Hypergeometric Functions and Arithmetic Properties of Algebraic Varieties

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Abstract

In this thesis, we investigate the relationship between special functions and arithmetic properties of algebraic varieties. More specifically, we use Greene's finite field hypergeometric functions to give point count formulas for families of algebraic varieties over finite fields. We demonstrate that this is possible for a family of genus 3 curves and for families of higher dimensional varieties called Dwork hypersurfaces. We work out the calculations in great detail for Dwork K3 surfaces over fields whose order is congruent to 1 modulo 4. Furthermore, for K3 surfaces, we also give point count formulas in terms of finite field hypergeometric functions defined by McCarthy. This allows us to give formulas that hold for all primes.

Inspired by a result of Manin for curves, we study the relationship between certain period integrals and the trace of Frobenius of these varieties. We show that these can be expressed in terms of "matching" classical and finite field hypergeometric functions. Through congruences between classical and finite field hypergeometric functions that we prove, we show that the fundamental period is congruent to the trace of Frobenius for Dwork K3 surfaces and conjecture that this is true for higher dimensional Dwork hypersurfaces.

Contents

Chapter 1

Introduction

The motivation for this work comes from a particular family of elliptic curves. For $\lambda \neq 0, 1$ we define an elliptic curve in the Legendre family by

$$E_\lambda : y^2 = x(x-1)(x-\lambda).$$

We compute a period integral associated to the Legendre elliptic curve given by integrating the nowhere vanishing holomorphic 1-form $\omega = \frac{dx}{y}$ over a 1-dimensional cycle containing λ. This period is a solution to a hypergeometric differential equation and can be expressed as the classical hypergeometric series

$$\pi = \int_0^\lambda \frac{dx}{y} = {}_2F_1\left(\begin{array}{cc|} \frac{1}{2} & \frac{1}{2} \\ & 1 \end{array}\ \lambda\right).$$

See the exposition in [6] for more details on this.

If we specialize to the case where $\lambda \in \mathbb{Q} \setminus \{0, 1\}$, Koike [30, Section 4] showed that, for all odd primes p, the trace of Frobenius for curves in this family can be expressed in terms of Greene's hypergeometric function

$$a_{E_\lambda}(p) = -\phi(-1)p \cdot {}_2F_1\left(\begin{array}{cc|} \phi & \phi \\ & \epsilon \end{array}\ \lambda\right)_p,$$

where ϵ is the trivial character and ϕ is a quadratic character modulo p.

Note the similarity between the period and trace of Frobenius expressions: the period is given by a classical hypergeometric series whose arguments are the fractions with denominator 2 and the trace of Frobenius is given by a finite field hypergeometric function whose arguments are characters of order 2. This similarity is to be expected for curves. Manin proved in [34] that the rows of the Hasse-Witt matrix of an algebraic curve are solutions to the differential equations of the periods. In the case where the genus is 1, the Hasse-Witt matrix has a single entry: the trace of Frobenius. Igusa showed in [24] that the trace of Frobenius is congruent modulo p to the classical hypergeometric expression

$$(-1)^{\frac{p-1}{2}}{}_2F_1\left(\begin{array}{cc|} \frac{1}{2} & \frac{1}{2} \\ & 1 \end{array}\ \lambda\right)$$

for odd primes p (see the exposition in Clemens' book [6]). Furthermore, in Corollary 3.1.2 we show that these classical and finite field ${}_2F_1$ hypergeometric expressions are congruent modulo p for odd primes. This result would imply merely a congruence between the finite field hypergeometric function expression and the point count over $\mathbb{F}_p$. The fact that Koike showed that we actually have an equality is very intriguing and leads us to wonder for what other varieties this type of equality holds.

Further examples of this correspondence have been observed for algebraic curves [8, 17, 33, 41] and for particular Calabi-Yau threefolds [1, 36]. For example, Fuselier [17] gave a finite field hypergeometric trace of Frobenius formula for elliptic curves with j-invariant $\frac{1728}{t}$, where $t \in \mathbb{F}_p \setminus \{0, 1\}$. Lennon [33] extended this by giving a hypergeometric trace of Frobenius formula that does not depend on the Weierstrass model chosen for the elliptic curve. In [1], Ahlgren and Ono gave a formula for the number of $\mathbb{F}_p$ points on a modular Calabi-Yau threefold. We extend these works to Dwork hypersurfaces, largely focusing on results that hold for Dwork K3 surfaces. Recall that the family of Dwork K3 surfaces is defined by

$$X_\lambda^4: \quad x_1^4 + x_2^4 + x_3^4 + x_4^4 = 4\lambda x_1 x_2 x_3 x_4.$$

We show that the number of points on the family of Dwork K3 surfaces over finite fields can be expressed in terms of Greene's finite field hypergeometric functions. The following is proved in Chapter 5.

Theorem 1.0.1. *Let $q = p^e$ be a prime power such that $q \equiv 1 \pmod 4$, $t = \frac{q-1}{4}$, and T be a generator for $\widehat{\mathbb{F}_q^\times}$. When $\lambda^4 = 1$ we have*

$$\#X_\lambda^4(\mathbb{F}_q) = \frac{q^3-1}{q-1} + 3qT^t(-1) + q^2 {}_3F_2\left(\begin{matrix} T^t & T^{2t} & T^{3t} \\ & \epsilon & \epsilon \end{matrix} \middle| 1\right)_q.$$

More generally, for $\lambda \neq 0$,

$$\begin{aligned}\#X_\lambda^4(\mathbb{F}_q) = {} & \frac{q^3-1}{q-1} + 12qT^t(-1)T^{2t}(1-\lambda^4) \\ & + q^2 {}_3F_2\left(\begin{matrix} T^t & T^{2t} & T^{3t} \\ & \epsilon & \epsilon \end{matrix} \middle| \frac{1}{\lambda^4}\right)_q + 3q^2\binom{T^{3t}}{T^t} {}_2F_1\left(\begin{matrix} T^{3t} & T^t \\ & T^{2t} \end{matrix} \middle| \frac{1}{\lambda^4}\right)_q,\end{aligned}$$

for all prime powers $q \equiv 1 \pmod 4$ away from the primes of λ.

Remark. Salerno [46] used a finite field hypergeometric function defined by Katz [26] to develop a point count formula for a larger class of diagonal hypersurfaces that could specialize to Dwork K3 surfaces. Theorem 5.5 of Salerno's paper gives a congruence between the number of points on diagonal surfaces and classical truncated hypergeometric series. This specializes to a single hypergeometric term in the Dwork K3 surface case (see Section 5.4 of Salerno's paper). Our result in Theorem 1.0.1 gives an exact formula for the point count, not just a congruence. Furthermore, in Chapter 3 of this thesis we give a congruence between the ${}_3F_2$ finite field hypergeometric function of Theorem 1.0.1 and classical truncated hypergeometric series that appears in Section 5.4 of Salerno's paper. We later use this congruence to prove a result that gives the relationship between the trace of Frobenius and certain periods associated to Dwork K3 surfaces, so for our purposes, Greene's finite field hypergeometric function is a natural choice of functions to work with.

Note that the character T^t is only defined over $\mathbb{F}_q$ when $q \equiv 1 \pmod 4$. We would like to develop a point count formula to use for fields $\mathbb{F}_p$, where $p \equiv 3 \pmod 4$ is prime. In this case, it seems unlikely that we will be able to develop a finite field hypergeometric formula. However, McCarthy [38] defined a p-adic version of these hypergeometric functions which we will use to write a concise formula to calculate the number of points on Dwork K3 surfaces over these fields.

Theorem 1.0.2. *When $p \equiv 3 \pmod 4$ and $\lambda \neq 0$, the point count is given by*

$$\#X_\lambda^4(\mathbb{F}_p) = \frac{p^3-1}{p-1} + {}_3G_3\left[\begin{array}{ccc|c} 1/4 & 2/4 & 3/4 & \\ 0 & 0 & 0 & \end{array} \lambda^4\right]_p - 3p\,{}_2G_2\left[\begin{array}{cc|c} 3/4 & 1/4 & \\ 0 & 1/2 & \end{array} \lambda^4\right]_p,$$

for all primes $p \equiv 3 \pmod 4$ away from the primes of λ.

Though we have already developed a hypergeometric point count formula that holds for primes $p \equiv 1 \pmod 4$ in Theorem 1.0.1, we show that we can also give a p-adic hypergeometric formula for these primes. Note the similarity between this formula and that of Theorem 1.0.2.

Theorem 1.0.3. *When $p \equiv 1 \pmod 4$ and $\lambda \neq 0$, the point count is given by*

$$\begin{aligned}\#X_\lambda^4(\mathbb{F}_p) = \frac{p^3-1}{p-1} &+ 12pT^t(-1)T^{2t}(1-\lambda^4) \\ &+ {}_3G_3\left[\begin{array}{ccc|c} 1/4 & 2/4 & 3/4 & \\ 0 & 0 & 0 & \end{array} \lambda^4\right]_p + 3p\,{}_2G_2\left[\begin{array}{cc|c} 3/4 & 1/4 & \\ 0 & 2/4 & \end{array} \lambda^4\right]_p,\end{aligned}$$

for all primes $p \equiv 1 \pmod 4$ away from the primes of λ.

We observe an interesting phenomenon with certain periods associated to the Dwork hypersurfaces we have studied. For an n-dimensional Dwork hypersurface, we calculate a period integral, obtained by choosing dual bases of the space of holomorphic $(n,0)$-differentials and the space of cycles $H^n(X_\lambda^{n+2}, \mathscr{O})$ and integrating the differentials over each cycle. The natural choice for a basis of differentials would be the nowhere vanishing holomorphic n-form. We are interested in periods, which we will call fundamental periods, that are solutions to the Picard-Fuchs equation

$$\left(\vartheta^{d-1} - z(\vartheta + \tfrac{1}{d}) \cdots (\vartheta + \tfrac{d-1}{d})\right) \pi = 0$$

where $\vartheta = z\frac{d}{dz}$ and $z = \lambda^{-d}$. These periods can be written in terms of classical hypergeometric series, a fact that was first noted by Dwork in [11]. Interestingly, the hypergeometric expressions for the periods and the point counts "match" in the sense that fractions with denominator a in the classical series coincide with characters of order a in the finite field hypergeometric functions. We use these matching expressions to prove the following result for Dwork K3 surfaces in Section 5.1.3.

Theorem 1.0.4. *For the Dwork K3 surface*

$$X_\lambda^4: \quad x_1^4 + x_2^4 + x_3^4 + x_4^4 = 4\lambda x_1 x_2 x_3 x_4,$$

the trace of Frobenius over $\mathbb{F}_p$ and the fundamental period of the surface are congruent modulo p when $p \equiv 1 \pmod 4$.

We conjecture an analogous result for higher dimensional Dwork hypersurfaces

$$X_\lambda^d: \quad x_1^d + x_2^d + \ldots + x_d^d = d\lambda x_1 x_2 \cdots x_d$$

in Section 5.3.

The remainder of this thesis is organized as follows. In Chapter 2 we give some necessary background information that will be used throughout the paper. Chapter 3 gives some congruences between classical hypergeometric series and finite field hypergeometric functions that we will use in our period and trace of Frobenius results. Chapter 4 gives results for curves, focusing on families of generalized Legendre curves. The final chapter of this thesis focuses on higher dimensional varieties – in particular, on families of Dwork hyeprsurfaces. We prove Theorem 1.0.1 in Section 5.1.1, using results that are proved in Section 5.1. In Section 5.1.2 we prove the p-adic point count formulas of Theorems 1.0.2 and 1.0.3. We prove Theorem 1.0.4 in Section 5.1.3. Finally, we extend these Dwork K3 surface results to higher dimensional hypersurfaces in Sections 5.2 and 5.3.

Chapter 2

Background Information

2.1 Calabi-Yau Manifolds

We start with some material that, unless stated otherwise, can be found in [7]. The main geometric objects that we are interested in are certain varieties called Calabi-Yau manifolds. These are Kähler manifolds that have a nowhere vanishing n-form, where n is the (complex) dimension of the Calabi-Yau manifold. A well-known and well-understood example of Calabi-Yau manifolds is the dimension 1 case of elliptic curves. Other examples are Calabi-Yau manifolds of dimension 2 (K3 surfaces) and of dimension 3 (threefolds). More generally, any n-dimensional algebraic variety with a nowhere vanishing n-form is a Calabi-Yau manifold. In Chapter 4 we will also discuss higher genus curves which are not Calabi-Yau manifolds.

The Modularity Theorem, formerly called the Shimura-Taniyama-Weil conjecture, for elliptic curves gives a relationship between modular forms and elliptic curves. For an elliptic curve E defined over $\mathbb{Q}$ with conductor N (given by Equation 2.1.1), let

$$a_E(p) = p + 1 - \#\{\text{the set of points on } E \text{ over } \mathbb{F}_p\}.$$

Then the Modularity Theorem tells us that there exists a weight 2, level N cusp form f, with Fourier coefficients $a_f(n)$ such that

$$a_f(p) = a_E(p)$$

for all primes p. This important theorem was proved recently by Breuil, Conrad, Diamand, and Taylor [4]. Modularity of higher dimensional Calabi-Yau manifolds has been conjectured for $d = 2, 3$, but not proven. However, there are some specific examples of K3 surfaces and threefolds that are known to be modular.

2.1.1 Elliptic Curves

We largely follow the exposition given by Silverman in [48, Chapter 3].

An elliptic curve is a genus one curve having a specified basepoint. Every elliptic curve defined over a field K can be written as the locus in $\mathbb{P}^2$ of a cubic equation with only one point (the basepoint) on the line at ∞:

$$Y^2Z + a_1XYZ + a_3YZ^2 = X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3,$$

where the coefficients $a_1, \ldots, a_6$ are in K. The basepoint is given by $O = [0, 1, 0]$. Every elliptic curve can be written as a plane cubic using the nonhomogenous coordinates $x = X/Z$ and $y = Y/Z$,

$$E : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6,$$

with the extra point $O = [0, 1, 0]$ out at infinity. This equation is referred to as the Weierstrass equation of the curve E. After a substitution and change of variables, we get the following equation for E (if $\mathrm{char}(\overline{K}) \neq 2$):

$$E : y^2 = 4x^3 + b_2x^2 + 2b_4x + b_6.$$

The discriminant of an elliptic curve is defined as a product of differences of the roots of the curve. We can obtain the value of the discriminant of E from the above equation by

$$\Delta(E) = -b_2^2 b_8 - 8b_4^3 - 27b_6^2 + 9b_2 b_4 b_6.$$

Curves with discriminant $\Delta = 0$ are called singular, as their complex points form a singular variety. For non-singular curves E we define the j-invariant to be

$$j(E) = \frac{(b_2^2 - 24b_4)^3}{\Delta}.$$

Two elliptic curves are isomorphic over $\overline{K}$ if and only if the both have the same j-invariant. Thus, this quantity is an invariant of the isomorphism class of the curve.

If a prime p divides Δ, then E is singular over the finite field $\mathbb{F}_p$. Thus we say that the primes dividing the discriminant Δ are the primes for which E has "bad" reduction; the two types of bad reduction are multiplicative, in which case the nonsingular part of the Weierstrass equation is isomorphic to the multiplicative group $\overline{K}^*$, and additive, in which case the nonsingular part is isomorphic to the additive group $\overline{K}^+$. The conductor of E is defined by

$$N_E := \prod_{p \text{ prime}} p^{f_p(E)}, \tag{2.1.1}$$

where, for $p \neq 2, 3$,

$$f_p(E) = \begin{cases} 0 & \text{if } E \text{ has good reduction at } p, \\ 1 & \text{if } E \text{ has multiplicative reduction at } p, \\ 2 & \text{if } E \text{ has additive reduction at } p. \end{cases}$$

See [48, Chapter 8] for more on when $p = 2$ or 3. An elliptic curve with conductor N will be, by the Modularity Theorem, associated to a level N modular form.

Let $L \subset \mathbb{P}^2$ be a line; any line must intersect an elliptic curve E at exactly three points (not necessarily distinct), say P, Q, and R. The composition law $\oplus$ on E is defined so that

$P \oplus Q \oplus R = O$. The composition law makes E into a group with identity element O. For a positive integer m, the m-torsion subgroup of E, denoted by $E[m]$, is the set of all points of order m:

$$E[m] = \{P \in E : mP = O\},$$

where mP is obtained by adding P to itself m times under the composition law.

