{c}{1-q_2}$, $p_2^* = v - \delta + \frac{c}{1-q_2} + \frac{c}{q_1}$, $r_{1,2}^* = v - \delta + \frac{1-q}{1-2q+\beta}c + \frac{q^2}{\beta}c$.

- If $\delta < \frac{c}{2q_1(1-q_1)}$, then $p_1^* = v - \delta + \frac{c}{1-q_2}$, $p_2^* = v - \delta + \frac{c}{1-q_2} + 2(1-q_1)\delta$, $r_{1,2}^* = v - \delta + \frac{1-q}{1-2q+\beta}c + 2(q-\beta)\delta$.

If $\delta < \frac{c}{2q_2(1-q_2)}$,

- If $\delta \geq \frac{c}{2q_1(1-q_1)}$, then $p_1^* = v - \delta + 2q_2\delta$, $p_2^* = v - \delta + 2q_2\delta + \frac{c}{q_1}$, $r_{1,2}^* = v - \delta + 2\frac{q-\beta}{1-q}\delta + \frac{q^2}{\beta}c$.

- If $\delta < \frac{c}{2q_1(1-q_1)}$, then $p_1^* = v - \delta + 2q_2\delta$, $p_2^* = v - \delta + 2q_2\delta + 2(1-q_1)\delta$, $r_{1,2}^* = v - \delta + 2\frac{q-\beta}{1-q}\delta + 2(q-\beta)\delta$.

(ii) Suppose negative correlation. That is, $\beta < q^2 \Rightarrow q_1 < q_2$. The optimal solution is as follows:

If $\delta \geq \frac{c}{2q_2(1-q_2)}$,

- If $\delta \geq \frac{c}{2q_1(1-q_1)}$, then $p_1^* = v - \delta + \frac{c}{1-q_2}$,

$$p_2^* = \begin{cases} v + \delta & \text{if } \delta \leq \frac{c}{2}(\frac{1}{q_1} + \frac{1}{1-q_2}) \\ v - \delta + \frac{c}{1-q_2} + \frac{c}{q_1} & \text{if } \delta > \frac{c}{2}(\frac{1}{q_1} + \frac{1}{1-q_2}) \end{cases},$$

$$r_{1,2}^* = \begin{cases} v + (2q-1)\delta + \frac{(1-q)^2}{1-2q+\beta}c & \text{if } \delta \leq \frac{c}{2}(\frac{1}{q_1} + \frac{1}{1-q_2}) \\ v - \delta + \frac{1-q}{1-2q+\beta}c + \frac{q^2}{\beta}c & \text{if } \delta > \frac{c}{2}(\frac{1}{q_1} + \frac{1}{1-q_2}) \end{cases}.$$

- If $\delta < \frac{c}{2q_1(1-q_1)}$, then $p_1^* = v-\delta+\frac{c}{1-q_2}$, $p_2^* = v+\delta$, $r_{1,2}^* = v+(2q-1)\delta+\frac{(1-q)^2}{1-2q+\beta}c$.

If $\delta < \frac{c}{2q_2(1-q_2)}$, then $p_1^* = v - \delta + 2q_2\delta$, $p_2^* = v + \delta$, $r_{1,2}^* = v + (2q-1)\delta + 2(q-\beta)\delta$.

Case 1+3 ($b2 + sb$):

$$\max_{\boldsymbol{p}} qp_2 + (1-q)[q_2p_1 + (1-q_2)p_2] = (1-q+\beta)p_2 + (q-\beta)p_1$$

$$s.t.\quad \begin{cases} p_1 - p_2 \geq 2(q_1-1)\delta \\ p_1 - p_2 \geq -\dfrac{c}{q_1} \\ q_1p_1 - p_2 \geq -c-(1-q_1)(v+\delta) \\ p_2 \leq v+\delta \end{cases}$$

$$\begin{cases} p_1 - p_2 \leq 2\delta - \dfrac{c}{q_2} \\ p_1 - p_2 \geq \dfrac{c}{1-q_2} \\ p_2 \leq v-\delta \\ q_2p_1 + (1-q_2)p_2 \leq v+(2q_2-1)\delta - c \end{cases}$$

$\Rightarrow$

$$s.t. \begin{cases} \max\{-2(1-q_1)\delta, -\dfrac{c}{q_1}, \dfrac{c}{1-q_2}\} \leq p_1 - p_2 \leq 2\delta - \dfrac{c}{q_2} \\ p_2 \leq v-\delta \\ q_1p_1 - p_2 \geq -c-(1-q_1)(v+\delta) \\ q_2p_1 + (1-q_2)p_2 \leq v+(2q_2-1)\delta - c \end{cases}$$

The optimal solution is as follows,

If $\delta \geq \frac{c}{2q_2(1-q_2)}$, then $p_1^* = v+\delta-\frac{c}{q_2}$, $p_2^* = v-\delta$, $r_{1,3}^* = v+(2q-2\beta-1)\delta-(1-q)c$.

If $\delta < \frac{c}{2q_2(1-q_2)}$, infeasible.

Case 1+4 ($b2+s0$):

$$\max_{\boldsymbol{p}} qp_2 + (1-q)q_2p_1 = qp_2 + (q-\beta)p_1$$

$$
s.t.\quad \begin{cases} p_1 - p_2 \geq 2(q_1 - 1)\delta \\ p_1 - p_2 \geq -\dfrac{c}{q_1} \\ q_1 p_1 - p_2 \geq -c - (1 - q_1)(v + \delta) \\ p_2 \leq v + \delta \end{cases}
$$

$$
\begin{cases} q_2 p_1 - p_2 \leq (1 + q_2)\delta - (1 - q_2)v - c \\ p_1 \geq v - \delta + \dfrac{c}{1 - q_2} \\ p_2 \geq v - \delta \\ p_1 \leq v + \delta - \dfrac{c}{q_2} \end{cases}
$$

$\Rightarrow$

$$
s.t. \begin{cases} p_1 - p_2 \geq \max\{-2(1 - q_1)\delta, -\dfrac{c}{q_1}\} \\ v - \delta \leq p_2 \leq v + \delta \\ v - \delta + \dfrac{c}{1 - q_2} \leq p_1 \leq v + \delta - \dfrac{c}{q_2} \\ q_1 p_1 - p_2 \geq -c - (1 - q_1)(v + \delta) \\ q_2 p_1 - p_2 \leq (1 + q_2)\delta - (1 - q_2)v - c \end{cases}
$$

(i) Suppose positive correlation. That is, $\beta \geq q^2 \Rightarrow q_1 \geq q_2$. The optimal solution is as follows.

If $\delta \geq \frac{c}{2q_2(1-q_2)}$,

- If $\delta \geq \frac{c}{2q_1(1-q_1)}$, then $p_1^* = v + \delta - \frac{c}{q_2}$, $p_2^* = v + \delta - \frac{c}{q_2} + \frac{c}{q_1}$, $r_{1,4}^* = (2q - \beta)(v + \delta) - \frac{q(1-q)}{q-\beta}c - (1 - q)c + \frac{q^2}{\beta}c$.

- If $\delta < \frac{c}{2q_1(1-q_1)}$, then $p_1^* = v + \delta - \frac{c}{q_2}$, $p_2^* = v + \delta - \frac{c}{q_2} + 2(1 - q_1)\delta$, $r_{1,4}^* = (2q - \beta)(v + \delta) - \frac{q(1-q)}{q-\beta}c - (1 - q)c + 2(q - \beta)\delta$.

If $\delta < \frac{c}{2q_2(1-q_2)}$, then infeasible.

(ii) Suppose negative correlation. That is, $\beta < q^2 \Rightarrow q_1 < q_2$. The optimal solution is as follows.

If $\delta \geq \frac{c}{2q_2(1-q_2)}$, then $p_1^* = v + \delta - \frac{c}{q_2}$, $p_2^* = v + \delta$, $r_{1,4}^* = (2q - \beta)(v + \delta) - (1 - q)c$.