2.1.2 Bounds and Distributions

For an elliptic curve E defined over a finite field $\mathbb{F}_q$, the Frobenius endomorphism $\pi : E \to E$ is given by $\pi(x, y) = (x^q, y^q)$. If we let $a_E(q)$ denote the trace of Frobenius, then we have

$$a_E(q) = q + 1 - \#\{\text{the set of points on } E \text{ over } \mathbb{F}_q\}.$$

Note that this is the same $a_E(q)$ that was mentioned when discussing modularity in Section 2.1. A theorem of Hasse gives the following bound for the trace of Frobenius

$$|a_E(q)| \leq 2\sqrt{q}.$$

When $q = p$, let $x_p = a_E(p)/\sqrt{p}$ denote the normalized trace. We can look at the distribution of the x_p as $p \to \infty$. Surprisingly, the values are not uniformly distributed over the interval $[-2, 2]$. In the 1960s, Mikio Sato and John Tate independently conjectured that, for curves defined over $\mathbb{Q}$ without complex multiplication, the normalized traces fall along the curve $y = \frac{1}{2\pi}\sqrt{4 - x^2}$. One can think of this as an equidistribution of the sequence of traces with respect to the Haar measure $\mu = \frac{1}{2\pi}\sqrt{4 - t^2}dt$. Barnet-Lamb, Geraghty, Harris, Shepard-Barron, and Taylor [3, 21] recently proved this conjecture for elliptic curves without complex multiplication. Proving the Sato-Tate Conjecture merged three massive mathematical theories: L-functions, automorphic forms, and Galois representations.

Traces of elliptic curves with complex multiplication and many higher genus curves also seem to have Sato-Tate-like distributions and determining these distributions is the source of ongoing work (see, for example, [14, 15]).

2.2 Hypergeometric Series and Functions

We start by recalling the definition of the classical hypergeometric series

$$ {}_{n+1}F_n\left(\begin{array}{cccc} a_0 & a_1 & \dots & a_n \\ & b_1 & \dots, & b_n \end{array}\middle|\, x\right) = \sum_{k=0}^{\infty} \frac{(a_0)_k \dots (a_n)_k}{(b_1)_k \dots (b_n)_k k!} x^k, \tag{2.2.1} $$

where $(a)_0 = 1$ and $(a)_k = a(a+1)(a+2)\dots(a+k-1)$. In Section 3 we will also be interested in truncated hypergeometric series. For a positive integer, m, define the hypergeometric series truncated at m to be

$$ {}_{n+1}F_n\left(\begin{array}{cccc} a_0 & a_1 & \dots & a_n \\ & b_1 & \dots, & b_n \end{array}\middle|\, x\right)_{\mathrm{tr}(m)} = \sum_{k=0}^{m-1} \frac{(a_0)_k \dots (a_n)_k}{k!(b_1)_k \dots (b_n)_k} x^k. \tag{2.2.2} $$

In his 1987 paper [19], Greene introduced a finite field, character sum analogue of classical hypergeometric series that satisfies similar summation and transformation properties. Let $\mathbb{F}_q$ be the finite field with q elements, where q is a power of an odd prime p. If χ is a multiplicative character of $\widehat{\mathbb{F}_q^\times}$, extend it to all of $\mathbb{F}_q$ by setting $\chi(0) = 0$. For any two characters A, B of $\widehat{\mathbb{F}_q^\times}$ we define the normalized Jacobi sum by

$$ \binom{A}{B} := \frac{B(-1)}{q} \sum_{x \in \mathbb{F}_q} A(x)\overline{B}(1-x) = \frac{B(-1)}{q} J(A, \bar{B}), \tag{2.2.3} $$

where $J(A, B) = \sum_{x \in \mathbb{F}_q} A(x)B(1-x)$ is the usual Jacobi sum.

For any positive integer n and characters $A_0, \dots, A_n, B_1, \dots, B_n$ in $\widehat{\mathbb{F}_q^\times}$, Greene defined the finite field hypergeometric function ${}_{n+1}F_n$ over $\mathbb{F}_q$ by

$$ {}_{n+1}F_n\left(\begin{matrix} A_0, & A_1, & \ldots, & A_n \\ & B_1, & \ldots, & B_n \end{matrix} \;\middle|\; x\right)_q = \frac{q}{q-1}\sum_{\chi}\binom{A_0\chi}{\chi}\binom{A_1\chi}{B_1\chi}\cdots\binom{A_n\chi}{B_n\chi}\chi(x). \quad (2.2.4) $$

In the case where $n = 1$, an alternate definition, which is in fact Greene's original definition, is given by

$$ {}_2F_1\left(\begin{matrix} A & B \\ & C \end{matrix} \;\middle|\; x\right)_q = \epsilon(x)\frac{BC(-1)}{q}\sum_{y} B(y)\overline{B}C(1-y)\overline{A}(1-xy). \quad (2.2.5) $$

2.2.1 Bounds and Distributions

Recall Koike's result given in Chapter 1 that, for all odd primes p, the trace of Frobenius for curves in the Legendre family can be expressed in terms of Greene's hypergeometric function

$$ a_{E_\lambda}(p) = -\phi(-1)p\cdot {}_2F_1\left(\begin{matrix} \phi & \phi \\ & \epsilon \end{matrix} \;\middle|\; \lambda\right)_p, $$

where ϵ is the trivial character and ϕ is a quadratic character modulo p. Combining this result with the work in Section 2.1.2 gives us information about the bounds and distributions of these ${}_2F_1$ hypergeometric functions.

Hasse's bound on the trace of Frobenius tells us that

$$ \left| p\cdot {}_2F_1\left(\begin{matrix} \phi & \phi \\ & \epsilon \end{matrix} \;\middle|\; \lambda\right)_p \right| \leq 2p^{1/2}. $$

Furthermore, when the associated Legendre elliptic curves does not have complex multiplication, the sequence of hypergeometric expressions

$$ -\phi(-1)p^{1/2}\cdot {}_2F_1\left(\begin{matrix} \phi & \phi \\ & \epsilon \end{matrix} \;\middle|\; \lambda\right)_p $$

as $p \to \infty$ is equidistributed with repsect to the Haar measure $\mu = \frac{1}{2\pi}\sqrt{4-t^2}dt$.

It would be interesting to investigate bounds and distributions for other hypergeometric functions. For example, one could use Serre's refinement of Hasse's bounds [47] to give bounds for the hypergeometric functions that appear in the point count expression of the curves in Chapter 4. Furthermore, determining bounds and distributions for hypergeometric functions more generally could lead to bound and Sato-Tate distribution results for new families of curves.

2.3 Gauss and Jacobi Sums

Unless otherwise stated, information in this section can be found in Ireland and Rosen's text [25, Chapter 8].

We define the standard trace map $\text{tr} : \mathbb{F}_q \to \mathbb{F}_p$ by

$$\text{tr}(x) = x + x^p + \ldots + x^{p^{e-1}}.$$

Let $\pi \in \mathbb{C}_p$ be a fixed root of $x^{p-1} + p = 0$ and let ζ_p be the unique p^{th} root of unity in $\mathbb{C}_p$ such that $\zeta_p \equiv 1 + \pi \pmod{\pi^2}$. Then for $\chi \in \widehat{\mathbb{F}_q^{\times}}$ we define the Gauss sum $g(\chi)$ to be

$$g(\chi) := \sum_{x \in \mathbb{F}_q} \chi(x)\theta(x), \tag{2.3.1}$$

where we define the additive character θ by $\theta(x) = \zeta_p^{\text{tr}(x)}$. Note that if χ is nontrivial then $g(\chi)g(\overline{\chi}) = \chi(-1)q$.

We have the following connection between Gauss sums and Jacobi sums. For non-trivial characters χ and ψ on $\mathbb{F}_q$ whose product is also non-trivial,

$$J(\chi, \psi) = \frac{g(\chi)g(\psi)}{g(\chi\psi)}.$$

More generally, for non-trivial characters $\chi_1, \ldots, \chi_n$ on $\mathbb{F}_q$ whose product is also non-trivial,

$$J(\chi_1, \ldots, \chi_n) = \frac{g(\chi_1)\cdots g(\chi_n)}{g(\chi_1 \cdots \chi_n)}.$$

Another important product formula is the Hasse-Davenport formula.

Theorem 2.3.1. *[31, Theorem 10.1] Let m be a positive integer and let q be a prime power such that $q \equiv 1 \pmod{m}$. For characters $\chi, \psi \in \widehat{\mathbb{F}_q^\times}$ we have*

$$\prod_{i=0}^{m-1} g(\chi^i \psi) = -g(\psi^m)\psi^{-m}(m) \prod_{i=0}^{m-1} g(\chi^i).$$

In Section 5.1 we will use the following specializations of this.

Corollary 2.3.2.

$$g(T^{4j}) = \frac{\prod_{i=0}^{3} g(T^{it+j})}{qT^{-4j}(4)T^t(-1)g(T^{2t})}.$$

Proof. This follows from Theorem 2.3.1 using $m = 4, \chi = T^t$, and $\psi = T^j$. □

Corollary 2.3.3. *More generally,*

$$g(T^{dj}) = \frac{\prod_{i=0}^{d-1} g(T^{it+j})}{T^{-dj}(d)\prod_{i=1}^{d-1} g(T^{it})},$$

where $q \equiv 1 \pmod{d}$, $t = \frac{q-1}{d}$, and T is a generator for $\widehat{\mathbb{F}_q^\times}$.

Though there are other relations for Gauss sum expressions (see, for example, [13, 49]), the Hasse-Davenport formula is our main tool for simplifying the expressions that appear in this paper's results. However, the following theorem of Helverson-Pasotto will be useful for rewriting an expression that appears in Section 5.1.

Theorem 2.3.4. *[22, Theorem 2]*

$$\frac{1}{q-1}\sum_{\chi} g(A\chi)g(B\overline{\chi})g(C\chi)g(D\overline{\chi}) = \frac{g(AB)g(AD)g(BC)g(CD)}{g(ABCD)} + q(q-1)AC(-1)\delta(ABCD),$$

where $\delta(ABCD) = 1$ if $ABCD$ is the trivial character and 0 otherwise.

The following is a generalization of this result that will also be helpful for simplifying formulas in Section 5.1.

Proposition 2.3.5. *Let q be a prime power such that $q \equiv 1 \pmod 4$, $t = \frac{q-1}{4}$, and T be a generator for $\widehat{\mathbb{F}_q^\times}$. Let a, b be multiples of t satisfying $a + b = 2t$. Then*

$$g(T^{2t}) \sum_{j=0}^{q-2} g(T^{j+a}) g(T^{-j+b}) T^j(-1) T^{4j}(\lambda) = q(q-1) T^b(-1) T^{2t}(1-\lambda^4).$$

Proof. We start by using Equation 2.3.1 to write

$$\begin{aligned}
\sum_{j=0}^{q-2} g(T^{j+a}) g(T^{-j+b}) T^j(-1) T^{4j}(\lambda) &= \sum_{j=0}^{q-2} T^j(-\lambda^4) \left(\sum_{x \in \mathbb{F}_q} T^{j+a}(x)\theta(x) \right) \left(\sum_{y \in \mathbb{F}_q} T^{-j+b}(y)\theta(y) \right) \\
&= \sum_{j=0}^{q-2} T^j(-\lambda^4) \sum_{x,y \in \mathbb{F}_q} T^{j+a}(x) T^{-j+b}(y) \theta(x+y) \\
&= \sum_{j=0}^{q-2} T^j(-\lambda^4) \sum_{x,y \in \mathbb{F}_q^\times} T^j(x/y) T^a(x) T^b(y) \theta(x+y) \\
&= \sum_{x,y \in \mathbb{F}_q^\times} T^a(x) T^b(y) \theta(x+y) \sum_{j=0}^{q-2} T^j \left(-\frac{\lambda^4 x}{y} \right).
\end{aligned}$$

Note that $\sum_{j=0}^{q-2} T^j \left(-\frac{\lambda^4 x}{y} \right) = 0$ unless $-\frac{\lambda^4 x}{y} = 1$, in which case the sum equals $q-1$. So, we let $x = -\frac{y}{\lambda^4}$ to get

$$\begin{aligned}
\sum_{j=0}^{q-2} g(T^{j+a}) g(T^{-j+b}) T^j(-1) T^{4j}(\lambda) &= (q-1) \sum_{y \in \mathbb{F}_q^\times} T^a \left(-\frac{y}{\lambda^4} \right) T^b(y) \theta \left(-\frac{y}{\lambda^4} + y \right) \\
&= (q-1) \sum_{y \in \mathbb{F}_q^\times} T^a \left(-\frac{y}{\lambda^4} \right) T^b(y) \theta \left(y(-\frac{1}{\lambda^4} + 1) \right).
\end{aligned}$$

We now consider two cases. First, suppose $\lambda^4 = 1$. Then we have

$$\sum_{y \in \mathbb{F}_q^\times} T^a(-y) T^b(y) \theta(0) = T^a(-1) \sum_{y \in \mathbb{F}_q^\times} T^{2t}(y) = 0$$

since T^{2t} is not a trivial character. Now suppose $\lambda^4 \neq 1$. Then we perform the change of variables $y \to y(-1/\lambda^4+1)^{-1}$ to get

$$\begin{aligned}\sum_{y\in\mathbb{F}_q^\times} T^a\left(\tfrac{-y}{\lambda^4-1}\right)T^b\left(\tfrac{y}{-1/\lambda^4+1}\right)\theta(y) &= T^{-a}(1-\lambda^4)T^{-b}\left(\tfrac{-1}{\lambda^4}+1\right)\sum_{y\in\mathbb{F}_q^\times} T^{a+b}(y)\theta(y)\\ &= T^{b-2t}(1-\lambda^4)T^{-b}\left(\tfrac{\lambda^4-1}{\lambda^4}\right)\sum_{y\in\mathbb{F}_q^\times} T^{2t}(y)\theta(y)\\ &= T^b(-\lambda^4)T^{-2t}(1-\lambda^4)g(T^{2t}).\end{aligned}$$

Note that if $\lambda^4 = 1$ then $T^b(-\lambda^4)T^{-2t}(1-\lambda^4)g(T^{2t}) = 0$, so we can use this expression for all λ. Hence,

$$\begin{aligned}g(T^{2t})\sum_{j=0}^{q-2} g(T^{j+a})g(T^{-j+b})T^j(-1)T^{4j}(\lambda) &= g(T^{2t})\cdot(q-1)T^b(-\lambda^4)T^{-2t}(1-\lambda^4)g(T^{2t})\\ &= qT^{2t}(-1)\cdot(q-1)T^b(-\lambda^4)T^{-2t}(1-\lambda^4)\\ &= q(q-1)T^b(-1)T^{2t}(1-\lambda^4),\end{aligned}$$

where the last equation holds because b is a multiple of t and $T^{4t}(\lambda) = 1$.

□

This proposition generalizes nicely for Gauss sum expressions of a particular form. We first note that by combining Theorem 3.13 and Definition 3.5 of [19], we can express finite field hypergeometric functions in the following way. For characters $A_0, \ldots, A_n, B_1, \ldots, B_n$ over $\mathbb{F}_q$ and $x \in \mathbb{F}_q^\times$,

$$\begin{aligned}{}_{n+1}F_n\left(\left.\begin{matrix} A_0 & A_1 & \ldots & A_n \\ & B_1 & \ldots & B_n\end{matrix}\,\right|\, x_0\right)_q &= \frac{\prod_{j=1}^n A_jB_j(-1)}{q^n}\\ &\cdot\sum_{x_i} A_1(x_1)\overline{A_1}B_1(1-x_1)\cdots A_n(x_n)\overline{A_n}B_n(1-x_n)\overline{A_0}(1-x_0x_1\cdots x_n) \qquad (2.3.2)\end{aligned}$$

Theorem 2.3.6. *Let q be a prime power such that $q \equiv 1 \pmod{k}$ and $t = \frac{q-1}{k}$. For a positive integer n and for $0 \leq i \leq n$, let a_i, b_i be integer multiples of t, not all 0, such that $\sum_i a_i + \sum_i b_i \equiv 0 \pmod{q-1}$. Then, for $\lambda \neq 0$,*

$$\frac{1}{q-1}\sum_{j=0}^{q-2}\left(\prod_{i=1}^{n} g(T^{a_i+j})\prod_{i=1}^{n} T^{-b_i+j}(-1)g(T^{b_i-j})\right)T^j(\lambda^k)$$
$$= T^m(-1)G\cdot q^{n-1}{}_nF_{n-1}\left(\begin{matrix} T^{b_n+a_1} & T^{b_1+a_1} & \dots & T^{b_{n-1}+a_1} \\ & T^{a_1-a_2} & \dots & T^{a_1-a_n}\end{matrix}\,\middle|\,\lambda^k\right)_q,$$

where $m = \sum_1^n a_i - \sum_1^{n-1}(b_i + a_1)$ and $G = g(T^{a_2+b_1})\cdots g(T^{a_n+b_{n-1}})g(T^{b_n+a_1})$.

Remark. The Gauss sum expression G will likely simplify as there will often be pairs of inverses $g(\chi)g(\overline{\chi}) = \chi(-1)q$. Additionally, in Proposition 5.1.6 we see that G can be expressed in terms of normalized Jacobi sums.

Remark. This result is likely relevant to point count formulas for Dwork hypersurfaces of higher dimensions. In using Koblitz's point count formula [28] and the Hasse-Davenport relation, we end up with Gauss sum expressions like those on the left side of the above equation. This theorem would allow us to give a point count formula that is almost entirely in terms of finite field hypergeometric functions.

Proof. We start by assuming $a_1 \leq \ldots \leq a_n$ and $b_1 \leq \ldots \leq b_n$ since the above Gauss sum expression is independent of this ordering. We also assume that $a_i + b_{i+1} \not\equiv 0 \pmod{q-1}$ for all i. Recalling that $g(\chi) = \sum_x \chi(x)\theta(x)$ we can write

$$\frac{1}{q-1}\sum_{j=0}^{q-2}\left(\prod_{i=1}^{n} g(T^{a_i+j})\prod_{i=1}^{n} T^{-b_i+j}(-1)g(T^{b_i-j})\right)T^j(\lambda^k)$$
$$= \frac{T^{m'}(-1)}{q-1}\sum_{x_i,y_i} T^{a_1}(x_1)\cdots T^{a_n}(x_n)T^{b_1}(y_1)\cdots T^{b_n}(y_n)$$
$$\cdot\,\theta\left(\sum x_i + \sum y_i\right)\sum_{j=0}^{q-2} T^j\left(\tfrac{(-1)^n x_1\cdots x_n\lambda^k}{y_1\cdots y_n}\right),$$

where $x_i, y_i \neq 0$ and $m' = -(b_1 + \ldots + b_n)$. Note that $\sum T^j\left(\frac{(-1)^n x_1 \cdots x_n \lambda^k}{y_1 \cdots y_n}\right) = q-1$ if $(-1)^n x_1 \cdots x_n \lambda^k / y_1 \cdots y_n = 1$ and equals 0 otherwise. Letting $x_1 = \frac{(-1)^n y_1 \cdots y_n}{x_2 \cdots x_n \lambda^k}$ and recalling that $T^{kt}(\lambda) = 1$ yields the following expression

$$T^{m'+a_1}(-1) \sum_{x_i, y_i} T^{a_2 - a_1}(x_2) \cdots T^{a_n - a_1}(x_n) T^{b_1 + a_1}(y_1) \cdots T^{b_n + a_1}(y_n)$$
$$\cdot\, \theta\left(\tfrac{(-1)^n y_1 \cdots y_n}{x_2 \cdots x_n \lambda^k} + x_2 + \ldots + x_n + y_1 + \ldots + y_n\right),$$

where we sum over all x_i, y_i except x_1.

Our goal now is to get the above expression in terms of Gauss sums and multiplicative characters. We perform the following changes of variables.

$$y_1 \to y_1 x_2, \quad y_2 \to y_2 x_3, \quad \ldots \quad , \quad y_{n-1} \to (-1)^n y_{n-1} x_n \lambda^k.$$

This yields the expression

$$T^{m'+a_1}(-1) \sum_{x_i, y_i} T^{a_2 + b_1}(x_2) \cdots T^{a_n + b_{n-1}}(x_n) T^{b_1 + a_1}(y_1) \cdots T^{b_n + a_1}(y_n)$$
$$\cdot\, \theta(y_1 \cdots y_n + x_2 + \ldots + x_n + y_1 x_2 + y_2 x_3 + \ldots + (-1)^n y_{n-1} x_n \lambda^k + y_n).$$

To further simplify this expression, we rewrite the argument of the additive character θ and perform another change of variables. Factoring yields

$$y_1 \cdots y_n + x_2 + \ldots + x_n + y_1 x_2 + y_2 x_3 + \ldots + (-1)^n y_{n-1} x_n \lambda^k + y_n$$
$$= y_n(1 + y_1 \cdots y_{n-1}) + x_2(1 + y_1) + x_3(1 + y_2) + \ldots + x_n(1 + (-1)^n y_{n-1} \lambda^k).$$

If any of $y_1, \ldots, y_{n-2} = -1, y_1 \cdots y_{n-1} = -1$, or $y_{n-1} = (-1)^{n+1}/\lambda^k$, then the entire sum is 0. To see this, note that if, for example, $y_1 = -1$, then the expression becomes

$$T^{m'+a_1}(-1) \sum_{x_i, y_i} T^{a_3 + b_1}(x_3) \cdots T^{a_n + b_{n-1}}(x_n) T^{b_1 + a_1}(y_1) \cdots T^{b_n + a_1}(y_n)$$
$$\cdot\, \theta(y_n(1 + y_1 \cdots y_{n-1}) + x_3(1 + y_2) + \ldots + x_n(1 + (-1)^n y_{n-1} \lambda^k)) \cdot \sum_{x_2} T^{a_2 + b_1}(x_2),$$

and $\sum_{x_2} T^{a_2+b_1}(x_2) = 0$ when $a_2 + b_1 \neq 0$.

For all other values, we perform the following changes of variables.

$$y_n \to y_n/(1 + y_1 \cdots y_{n-1}), \quad x_2 \to x_2/(1 + y_1), \quad \ldots, \quad x_n \to x_n/(1 + (-1)^n y_{n-1}\lambda^k).$$

This yields the following expression.

$$\begin{aligned} T^{m'+a_1}(-1) \sum_{x_i, y_i} & T^{a_2+b_1}(x_2) \cdots T^{a_n+b_{n-1}}(x_n) T^{b_n+a_1}(y_n) \cdot \theta(y_n + x_2 + \ldots + x_n) \\ & T^{b_1+a_1}(y_1) T^{-(a_2+b_1)}(1 + y_1) T^{b_2+a_1}(y_2) T^{-(a_3+b_2)}(1 + y_2) \\ & \cdots T^{b_{n-1}+a_1}(y_{n-1}) T^{-(a_n+b_{n-1})}(1 + (-1)^n y_{n-1}\lambda^k) T^{-(b_n+a_1)}(1 + y_1 \cdots y_{n-1}). \end{aligned}$$

Note that the summand equals 0 whenever $y_1, \ldots, y_{n-2} = -1$, $y_1 \cdots y_{n-1} = -1$, or $y_{n-1} = (-1)^{n+1}/\lambda^k$, so we can include those values back in the sum. The first part of this summand becomes a product of Gauss sums:

$$\begin{aligned} G &:= \sum_{x_i, y_n} T^{a_2+b_1}(x_2) \cdots T^{a_n+b_{n-1}}(x_n) T^{b_n+a_1}(y_n) \cdot \theta(y_n + x_2 + \ldots + x_n) \\ &= g(T^{a_2+b_1}) \cdots g(T^{a_n+b_{n-1}}) g(T^{b_n+a_1}). \end{aligned}$$

So, we write our expression as

$$\begin{aligned} T^{m'+a_1}(-1) G \sum_{y_i} & T^{b_1+a_1}(y_1) T^{-(a_2+b_1)}(1 + y_1) T^{b_2+a_1}(y_2) T^{-(a_3+b_2)}(1 + y_2) \\ & \cdots T^{b_{n-1}+a_1}(y_{n-1}) T^{-(a_n+b_{n-1})}(1 + (-1)^n y_{n-1}\lambda^k) T^{-(b_n+a_1)}(1 + y_1 \cdots y_{n-1}). \end{aligned}$$

In order to get the remaining expression to match Equation 2.3.2 We need to perform more changes of variables. First, let $y_{n-1} \to (-1)^n y_{n-1}/\lambda^k$ to get

$$\begin{aligned} T^{m'+a_1}(-1) G \sum_{y_i} & T^{b_1+a_1}(y_1) T^{-(a_2+b_1)}(1 + y_1) T^{b_2+a_1}(y_2) T^{-(a_3+b_2)}(1 + y_2) \\ & \cdots T^{b_{n-1}+a_1}(y_{n-1}) T^{-(a_n+b_{n-1})}(1 + y_{n-1}) T^{-(b_n+a_1)}(1 + (-1)^n y_1 \cdots y_{n-1}\lambda^{-k}). \end{aligned}$$

We now let $y_i \to -y_i$ for all i and, noting that $(-1)^{n-1}(-1)^n = -1$, get

$$T^{m''}(-1)G\sum_{y_i} T^{b_1+a_1}(y_1)T^{-(a_2+b_1)}(1-y_1)T^{b_2+a_1}(y_2)T^{-(a_3+b_2)}(1-y_2)$$
$$\cdots T^{b_{n-1}+a_1}(y_{n-1})T^{-(a_n+b_{n-1})}(1-y_{n-1})T^{-(b_n+a_1)}(1-y_1\cdots y_{n-1}\lambda^{-k}),$$

where

$$\begin{aligned} m'' &= m' + a_1 + (b_1+a_1) + (b_2+a_1) + \ldots + (b_{n-1}+a_1) \\ &= na_1. \end{aligned}$$

Finally, applying Equation 2.3.2 to this yields

$$T^m(-1)G\cdot q^{n-1}{}_nF_{n-1}\left(\begin{matrix} T^{b_n+a_1} & T^{b_1+a_1} & \ldots & T^{b_{n-1}+a_1} \\ & T^{a_1-a_2} & \ldots & T^{a_1-a_n} \end{matrix}\,\middle|\, \lambda^k\right)_q,$$

where

$$\begin{aligned} m &= na_1 - ((b_1+a_1)+\ldots+(b_{n-1}+a_1)+(a_1-a_2)+\ldots+(a_1-a_n)) \\ &= \sum_1^n a_i - \sum_1^{n-1}(b_i+a_1). \end{aligned}$$

□

2.4 p-adic Gamma Function

Throughout this section let $q = p^e$ be a power of an odd prime p and let $\mathbb{Z}_p$ denote the ring of p-adic integers. Also, let $\mathbb{Q}_p$ denote the field of p-adic numbers and $\mathbb{C}_p$ be its p-adic completion. The following facts can be found in, for example, [41], [1], and [20]. We define the p-adic Gamma function $\Gamma_p : \mathbb{Z}_p \to \mathbb{Z}_p^*$ by

$$\Gamma_p(n) := (-1)^n \prod_{j<n,\, p\nmid j} j \tag{2.4.1}$$

for numbers $n \in \mathbb{N}$. We extend this to all x in $\mathbb{Z}_p$ by

$$\Gamma_p(x) := \lim_{n \to x} \Gamma_p(n),$$

where in the limit we take any sequence of positive integers that approaches x p-adically.

Proposition 2.4.1. *[41, Proposition 4.2] If $p \geq 5$ is prime, $x, y \in \mathbb{Z}_p$, and $z \in p\mathbb{Z}_p$, then*

1. $\Gamma_p(x+1) = \begin{cases} -x\Gamma_p(x) & \text{if } x \in \mathbb{Z}_p^*, \\ -\Gamma_p(x) & \text{if } x \in p\mathbb{Z}_p. \end{cases}$
2. *If $n \geq 1$ and $x \equiv y \pmod{p^n}$ then $\Gamma_p(x) \equiv \Gamma_p(y) \pmod{p^n}$.*
3. $\Gamma_p'(x+z) \equiv \Gamma_p'(x) \pmod{p}$.
4. $|\Gamma_p(x)| = 1$.
5. *Let $x_0 \in \{1, \ldots, p\}$ be the constant term in the p-adic expansion of x. Then* $\Gamma_p(x)\Gamma_p(1-x) = (-1)^{x_0}$.

The following proposition relates the p-adic Gamma function to the Pochhammer symbol.

Proposition 2.4.2. *[41, Proposition 5.1] Let m and d be integers with $1 \leq m < d$. If $p \equiv 1 \pmod{d}$ is a prime, then define t such that $t = \frac{p-1}{d}$. If $0 \leq j \leq mt$, then*

$$\frac{\Gamma_p\left(\frac{m}{d}+j\right)\Gamma_p\left(1-\frac{m}{d}+j\right)}{\Gamma_p\left(1+j\right)^2} = (-1)^{mt+1}\frac{\left(\frac{m}{d}\right)_j\left(\frac{d-m}{d}\right)_j}{j!^2}.$$

We will use this proposition in the proof of Theorem 3.1.1 in Chapter 3.

We now state a relationship between Gauss sums and the p-adic Gamma function. Recall that $\pi \in \mathbb{C}_p$ is the fixed root of $x^{p-1} + p = 0$ given in Section 2.3. Define the Teichmüller character, ω, to be the primitive character on $\mathbb{F}_p$ that is uniquely defined by the property $\omega(x) \equiv x \pmod{p}$ for all x in $\{0, \ldots, p-1\}$. Then the Gross-Koblitz formula, specialized to the case where the field is $\mathbb{F}_p$, is the following.

Theorem 2.4.3. *[20, Theorem 1.7] For j in $\{0, \ldots, p-1\}$,*

$$g(\overline{\omega}^j) = -\pi^j \Gamma_p \left(\frac{j}{p-1} \right).$$

2.4.1 p-adic Hypergeometric Function

In Section 2.2 we defined finite field hypergeometric functions, which will be used in many of the results in this thesis. These hypergeometric functions, however, are limited to primes in which the characters are defined. For example, in Theorem 1.0.1 our point count result is limited to primes and prime powers congruent to 1 mod 4. In order to extend our results to primes in other congruence classes, we use a p-adic hypergeometric function developed by McCarthy in [38].

For $x \in \mathbb{Q}$ we let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x and let $\langle x \rangle$ denote the fractional part of x, i.e. $x - \lfloor x \rfloor$.

Definition 2.4.4. *[38, Definition 1.1] Let p be an odd prime and let $t \in \mathbb{F}_p$. For $n \in \mathbb{Z}^+$ and $1 \leq i \leq n$, let $a_i, b_i \in \mathbb{Q} \cap \mathbb{Z}_p$. Then we define*

$$\begin{aligned} {}_nG_n \left[\begin{matrix} a_1 & a_2 & \ldots & a_n \\ b_1 & b_2 & \ldots & b_n \end{matrix} \middle| t \right]_p &:= \frac{-1}{p-1} \sum_{j=0}^{p-2} (-1)^{jn} \overline{\omega}^j(t) \\ &\times \prod_{i=1}^{n} \frac{\Gamma_p\left(\left\langle a_i - \frac{j}{p-1} \right\rangle\right)}{\Gamma_p\left(\langle a_i \rangle\right)} \frac{\Gamma_p\left(\left\langle -b_i + \frac{j}{p-1} \right\rangle\right)}{\Gamma_p\left(\langle -b_i \rangle\right)} (-p)^{-\lfloor \langle a_i \rangle - \frac{j}{p-1} \rfloor - \lfloor \langle -b_i \rangle + \frac{j}{p-1} \rfloor}. \end{aligned}$$

Chapter 3

Hypergeometric Series Congruences

In this section we prove congruences between classical truncated hypergeometric series and finite field hypergeometric functions. This builds on results of Mortenson [41, 42] by considering hypergeometric functions evaluated away from 1, though our results hold mod p instead of p^2.

3.1 ${}_2F_1$ Congruence

The first result is for ${}_2F_1$ hypergeometric functions.

Theorem 3.1.1. *Let m and d be integers with $1 \leq m < d$. If $p \equiv 1 \pmod{d}$ and T is a generator for the character group $\widehat{\mathbb{F}_p^{\times}}$ then, for $x \neq 0$,*

$$ {}_2F_1\left(\begin{matrix} \frac{m}{d} & \frac{d-m}{d} \\ & 1 \end{matrix} \middle| \, x \right)_{tr(p)} \equiv -p \; {}_2F_1\left(\begin{matrix} T^{mt} & \overline{T}^{mt} \\ & \epsilon \end{matrix} \middle| \, x \right)_{p} \pmod{p}, $$

where $t = \frac{p-1}{d}$.

The following corollary applies to the hypergeometric functions that appear in Clemens' and Koike's trace of Frobenius expressions for Legendre elliptic curves.