If $\delta < \frac{c}{2q_2(1-q_2)}$, then infeasible.

<u>Case 1+5 ($b2 + 0$):</u>

$$\max_{\boldsymbol{p}} qp_2$$

$$s.t. \quad \begin{cases} p_1 - p_2 \geq 2(q_1 - 1)\delta \\ p_1 - p_2 \geq -\dfrac{c}{q_1} \\ q_1 p_1 - p_2 \geq -c - (1 - q_1)(v + \delta) \\ p_2 \leq v + \delta \end{cases}$$

$$\begin{cases} p_2 \geq v - \delta \\ p_1 \geq v - \delta + 2q_2\delta \\ q_2 p_1 + (1 - q_2)p_2 \geq v + (2q_2 - 1)\delta - c \\ p_1 \geq v + \delta - \dfrac{c}{q_2} \end{cases}$$

$\Rightarrow$

$$s.t. \begin{cases} p_1 - p_2 \geq \max\{-2(1 - q_1)\delta, -\dfrac{c}{q_1}\} \\ v - \delta \leq p_2 \leq v + \delta \\ p_1 \geq \max\{v + \delta - \dfrac{c}{q_2}, v - \delta + 2q_2\delta\} \\ q_1 p_1 - p_2 \geq -c - (1 - q_1)(v + \delta) \\ q_2 p_1 + (1 - q_2)p_2 \geq v + (2q_2 - 1)\delta - c \end{cases}$$

It is easy to see the optimal solution is

$$\begin{cases} p_2^* = v + \delta \\ r_{1,5}^* = q(v + \delta) \end{cases}$$

<u>Case 2+1 $(b1 + b2)$:</u>

$$\max_{\boldsymbol{p}} qp_1 + (1-q)p_2$$

$$s.t. \quad \begin{cases} p_1 - p_2 \leq 2(q_1 - 1)\delta ......(*) \\ p_1 - p_2 \leq -2\delta + \dfrac{c}{1-q_1} \\ p_1 \leq v - \delta + \dfrac{c}{1-q_1} \\ p_1 \leq v - \delta + 2q_1\delta \end{cases}$$

$$\begin{cases} p_1 - p_2 \geq 2q_2\delta ......(**) \\ p_1 - p_2 \geq 2\delta - \dfrac{c}{q_2} \\ q_2 p_1 - p_2 \geq (1+q_2)\delta - (1-q_2)v - c \\ p_2 \leq v - \delta \end{cases}$$

Infeasible because of $(*)$ and $(**)$.

<u>Case 2+2 $(b1 + b1)$:</u>

$$\max_{\boldsymbol{p}} p_1$$

$$s.t. \quad \begin{cases} p_1 - p_2 \leq 2(q_1 - 1)\delta \\ p_1 - p_2 \leq -2\delta + \dfrac{c}{1-q_1} \\ p_1 \leq v - \delta + \dfrac{c}{1-q_1} \\ p_1 \leq v - \delta + 2q_1\delta \end{cases}$$

$$\begin{cases} p_1 - p_2 \leq 2q_2\delta \\ p_1 - p_2 \leq \dfrac{c}{1-q_2} \\ \qquad p_1 \leq v - \delta + \dfrac{c}{1-q_2} \\ \qquad p_1 \leq v - \delta + 2q_2\delta \end{cases}$$

$\Rightarrow$

$$s.t. \begin{cases} p_1 - p_2 \leq \min\{-2(1-q_1)\delta, -2\delta + \dfrac{c}{1-q_1}, 2q_2\delta, \dfrac{c}{1-q_2}\} \\ \qquad p_1 \leq \min\{v - \delta + \dfrac{c}{1-q_2}, v - \delta + \dfrac{c}{1-q_1}, v - \delta + 2q_1\delta, v - \delta + 2q_2\delta\} \end{cases}$$

(i) Suppose positive correlation. $q_1 \geq q_2$.

If $\delta \geq \frac{c}{2q_2(1-q_2)}$, then $p_1^* = v - \delta + \frac{c}{1-q_2}$, $r_{2,2}^* = v - \delta + \frac{1-q}{1-2q+\beta}c$.

If $\delta < \frac{c}{2q_2(1-q_2)}$, then $p_1^* = v - \delta + 2q_2\delta$, $r_{2,2}^* = v - \delta + 2\frac{q-\beta}{1-q}\delta$.

(ii) Suppose negative correlation. $q_1 < q_2$.

If $\delta \geq \frac{c}{2q_1(1-q_1)}$, then $p_1^* = v - \delta + \frac{c}{1-q_1}$, $r_{2,2}^* = v - \delta + \frac{q}{q-\beta}c$.

If $\delta < \frac{c}{2q_1(1-q_1)}$, then $p_1^* = v - \delta + 2q_1\delta$, $r_{2,2}^* = v - \delta + 2\frac{\beta}{q}\delta$.

<u>Case 2+3 ($b1 + sb$):</u>

$$\max_{\boldsymbol{p}} (2q - \beta)p_1 + (1 - 2q + \beta)p_2$$

$$s.t. \quad \begin{cases} p_1 - p_2 \leq 2(q_1 - 1)\delta ......(*) \\ p_1 - p_2 \leq -2\delta + \dfrac{c}{1-q_1} \\ \qquad p_1 \leq v - \delta + \dfrac{c}{1-q_1} \\ \qquad p_1 \leq v - \delta + 2q_1\delta \end{cases}$$

$$\begin{cases} p_1 - p_2 \leq 2\delta - \dfrac{c}{q_2} \\ p_1 - p_2 \geq \dfrac{c}{1-q_2} ......(**) \\ p_2 \leq v - \delta \\ q_2 p_1 + (1-q_2) p_2 \leq v + (2q_2 - 1)\delta - c \end{cases}$$

Infeasible because of $(*)$ and $(**)$.

<u>Case 2+4 ($b1 + s0$):</u>

$$\max_{\boldsymbol{p}} qp_1 + (1-q)q_2 p_1 = (2q - \beta)p_1$$

$$s.t. \quad \begin{cases} p_1 - p_2 \leq 2(q_1 - 1)\delta \\ p_1 - p_2 \leq -2\delta + \dfrac{c}{1-q_1} \\ p_1 \leq v - \delta + \dfrac{c}{1-q_1} \\ p_1 \leq v - \delta + 2q_1\delta \end{cases}$$

$$\begin{cases} q_2 p_1 - p_2 \leq (1+q_2)\delta - (1-q_2)v - c \\ p_1 \geq v - \delta + \dfrac{c}{1-q_2} \\ p_2 \geq v - \delta \\ p_1 \leq v + \delta - \dfrac{c}{q_2} \end{cases}$$

$\Rightarrow$

$$s.t. \begin{cases} p_1 - p_2 \leq \min\{-2(1-q_1)\delta, -2\delta + \dfrac{c}{1-q_1}\} \\ v - \delta + \dfrac{c}{1-q_2} \leq p_1 \leq \min\{v - \delta + \dfrac{c}{1-q_1}, v - \delta + 2q_1\delta, v + \delta - \dfrac{c}{q_2}\} \\ ......(*) \\ p_2 \geq v - \delta \\ q_2 p_1 - p_2 \leq (1+q_2)\delta - (1-q_2)v - c \end{cases}$$

(i) Suppose positive correlation. $q_1 \geq q_2$.

If $\delta \geq \frac{c}{2q_2(1-q_2)}$,

- If $\delta \geq \frac{c}{2q_1(1-q_1)}$, then $p_1^* = \begin{cases} v - \delta + \dfrac{c}{1-q_1} & \text{if} \quad \delta \geq \dfrac{c}{2}(\dfrac{1}{1-q_1} + \dfrac{1}{q_2}) \\ v + \delta - \dfrac{c}{q_2} & \text{if} \quad \delta \leq \dfrac{c}{2}(\dfrac{1}{1-q_1} + \dfrac{1}{q_2}) \end{cases}$,

$$r_{2,4}^* = \begin{cases} (2q-\beta)(v-\delta) + \dfrac{q(2q-\beta)}{q-\beta}c & \text{if} \quad \delta \geq \dfrac{c}{2}(\dfrac{1}{1-q_1} + \dfrac{1}{q_2}) \\ (2q-\beta)(v+\delta) - \dfrac{(1-q)(2q-\beta)}{q-\beta}c & \text{if} \quad \delta \leq \dfrac{c}{2}(\dfrac{1}{1-q_1} + \dfrac{1}{q_2}) \end{cases}.$$

- If $\delta < \frac{c}{2q_1(1-q_1)}$, then $p_1^* = v + \delta - \frac{c}{q_2}$, $r_{2,4}^* = (2q-\beta)(v+\delta) - \frac{(1-q)(2q-\beta)}{q-\beta}c$.

If $\delta < \frac{c}{2q_2(1-q_2)}$, then infeasible.

(ii) Suppose negative correlation. $q_1 < q_2$. Infeasible because of $(*)$.

Case 2+5 $(b1+0)$:

$$\max_{\boldsymbol{p}} qp_1$$

$$s.t. \quad \begin{cases} p_1 - p_2 \leq 2(q_1 - 1)\delta \\ p_1 - p_2 \leq -2\delta + \dfrac{c}{1-q_1} \\ p_1 \leq v - \delta + \dfrac{c}{1-q_1} \\ p_1 \leq v - \delta + 2q_1\delta \end{cases}$$

$$\begin{cases} p_2 \geq v - \delta \\ p_1 \geq v - \delta + 2q_2\delta \\ q_2p_1 + (1-q_2)p_2 \geq v + (2q_2 - 1)\delta - c \\ p_1 \geq v + \delta - \dfrac{c}{q_2} \end{cases}$$

$\Rightarrow$

$$s.t. \begin{cases} p_1 - p_2 \le \min\{2(q_1-1)\delta, -2\delta + \dfrac{c}{1-q_1}\} \\ p_2 \ge v - \delta \\ \max\{v - \delta + 2q_2\delta, v + \delta - \dfrac{c}{q_2}\} \le p_1 \le \min\{v - \delta + \dfrac{c}{1-q_1}, v - \delta + 2q_1\delta\} \\ ......(*) \\ q_2 p_1 + (1-q_2)p_2 \ge v + (2q_2 - 1)\delta - c \end{cases}$$

(i) Suppose positive correlation. $q_1 \ge q_2$.

If $\frac{c}{2q_1(1-q_1)} \le \delta \le \frac{c}{2}[\frac{1}{q_2} + \frac{1}{1-q_1}]$, then $p_1^* = v - \delta + \frac{c}{1-q_1}$, $r_{2,5}^* = q(v-\delta) + \frac{q^2}{q-\beta}c$.

If $\delta < \frac{c}{2q_1(1-q_1)}$, then $p_1^* = v - \delta + 2q_1\delta$, $r_{2,5}^* = q(v-\delta) + 2\beta\delta$.

Otherwise, infeasible.

(ii) Suppose negative correlation. $q_1 < q_2$. Infeasible because of $(*)$.

Case 3+1 ($sb + b2$):

$$\max_{\boldsymbol{p}} \beta p_1 + (1-\beta) p_2$$

$$s.t. \quad \begin{cases} p_1 - p_2 \le -\dfrac{c}{q_1}......(*) \\ p_1 - p_2 \ge -2\delta + \dfrac{c}{1-q_1} \\ p_2 \le v + \delta \\ q_1 p_1 + (1-q_1)p_2 \le v + \delta - c \end{cases}$$

$$\begin{cases} p_1 - p_2 \ge 2q_2\delta......(**) \\ p_1 - p_2 \ge 2\delta - \dfrac{c}{q_2} \\ q_2 p_1 - p_2 \ge (1+q_2)\delta - (1-q_2)v - c \\ p_2 \le v - \delta \end{cases}$$

Infeasible because of (∗) and (∗∗).

<u>Case 3+2 ($sb + b1$):</u>

$$\max_{\boldsymbol{p}} q[q_1p_1 + (1-q_1)p_2] + (1-q)p_1 = (1+\beta-q)p_1 + (q-\beta)p_2$$

$$s.t. \begin{cases} p_1 - p_2 \leq -\dfrac{c}{q_1}......(*) \\ p_1 - p_2 \geq -2\delta + \dfrac{c}{1-q_1} \\ p_2 \leq v + \delta \\ q_1p_1 + (1-q_1)p_2 \leq v + \delta - c \end{cases}$$

$$\begin{cases} p_1 - p_2 \leq 2q_2\delta \\ p_1 - p_2 \leq \dfrac{c}{1-q_2} \\ p_1 \leq v - \delta + \dfrac{c}{1-q_2} \\ p_1 \leq v - \delta + 2q_2\delta \end{cases}$$

$\Rightarrow$

$$s.t. \begin{cases} -2\delta + \dfrac{c}{1-q_1} \leq p_1 - p_2 \leq -\dfrac{c}{q_1}........(*) \\ p_1 \leq \min\{v - \delta + \dfrac{c}{1-q_2}, v - \delta + 2q_2\delta\} \\ p_2 \leq v + \delta \\ q_1p_1 + (1-q_1)p_2 \leq v + \delta - c \end{cases}$$

(i) Suppose positive correlation. $q_1 \geq q_2$.

If $\delta \geq \frac{c}{2q_2(1-q_2)}$,

- If $\delta \geq \frac{c}{2q_1(1-q_1)}$, then $p_1^* = v - \delta + \frac{c}{1-q_2}$, $p_2^* = v + \delta + \frac{c}{1-q_2} - \frac{c}{1-q_1}$, $r_{3,2}^* = v - \delta + \frac{1-q}{1-2q+\beta}c + 2\delta(q-\beta) - qc$.

- If $\delta < \frac{c}{2q_1(1-q_1)}$, then infeasible by (∗).

If $\delta < \frac{c}{2q_2(1-q_2)}$,

- If $\delta \geq \frac{c}{2q_1(1-q_1)}$, then $p_1^* = v - \delta + 2q_2\delta$, $p_2^* = v + \delta + 2q_2\delta - \frac{c}{1-q_1}$, $r_{3,2}^* = v - \delta + 2\frac{q-\beta}{1-q}\delta + 2\delta(q-\beta) - qc$.

- If $\delta < \frac{c}{2q_1(1-q_1)}$, then infeasible by $(*)$.

(ii) Suppose negative correlation. $q_1 < q_2$.

If $\delta \geq \frac{c}{2q_2(1-q_2)}$,

- If $\delta \geq \frac{c}{2q_1(1-q_1)}$,
  - If $\delta \geq \frac{c}{2}(\frac{1}{q_1} + \frac{1}{1-q_2})$, then $p_1^* = v - \delta + \frac{c}{1-q_2}$, $p_2^* = v + \delta$, $r_{3,2}^* = v + \delta + \frac{(1-q)(1+\beta-q)}{1-2q+\beta}c - 2\delta(1+\beta-q)$.
  - If $\delta \geq \frac{c}{2}(\frac{1}{q_1} + \frac{1}{1-q_2})$, then $p_1 = v - \delta + \frac{c}{1-q_2}$, $p_2 = v + \frac{1+q_1}{1-q_1}\delta - \frac{c}{1-q_1} - \frac{q_1}{(1-q_1)(1-q_2)}c$, $r_{3,2}^* \leq v - (1-2q)\delta + \frac{1-3q+3q^2-q\beta}{1-2q+\beta}c$ is an upperbound.
  - If $\delta < \frac{c}{2q_1(1-q_1)}$, then infeasible by $(*)$.

If $\delta < \frac{c}{2q_2(1-q_2)}$,

- If $\delta \geq \frac{c}{2q_1(1-q_1)}$, then $p_1^* = v + \delta - \frac{c}{q_1}$, $p_2^* = v + \delta$, $r_{3,2}^* = v + \delta - \frac{q(1+\beta-q)}{\beta}c$.