Corollary 3.1.2. *If p is an odd prime and ϕ is a quadratic character in $\widehat{\mathbb{F}_p^\times}$, then, for* $\lambda \neq 0$,

$$(-1)^{\frac{p-1}{2}} {}_2F_1\left(\begin{array}{cc|} \frac{1}{2} & \frac{1}{2} \\ & 1 \end{array}\ \lambda\right)_{tr(p)} \equiv -\phi(-1)p\ {}_2F_1\left(\begin{array}{cc|} \phi & \phi \\ & \epsilon \end{array}\ \lambda\right)_p \pmod p.$$

Proof of Theorem 3.1.1. We use Equation 2.2.4 to write

$$-p\ {}_2F_1\left(\begin{array}{cc|} T^{mt} & \overline{T}^{mt} \\ & \epsilon \end{array}\ x\right)_p = \frac{p^2}{1-p}\sum_\chi \binom{T^{mt}\chi}{\chi}\binom{\overline{T}^{mt}\chi}{\epsilon\chi}\chi(x)$$

Using properties of Gauss and Jacobi sums from Sections 2.2 and 2.3 we break this down to

$$\begin{aligned}\frac{p^2}{1-p}\sum_\chi \frac{\chi(-1)}{p}J(T^{mt}\chi,\overline{\chi})\frac{\chi(-1)}{p}J(\overline{T}^{mt}\chi,\overline{\chi})\chi(x)& \\ &= \frac{1}{1-p}\sum_\chi \frac{g(T^{mt}\chi)g(\overline{\chi})}{g(T^{mt})}\frac{g(\overline{T}^{mt}\chi)g(\overline{\chi})}{g(\overline{T}^{mt})}\chi(x) \\ &= \frac{1}{1-p}\sum_\chi \frac{g(T^{mt}\chi)g(\overline{T}^{mt}\chi)g(\overline{\chi})^2}{T^{mt}(-1)p}\chi(x) \\ &= \frac{\overline{T}^{mt}(-1)}{p(1-p)}\sum_\chi g(T^{mt}\chi)g(\overline{T}^{mt}\chi)g(\overline{\chi})^2\chi(x).\end{aligned}$$

We rewrite this in terms of the Teichmüller character ω defined in Section 2.4 by letting $T = \overline{\omega}$ and $\chi = \overline{\omega}^{-j}$. Furthermore, we split up the sum as Mortenson does in [41]. This yields the expression

$$\frac{\omega^{mt}(-1)}{p(1-p)}\sum_{\chi} g(T^{mt}\chi)g(\overline{\chi})^2\chi(x) = \frac{\overline{T}^{mt}(-1)}{p(1-p)}\left[\sum_{j=0}^{mt} g(\overline{\omega}^{mt-j})g(\overline{\omega}^{p-1-mt-j})g(\overline{\omega}^{j})^2\omega^j(x)\right.$$
$$+\sum_{j=mt+1}^{p-1-mt} g(\overline{\omega}^{p-1+mt-j})g(\overline{\omega}^{p-1-mt-j})g(\overline{\omega}^{j})^2\omega^j(x)$$
$$\left.+\sum_{j=p-mt}^{p-2} g(\overline{\omega}^{p-1+mt-j})g(\overline{\omega}^{2(p-1)-mt-j})g(\overline{\omega}^{j})^2\omega^j(x)\right].$$

We use the Gross-Koblitz formula given in Theorem 2.4.3 to write this expression in terms of the p-adic Gamma function

$$\frac{\omega^{mt}(-1)}{p(1-p)}\left[\sum_{j=0}^{mt}\pi^{p-1}\Gamma_p\left(\tfrac{mt-j}{p-1}\right)\Gamma_p\left(\tfrac{p-1-mt-j}{p-1}\right)\Gamma_p\left(\tfrac{j}{p-1}\right)^2\omega^j(x)\right.$$
$$+\sum_{j=mt+1}^{p-1-mt}\pi^{2(p-1)}\Gamma_p\left(\tfrac{p-1+mt-j}{p-1}\right)\Gamma_p\left(\tfrac{p-1-mt-j}{p-1}\right)\Gamma_p\left(\tfrac{j}{p-1}\right)^2\omega^j(x)$$
$$\left.+\sum_{j=p-mt}^{p-2}\pi^{3(p-1)}\Gamma_p\left(\tfrac{p-1+mt-j}{p-1}\right)\Gamma_p\left(\tfrac{2(p-1)-mt-j}{p-1}\right)\Gamma_p\left(\tfrac{j}{p-1}\right)^2\omega^j(x)\right].$$

Recalling that π is a solution to $x^{p-1}+p=0$, we have that $\pi^{p-1}=-p$. We use this to rewrite the sum as

$$\frac{\omega^{mt}(-1)}{p(1-p)}\left[\sum_{j=0}^{mt}-p\Gamma_p\left(\tfrac{mt-j}{p-1}\right)\Gamma_p\left(\tfrac{p-1-mt-j}{p-1}\right)\Gamma_p\left(\tfrac{j}{p-1}\right)^2\omega^j(x)\right.$$
$$+\sum_{j=mt+1}^{p-1-mt}p^2\Gamma_p\left(\tfrac{p-1+mt-j}{p-1}\right)\Gamma_p\left(\tfrac{p-1-mt-j}{p-1}\right)\Gamma_p\left(\tfrac{j}{p-1}\right)^2\omega^j(x)$$
$$\left.+\sum_{j=p-mt}^{p-2}-p^3\Gamma_p\left(\tfrac{p-1+mt-j}{p-1}\right)\Gamma_p\left(\tfrac{2(p-1)-mt-j}{p-1}\right)\Gamma_p\left(\tfrac{j}{p-1}\right)^2\omega^j(x)\right].$$

In Section 2.4 we define the Teichmüller character by the property $\omega(x)\equiv x \pmod p$ for all x in $\{0,\ldots,p-1\}$. Using this we prove that the above sum is congruent modulo p to the expression

$$\frac{(-1)^{mt}}{p(1-p)}\left[\sum_{j=0}^{mt}-p\Gamma_p\left(\tfrac{mt-j}{p-1}\right)\Gamma_p\left(\tfrac{p-1-mt-j}{p-1}\right)\Gamma_p\left(\tfrac{j}{p-1}\right)^2x^j\right.$$
$$+\sum_{j=mt+1}^{p-1-mt}p^2\Gamma_p\left(\tfrac{p-1+mt-j}{p-1}\right)\Gamma_p\left(\tfrac{p-1-mt-j}{p-1}\right)\Gamma_p\left(\tfrac{j}{p-1}\right)^2x^j$$
$$\left.+\sum_{j=p-mt}^{p-2}-p^3\Gamma_p\left(\tfrac{p-1+mt-j}{p-1}\right)\Gamma_p\left(\tfrac{2(p-1)-mt-j}{p-1}\right)\Gamma_p\left(\tfrac{j}{p-1}\right)^2x^j\right].$$

We simplify this expression to get that this is congruent modulo p to the expression

$$\frac{(-1)^{mt}}{(1-p)}\left[\sum_{j=0}^{mt}-\Gamma_p\left(\tfrac{mt-j}{p-1}\right)\Gamma_p\left(\tfrac{p-1-mt-j}{p-1}\right)\Gamma_p\left(\tfrac{j}{p-1}\right)^2x^j\right.$$
$$+\sum_{j=mt+1}^{p-1-mt}p\Gamma_p\left(\tfrac{p-1+mt-j}{p-1}\right)\Gamma_p\left(\tfrac{p-1-mt-j}{p-1}\right)\Gamma_p\left(\tfrac{j}{p-1}\right)^2x^j$$
$$\left.+\sum_{j=p-mt}^{p-2}-p^2\Gamma_p\left(\tfrac{p-1+mt-j}{p-1}\right)\Gamma_p\left(\tfrac{2(p-1)-mt-j}{p-1}\right)\Gamma_p\left(\tfrac{j}{p-1}\right)^2x^j\right].$$

Note that since $\Gamma_p : \mathbb{Z}_p \to \mathbb{Z}_p^*$, the last two summands are congruent to 0 modulo p. Hence, we are left with

$$\frac{(-1)^{mt}}{(1-p)}\left[\sum_{j=0}^{mt}-\Gamma_p\left(\tfrac{mt-j}{p-1}\right)\Gamma_p\left(\tfrac{p-1-mt-j}{p-1}\right)\Gamma_p\left(\tfrac{j}{p-1}\right)^2x^j\right] \pmod{p}.$$

This much simpler expression can be broken down even further. Note that $\frac{mt}{p-1}=\frac{m}{d}$. Then the above expression is congruent, modulo p, to

$$-(-1)^{mt}\sum_{j=0}^{mt}\Gamma_p\left(\tfrac{m}{d}+j\right)\Gamma_p\left(1-\tfrac{m}{d}+j\right)\Gamma_p\left(-j\right)^2x^j. \tag{3.1.1}$$

We then use part 5 of Proposition 2.4.1 to get

$$\Gamma_p(y)^2=\frac{1}{\Gamma_p(1-y)^2}.$$

We use this to write that Equation 3.1.1 is congruent modulo p to the expression

$$(-1)^{mt+1}\sum_{j=0}^{mt}\frac{\Gamma_p\left(\frac{m}{d}+j\right)\Gamma_p\left(1-\frac{m}{d}+j\right)}{\Gamma_p\left(1+j\right)^2}x^j.$$

We now use Proposition 2.4.2 to write that Equation 3.1.1 is congruent modulo p to the expression

$$\sum_{j=0}^{mt}\frac{\left(\frac{m}{d}\right)_j\left(\frac{d-m}{d}\right)_j}{j!^2}x^j.$$

Note that we are summing to $mt=\frac{m(p-1)}{d}$. If $j>mt$ then the rising factorial $\left(\frac{m}{d}\right)_j$ has as a factor

$$\begin{aligned}\frac{m}{d}+\left(\frac{m(p-1)}{d}+1\right)-1&=\frac{m}{d}+\frac{m(p-1)}{d}\\&=p\cdot\frac{m}{d}.\end{aligned}$$

Thus,

$$\sum_{j=0}^{mt}\frac{\left(\frac{m}{d}\right)_j\left(\frac{d-m}{d}\right)_j}{j!^2}x^j\equiv\sum_{j=0}^{p-1}\frac{\left(\frac{m}{d}\right)_j\left(\frac{d-m}{d}\right)_j}{j!^2}x^j\pmod{p},$$

since each missing term in the summand on the left has a factor of p in it.

Noting that $(1)_j=j!$, we use Equation 2.2.2 to identify this as a truncated hypergeometric series

$$\sum_{j=0}^{p-1}\frac{\left(\frac{m}{d}\right)_j\left(\frac{d-m}{d}\right)_j}{j!^2}x^j={}_2F_1\left(\begin{matrix}\frac{m}{d} & \frac{d-m}{d}\\ & 1\end{matrix}\,\middle|\,x\right)_{\mathrm{tr}(p)}.$$

Thus, we have proved that

$${}_2F_1\left(\begin{matrix}\frac{m}{d} & \frac{d-m}{d}\\ & 1\end{matrix}\,\middle|\,x\right)_{\mathrm{tr}(p)}\equiv -p\;{}_2F_1\left(\begin{matrix}T^{mt} & \overline{T}^{mt}\\ & \epsilon\end{matrix}\,\middle|\,x\right)_p\pmod{p}.$$

□

3.2 Dwork ${}_{d-1}F_{d-2}$ Congruence

Our next congruence result holds for hypergeometric functions that appear in our work with Dwork hypersurfaces. The proof is similar to that of Theorem 3.1.1 and so we omit some steps.

Theorem 3.2.1. *Let $p \equiv 1 \pmod d$ and T be a generator for the character group $\widehat{\mathbb{F}_p^\times}$ then,*

$$p^{d-2}\ {}_{d-1}F_{d-2}\left(\begin{matrix} T^t & T^{2t} & \dots & T^{(d-1)t} \\ & \epsilon & \dots & \epsilon \end{matrix}\,\middle|\, x\right)_p$$
$$\equiv (-1)^d{}_{d-1}F_{d-2}\left(\begin{matrix} \frac{1}{d} & \frac{2}{d} & \dots & \frac{d-1}{d} \\ & 1 & \dots & 1 \end{matrix}\,\middle|\, x\right)_{tr(p)} \pmod p,$$

where $t = \frac{p-1}{d}$.

We will use the following corollary in our work with Dwork K3 surfaces in Section 5.1.3.

Corollary 3.2.2. *Let $p \equiv 1 \pmod 4$ and T be a generator for the character group $\widehat{\mathbb{F}_p^\times}$ then,*

$${}_3F_2\left(\begin{matrix} \frac{1}{4} & \frac{2}{4} & \frac{3}{4} \\ & 1 & 1 \end{matrix}\,\middle|\, x\right)_{tr(p)} \equiv p^2\ {}_3F_2\left(\begin{matrix} T^t & T^{2t} & T^{3t} \\ & \epsilon & \epsilon \end{matrix}\,\middle|\, x\right)_p \pmod p, \tag{3.2.1}$$

where $t = \frac{p-1}{4}$.

Proof of Theorem 3.2.1. We start by using Equation 2.2.4 to write

$$p^{d-2}\ {}_{d-1}F_{d-2}\left(\begin{matrix} T^t & T^{2t} & \dots & T^{(d-1)t} \\ & \epsilon & \dots & \epsilon \end{matrix}\,\middle|\, x\right)_p = \frac{p^{d-1}}{p-1}\sum_\chi \binom{T^t\chi}{\chi}\binom{T^{2t}\chi}{\dots}\epsilon\chi\binom{T^{(d-1)t}\chi}{\epsilon\chi}\chi(x).$$

Using properties of Gauss and Jacobi sums from Sections 2.2 and 2.3 we break this down to

$$\frac{1}{p-1}\sum_\chi J(T^t\chi,\overline{\chi})J(T^{2t}\chi,\overline{\chi})\cdots J(T^{(d-1)t}\chi,\overline{\chi})\chi((-1)^{d-1}x)$$
$$= \frac{1}{p-1}\sum_\chi \frac{g(T^t\chi)g(T^{2t}\chi)\cdots g(T^{(d-1)t}\chi)g(\overline{\chi})^{d-1}}{g(T^t)g(T^{2t})\cdots g(T^{(d-1)t})}\chi((-1)^{d-1}x).$$

We are able to simplify the denominator, but the result will be different depending on the parity of d. We split into two cases: d even and d odd.