- If $\delta < \frac{c}{2q_1(1-q_1)}$, then infeasible by $(*)$.

<u>Case 3+3 ($sb + sb$):</u>

$$\max_{\boldsymbol{p}} q[q_1p_1 + (1-q_1)p_2] + (1-q)[q_2p_1 + (1-q_2)p_2] = qp_1 + (1-q)p_2$$

$$s.t. \quad \begin{cases} p_1 - p_2 \leq -\dfrac{c}{q_1} \text{......}(*) \\ p_1 - p_2 \geq -2\delta + \dfrac{c}{1-q_1} \\ p_2 \leq v + \delta \\ q_1p_1 + (1-q_1)p_2 \leq v + \delta - c \end{cases}$$

$$\begin{cases} p_1 - p_2 \leq 2\delta - \dfrac{c}{q_2} \\ p_1 - p_2 \geq \dfrac{c}{1-q_2} ......(**) \\ p_2 \leq v - \delta \\ q_2 p_1 + (1-q_2)p_2 \leq v + (2q_2 - 1)\delta - c \end{cases}$$

Infeasible by $(*)$ and $(**)$.

<u>Case 3+4 ($sb + s0$):</u>

$$\max_{\boldsymbol{p}} q[q_1 p_1 + (1-q_1)p_2] + (1-q)q_2 p_1 = qp_1 + (q-\beta)p_2$$

$$s.t. \quad \begin{cases} p_1 - p_2 \leq -\dfrac{c}{q_1} \\ p_1 - p_2 \geq -2\delta + \dfrac{c}{1-q_1} \\ p_2 \leq v + \delta \\ q_1 p_1 + (1-q_1)p_2 \leq v + \delta - c \end{cases}$$

$$\begin{cases} q_2 p_1 - p_2 \leq (1+q_2)\delta - (1-q_2)v - c \\ p_1 \geq v - \delta + \dfrac{c}{1-q_2} \\ p_2 \geq v - \delta \\ p_1 \leq v + \delta - \dfrac{c}{q_2} \end{cases}$$

$\Rightarrow$

$$s.t. \begin{cases} -2\delta + \dfrac{c}{1-q_1} \leq p_1 - p_2 \leq -\dfrac{c}{q_1} \\ v - \delta + \dfrac{c}{1-q_2} \leq p_1 \leq v + \delta - \dfrac{c}{q_2} \\ v - \delta \leq p_2 \leq v + \delta \\ q_1 p_1 + (1-q_1)p_2 \leq v + \delta - c \\ q_2 p_1 - p_2 \leq (1+q_2)\delta - (1-q_2)v - c \end{cases}$$

(i) Suppose positive correlation. $q_1 \geq q_2$.

If $\delta \geq \frac{c}{2q_2(1-q_2)}$ and $\delta \geq \frac{c}{2q_1(1-q_1)}$, then $p_1^* = v + \delta - \frac{c}{q_2}$, $p_2^* = \min\{v+\delta, v+\delta+2\delta - \frac{c}{q_2} - \frac{c}{1-q_1}\}$, $r_{3,4}^* = \min\{(2q-\beta)(v+\delta) - \frac{q(1-q)}{q-\beta}c, (2q-\beta)(v+\delta) - \frac{(2q-\beta)(1-q)}{q-\beta}c + 2(q-\beta)\delta - qc\}$.

Otherwise, infeasible.

(ii) Suppose negative correlation. $q_1 < q_2$. By the second and third constraint, we have $-2\delta + \frac{c}{1-q_2} \leq p_1 - p_2 \leq 2\delta - \frac{c}{q_2}$. Combining with the first constraint, we have $\max\{-2\delta + \frac{c}{1-q_2}, -2\delta + \frac{c}{1-q_1}\} \leq p_1 - p_2 \leq \min\{-\frac{c}{q_1}, 2\delta - \frac{c}{q_2}\}$. Since $q_1 < q_2$, then it becomes $-2\delta + \frac{c}{1-q_2} \leq p_1 - p_2 \leq -\frac{c}{q_1}$. Therefore, a necessary condition is

$$\delta \geq \frac{1}{2}[\frac{c}{1-q_2} + \frac{c}{q_1}]$$

It is not hard to see $p_2 = v + \delta$, $p_1 - p_2 = -\frac{c}{q_1}$ is an upperbound.

If $\delta \geq \frac{1}{2}[\frac{c}{1-q_2} + \frac{c}{q_1}]$, then $p_1 = v + \delta - \frac{c}{q_1}$, $p_2 = v + \delta$, $r_{3,4}^* \leq (2q-\beta)(v+\delta) - \frac{q^2}{\beta}c$.

Otherwise, infeasible.

Case 3+5 ($sb + 0$):

$$\max_{\boldsymbol{p}} q[q_1p_1 + (1-q_1)p_2] = \beta p_1 + (q-\beta)p_2$$

$$s.t. \begin{cases} p_1 - p_2 \leq -\dfrac{c}{q_1} ......(*) \\ p_1 - p_2 \geq -2\delta + \dfrac{c}{1-q_1} \\ p_2 \leq v + \delta ......(**) \\ q_1p_1 + (1-q_1)p_2 \leq v + \delta - c \end{cases}$$

$$\begin{cases} p_2 \geq v - \delta \\ p_1 \geq v - \delta + 2q_2\delta \\ q_2p_1 + (1-q_2)p_2 \geq v + (2q_2-1)\delta - c \\ p_1 \geq v + \delta - \dfrac{c}{q_2} ....(***) \end{cases}$$

$\Rightarrow$

$$s.t. \begin{cases} -2\delta + \dfrac{c}{1-q_1} \leq p_1 - p_2 \leq -\dfrac{c}{q_1} \\ p_1 \geq \max\{v + \delta - \dfrac{c}{q_2}, v - \delta + 2q_2\delta\} \\ v - \delta \leq p_2 \leq v + \delta \\ q_1p_1 + (1-q_1)p_2 \leq v + \delta - c \\ q_2p_1 + (1-q_2)p_2 \geq v + (2q_2-1)\delta - c \end{cases}$$

(i) Suppose positive correlation. $q_1 \geq q_2$.

If $\delta \geq \frac{c}{2q_1(1-q_1)}$, then $p_1^* = v + \delta - \frac{c}{q_1}$, $p_2^* = v + \delta$, $r_{3,5}^* = q(v+\delta) - qc$. Otherwise, infeasible.

(ii) Suppose negative correlation. $q_1 < q_2$.

By $(**)$ and $(***)$, we have $p_1 - p_2 \geq -\frac{c}{q_2}$, which contradicts $(*)$. Hence, infeasible.

Case 4+1 ($s0 + b2$):

$$\max_{\boldsymbol{p}} qq_1p_1 + (1-q)p_2 = \beta p_1 + (1-q)p_2$$

$$s.t. \quad \begin{cases} q_1p_1 - p_2 \leq -c - (1-q_1)(v+\delta) \\ p_2 \geq v + \delta ......(*) \\ p_1 \geq v - \delta + \dfrac{c}{1-q_1} \\ p_1 \leq v + \delta - \dfrac{c}{q_1} \end{cases}$$

$$\begin{cases} p_1 - p_2 \geq 2q_2\delta \\ p_1 - p_2 \geq 2\delta - \dfrac{c}{q_2} \\ q_2p_1 - p_2 \geq (1+q_2)\delta - (1-q_2)v - c \\ p_2 \leq v - \delta ......(**) \end{cases}$$

Infeasible because of $(*)$ and $(**)$.

<u>Case 4+2 ($s0 + b1$):</u>

$$\max_{\boldsymbol{p}} qq_1p_1 + (1-q)p_1 = (1+\beta-q)p_1$$

$$s.t. \quad \begin{cases} q_1p_1 - p_2 \leq -c - (1-q_1)(v+\delta) \\ p_2 \geq v + \delta \\ p_1 \geq v - \delta + \dfrac{c}{1-q_1} ......(*) \\ p_1 \leq v + \delta - \dfrac{c}{q_1} \end{cases}$$

$$\begin{cases} p_1 - p_2 \leq 2q_2\delta \\ p_1 - p_2 \leq \dfrac{c}{1-q_2} \\ p_1 \leq v - \delta + \dfrac{c}{1-q_2} ......(**) \\ p_1 \leq v - \delta + 2q_2\delta \end{cases}$$

$\Rightarrow$

$$s.t. \begin{cases} p_1 - p_2 \leq \min\{2q_2\delta, \dfrac{c}{1-q_2}\} \\ v - \delta + \dfrac{c}{1-q_1} \leq p_1 \leq \min\{v + \delta - \dfrac{c}{q_1}, v - \delta + \dfrac{c}{1-q_2}, v - \delta + 2q_2\delta\} \\ p_2 \geq v + \delta \\ q_1p_1 - p_2 \leq -c - (1-q_1)(v+\delta) \end{cases}$$

(i) Suppose positive correlation. $q_1 \geq q_2$.

Infeasible because of $(*)$ and $(**)$.

(ii) Suppose negative correlation. $q_1 < q_2$.

If $\delta \geq \frac{c}{2q_1(1-q_1)}$,

- If $\delta \geq \frac{c}{2q_2(1-q_2)}$, then $p_1^* = \begin{cases} v + \delta - \dfrac{c}{q_1} & \text{if } \delta \leq \dfrac{c}{2}(\dfrac{1}{q_1} + \dfrac{1}{1-q_2}) \\ v - \delta + \dfrac{c}{1-q_2} & \text{if } \delta \geq \dfrac{c}{2}(\dfrac{1}{q_1} + \dfrac{1}{1-q_2}) \end{cases}$,

$$r_{4,2}^* = \begin{cases} (1+\beta-q)(v+\delta-\dfrac{q}{\beta}c) & \text{if } \delta \leq \dfrac{c}{2}(\dfrac{1}{q_1} + \dfrac{1}{1-q_2}) \\ (1+\beta-q)(v-\delta+\dfrac{1-q}{1-2q+\beta}c) & \text{if } \delta \geq \dfrac{c}{2}(\dfrac{1}{q_1} + \dfrac{1}{1-q_2}) \end{cases}.$$

- If $\delta < \frac{c}{2q_2(1-q_2)}$, then $p_1^* = v + \delta - \frac{c}{q_1}$, $r_{4,2}^* = (1+\beta-q)(v+\delta-\frac{q}{\beta}c)$.

If $\delta < \frac{c}{2q_1(1-q_1)}$, then infeasible.

<u>Case 4+3 ($s0 + sb$):</u>

$$\max_{\boldsymbol{p}} qq_1p_1 + (1-q)[q_2p_1 + (1-q_2)p_2] = qp_1 + (1-2q+\beta)p_2$$

$$s.t. \quad \begin{cases} q_1p_1 - p_2 \leq -c - (1-q_1)(v+\delta) \\ p_2 \geq v + \delta ......(*) \\ p_1 \geq v - \delta + \dfrac{c}{1-q_1} \\ p_1 \leq v + \delta - \dfrac{c}{q_1} \end{cases}$$

$$\begin{cases} p_1 - p_2 \leq 2\delta - \dfrac{c}{q_2} \\ p_1 - p_2 \geq \dfrac{c}{1-q_2} \\ p_2 \leq v - \delta ....(**) \\ q_2p_1 + (1-q_2)p_2 \leq v + (2q_2-1)\delta - c \end{cases}$$

Infeasible because of $(*)$ and $(**)$.

Case 4+4 ($s0 + s0$):

$$\max_{\boldsymbol{p}} qq_1p_1 + (1-q)q_2p_1 = qp_1$$

$$s.t. \quad \begin{cases} q_1p_1 - p_2 \le -c - (1-q_1)(v+\delta) \\ p_2 \ge v + \delta ......(*) \\ p_1 \ge v - \delta + \dfrac{c}{1-q_1} \\ p_1 \le v + \delta - \dfrac{c}{q_1} \end{cases}$$

$$\begin{cases} q_2p_1 - p_2 \le (1+q_2)\delta - (1-q_2)v - c \\ p_1 \ge v - \delta + \dfrac{c}{1-q_2} \\ p_2 \ge v - \delta \\ p_1 \le v + \delta - \dfrac{c}{q_2} \end{cases}$$

(i) Suppose positive correlation. $q_1 \ge q_2$.

If $\delta \ge \frac{c}{2}[\frac{1}{q_2} + \frac{1}{1-q_1}]$, then $p_1^* = v + \delta - \frac{c}{q_2}$, $r_{4,4}^* = q(v+\delta) - \frac{q(1-q)}{q-\beta}c$.

Otherwise, infeasible.

(ii) Suppose negative correlation. $q_1 < q_2$.

If $\delta \ge \frac{c}{2}[\frac{1}{q_1} + \frac{1}{1-q_2}]$, then $p_1^* = v + \delta - \frac{c}{q_1}$, $r_{4,4}^* = q(v+\delta) - \frac{q^2}{\beta}c$.

Otherwise, infeasible.

Case 4+5 ($s0 + 0$):

$$\max_{\boldsymbol{p}} qq_1p_1 = \beta p_1$$

$$s.t.\quad \begin{cases} q_1p_1 - p_2 \leq -c - (1-q_1)(v+\delta) \\ p_2 \geq v+\delta \\ p_1 \geq v - \delta + \dfrac{c}{1-q_1} \\ p_1 \leq v+\delta - \dfrac{c}{q_1} ......(*) \end{cases}$$

$$\begin{cases} p_2 \geq v - \delta \\ p_1 \geq v - \delta + 2q_2\delta \\ q_2p_1 + (1-q_2)p_2 \geq v + (2q_2-1)\delta - c \\ p_1 \geq v + \delta - \dfrac{c}{q_2} ....(**) \end{cases}$$

(i) Suppose positive correlation. $q_1 \geq q_2$.

If $\delta \geq \frac{c}{2q_1(1-q_1)}$, then $p_1^* = v + \delta - \frac{c}{q_1}$, $r_{4,5}^* = \beta(v+\delta) - qc$.

Otherwise, infeasible.

(ii) Suppose negative correlation. $q_1 < q_2$.

Infeasible because of $(*)$ and $(**)$.

<u>Case 5+1 ($0 + b2$):</u>

$$\max_{\boldsymbol{p}} (1-q)p_2$$

$$s.t.\quad \begin{cases} p_2 \geq v+\delta ......(*) \\ p_1 \geq v + (2q_1-1)\delta \\ q_1p_1 + (1-q_1)p_2 \geq v + \delta - c \\ p_1 \geq v + \delta - \dfrac{c}{q_1} \end{cases}$$

$$\begin{cases} p_1 - p_2 \geq 2q_2\delta \\ p_1 - p_2 \geq 2\delta - \dfrac{c}{q_2} \\ q_2p_1 - p_2 \geq (1+q_2)\delta - (1-q_2)v - c \\ p_2 \leq v - \delta ......(**) \end{cases}$$

Infeasible because of $(*)$ and $(**)$.

Case 5+2 $(0 + b1)$:

$$\max_{\boldsymbol{p}} (1-q)p_1$$

$$s.t. \quad \begin{cases} p_2 \geq v + \delta \\ p_1 \geq v + (2q_1 - 1)\delta ......(*) \\ q_1p_1 + (1-q_1)p_2 \geq v + \delta - c \\ p_1 \geq v + \delta - \dfrac{c}{q_1} \end{cases}$$

$$\begin{cases} p_1 - p_2 \leq 2q_2\delta \\ p_1 - p_2 \leq \dfrac{c}{1-q_2} \\ p_1 \leq v - \delta + \dfrac{c}{1-q_2} \\ p_1 \leq v - \delta + 2q_2\delta ......(**) \end{cases}$$

(i) Suppose positive correlation. $q_1 \geq q_2$.