We start by assuming d is even. In this case our expression becomes

$$= \frac{1}{p-1} \sum_{\chi} \frac{g(T^t\chi)g(T^{2t}\chi)\cdots g(T^{(d-1)t}\chi)g(\overline{\chi})^{d-1}}{g(T^t)g(T^{2t})\cdots g(T^{(d-1)t})} \chi(-x).$$

Note that there are $d-1$ terms in the denominator, which is an odd number. We will have $\frac{d-2}{2}$ pairings of the form $g(T^{mt})g(T^{(d-m)t}) = pT^{mt}(-1)$. The remaining term that does not get paired off is $g(T^{td/2})$. Thus, our expression can be written as

$$= \frac{T^a(-1)}{p^{(d-2)/2}(p-1)g(T^{td/2})} \sum_{\chi} g(T^t\chi)g(T^{2t}\chi)\cdots g(T^{(d-1)t}\chi)g(\overline{\chi})^{d-1}\chi(-x),$$

where $a = t + 2t + \ldots + \frac{d-2}{2}t$. We now rewrite this in terms of the Teichmüller character ω by letting $T = \overline{\omega}$ and $\chi = \overline{\omega}^{-j}$. Furthermore, we can split up the sum in a similar manner to how we did in the proof of Theorem 3.1.1. We will focus on the first summand only, with $0 \le j \le t$, since, as in the previous proof, the remaining summands are congruent to 0 modulo p. Thus, we have

$$\frac{\overline{\omega}^a(-1)}{p^{(d-2)/2}(p-1)g(\overline{\omega}^{td/2})} \sum_{j=0}^{t} g(\overline{\omega}^{t-j})g(\overline{\omega}^{2t-j})\ldots g(\overline{\omega}^{(d-1)t-j})g(\overline{\omega}^j)^{d-1}\omega^j(-x).$$

We use the Gross-Koblitz formula to write this expression in terms of the p-adic Gamma function. Note that the resulting power of π cancels with the $p^{(d-2)/2}$ in the denominator.

$$= \frac{-\overline{\omega}^a(-1)}{(-1)^{(d-2)/2}(p-1)\Gamma_p\left(\frac{td/2}{p-1}\right)} \sum_{j=0}^{t} \Gamma_p\left(\tfrac{t-j}{p-1}\right)\Gamma_p\left(\tfrac{2t-j}{p-1}\right)\ldots\Gamma_p\left(\tfrac{(d-1)t-j}{p-1}\right)\Gamma_p\left(\tfrac{j}{p-1}\right)^{d-1}\omega^j(-x).$$

We now reduce this sum modulo p and simplify using the same techniques as in the proof of Theorem 3.1.1. We recall that $\omega(x) \equiv x \pmod p$ for all x in $\{0, \ldots, p-1\}$ and $p-1 \equiv -1 \pmod p$. Furthermore, by Equation 2.4.1 we have that $\Gamma_p(1+j) = (-1)^{1+j}j!$. We combine this with part 5 of Proposition 2.4.1 to get

$$\Gamma_p(-j)^{d-1} = \frac{1}{j!^{d-1}}.$$

Lastly, if $0 \leq j \leq t$, then $\Gamma_p\left(\frac{m}{d}+j\right) = (-1)^j \left(\frac{m}{d}\right)_j \Gamma_p\left(\frac{m}{d}\right)$, for $m = 1, 2, \ldots, d-1$. Thus, our expression is congruent modulo p to

$$\frac{(-1)^{a-(d-2)/2}}{\Gamma_p\left(\frac{d/2}{d}\right)}\left[\sum_{j=0}^{t} \frac{\left(\frac{1}{d}\right)_j \left(\frac{2}{d}\right)_j \cdots \left(\frac{d-1}{d}\right)_j \Gamma_p\left(\frac{1}{d}\right)\Gamma_p\left(\frac{2}{d}\right)\cdots\Gamma_p\left(\frac{d-1}{d}\right)}{j!^{d-1}}(x)^j\right].$$

We use part 5 of Proposition 2.4.1 to simplify the p-adic Gamma pairs

$$\Gamma_p\left(\frac{m}{d}\right)\Gamma_p\left(1-\frac{m}{d}\right) = (-1)^{(d-m)t+1}.$$

The resulting exponent of -1 from this will be $b+\frac{d-2}{2}$, where $b = (d-1)t+\ldots+\left(d-\frac{d-2}{2}\right)t$. Thus, our expression is congruent modulo p to

$$\equiv (-1)^{a+b}\sum_{j=0}^{t} \frac{\left(\frac{1}{d}\right)_j \left(\frac{2}{d}\right)_j \cdots \left(\frac{d-1}{d}\right)_j}{j!^{d-1}}(x)^j \pmod{p},$$

Note that

$$\begin{aligned} a+b &= t+2t+\ldots+\tfrac{d-2}{2}t+(d-1)t+\ldots+\left(d-\tfrac{d-2}{2}\right)t \\ &= dt+\ldots+dt \\ &= dt\cdot\tfrac{d-2}{2} \\ &= (p-1)\cdot\tfrac{d-2}{2}, \end{aligned}$$

which is even. Thus we can write that $(-1)^{a+b} = (-1)^d$.

Now suppose that d is odd. In this case our expression becomes

$$\frac{1}{p-1}\sum_{\chi}\frac{g(T^t\chi)g(T^{2t}\chi)\cdots g(T^{(d-1)t}\chi)g(\overline{\chi})^{d-1}}{g(T^t)g(T^{2t})\cdots g(T^{(d-1)t})}\chi(x).$$

The rest of the proof is similar to the case when d is even, but slightly easier. There are $d-1$ terms in the denominator, which is an even number. We will have $\frac{d-1}{2}$ pairings of the form $g(T^{mt})g(T^{(d-m)t}) = pT^{mt}(-1)$. Thus, our expression can be written as

$$\frac{T^{a'}(-1)}{p^{(d-1)/2}(p-1)}\sum_{\chi} g(T^t\chi)g(T^{2t}\chi)\cdots g(T^{(d-1)t}\chi)g(\overline{\chi})^{d-1}\chi(x),$$

where $a' = t + 2t + \ldots + \frac{d-1}{2}t$.

We now rewrite this in terms of the Teichmüller character ω by letting $T = \overline{\omega}$ and $\chi = \overline{\omega}^{-j}$. Furthermore, we can split up the sum in a similar manner to how we did when d was even. We will focus on the first summand only, with $0 \leq j \leq t$, since, as in the previous proof, the remaining summands are congruent to 0 modulo p. Thus, we have

$$\frac{\overline{\omega}^{a'}(-1)}{p^{(d-1)/2}(p-1)} \sum_{j=0}^{t} g(\overline{\omega}^{t-j})g(\overline{\omega}^{2t-j})\ldots g(\overline{\omega}^{(d-1)t-j})g(\overline{\omega}^{j})^{d-1}\omega^{j}(x).$$

We use the Gross-Koblitz formula to write this expression in terms of the p-adic Gamma function

$$= \frac{\overline{\omega}^{a'}(-1)}{(-1)^{(d-1)/2}(p-1)} \sum_{j=0}^{t} \Gamma_p\left(\tfrac{t-j}{p-1}\right)\Gamma_p\left(\tfrac{2t-j}{p-1}\right)\ldots\Gamma_p\left(\tfrac{(d-1)t-j}{p-1}\right)\Gamma_p\left(\tfrac{j}{p-1}\right)^{d-1}\omega^{j}(x).$$

Note that the resulting power of π canceled with the $p^{(d-1)/2}$ in the denominator. Furthermore we reduce the sum modulo p and simplify using the same techniques as in the proof for d even. Our expression is congruent modulo p to

$$(-1)^{a'-(d-1)/2+1}\left[\sum_{j=0}^{t} \frac{\left(\frac{1}{d}\right)_j\left(\frac{2}{d}\right)_j\cdots\left(\frac{d-1}{d}\right)_j\Gamma_p\left(\frac{1}{d}\right)\Gamma_p\left(\frac{2}{d}\right)\cdots\Gamma_p\left(\frac{d-1}{d}\right)}{j!^{d-1}}(x)^j\right].$$

We use part 5 of Proposition 2.4.1 to simplify each term of the form

$$\Gamma_p\left(\frac{m}{d}\right)\Gamma_p\left(1-\frac{m}{d}\right) = (-1)^{(d-m)t+1}.$$

The resulting exponent of -1 from this will be $b' + \frac{d-2}{2}$, where $b' = (d-1)t + \ldots + \left(d - \frac{d-2}{2}\right)t$. Thus, our expression is congruent modulo p to

$$(-1)^{a'+b'+1}\sum_{j=0}^{t}\frac{\left(\frac{1}{d}\right)_j\left(\frac{2}{d}\right)_j\cdots\left(\frac{d-1}{d}\right)_j}{j!^{d-1}}(x)^j.$$

Note that $a' + b' + 1 = \frac{d-1}{2}(p-1) + 1$, which is an odd number. Thus, $(-1)^{a'+b'+1} = (-1)^d$, which matches the exponent of -1 in the case where d was even.

We now bring the two cases together. For both even and odd d we have

$$p^{d-2}\ {}_{d-1}F_{d-2}\left(\begin{array}{cccc|c} T^{t} & T^{2t} & \dots & T^{(d-1)t} & \\ & \epsilon & \dots & \epsilon & x \end{array}\right)_p \equiv (-1)^d \sum_{j=0}^{t} \frac{\left(\frac{1}{d}\right)_j \left(\frac{2}{d}\right)_j \cdots \left(\frac{d-1}{d}\right)_j}{j!^{d-1}} (x)^j \pmod{p}.$$

We use Equation 2.2.2 to identify this as a truncated hypergeometric series

$$\equiv (-1)^d\ {}_{d-1}F_{d-2}\left(\begin{array}{cccc|c} \frac{1}{d} & \frac{2}{d} & \dots & \frac{d-1}{d} & \\ & 1 & \dots & 1 & x \end{array}\right)_{\mathrm{tr}(p)} \pmod{p},$$

where, as in the proof of Theorem 3.1.1, the terms with $j > t$ are congruent to 0 mod p.

$\square$

Chapter 4

Curves

The first two sections of this chapter describe the relationship between arithmetic and analytic properties of curves. We then focus on the family of genus 3 generalized Legendre curves.

4.1 Using the Lefschetz Number

The Lefschetz Number associated to a map from a manifold to itself essentially keeps track of the number of fixed points of the map. Let $f : M \to M$ be a differentiable map on the compact differentiable manifold M such that the graph of f meets the diagonal transversely (think: not tangentally). Then the Lefschetz number $L(f)$ can be computed in two ways:

$$L(f) = \sum_{p \in M} \sigma_p(f) = \sum_{n=0}^{\infty} (-1)^n \mathrm{tr}[f^* : H^n(M, \mathbb{C}) \to H^n(M, \mathbb{C})], \tag{4.1.1}$$

where

$$\sigma_p(f) = \begin{cases} 0 & : f(p) \neq p \\ \pm 1 & : (\text{graph } f) \text{ meets diagonal with positive/negative orientation.} \end{cases}$$

When the map f is the Frobenius map on a curve, then $L(f)$ measures the number of points on the curve over a finite field $\mathbb{F}_q$. In this field we have $(x, y) = (x^q, y^q)$, so that any point on the curve will be a fixed point of the map.

We will rewrite both expressions for the Lefschetz number in order to show the relationship between the period associated to a curve and its point count. We follow the work of Clemens [6, Chapter 2].

We start by rewriting $\sum_{p\in M}\sigma_p(f)$. Let $J_p(f)$ be the Jacobian of f at the point p. The transversality of f at p implies that (identity $-\ f$) has maximal rank at p. This is the rank of $I - J_p(f)$ at the point p, which is a matrix that gives us information about the orientation of the map f. Thus, we can write $\sigma_p(f) = \text{sign}\ \det(I - J_p(f))$. Clemens shows that this determinant can also be expressed as

$$\det(I - J_p(f)) = \sum_{r=0}^{n}(-1)^r \text{tr}(\wedge^r J_p(f))$$

so that we can write

$$\sum_{p\in M}\sigma_p(f) = \sum_{p,r}(-1)^r \frac{\text{tr}(\wedge^r J_p(f))}{|\det(I - J_p(f))|}.$$

Denote the restrictions of $J_p(f)$ to type (1,0) (holomorphic) and type (0,1) (antiholomorphic) parts of $J_p(f)$ by $J_p'(f)$ and $J_p''(f)$, respectively. Clemens notes that if the manifold M is a Kähler manifold then we can replace the de Rham complex by the Dolbeault complex on M. Thus Equation 4.1.1 becomes

$$\sum_{p,r}(-1)^r \frac{\text{tr}(\wedge^r J_p(f))}{|\det(I - J_p(f))|} = \sum_{n=0}^{\infty}(-1)^n \text{tr}[f^*|_{H^n(M,\mathscr{O})}], \tag{4.1.2}$$

We also have that

$$\sum_{r}(-1)^r \text{tr}(\wedge^r J_p(f)) = \det(I - J_p''(f))$$

and

$$\det(I - J_p(f)) = \det(I - J_p'(f))\det(I - J_p''(f)).$$

Hence,

$$\begin{aligned}\sum_{p,r}(-1)^r \frac{\operatorname{tr}(\wedge^r J_p(f))}{|\det(I - J_p(f))|} &= \sum_p \frac{\det(I - J''_p(f))}{|\det(I - J'_p(f))\det(I - J''_p(f))|} \\ &= \sum_p \frac{1}{|\det(I - J'_p(f))|} \\ &= \sum_{p \text{ fixed}} \frac{1}{|\det(I - J'_p(f))|}.\end{aligned}$$

Thus, $\sum \sigma_p(f)$ can be expressed in terms of the holomorphic part of $J_p(f)$.

We consider the case where f is the Frobenius map and the manifold is an algebraic curve M. We have that $J_p(f) = 0$ since $d(x^p)/dx = px^{p-1} = 0$ in $\mathbb{F}_p$. Hence, $|\det(I - J'_p(f))| = 1$ and $\sum_{p \in M} \sigma_p(f) =$ the number of fixed points of f. Since f is a map on M and $x^p = x$ if and only if $x \in \mathbb{F}_p$, the number of fixed points of f will be exactly the number of points on M plus the point at infinity. Thus,

$$\sum_{p \in M} \sigma_p(f) = 1 + \text{ the number of points on } M.$$

We now rewrite the expression $\sum_{n=0}^{\infty}(-1)^n \operatorname{tr}[f^*|_{H^n(M,\mathscr{O})}]$ for the case we are considering. Recall that $H^n(M,\mathscr{O})) = 0$ whenever $n > \dim(M)$. Equation 4.1.2 then becomes

$$1 + \text{ the number of points on } M = 1 - \operatorname{tr}[f^*|_{H^1(M,\mathscr{O})}],$$

i.e.

$$\text{the number of points on } M = -\operatorname{tr}[f^*|_{H^1(M,\mathscr{O})}]. \tag{4.1.3}$$

In the next section we will see that the right-hand-side of this equation is related to the periods associated to an algebraic curve.

4.2 The Hasse-Witt Matrix

Let g be the genus of the algebraic curve M. The Hasse-Witt matrix of M is the $g \times g$ matrix of the Frobenius map with respect to a basis of regular differentials of the first kind.

Thus, the trace of this matrix will give us $\mathrm{tr}[f^*|_{H^1(M,\mathscr{O})}]$ (trace is independent of basis). Furthermore, Manin [34, Theorem 2] shows that the rows of the Hasse-Witt matrix satisfy the system of differential equations that is satisfied by the periods associated to the curve. In this section we aim to describe the Hasse-Witt matrix in greater detail.

The genus g of the curve is equal to both the dimension of the space $H^1(M,\mathscr{O})$ and the dimension of the space of regular 1-forms on M. We will choose dual bases for these two spaces (dual with respect to a residue pairing). Let $P_1, \ldots, P_g$ be a set of distinct points on M such that the divisor $D = \sum P_i$ is nonspecial. It is noted by Manin [34, page 252] that we may identify $H^1(M,\mathscr{O})$ with the space of functions that have poles at worst at the points $P_1, \ldots, P_g$. Thus, we can choose a basis $h_1, \ldots, h_g$ for $H^1(M,\mathscr{O})$, where each h_i is a function with a simple pole at P_i and no other poles (except at infinity). We express each h_i as in [6, Chapter 2, Section 12]:

$$h_i = \frac{1}{x - x_i} + \sum_{l \geq 0} b_{i,l}(x - x_i)^l,$$

where $P_i = (x_i, y_i)$ is a point on M as above. Similarly, to each point $P_i = (x_i, y_i)$ we can associate a differential ω_i, which is to say we can write ω_i locally at the point P_i:

$$\omega_i = dx + \sum_{r \geq 1} a_{i,r}(x - x_i)^r dx.$$

The bases $\{\omega_i\}_i$ and $\{h_i\}_i$ are dual with respect to the pairing $(\omega_i, h_j) = \mathrm{res}_{P_i}(h_j\omega_i)$ since

$$\mathrm{res}_{P_i}(h_j\omega_i) = \begin{cases} 1 & : i = j \\ 0 & : i \neq j. \end{cases}$$

Let K be the matrix of scalar products $[(\omega_i, h_j)]$. We can write the Hasse-Witt matrix H as

$$H = KH = [(\omega_i, f^*h_j)],$$

where the map f^* sends each $h_i(x)$ to

$$h_i(x^p) = \frac{1}{(x-x_i)^p} + \sum_{l \geq 0} b_{i,l}(x-x_i)^{pl}.$$

Thus, $\mathrm{tr}[f^*|_{H^1(M,\mathscr{O})}] = \sum_{i=1}^{g}(\omega_i, f^*h_i)$. In fact we can say even more about this matrix. Note that if $i \neq j$ then

$$(\omega_i, f^*h_j) = \mathrm{res}_{P_i}(f^*h_j\omega_i) = 0$$

since h_j, and therefore f^*h_j, is holomorphic at the point P_i. Thus, the Hasse-Witt matrix is a diagonal matrix with this choice of basis.

These diagonal entries can be expressed in terms of coefficients in the expansions of the differentials. We have that $\mathrm{res}_{P_i}(f^*h_i\omega_i)$ is the coefficient of $1/(x-x_i)$ in the expansion

$$f^*h_i\omega_i = \left(\frac{1}{(x-x_i)^p} + \sum_{l \geq 0} b_{i,l}(x-x_i)^{pl}\right)\left(1 + \sum_{r \geq 1} a_{i,r}(x-x_i)^r\right) dx.$$

Thus, $(\omega_i, f^*h_i) = a_{i,p-1}$, so that $\mathrm{tr}[f^*|_{H^1(M,\mathscr{O})}] = \sum_{i=1}^{g} a_{i,p-1}$.

In the next section we apply this theory to a particular family of curves.