Infeasible because of $(*)$ and $(**)$.

(ii) Suppose negative correlation. $q_1 < q_2$.

If $\frac{c}{2q_2(1-q_2)} \leq \delta \leq \frac{c}{2}[\frac{1}{q_1} + \frac{1}{1-q_2}]$, then $p_1^* = v - \delta + \frac{c}{1-q_2}$, $r_{5,2}^* = (1-q)(v-\delta) + \frac{(1-q)^2}{1-2q+\beta}c$.

If $\delta \leq \frac{c}{2q_2(1-q_2)}$, then $p_1^* = v - \delta + 2q_2\delta$, $r_{5,2}^* = (1-q)(v-\delta) + 2(q-\beta)\delta$.

Otherwise, infeasible.

Case 5+3 $(0+sb)$:

$$\max_{\boldsymbol{p}}(1-q)[q_2p_1+(1-q_2)p_2]=(q-\beta)p_1+(1-2q+\beta)p_2$$

$$s.t.\quad \begin{cases} p_2 \geq v+\delta......(*) \\ p_1 \geq v+(2q_1-1)\delta \\ q_1p_1+(1-q_1)p_2 \geq v+\delta-c \\ p_1 \geq v+\delta-\dfrac{c}{q_1} \end{cases}$$

$$\begin{cases} p_1-p_2 \leq 2\delta-\dfrac{c}{q_2} \\ p_1-p_2 \geq \dfrac{c}{1-q_2} \\ p_2 \leq v-\delta......(**) \\ q_2p_1+(1-q_2)p_2 \leq v+(2q_2-1)\delta-c \end{cases}$$

Infeasible because of $(*)$ and $(**)$.

Case 5+4 $(0+s0)$:

$$\max_{\boldsymbol{p}}(1-q)q_2p_1=(q-\beta)p_1$$

$$s.t.\quad \begin{cases} p_2 \geq v+\delta \\ p_1 \geq v+(2q_1-1)\delta \\ q_1p_1+(1-q_1)p_2 \geq v+\delta-c \\ p_1 \geq v+\delta-\dfrac{c}{q_1}......(*) \end{cases}$$

$$\begin{cases} q_2p_1 - p_2 \leq (1+q_2)\delta - (1-q_2)v - c \\ p_1 \geq v - \delta + \dfrac{c}{1-q_2} \\ p_2 \geq v - \delta \\ p_1 \leq v + \delta - \dfrac{c}{q_2} ......(**) \end{cases}$$

(i) Suppose positive correlation. $q_1 \geq q_2$.

Infeasible because of $(*)$ and $(**)$.

(ii) Suppose negative correlation. $q_1 < q_2$.

If $\delta \geq \frac{c}{2q_2(1-q_2)}$, then $p_1^* = v + \delta - \frac{c}{q_2}$, $r_{5,4}^* = (q-\beta)(v+\delta) - (1-q)c$.

Otherwise, infeasible.

Case 5+5 $(0+0)$: $r_{5,5}^* = 0$.

Summary

Next, let's derive the optimal solution.

Part 1: Positive Correlation $(q_1 \geq q_2)$

(a) If $\delta \geq \max\{\frac{c}{2q_1(1-q_1)}, \frac{c}{2q_2(1-q_2)}\}$, then we have the Table 5.4. (Note that "IF" denotes infeasible, "<" denotes its revenue is smaller than some other cases, "*" denotes a possible optimal solution.)

Table 5.4: Part 1 (a)

		Group 2's action				
		$1(b2)$	$2(b1)$	$3(sb)$	$4(s0)$	$5(0)$
Group 1's action	$1(b2)$	<	*	<	*	*
	$2(b1)$	IF	<	IF	<	<
	$3(sb)$	IF	*	IF	*	<
	$4(s0)$	IF	IF	IF	<	<
	$5(0)$	IF	IF	IF	IF	0

We can verify that

$$r^*_{1,1} < r^*_{1,2}, r^*_{1,3} < r^*_{3,2}, r^*_{2,2} < r^*_{1,2}, r^*_{2,4} < r^*_{1,4}$$

$$r^*_{2,5} < r^*_{1,5}, r^*_{3,5} < r^*_{1,5}, r^*_{4,4} < r^*_{1,5}, r^*_{4,5} < r^*_{1,5}.$$

Hence, there are only 5 possible cases left which are identified by "$*$". Thus, $R^*_{UC} = \max\{v - \delta + \frac{1-q}{1-2q+\beta}c + \frac{q^2}{\beta}c, (2q-\beta)(v+\delta) - \frac{(1-q)(2q-\beta)}{q-\beta}c + \frac{q^2}{\beta}c, q(v+\delta), v - \delta + \frac{1-q}{1-2q+\beta}c + 2\delta(q-\beta) - qc, \min\{(2q-\beta)(v+\delta) - \frac{q(1-q)}{q-\beta}c, (2q-\beta)(v+\delta) - \frac{(2q-\beta)(1-q)}{q-\beta}c + 2(q-\beta)\delta - qc\}\}$.

(b) If $\frac{c}{2q_2(1-q_2)} \leq \delta < \frac{c}{2q_1(1-q_1)}$ (can hold only when $q \geq 1/2$), then we have the Table 5.5.

Table 5.5: Part 1 (b)

		Group 2's action				
		$1(b2)$	$2(b1)$	$3(sb)$	$4(s0)$	$5(0)$
Group 1's action	$1(b2)$	$<$	$*$	$<$	$*$	$*$
	$2(b1)$	IF	$<$	IF	$<$	$<$
	$3(sb)$	IF	IF	IF	IF	IF
	$4(s0)$	IF	IF	IF	IF	IF
	$5(0)$	IF	IF	IF	IF	0

We can verify that

$$r^*_{1,1} < r^*_{1,2}, r^*_{1,3} < r^*_{1,2}, r^*_{2,2} < r^*_{1,2}$$

$$r^*_{2,4} < r^*_{1,4}, r^*_{2,5} < r^*_{1,5}.$$

Hence, there are only 3 possible cases left which are identified by "$*$". Thus, $R^*_{UC} = \max\{v - \delta + \frac{1-q}{1-2q+\beta}c + 2(q-\beta)\delta, (2q-\beta)(v+\delta) - \frac{(1-q)(2q-\beta)}{q-\beta}c + 2(q-\beta)\delta, q(v+\delta)\}$.

(c) If $\frac{c}{2q_1(1-q_1)} \leq \delta < \frac{c}{2q_2(1-q_2)}$ (can hold only when $q < 1/2$), then we have the Table 5.6.

Table 5.6: Part 1 (c)

		Group 2's action				
		1(b2)	2(b1)	3(sb)	4(s0)	5(0)
Group 1's action	1(b2)	<	*	IF	IF	*
	2(b1)	IF	<	IF	IF	<
	3(sb)	IF	*	IF	IF	<
	4(s0)	IF	IF	IF	IF	<
	5(0)	IF	IF	IF	IF	0

We can verify that

$$r^*_{1,1} < r^*_{1,2}, r^*_{2,2} < r^*_{1,2}, r^*_{2,5} < r^*_{1,5}$$

$$r^*_{3,5} < r^*_{1,5}, r^*_{4,5} < r^*_{1,5}.$$

Hence, there are only 3 possible cases left which are identified by "$*$". Thus, $R^*_{UC} = \max\{v - \delta + \frac{2(q-\beta)}{1-q}\delta + \frac{q^2}{\beta}c, q(v+\delta), v - \delta + 2\frac{q-\beta}{1-q}\delta + 2\delta(q-\beta) - qc\}$.

(d) If $\delta < \min\{\frac{c}{2q_2(1-q_2)}, \frac{c}{2q_1(1-q_1)}\}$, then we have the Table 5.7.