4.3 Generalized Legendre Curves

We look at a specific case of the generalized Legendre curves given by

$$C_\lambda^4 : y^4 = x(x-1)(x-\lambda).$$

Viewed as a projective curve it is given by the homogeneous equation

$$Y^4 = ZX(X-Z)(X-\lambda Z)$$

by sending $(x, y) \to (X/Z, Y/Z)$. When written in this form we see that the curve is nonsingular in $\mathbb{P}^2$. Thus, by a well-known genus formula for nonsingular curves, the genus of C_λ^4 is $g = \frac{(d-1)(d-2)}{2} = 3$.

4.3.1 Period Computation

In this section we give a formula for certain period integrals associated to genus 3 generalized Legendre curves. The periods we are interested in are obtained by choosing dual bases of the space of holomorphic differentials and the space of cycles $H^1(C_\lambda^4, \mathscr{O})$ and integrating the differentials over each cycle. Note that since the curves are genus $g = 3$, the dimension of both spaces is 3 and, so, we get three period integrals.

Theorem 4.3.1. *The periods associated to the above curve are given by*

$$\pi_1 = {}_2F_1\left(\begin{matrix} 1/4 & 3/4 \\ & 1/2 \end{matrix}\middle| \lambda\right), \pi_2 = {}_2F_1\left(\begin{matrix} 1/2 & 1/2 \\ & 1 \end{matrix}\middle| \lambda\right), \text{ and } \pi_3 = {}_2F_1\left(\begin{matrix} 3/4 & 5/4 \\ & 3/2 \end{matrix}\middle| \lambda\right).$$

Proof. Using a method found in Miranda's text [40, Chapter 4, Section 1], we find a basis for the space of differentials formed by

$$\omega_1 = \frac{xdx}{y^3}, \omega_2 = \frac{dx}{y^2}, \omega_3 = \frac{dx}{y^3}.$$

As noted in the previous section, we can write each ω_i locally at a distinct point P_i. We compute the periods π_1, π_2, π_3 associated to C_λ^4 by integrating each differential ω_i over a cycle in $H^1(C_\lambda^4, \mathscr{O})$ that contains the point P_i and not the other P_j. Such a path exists since the chosen points are distinct.

We follow the work of Clemens [6, Chapter 2, Section 10] to find differential equations satisfied by the periods and then give combinatorial expressions for them. We show the computation for π_3. Starting with the differential

$$\omega_3 = \frac{dx}{y^3} = (x(x-1)(x-\lambda))^{-3/4}dx,$$

we take derivatives with respect to λ to get

$$\begin{aligned} \frac{\partial}{\partial\lambda}((x(x-1)(x-\lambda))^{-3/4}) &= -\tfrac{3}{4}x^{-3/4}(x-1)^{-3/4}(x-\lambda)^{-7/4} \\ \frac{\partial^2}{\partial\lambda^2}((x(x-1)(x-\lambda))^{-3/4}) &= \tfrac{21}{16}x^{-3/4}(x-1)^{-3/4}(x-\lambda)^{-11/4}. \end{aligned}$$

We wish to find a linear combination of ω_3 and its derivatives that gives an exact differential. To do this, we rewrite the following differential.

$$
\begin{aligned}
d\left(\frac{x^{1/4}(x-1)^{1/4}(x-\lambda)^{1/4}}{(x-\lambda)^2}\right) &= d\left(x^{1/4}(x-1)^{1/4}(x-\lambda)^{-7/4}\right) \\
&= \Big[\frac{1}{4}x^{-3/4}(x-1)^{1/4}(x-\lambda)^{-7/4} \\
&\quad +\frac{1}{4}x^{1/4}(x-1)^{-3/4}(x-\lambda)^{-7/4} - \frac{7}{4}x^{1/4}(x-1)^{1/4}(x-\lambda)^{-11/4}\Big] \\
&= \frac{1}{3}(x-1)\frac{d\omega}{d\lambda} + \frac{1}{3}x\frac{d\omega}{d\lambda} - \frac{4}{3}x(x-1)\frac{d^2\omega}{d\lambda^2} \\
&= -\frac{5}{4}\omega - 2(2\lambda+1)\frac{d\omega}{d\lambda} - \frac{4}{3}\lambda(\lambda-1)\frac{d^2\omega}{d\lambda^2}.
\end{aligned}
$$

Integrating both sides and then multiplying by $\frac{3}{4}$ gives us the Picard-Fuchs equation

$$
-\frac{15}{16}\pi_3 + (3/2-3\lambda)\frac{d\pi_3}{d\lambda} + \lambda(1-\lambda)\frac{d^2\pi_3}{d\lambda^2} = 0. \tag{4.3.1}
$$

Formula 5.8 in [23, Chapter 9] tells us that

$$
F = {}_2F_1\left(\begin{array}{cc|c} a & b & \\ & & \lambda \\ & c & \end{array}\right)
$$

satisfies

$$
-abF + (c-(a+b+1)\lambda)\frac{d}{d\lambda}F + \lambda(1-\lambda)\frac{d^2}{d\lambda^2}F = 0.
$$

We solve for a, b, c in Equation 4.3.1 and find that $a = 3/4, b = 5/4$ (or vice versa) and $c = 3/2$. This gives us the following expression for the period

$$
\pi_3 = {}_2F_1\left(\begin{array}{cc|c} 3/4 & 5/4 & \\ & & \lambda \\ & 3/2 & \end{array}\right).
$$

Similarly, we find that π_1 satisfies

$$
-\frac{3}{16}\pi_1 + (1/2-2\lambda)\frac{d\pi_1}{d\lambda} + \lambda(1-\lambda)\frac{d^2\pi_1}{d\lambda^2} = 0 \tag{4.3.2}
$$

and can be expressed as

$$\pi_1 = {}_2F_1\left(\begin{array}{cc|c} 1/4 & 3/4 & \\ & 1/2 & \lambda \end{array}\right),$$

and that π_2 satisfies

$$-\frac{1}{4}\pi_2 + (1-2\lambda)\frac{d\pi_2}{d\lambda} + \lambda(1-\lambda)\frac{d^2\pi_2}{d\lambda^2} = 0 \tag{4.3.3}$$

and can be expressed as

$$\pi_2 = {}_2F_1\left(\begin{array}{cc|c} 1/2 & 1/2 & \\ & 1 & \lambda \end{array}\right).$$

□

4.3.2 Point Count

In this section we will compute the number of points on the curve C_λ^4 in two ways. We first specify the work in Section 4.2 to the curve C_λ^4. Then, we will compute the number of points using character sums.

We find a basis of functions h_i with simple poles at the points P_i, thus giving us a dual basis to $\{\omega_i\}$. As in Section 4.2, the diagonal entries of the Hasse-Witt matrix are given by the coefficients $a_{i,p-1}$ in the series expansions of the differentials at the points P_i. We show that these values satisfy the Picard-Fuchs equations associated to the periods of the curve C_λ^4. Let

$$\left(f_{i,0} + f_{i,1}\tfrac{\partial}{\partial\lambda} + f_{i,2}\tfrac{\partial^2}{\partial\lambda^2}\right)\omega_i = d\,(\text{series expansion of a function around } P_i)\,.$$

Matching coefficients of the two sides gives us the relation

$$\left(f_{i,0} + f_{i,1}\tfrac{\partial}{\partial\lambda} + f_{i,2}\tfrac{\partial^2}{\partial\lambda^2}\right)\left(a_{i,p-1}(x-x_i)^{p-1}\right) = d\left(c_{i,p}(x-x_i)^p\right),$$

where $c_{i,p-1}$ is the $(p-1)$st coefficient of the series expansion. The right side of this equation equals 0 in $\mathbb{F}_p$, which tells us that $a_{i,p-1}$ satisfies the Picard-Fuchs equation associated to π_i.

Thus, since the trace of the Hasse-Witt matrix is given by $\sum_{i=1}^{3} a_{i,p-1}$, we see that

$$\mathrm{tr}[f^*|_{H^1(M,\mathscr{O})}] \equiv \sum_{i=1}^{3} \pi_i \pmod{p}.$$

Using Equation 4.1.3 we can conclude that

$$\text{number of points on } C_\lambda^4 \equiv -\sum_{i=1}^{3} \pi_i \pmod{p},$$

which is sum of classical hypergeometric series. In fact each of the classical hypergeometric series are congruent to truncated series when we reduce mod p.

We can also compute the number of points on C_λ^4 by using character sums. In doing so, we obtain a congruence between certain classical hypergeometric series and Greene's finite field hypergeometric function.

Theorem 4.3.2. *Let q be a prime power such that $q \equiv 1 \pmod 4$. Let $T \in \widehat{\mathbb{F}_q^\times}$ be a generator of the character group and let $\psi = T^{\frac{p-1}{4}}$. Then*

$$\#C_\lambda^4 = q + 1 + q\epsilon(\lambda) \sum_{m=1}^{3} \psi^m(-1) \cdot {}_2F_1\left(\begin{matrix} \psi^{-m} & \psi^m \\ & \psi^{2m} \end{matrix} \,\middle|\, \lambda \right)_q .$$

Remark. This result may follow from Theorem 11 in [8], though our equation for the generalized Legendre curve is written in a slightly different form. In [8], the generalized Legendre curve would be written as

$$y^4 = x(1-x)(1-\lambda x).$$

The resulting point count formulas are identical, so we should be able to find a transformation between the two curves.

Proof. To prove the result, we first express the number of points as a sum of characters over the finite field $\mathbb{F}_q$.

$$\begin{aligned}
\#C_\lambda^4(\mathbb{F}_q) &= \sum_{x\in\mathbb{F}_q} \#\left\{y\in\mathbb{F}_q \,\middle|\, y^4 = x(x-1)(x-\lambda)\right\} + 1 \\
&= \sum_{x\in\mathbb{F}_q-\{0,1,\lambda\}} \left(\sum_{m=0}^{3} \psi^m(x(x-1)(x-\lambda))\right) + 1 + 3 \\
&= \sum_{x\in\mathbb{F}_q-\{0,1,\lambda\}} \epsilon(x(x-1)(x-\lambda)) + \sum_{x\in\mathbb{F}_q-\{0,1,\lambda\}} \left(\sum_{m=1}^{3} \psi^m(x(x-1)(x-\lambda))\right) + 4 \\
&= q - 3 + \sum_{x\in\mathbb{F}_q-\{0,1,\lambda\}} \left(\sum_{m=1}^{3} \psi^m(x(x-1)(x-\lambda))\right) + 4
\end{aligned}$$

For each m we have

$$\sum_{x\in\mathbb{F}_q-\{0,1,\lambda\}} \psi^m(x(x-1)(x-\lambda)) = \sum_{x\in\mathbb{F}_q-\{0,1,\lambda\}} \psi^m(x)\psi^m(x-1)\psi^m(x-\lambda).$$

We work to rewrite the summand and get

$$\begin{aligned}
&= \sum_{x\in\mathbb{F}_q-\{0,1,\lambda\}} \psi^m(x)\psi^m(1-x)\psi^m(\lambda-x)\psi^m(-1)\psi^m(-1) \\
&= \sum_{x\in\mathbb{F}_q-\{0,1,\lambda\}} \psi^m(x)\psi^m(1-x)\psi^m(1-\frac{1}{\lambda}x)\psi^m(\lambda) \\
&= \psi^m(\lambda) \sum_{x\in\mathbb{F}_q-\{0,1,\lambda\}} \psi^m(x)\psi^m(1-x)\psi^m(1-\frac{1}{\lambda}x) \\
&= \psi^m(\lambda) \sum_{x\in\mathbb{F}_q-\{0,1,\lambda\}} \psi^m(x)\psi^{-m}\psi^{2m}(1-x)\psi^m(1-\frac{1}{\lambda}x),
\end{aligned}$$

which we recognize as being the hypergeometric function expression

$$= \psi^m(\lambda) \cdot \frac{q}{\epsilon\left(\frac{1}{\lambda}\right)\psi^{3m}(-1)} \cdot {}_2F_1\left(\begin{array}{cc|c} \psi^{-m} & \psi^m & \frac{1}{\lambda} \\ & \psi^{2m} & \end{array}\right)_q$$

$$= \psi^m(-\lambda) \cdot \frac{q}{\epsilon\left(\frac{1}{\lambda}\right)} \cdot {}_2F_1\left(\begin{array}{cc|c} \psi^{-m} & \psi^m & \frac{1}{\lambda} \\ & \psi^{2m} & \end{array}\right)_q.$$

We use Theorem 4.2 (ii) in Greene [19] to write

$${}_2F_1\left(\begin{array}{cc|c} \psi^{-m} & \psi^m & \frac{1}{\lambda} \\ & \psi^{2m} & \end{array}\right)_q = \psi^{-m+m+2m}(-1)\psi^m\left(\frac{1}{\lambda}\right) \cdot {}_2F_1\left(\begin{array}{cc|c} \psi^{-m} & \psi^{-m+2m} & \lambda \\ & \psi^{-m-m} & \end{array}\right)_q$$

$$= \psi^m\left(\frac{1}{\lambda}\right) \cdot {}_2F_1\left(\begin{array}{cc|c} \psi^{-m} & \psi^m & \lambda \\ & \psi^{-2m} & \end{array}\right)_q$$

$$= \psi^m\left(\frac{1}{\lambda}\right) \cdot {}_2F_1\left(\begin{array}{cc|c} \psi^{-m} & \psi^m & \lambda \\ & \psi^{2m} & \end{array}\right)_q.$$

Thus, for each m we have

$$\psi^m(-\lambda) \cdot \frac{q}{\epsilon\left(\frac{1}{\lambda}\right)} \cdot {}_2F_1\left(\begin{array}{cc|c} \psi^{-m} & \psi^m & \frac{1}{\lambda} \\ & \psi^{2m} & \end{array}\right)_q = \psi^m(-\lambda) \cdot \frac{q}{\epsilon\left(\frac{1}{\lambda}\right)} \cdot \psi^m\left(\frac{1}{\lambda}\right) \cdot {}_2F_1\left(\begin{array}{cc|c} \psi^{-m} & \psi^m & \lambda \\ & \psi^{2m} & \end{array}\right)_q$$

$$= q \cdot \psi^m(-1)\epsilon(\lambda) \cdot {}_2F_1\left(\begin{array}{cc|c} \psi^{-m} & \psi^m & \lambda \\ & \psi^{2m} & \end{array}\right)_q.$$

Putting this back into the formula for $\#C_\lambda^4$ gives

$$\#C_\lambda^4(\mathbb{F}_q) = q - 3 + \sum_{m=1}^{3} q \cdot \psi^m(-1)\epsilon(\lambda) \cdot {}_2F_1\left(\begin{matrix} \psi^{-m} & \psi^m \\ & \psi^{2m} \end{matrix} \middle| \lambda \right)_q + 4$$
$$= q + 1 + q\epsilon(\lambda) \sum_{m=1}^{3} \psi^m(-1) \cdot {}_2F_1\left(\begin{matrix} \psi^{-m} & \psi^m \\ & \psi^{2m} \end{matrix} \middle| \lambda \right)_q .$$

□

4.3.3 Period - Point Count Connection

We notice two phenomena here that also occur when we are computing periods and point counts for (elliptic) Legendre curves C_λ^2. The first is that, remarkably, the number of points on the curve can be expressed in terms of finite field hypergeometric functions with input given by λ. In fact we get equality, not just a congruence, between the number of points and a finite field hypergeometric expression. This phenomenon also occurs for families of curves not expressible in Legendre form (with $\lambda \in \mathbb{Q}$; examples of this can be found in [16, 17, 33]). In fact, this phenomenon seems to extend to some higher dimensional Calabi-Yau manifolds as is shown in Chapter 5 and in [1, 32, 35, 45]) leading us to wonder if this will be the case for a large class of algebraic varieties.

The second phenomenon is that in computing the point count in two different ways, we get a congruence between classical and finite field hypergeometric expressions. We can say a bit more on this: it seems as though we can identify congruences between particular summands that "match". For example, we saw in Section 4.3.1 that one period of the curve C_λ^4 can be expressed as

$$\pi_2 = {}_2F_1\left(\begin{matrix} 1/2 & 1/2 \\ & 1 \end{matrix} \middle| \lambda \right).$$

We saw in Section 4.3.2 that one of the summands in the point count for C_λ^4 is

$$pT^{\frac{q-1}{2}}(-1){}_2F_1\left(\begin{array}{cc} T^{\frac{q-1}{2}} & T^{\frac{q-1}{2}} \\ & \epsilon \end{array}\middle|\,\lambda\right)_q.$$

Note that since $p \equiv 1 \pmod 4$, $T^{\frac{q-1}{2}}(-1) = 1$. In Section 3.1 we proved that, in fact, this expression is congruent modulo p to the negative of the classical hypergeometric series π_2. The classical and finite field hypergeometric expressions – including the two that are not covered by Theorem 3.1.1 – for the generalized Legendre curves "match" in the same way that period and trace of Frobenius expressions match for elliptic curves: we replace the fraction $\frac{a}{b}$ with a character of order b raised to the ath power. This phenomenon also seems to extend to some other curves (see [41]) and to higher dimensional Calabi-Yau manifolds (see, for example, [27, 32, 42, 35] and Chapters 3 and 5 of this thesis). By testing values in Sage [9], we know that it is not the case that congruences exist between arbitrary truncated hypergeometric series and finite field hypergeometric functions (paired up in the usual manner). This leads us to wonder when we can expect to have a congruence between these two types of series.

Chapter 5

Dwork Hypersurfaces

Most of this Chapter deals with Dwork K3 Surfaces, though in later sections we discuss results and conjectures for higher dimensional Dwork hypersurfaces.

5.1 Dwork K3 Surfaces

Koblitz [28] developed a formula for the number of points on diagonal hypersurfaces in the Dwork family in terms of Gauss sums. We specialize this formula to the family of Dwork K3 surfaces, i.e. to the case when $d, n = 4, h = 1$.

Let W be the set of all 4-tuples $w = (w_1, w_2, w_3, w_4)$ satisfying $0 \leq w_i < 4$ and $\sum w_i \equiv 0 \pmod 4$. Denote the points on the diagonal hypersurface

$$x_1^4 + x_2^4 + x_3^4 + x_4^4 = 0$$

by $N_q(0) := \sum N_q(0, w)$, where

$$N_q(0,w) = \begin{cases} 0 & \text{if some but not all } w_i = 0, \\ \frac{q^3-1}{q-1} & \text{if all } w_i = 0, \\ -\frac{1}{q} J\left(T^{\frac{w_1}{4}}, T^{\frac{w_2}{4}}, T^{\frac{w_3}{4}}, T^{\frac{w_4}{4}}\right) & \text{if all } w_i \neq 0. \end{cases}$$

Theorem 5.1.1. *[28, Theorem 2] The number of points on the Dwork K3 surface X_λ^4 is given by*

$$\#X_\lambda^4(\mathbb{F}_q) = N_q(0) + \frac{1}{q-1}\sum \frac{\prod_{i=1}^4 g\left(T^{w_i t+j}\right)}{g(T^{4j})} T^{4j}(4\lambda)$$

where the sum is taken over $j \in \{0, \dots, q-2\}$ and $w = (w_1, w_2, w_3, w_4)$ in $W/\sim$ (defined below).

We wish to simplify this formula. We start by considering the term $N_q(0)$.

Proposition 5.1.2. $N_q(0) = q^2 + 7q + 1 + \frac{1}{q}\sum_{i=1}^3 g(T^{it})^4 + 12qT^t(-1)$.

Proof. The list of possible 4-tuples (up to reordering since the order doesn't matter in the Jacobi sum) is

$$W^* = \{(1,1,1,1)^1, (2,2,2,2)^1, (3,3,3,3)^1, (1,1,3,3)^6, (1,2,2,3)^{12}\},$$

$w = (0,0,0,0)$, and all of the 4-tuples where some, not all, of the $w_i = 0$ (we exclude this list since $N_q(0, w) = 0$ for these tuples). Letting $t = \frac{q-1}{4}$ we see that

$$J\left(T^{\frac{w_1}{4}}, T^{\frac{w_2}{4}}, T^{\frac{w_3}{4}}, T^{\frac{w_4}{4}}\right) = -\prod_i g(T^{w_i t}).$$

Thus, we have

$$\begin{aligned} N_q(0) &= \sum_w N_q(0, w) \\ &= \frac{q^3-1}{q-1} + \sum_{w \in W^*} N_q(0, w) \\ &= q^2 + q + 1 + \frac{1}{q}\left[\sum_{i=1}^3 g(T^{it})^4 + 6g(T^t)^2 g(T^{3t})^2 + 12g(T^t)g(T^{2t})^2 g(T^{3t})\right] \\ &= q^2 + q + 1 + \frac{1}{q}\left[\sum_{i=1}^3 g(T^{it})^4 + 6q^2 + 12q^2 T^t(-1)\right] \\ &= q^2 + 7q + 1 + \frac{1}{q}\sum_{i=1}^3 g(T^{it})^4 + 12qT^t(-1). \end{aligned}$$

□

Define the equivalence relation $\sim$ on W by $w \sim w'$ if $w - w'$ is a multiple of $(1,1,1,1)$. Up to permutation, there are three cosets (up to permutation) in $W/\sim$:

$$(0,0,0,0)^1, (0,1,1,2)^{12}, (0,0,2,2)^3.$$

The first coset contains the obvious four elements. The second set of 12 cosets is made up of permutations of the coset $\overline{(0,1,1,2)} = \{(0,1,1,2), (1,2,2,3), (2,3,3,0), (3,0,0,1)\}$. The third set of 3 cosets is made up of permutations of the coset $\overline{(0,0,2,2)} = \{(0,0,2,2), (1,1,3,3), (2,2,0,0), (3,3,1,1)\}$. This last set of cosets is different in that some permutations are in the same coset. For example, the element $(2,2,0,0)$ is a permutation of $(0,0,2,2)$ but they are also in the same coset.

Let

$$S_{[w]} = \frac{1}{q-1} \sum_{j=0}^{q-2} \frac{\prod_{i=1}^{4} g\left(T^{w_i t + j}\right)}{g\left(T^{4j}\right)} T^{4j}(4\lambda) \tag{5.1.1}$$

denote the summands corresponding to all $w' \in [w]$. Our main tool for simplifying terms of this form is the Hasse-Davenport relation for Gauss sums.

In the propositions that follow, we give explicit formulas for each $S_{[w]}$.

Proposition 5.1.3. *Let* $w = (0,0,0,0)$. *Then*

$$S_{[w]} = -\frac{1}{q} \sum_{i=1}^{3} g\left(T^{it}\right)^4 + q^2 {}_3F_2 \left(\begin{array}{ccc|c} T^t & T^{2t} & T^{3t} & \frac{1}{\lambda^4} \\ & \epsilon & \epsilon & \end{array} \right)_q$$

Remark. Note that the term $-\frac{1}{q}\sum_{i=1}^{3} g\left(T^{it}\right)^4$ negates a term in the overall point count (see Proposition 5.1.2).

Proof. By Eq. 5.1.1 we have

$$S_{(0,0,0,0)} = \frac{1}{q-1} \sum_{j=0}^{q-2} \frac{g\left(T^j\right)^4}{g\left(T^{4j}\right)} T^{4j}(4\lambda).$$

If $t \mid j$, i.e. if $j = t, 2t, 3t$, then

$$\frac{g\left(T^j\right)^4}{g\left(T^{4j}\right)} T^{4j}(4\lambda) = -g\left(T^j\right)^4.$$

Thus,

$$\begin{aligned}
S_{(0,0,0,0)} &= -\frac{1}{q-1}\sum_{i=1}^{3} g\left(T^{it}\right)^4 + \frac{1}{q-1}\sum_{j=0, t\nmid j}^{q-2} \frac{g\left(T^j\right)^4}{g\left(T^{4j}\right)} T^{4j}(4\lambda)\\
&= -\frac{1}{q-1}\sum_{i=1}^{3} g\left(T^{it}\right)^4 + \frac{1}{q-1}\sum_{j=0, t\nmid j}^{q-2} \frac{g\left(T^j\right)^4 g\left(T^{-4j}\right)}{T^{4j}(-1)q} T^{4j}(4\lambda)\\
&= -\frac{1}{q-1}\sum_{i=1}^{3} g\left(T^{it}\right)^4 + \frac{1}{q(q-1)}\sum_{j=0, t\nmid j}^{q-2} g\left(T^j\right)^4 g\left(T^{-4j}\right) T^{4j}(4\lambda).
\end{aligned}$$

Note that if $t \mid j$ then

$$g\left(T^j\right)^4 g\left(T^{-4j}\right) T^{4j}(4\lambda) = -g\left(T^j\right)^4.$$

Hence,

$$\begin{aligned}
S_{(0,0,0,0)} &= -\frac{1}{q-1}\sum_{i=1}^{3} g(T^{it})^4 + \frac{1}{q(q-1)}\sum_{i=1}^{3} g(T^{it})^4 + \frac{1}{q(q-1)}\sum_{j=0}^{q-2} g(T^j)^4 g(T^{-4j}) T^{4j}(4\lambda)\\
&= \frac{-q+1}{q(q-1)}\sum_{i=1}^{3} g(T^{it})^4 + \frac{1}{q(q-1)}\sum_{j=0}^{q-2} g(T^j)^4 g(T^{-4j}) T^{4j}(4\lambda)\\
&= -\frac{1}{q}\sum_{i=1}^{3} g(T^{it})^4 + \frac{1}{q(q-1)}\sum_{j=0}^{q-2} g(T^j)^4 g(T^{-4j}) T^{4j}(4\lambda).
\end{aligned}$$

Working from the other direction we see that

$$\begin{aligned}
{}_3F_2\left(\begin{array}{ccc|c} T^t & T^{2t} & T^{3t} & \frac{1}{\lambda^4} \\ & \epsilon & \epsilon & \end{array}\right)_q &= \frac{q}{q-1}\sum_{\chi} \binom{T^t\chi}{\chi}\binom{T^{2t}\chi}{\epsilon\chi}\binom{T^{3t}\chi}{\epsilon\chi}\chi(1/\lambda^4)\\
&= \frac{q}{q-1}\sum_{\chi}\left(\frac{\chi(-1)}{q}\right)^3 J(T^t\chi,\overline{\chi})J(T^{2t}\chi,\overline{\chi})J(T^{3t}\chi,\overline{\chi})\overline{\chi}(\lambda^4)\\
&= \frac{1}{q^2(q-1)}\sum_{\chi}\frac{g(T^t\chi)g(T^{2t}\chi)g(T^{3t}\chi)g(\overline{\chi})^3}{\prod_{i=1}^{3} g(T^{it})}\overline{\chi}(-\lambda^4).
\end{aligned}$$

Use the Hasse-Davenport relation

$$\prod_{i=1}^{3} \frac{g\left(T^{it}\chi\right)}{g\left(T^{it}\right)} = \frac{g(\chi^4)\chi^{-4}(4)}{g(\chi)}$$

to get

$$\begin{aligned} {}_3F_2\left(\begin{matrix} T^t & T^{2t} & T^{3t} \\ & \epsilon & \epsilon \end{matrix} \middle| \frac{1}{\lambda^4}\right)_q &= \frac{1}{q^2(q-1)} \sum_{\chi} \frac{g(\chi^4)\chi^{-4}(4)g(\overline{\chi})^3}{g(\chi)} \overline{\chi}(-\lambda^4) \\ &= \frac{1}{q^2(q-1)} \sum_{\chi} \frac{g(\chi^4)\chi^{-4}(4)g(\overline{\chi})^4}{\chi(-1)q} \overline{\chi}(-\lambda^4) \\ &= \frac{1}{q^3(q-1)} \sum_{j=0}^{q-2} g(T^j)^4 g(T^{-4j}) T^{4j}(4\lambda), \end{aligned}$$

which proves the desired result. □

Proposition 5.1.4. *Let* $w = (0, 1, 1, 2)$*. Then*

$$S_{[w]} = 12qT^t(-1)\left(T^{2t}(1-\lambda^4) - 1\right).$$

Remark. Note that the term $-12qT^t(-1)$ negates a term in the overall point count (see Proposition 5.1.2) Also note that $T^{2t}(1-\lambda^4) = 0$ when $\lambda^4 = 1$, so the final expression for the point count when $\lambda^4 = 1$ is as simple as we might expect.

Corollary 5.1.5. *Let* $w = (0, 1, 1, 2)$ *and* $\lambda^4 = 1$*. Then*

$$S_{[w]} = -12qT^t(-1).$$

Proof of Proposition 5.1.4. By Eq. 5.1.1 we have

$$S_{(0,1,1,2)} = \frac{12}{q-1} \sum_{j=0}^{q-2} \frac{g\left(T^j\right) g\left(T^{t+j}\right)^2 g\left(T^{2t+j}\right)}{g\left(T^{4j}\right)} T^{4j}(4\lambda).$$

If $j = t$ then

$$\begin{aligned} \frac{g\left(T^j\right) g\left(T^{t+j}\right)^2 g\left(T^{2t+j}\right)}{g\left(T^{4j}\right)} T^{4j}(4\lambda) &= \frac{g\left(T^t\right) g\left(T^{2t}\right)^2 g\left(T^{3t}\right)}{g\left(T^{4t}\right)} T^{4t}(4\lambda) \\ &= \frac{T^t(-1)q \cdot T^{2t}(-1)q}{-1} \\ &= -T^t(-1)q^2. \end{aligned}$$

For the terms with $j \neq t$ we use Corollary 2.3.2 to write

$$\begin{aligned}
\sum_{j=0}^{q-2} \frac{g\left(T^j\right) g\left(T^{t+j}\right)^2 g\left(T^{2t+j}\right)}{g\left(T^{4j}\right)} T^{4j}(4\lambda) &= g(T^{2t})T^t(-1)q \\
&\quad \times \sum_{j=0, j\neq t}^{q-2} \frac{g\left(T^j\right) g\left(T^{t+j}\right)^2 g\left(T^{2t+j}\right)}{g\left(T^j\right) g\left(T^{t+j}\right) g\left(T^{2t+j}\right) g\left(T^{3t+j}\right)} T^{4j}(\lambda) \\
&= g(T^{2t})T^t(-1)q \sum_{j=0, j\neq t}^{q-2} \frac{g\left(T^{t+j}\right)}{g\left(T^{3t+j}\right)} T^{4j}(\lambda) \\
&= g(T^{2t})T^t(-1)q \sum_{j=0, j\neq t}^{q-2} \frac{g\left(T^{t+j}\right) g\left(T^{t-j}\right)}{T^{3t+j}(-1)q} T^{4j}(\lambda) \\
&= g(T^{2t}) \sum_{j=0, j\neq t}^{q-2} g\left(T^{t+j}\right) g\left(T^{t-j}\right) T^j(-1) T^{4j}(\lambda).
\end{aligned}$$

Note that if $j = t$ then

$$\begin{aligned}
g(T^{2t}) \cdot g\left(T^{t+j}\right) g\left(T^{t-j}\right) T^j(-1) T^{4j}(\lambda) &= g\left(T^{2t}\right)^2 g\left(T^0\right) T^t(-1) T^{4t}(\lambda) \\
&= -T^t(-1)q.
\end{aligned}$$

Thus,

$$\begin{aligned}
&g(T^{2t}) \sum_{j=0, j\neq t}^{q-2} g(T^{t+j}) g(T^{t-j}) T^j(-1) T^{4j}(\lambda) \\
&\qquad = g(T^{2t}) \sum_{j=0}^{q-2} g(T^{t+j}) g(T^{t-j}) T^j(-1) T^{4j}(\lambda) + T^t(-1)q \\
&\qquad = q(q-1) T^t(-1) T^{2t}(1-\lambda^4) + T^t(-1)q
\end{aligned}$$

by Proposition 2.3.5. Hence,

$$\begin{aligned}
S_{(0,1,1,2)} &= \frac{12}{q-1}\left(q(q-1)T^t(-1)T^{2t}(1-\lambda^4) + T^t(-1)q - T^t(-1)q^2\right) \\
&= \frac{12}{q-1}\left(q(q-1)T^t(-1)T^{2t}(1-\lambda^4) - T^t(-1)q(q-1)\right) \\
&= 12qT^t(-1)\left(T^{2t}(1-\lambda^4) - 1\right).
\end{aligned}$$

□

The remaining piece of the point count formula for the Dwork K3 surface family is the term associated to $S_{[w]}$ with $w = (0,0,2,2)$. It's interesting that this term can be expressed in terms of a ${}_2F_1$ hypergeometric function. Note that the ${}_2F_1$ that appears is of a different form than what has been observed in other point-count formulas [1, 8, 17, 30, 33, 36, 41], where the lower characters are all the trivial character.