Table 5.7: Part 1 (d)

		Group 2's action				
		1(b2)	2(b1)	3(sb)	4(s0)	5(0)
Group 1's action	1(b2)	<	*	IF	IF	*
	2(b1)	IF	<	IF	IF	<
	3(sb)	IF	IF	IF	IF	IF
	4(s0)	IF	IF	IF	IF	IF
	5(0)	IF	IF	IF	IF	0

We can verify that

$$r^*_{1,1} < r^*_{1,2}, r^*_{2,2} < r^*_{1,2}, r^*_{2,5} < r^*_{1,5}.$$

Hence, there are only 2 possible cases left which are identified by "$*$". Thus, $R^*_{UC} = \max\{v - \delta + \frac{2(q-\beta)}{1-q}\delta + 2(q-\beta)\delta, q(v+\delta)\}$.

Part 2: Negative Correlation ($q_1 < q_2$)

(a) If $\delta \geq \frac{c}{2}[\frac{1}{q_1} + \frac{1}{1-q_2}] = \frac{q-2q^2+\beta}{2\beta(1-2q+\beta)}c$. We have the Table 5.8.

Table 5.8: Part 2 (a)

		Group 2's action				
		1($b2$)	2($b1$)	3(sb)	4($s0$)	5(0)
Group 1's action	1($b2$)	<	*	<	*	<
	2($b1$)	IF	<	IF	IF	IF
	3(sb)	IF	*	IF	<	IF
	4($s0$)	IF	<	IF	<	IF
	5(0)	IF	IF	IF	<	0

We can verify that

$$r^*_{1,1} < r^*_{1,2}, r^*_{1,3} < r^*_{3,2}, r^*_{1,5} < r^*_{1,4}, r^*_{2,2} < r^*_{1,2},$$

$$r^*_{3,4} < r^*_{1,4}, r^*_{4,2} < r^*_{1,2}, r^*_{4,4} < r^*_{1,5}, r^*_{5,4} < r^*_{1,4}.$$

Hence, there are only 3 possible cases left which are identified by "$*$". Thus, $R^*_{UC} = \max\{v - \delta + \frac{1-q}{1-2q+\beta}c + \frac{q^2}{\beta}c, (2q-\beta)(v+\delta) - (1-q)c, v + \delta + \frac{(1-q)(1+\beta-q)}{1-2q+\beta}c - 2\delta(1+\beta-q)\}$.

(b) If $\max\{\frac{c}{2q_1(1-q_1)}, \frac{c}{2q_2(1-q_2)}\} \leq \delta < \frac{c}{2}[\frac{1}{q_1} + \frac{1}{1-q_2}]$, that is, $\max\{\frac{c}{2(q-\beta)}\frac{q^2}{\beta}, \frac{c}{2(q-\beta)}\frac{(1-q)^2}{1-2q+\beta}\} \leq \delta \leq \frac{q-2q^2+\beta}{2\beta(1-2q+\beta)}c$. We have the Table 5.9.

We can verify that

$$r^*_{1,1} < r^*_{1,2}, r^*_{1,3} < r^*_{1,2}, r^*_{1,5} < r^*_{1,4}, r^*_{2,2} < r^*_{1,2},$$

Table 5.9: Part 2 (b)

		Group 2's action				
		1($b2$)	2($b1$)	3(sb)	4($s0$)	5(0)
Group 1's action	1($b2$)	$<$	*	$<$	*	$<$
	2($b1$)	IF	$<$	IF	IF	IF
	3(sb)	IF	$<$	IF	IF	IF
	4($s0$)	IF	$<$	IF	IF	IF
	5(0)	IF	$<$	IF	$<$	0

$$r^*_{3,2} < r^*_{1,2}, r^*_{4,2} < r^*_{1,2}, r^*_{5,2} < r^*_{1,2}, r^*_{5,4} < r^*_{1,4}.$$

Hence, there are only 2 possible cases left which are identified by "$*$". Thus, $R^*_{UC} = \max\{v + (2q-1)\delta + \frac{(1-q)^2}{1-2q+\beta}c, (2q-\beta)(v+\delta) - (1-q)c\}$.

(c) If $\frac{c}{2q_2(1-q_2)} \leq \delta < \frac{c}{2q_1(1-q_1)}$, that is, $\frac{c}{2(q-\beta)}\frac{(1-q)^2}{1-2q+\beta} \leq \delta < \frac{c}{2(q-\beta)}\frac{q^2}{\beta}$. We have the Table 5.10.

Table 5.10: Part 2 (c)

		Group 2's action				
		1($b2$)	2($b1$)	3(sb)	4($s0$)	5(0)
Group 1's action	1($b2$)	$<$	*	$<$	*	$<$
	2($b1$)	IF	$<$	IF	IF	IF
	3(sb)	IF	IF	IF	IF	IF
	4($s0$)	IF	IF	IF	IF	IF
	5(0)	IF	$<$	IF	$<$	0

We can verify that

$$r^*_{1,1} < r^*_{1,2}, r^*_{1,3} < r^*_{1,2}, r^*_{1,5} < r^*_{1,4},$$

$$r^*_{2,2} < r^*_{1,2}, r^*_{5,2} < r^*_{1,2}, r^*_{5,4} < r^*_{1,4}.$$

Hence, there are only 2 possible cases left which are identified by "$*$". Thus, $R^*_{UC} = \max\{v + (2q-1)\delta + \frac{(1-q)^2}{1-2q+\beta}c, (2q-\beta)(v+\delta) - (1-q)c\}$.

(d) If $\frac{c}{2q_1(1-q_1)} \leq \delta < \frac{c}{2q_2(1-q_2)}$, that is, $\frac{c}{2(q-\beta)}\frac{q^2}{\beta} \leq \delta < \frac{c}{2(q-\beta)}\frac{(1-q)^2}{1-2q+\beta}$. We have the Table 5.11.

Table 5.11: Part 2 (d)

		Group 2's action				
		1(b2)	2(b1)	3(sb)	4(s0)	5(0)
Group 1's action	1(b2)	<	*	IF	IF	<
	2(b1)	IF	<	IF	IF	IF
	3(sb)	IF	<	IF	IF	IF
	4(s0)	IF	<	IF	IF	IF
	5(0)	IF	<	IF	IF	0

We can verify that

$$r^*_{1,1} < r^*_{1,2}, r^*_{1,5} < r^*_{1,2}, r^*_{2,2} < r^*_{1,2},$$

$$r^*_{3,2} < r^*_{1,2}, r^*_{4,2} < r^*_{3,2}, r^*_{5,2} < r^*_{1,2}.$$

Hence, there is only 1 possible case left which are identified by " $*$ ". Thus, $R^*_{UC} = v + (2q-1)\delta + 2(q-\beta)\delta$.

(e) If $\delta \leq \min\{\frac{c}{2q_1(1-q_1)}, \frac{c}{2q_2(1-q_2)}\}$, that is, $\delta \leq \min\{\frac{c}{2(q-\beta)}\frac{(1-q)^2}{1-2q+\beta}, \frac{c}{2(q-\beta)}\frac{q^2}{\beta}\}$. We have the Table 5.12.

Table 5.12: Part 2 (e)

		Group 2's action				
		1(b2)	2(b1)	3(sb)	4(s0)	5(0)
Group 1's action	1(b2)	<	*	IF	IF	<
	2(b1)	IF	<	IF	IF	IF
	3(sb)	IF	IF	IF	IF	IF
	4(s0)	IF	IF	IF	IF	IF
	5(0)	IF	<	IF	IF	0

We can verify that

$$r^*_{1,1} < r^*_{1,2}, r^*_{1,5} < r^*_{1,2}, r^*_{2,2} < r^*_{1,2}, r^*_{5,2} < r^*_{1,2}.$$

Hence, there is only 1 possible case left which are identified by " $*$ ". Thus, $R^*_{UC} = v + (2q-1)\delta + 2(q-\beta)\delta$.

5.8.3 Optimal Solution of Case (C, C)

Solving CC

Case 1: If $p_1 - p_2 \geq 2\delta$, then we have $w^j_{b2}(\boldsymbol{p}) \geq w^j_{b1}(\boldsymbol{p})$ for all j. Therefore, $R^1(\boldsymbol{p}) = R^3(\boldsymbol{p}) = p_2 1_{\{p_2 \leq v+\delta\}}$, and $R^2(\boldsymbol{p}) = R^4(\boldsymbol{p}) = p_2 1_{\{p_2 \leq v-\delta\}}$. And

$$R_{CC}(\boldsymbol{p}) = \begin{cases} p_2 & \text{if } p_2 \leq v - \delta \\ qp_2 & \text{if } v - \delta < p_2 \leq v + \delta \\ 0 & \text{if } p_2 > v + \delta. \end{cases}$$

Therefore, the optimal solution in this case is

1. If $\delta \leq \frac{1-q}{1+q} v$, then $p^*_2 = v - \delta$, and $R^*_{CC} = v - \delta$.

2. Otherwise, then $p^*_2 = v + \delta$, and $R^*_{CC} = q(v + \delta)$.

Case 2: If $0 \leq p_1 - p_2 \leq 2\delta$, then we have $w^j_{b2}(\boldsymbol{p}) \geq w^j_{b1}(\boldsymbol{p})$ for $j \in \{1, 2, 4\}$, $w^3_{b2}(\boldsymbol{p}) \geq w^3_{b1}(\boldsymbol{p})$, $R^1(\boldsymbol{p}) = R^3(\boldsymbol{p}) = p_2 1_{\{p_2 \leq v+\delta\}}$, $R^2(\boldsymbol{p}) = p_1 1_{\{p_1 \leq v+\delta\}}$, and $R^4(\boldsymbol{p}) = p_2 1_{\{p_2 \leq v-\delta\}}$.

Moreover,

$$R_{CC}(\boldsymbol{p}) = \begin{cases} (1 - q + \beta)p_2 + (q - \beta)p_1 & \text{if } p_2 \leq v - \delta \\ qp_2 + (q - \beta)p_1 1_{\{p_1 \leq v+\delta\}} & \text{if } v - \delta < p_2 \leq v + \delta \\ 0 & \text{if } p_2 > v + \delta \end{cases}$$

Therefore, the optimal solution in this case is

1. If $\beta \leq \frac{1-2q+\beta}{1+\beta}v$, then $p_1^* = v + \delta$, $p_2^* = v - \delta$, and $R_{CC}^* = (q - \beta)(v + \delta) + (1 - q + \beta)(v - \delta)$.

2. Otherwise, $p_1^* = v + \delta$, $p_2^* = v + \delta$, and $R_{CC}^* = (2q - \beta)(v + \delta)$.

Case 3: If $-2\delta \leq p_1 - p_2 \leq 0$, by the symmetry of the two products, the optimal solution in this case is

1. If $\beta \leq \frac{1-2q+\beta}{1+\beta}v$, then $p_1^* = v - \delta$, $p_2^* = v + \delta$, and $R_{CC}^* = (q - \beta)(v + \delta) + (1 - q + \beta)(v - \delta)$.

2. Otherwise, $p_1^* = v + \delta$, $p_2^* = v + \delta$, and $R_{CC}^* = (2q - \beta)(v + \delta)$.

Case 4: If $p_1 - p_2 \leq -2\delta$, by the symmetry of the two products, the optimal solution in this case is

1. If $\delta \leq \frac{1-q}{1+q}v$, then $p_1^* = v - \delta$, and $R_{CC}^* = v - \delta$.

2. Otherwise, then $p_1^* = v + \delta$, and $R_{CC}^* = q(v + \delta)$.

Summary

Combining the above 4 cases, we find that there are only 4 possible optimal revenues, which is $\{v-\delta, q(v+\delta), (q-\beta)(v+\delta)+(1-q+\beta)(v-\delta), (2q-\beta)(v+\delta)\}$. It is easy to see that $v-\delta \leq (q-\beta)(v+\delta)+(1-q+\beta)(v-\delta)$ and $q(v+\delta) \leq (2q-\beta)(v+\delta)$. Therefore, $R_{CC}^* = \max\{(q-\beta)(v+\delta)+(1-q+\beta)(v-\delta), (2q-\beta)(v+\delta)\}$.

Chapter 6

Conclusion and Discussion

We consider several pricing problems with different consumer behavior which is more complex than typically assumed in traditional models. In Chapter 2, we consider dynamic pricing in the presence of patient customers. Our main result establishes that for arbitrary fixed patience levels and arbitrary valuation distributions, there is an optimal policy comprised of cycles of decreasing prices. We also provide bounds on the length of these cycles and present an algorithm for computing an optimal policy. Our work complements previous results that apply to problems with uniform valuation distributions in which patient customers are willing to wait one period to make a purchase. A direction for future research is to prove the conjecture that there are optimal policies with cycles of length $k + 1$ or $k + 2$, where k is the number of periods a patient customer is willing to wait to make a purchase. It would also be of interest to consider pricing problems with patient customers in which customer arrivals are stochastic. The models and policies discussed in this paper could potentially form the basis for heuristics in stochastic settings.

In Chapter 3, we study the same pricing problem as chapter 2 but with a continuous price set. Through a discretization approach, we proved that the decreasing cyclic policy is still optimal. Given the decreasing structure within each cycle, we proposed a new method to derive the optimal decreasing sequence for smaller value of k and uniform distribution. However, our approach to derive the optimal decreasing sequence works only for relatively small value of k. This is a limitation. The conjecture from the previous chapter that the length of optimal decreasing sequence is either $k+1$ or $k+2$ still remains. One future research direction is to find an approach that works for general k. Also our approach needs the uniform distribution assumption. A general valuation distribution problem is another direction.

We consider a learning and pricing problem in which the seller does not know the fraction of patient customers in Chapter 4. I propose some algorithms combining learning and pricing to minimize the revenue loss compared to the optimal revenue collected by a clairvoyant who knows the true fraction of patient customers in advance. By assuming a deterministic arrival stream of customers and homogeneous patience level, I derive a regret of $O(\sqrt{T})$, where T is the number of time periods. I will work to extend these results to the stochastic case or get an even better result of $O(log(T))$ for some special cases.

In Chapter 5, We study a problem in which a monopolist sells products for which customers have uncertain valuations. They can resolve this uncertainty by incurring research costs. In the single-product setting, the expected utility-maximizing customers need to decide whether or not to research the product. Based on customers' optimal action, we characterize the optimal pricing decisions for the revenue-maximizing seller and provide some comparative statics. It is perhaps surprising that the optimal revenue need not be decreasing in the valuation uncertainty and the research cost. This suggests that the seller should be careful when efforts are devoted to decrease the valuation uncertainty or alter the cost of research. In the two-product setting, the customers not only need to decide whether to research, but also need to decide in which order to research. We consider three scenarios: (1) a base case in which customers have uncertainty about both products and can resolve that uncertainty by incurring a cost or costs, and two other cases in which the seller makes information about (2) one or (3) both products freely available allowing customers to resolve their uncertainty about one or both products without cost. It is interesting to see that assigning research cost to only one of the two products can better differentiate the customers than assigning research cost to both products, thus giving a higher revenue. This result holds when the hidden features of the two products are negatively correlated, independent, and even weakly positively correlated. In addition, when the uncertainty is large, assigning no research cost to both products can generate the largest revenue and otherwise, assigning research cost to only one of the products is optimal. The managerial insight is as follows. When the monopolist sells two substitute products, it is optimal to disclose the information of both products to customers when there is a large valuation uncertainty. Otherwise, disclosing the information of only one product and hiding the other as secret can better differentiate the customers and thus improve the optimal revenue. Moreover, it is never optimal to hide the information of both products. In addition, consistent with intuition, the optimal revenue is decreasing in the correlation between the hidden features of the two products. In the future, we would like to see how sellers should manage the research cost in a competitive marketing environment.

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