Proposition 5.1.6. *Let $w = (0,0,2,2)$. For all $\lambda \not\equiv 0 \pmod{q}$,*

$$S_{[w]} = -6q + 3q^2 \binom{T^{3t}}{T^t} {}_2F_1 \left(\begin{matrix} T^{3t} & T^t \\ & T^{2t} \end{matrix} \middle| \frac{1}{\lambda^4} \right)_q .$$

Corollary 5.1.7. *As we will see in Eq. 5.1.4, this expression simplifies nicely when $\lambda^4 = 1$. In this case,*

$$S_{[w]} = -6q + 3qT^t(-1).$$

Proof of Proposition 5.1.6. The proof starts out similar to the proof of Proposition 5.1.4. By definition we have

$$S_{(0,0,2,2)} = \frac{3}{q-1} \sum_{j=0}^{q-2} \frac{g\left(T^j\right)^2 g\left(T^{2t+j}\right)^2}{g\left(T^{4j}\right)} T^{4j}(4\lambda). \tag{5.1.2}$$

We use Corollary 2.3.2 to write

$$\begin{aligned} S_{(0,0,2,2)} &= \frac{3g(T^{2t})T^t(-1)q}{q-1} \sum_{j=0}^{q-2} \frac{g\left(T^j\right)^2 g\left(T^{2t+j}\right)^2}{g\left(T^j\right) g\left(T^{t+j}\right) g\left(T^{2t+j}\right) g\left(T^{3t+j}\right)} T^{4j}(\lambda) \\ &= \frac{3g(T^{2t})T^t(-1)q}{q-1} \sum_{j=0}^{q-2} \frac{g\left(T^j\right) g\left(T^{2t+j}\right)}{g\left(T^{t+j}\right) g\left(T^{3t+j}\right)} T^{4j}(\lambda). \end{aligned}$$

If $j = t$ then

$$\begin{aligned} \frac{g\left(T^j\right) g\left(T^{2t+j}\right)}{g\left(T^{t+j}\right) g\left(T^{3t+j}\right)} T^{4j}(\lambda) &= \frac{g\left(T^t\right) g\left(T^{3t}\right)}{g\left(T^{2t}\right) g\left(T^{4t}\right)} T^{4t}(\lambda) \\ &= \frac{T^t(-1)q}{-g(T^{2t})}. \end{aligned}$$

Similarly, if $j = 3t$ then

$$\frac{g\left(T^j\right) g\left(T^{2t+j}\right)}{g\left(T^{t+j}\right) g\left(T^{3t+j}\right)} T^{4j}(\lambda) = \frac{T^t(-1)q}{-g(T^{2t})}.$$

For the terms with $j \neq t, 3t$ we have

$$\begin{aligned}\sum_{j=0, j\neq t,3t}^{q-2} \frac{g\left(T^j\right) g\left(T^{2t+j}\right)}{g\left(T^{t+j}\right) g\left(T^{3t+j}\right)} T^{4j}(\lambda) &= \sum_{j=0, j\neq t,3t}^{q-2} \frac{g\left(T^j\right) g\left(T^{2t+j}\right) g\left(T^{t-j}\right) g\left(T^{3t-j}\right)}{T^{t+j}(-1)q \cdot T^{3t+j}(-1)q} T^{4j}(\lambda) \\ &= \frac{1}{q^2} \sum_{j=0, j\neq t,3t}^{q-2} g\left(T^j\right) g\left(T^{2t+j}\right) g\left(T^{t-j}\right) g\left(T^{3t-j}\right) T^{4j}(\lambda).\end{aligned}$$

Note that if $j = t$ then

$$\begin{aligned}\frac{1}{q^2} g\left(T^j\right) g\left(T^{2t+j}\right) g\left(T^{t-j}\right) g\left(T^{3t-j}\right) T^{4j}(\lambda) &= \frac{1}{q^2} g\left(T^t\right) g\left(T^{3t}\right) g\left(T^0\right) g\left(T^{2t}\right) T^{4t}(\lambda) \\ &= -\frac{g(T^{2t})T^t(-1)}{q}.\end{aligned}$$

The same holds for $j = 3t$. Hence,

$$\begin{aligned}S_{(0,0,2,2)} = \frac{3g(T^{2t})T^t(-1)q}{q-1} \Bigg[&\frac{1}{q^2} \sum_{j=0}^{q-2} g(T^j)g(T^{2t+j})g(T^{t-j})g(T^{3t-j})T^{4j}(\lambda) \\ &+ \frac{2g(T^{2t})T^t(-1)}{q} + \frac{2T^t(-1)q}{-g(T^{2t})} \Bigg].\end{aligned}$$

Observe that

$$\begin{aligned}\frac{3g(T^{2t})T^t(-1)q}{q-1} \left[\frac{2g(T^{2t})T^t(-1)}{q} + \frac{2T^t(-1)q}{-g(T^{2t})} \right] &= \frac{6g(T^{2t})}{q-1} \left[g(T^{2t}) - \frac{q^2}{g(T^{2t})} \right] \\ &= \frac{6}{q-1} \left(T^{2t}(-1)q - q^2 \right) \\ &= -6q.\end{aligned}$$

To simplify

$$\frac{3g(T^{2t})T^t(-1)q}{q-1} \left[\frac{1}{q^2} \sum_{j=0}^{q-2} g(T^j)g(T^{2t+j})g(T^{t-j})g(T^{3t-j})T^{4j}(\lambda) \right], \tag{5.1.3}$$

we consider two cases. We first restrict to the case where $\lambda^4 = 1$ and apply Theorem 2.3.4 to our formula to get

$$\begin{aligned} \frac{1}{q-1}\sum_{j=0}^{q-2} g(T^j)g(T^{2t+j})g(T^{t-j})g(T^{3t-j}) &= \frac{g(T^t)g(T^{3t})g(T^{3t})g(T^t)}{g(T^{2t})} \\ &= \frac{q^2}{g(T^{2t})}. \end{aligned}$$

Hence, when $\lambda^4 = 1$ we have

$$\begin{aligned} S_{(0,0,2,2)} &= -6q + \frac{3g(T^{2t})T^t(-1)q}{q^2}\left(\frac{q^2}{g(T^{2t})}\right) \\ &= -6q + 3qT^t(-1). \end{aligned} \tag{5.1.4}$$

For all other $\lambda \neq 0$, we proceed as we did in the proof of Proposition 2.3.5. Recalling that $g(\chi) = \sum_x \chi(x)\theta(x)$ we can write

$$\begin{aligned} &\sum_{j=0}^{q-2} g(T^j)g(T^{2t+j})g(T^{t-j})g(T^{3t-j})T^{4j}(\lambda) \\ &\qquad = \sum_{x,y,z,w} T^{2t}(y)T^t(z)T^{3t}(w)\theta(x+y+z+w)\sum_{j=0}^{q-2} T^j\left(\tfrac{xy\lambda^4}{zw}\right), \end{aligned}$$

where $x, y, z, w \neq 0$. Note that $T^j\left(\frac{xy\lambda^4}{zw}\right) = q-1$ if $\frac{xy\lambda^4}{zw} = 1$ and equals 0 otherwise. Hence, letting $x = \frac{zw}{y\lambda^4}$, the sum simplifies to

$$(q-1)\sum_{y,z,w} T^{2t}(y)T^t(z)T^{3t}(w)\theta\left(\tfrac{zw}{y\lambda^4} + y + z + w\right).$$

Since y and λ are both nonzero, we can perform the change of variables $z \to zy\lambda^4$ and get

$$(q-1)\sum_{y,z,w} T^{2t}(y)T^t(zy\lambda^4)T^{3t}(w)\theta\left(zw + y + zy\lambda^4 + w\right).$$

Since $T^t(\lambda^4) = 1$, this equals

$$(q-1)\sum_{y,z,w} T^{3t}(y)T^t(z)T^{3t}(w)\theta\left(w(z+1) + y(1+z\lambda^4)\right).$$

Note that if $z=-1$, then the above expression equals

$$(q-1)\sum_{y,w} T^{3t}(y)T^{t}(z)T^{3t}(w)\theta\left(y(1-\lambda^4)\right) = (q-1)\sum_{y} T^{3t}(y)T^{t}(z)\theta\left(y(1-\lambda^4)\right)\sum_{w} T^{3t}(w),$$

which equals 0 since $T^{3t}\neq\epsilon$ implies $\sum_w T^{3t}(w)=0$. Similarly, the expression equals 0 when $z=-\lambda^{-4}$.

For $z\neq -1,-\lambda^{-4}$ we can perform the changes of variables $w\to\frac{w}{z+1}$ and $y\to\frac{y}{1+z\lambda^4}$. This portion of the sum then becomes

$$(q-1)\sum_{y,w} T^{3t}(y)T^{3t}(w)\theta(w+y)\sum_{z\neq-1,-\lambda^{-4}} T^{t}(z)T^{-3t}(1+z\lambda^4)T^{-3t}(z+1),$$

where

$$\sum_{z\neq-1,-\lambda^{-4}} T^{t}(z)T^{-3t}(1+z\lambda^4)T^{-3t}(z+1) = \sum_{z\neq-1,-\lambda^{-4}} T^{t}(z)T^{t}(1+z\lambda^4)T^{t}(z+1).$$

Note that if $z=-1$ or $z=-\lambda^{-4}$ then

$$T^{t}(z)T^{t}(1+z\lambda^4)T^{t}(z+1)=0$$

so that we can include these $z-$values in the sum to get

$$(q-1)\sum_{y,w} T^{3t}(y)T^{3t}(w)\theta(w+y)\sum_{z} T^{t}(z)T^{t}(1+z\lambda^4)T^{t}(z+1).$$

This expression reduces further since

$$\sum_{y,w} T^{3t}(y)T^{3t}(w)\theta(w+y) = g(T^{3t})g(T^{3t}).$$

Furthermore, using the change of variables $z\to -z$ we see that

$$\begin{aligned}\sum_{z} T^{t}(z)T^{t}(1+z\lambda^4)T^{t}(z+1) &= \sum_{z} T^{t}(-z)T^{t}(1-z\lambda^4)T^{t}(-z+1)\\ &= T^{t}(-1)\sum_{z} T^{t}(z)T^{t}(1-z\lambda^4)T^{t}(1-z)\\ &= \frac{q}{\epsilon(\lambda^4)}\,{}_2F_1\left(\begin{array}{cc|c} T^{3t} & T^{t} & \lambda^4\\ & T^{2t} & \end{array}\right)_q\end{aligned}$$

where the last expression is obtained by using Eq. 2.2.5 with $A = T^{3t}, B = T^t$, and $C = T^{2t}$.

Thus we have shown that the original Gauss sum expression can be written as

$$q(q-1)g(T^{3t})^2 \frac{q}{\epsilon(\lambda^4)} {}_2F_1 \left(\begin{matrix} T^{3t} & T^t \\ & T^{2t} \end{matrix} \middle| \lambda^4 \right)_q .$$

Putting all of this work together leads to

$$\begin{aligned} S_{(0,0,2,2)} &= -6q + \frac{3g(T^{2t})T^t(-1)q}{q-1} \cdot \frac{1}{q^2} \left[q(q-1)g(T^{3t})^2 \frac{q}{\epsilon(\lambda^4)} {}_2F_1 \left(\begin{matrix} T^{3t} & T^t \\ & T^{2t} \end{matrix} \middle| \lambda^4 \right)_q \right] \\ &= -6q + 3g(T^{2t})g(T^{3t})^2 T^t(-1) {}_2F_1 \left(\begin{matrix} T^{3t} & T^t \\ & T^{2t} \end{matrix} \middle| \lambda^4 \right)_q , \end{aligned}$$

for $\lambda^4 \neq 1$.

We now show that this expression may also be used in the case where $\lambda^4 = 1$. We use Eq. 2.2.5 and properties of Gauss and Jacobi sums to get

$$\begin{aligned} & 3g(T^{2t})g(T^{3t})^2 T^t(-1) {}_2F_1 \left(\begin{matrix} T^{3t} & T^t \\ & T^{2t} \end{matrix} \middle| 1 \right)_q \\ &= 3g(T^{2t})g(T^{3t})^2 T^t(-1) \cdot \frac{T^{3t}(-1)}{q} \sum_y T^t(y)T^t(1-y)T^t(1-y) \\ &= \frac{3g(T^{2t})g(T^{3t})^2}{q} \sum_y T^t(y)T^{2t}(1-y) \\ &= \frac{3g(T^{2t})g(T^{3t})^2}{q} J(T^t, T^{2t}) \\ &= \frac{3g(T^{2t})g(T^{3t})^2}{q} \frac{g(T^t)g(T^{2t})}{g(T^{3t})} \\ &= 3qT^t(-1). \end{aligned}$$

Hence, for all $\lambda \not\equiv 0 \pmod q$,

$$S_{[w]} = -6q + 3g(T^{2t})g(T^{3t})^2 T^t(-1)\, {}_2F_1\left(\begin{matrix} T^{3t} & T^t \\ & T^{2t} \end{matrix} \middle| \lambda^4\right)_q .$$

Note that $g(T^{3t})^2 = g(T^{2t})J(T^{3t}, T^{3t})$, hence

$$\begin{aligned} g(T^{2t})g(T^{3t})^2 T^t(-1) &= g(T^{2t})^2 J(T^{3t}, T^{3t}) T^t(-1) \\ &= qT^t(-1)J(T^{3t}, T^{3t}) \\ &= q^2 \binom{T^{3t}}{T^t}, \end{aligned}$$

where the last equality holds by Eq. 2.2.3. Thus we can write

$$S_{[w]} = -6q + 3q^2 \binom{T^{3t}}{T^t} {}_2F_1\left(\begin{matrix} T^{3t} & T^t \\ & T^{2t} \end{matrix} \middle| \lambda^4\right)_q .$$

We finish the proof by using Theorem 4.2 of [19] to rewrite the hypergeometric term.

$$\begin{aligned} {}_2F_1\left(\begin{matrix} T^{3t} & T^t \\ & T^{2t} \end{matrix} \middle| \lambda^4\right)_q &= T^{2t}(-1)T^{3t}(\lambda^4)\, {}_2F_1\left(\begin{matrix} T^{3t} & T^t \\ & T^{2t} \end{matrix} \middle| \frac{1}{\lambda^4}\right)_q \\ &= {}_2F_1\left(\begin{matrix} T^{3t} & T^t \\ & T^{2t} \end{matrix} \middle| \frac{1}{\lambda^4}\right)_q \end{aligned}$$

□

5.1.1 Proof of Theorem 1.0.1

Proof of Theorem 1.0.1. In Section 5.1 we found that

$$\#X_\lambda^4(\mathbb{F}_q) = N_q(0) + S_{(0,0,0,0)} + S_{(0,0,2,2)} + S_{(0,1,1,2)}.$$

Combining the results of Propositions 5.1.2, 5.1.3, and 5.1.6 gives us

$$\begin{aligned}
\#X_\lambda^4(\mathbb{F}_q) = q^2 + 7q + 1 + \frac{1}{q}\sum_{i=1}^{3} g(T^{it})^4 + 12qT^t(-1) - \frac{1}{q}\sum_{i=1}^{3} g\left(T^{it}\right)^4 \\
+ q^2 {}_3F_2\left(\begin{matrix} T^t & T^{2t} & T^{3t} \\ & \epsilon & \epsilon \end{matrix} \,\middle|\, \frac{1}{\lambda^4}\right)_q \\
- 6q + 3q^2\binom{T^{3t}}{T^t} {}_2F_1\left(\begin{matrix} T^{3t} & T^t \\ & T^{2t} \end{matrix} \,\middle|\, \frac{1}{\lambda^4}\right)_q \\
+ 12qT^t(-1)\left(T^{2t}(1-\lambda^4) - 1\right)
\end{aligned}$$

$$\begin{aligned}
= \frac{q^3-1}{q-1} + 12qT^t(-1)T^{2t}(1-\lambda^4) \\
+ q^2 {}_3F_2\left(\begin{matrix} T^t & T^{2t} & T^{3t} \\ & \epsilon & \epsilon \end{matrix} \,\middle|\, \frac{1}{\lambda^4}\right)_q + 3q^2\binom{T^{3t}}{T^t} {}_2F_1\left(\begin{matrix} T^{3t} & T^t \\ & T^{2t} \end{matrix} \,\middle|\, \frac{1}{\lambda^4}\right)_q .
\end{aligned}$$

In the case where $\lambda^4 = 1$, this, combined with Corollaries 5.1.5 and 5.1.7, gives us

$$\#X_\lambda^4(\mathbb{F}_q) = \frac{q^3-1}{q-1} + 3qT^t(-1) + q^2 {}_3F_2\left(\begin{matrix} T^t & T^{2t} & T^{3t} \\ & \epsilon & \epsilon \end{matrix} \,\middle|\, 1\right)_q .$$

□

5.1.2 Proof of Theorems 1.0.2 and 1.0.3

In this section we prove our two $p-$adic hypergeometric point count formulas.

Proof of Theorem 1.0.2. Let $N_p^A(\lambda)$ denote the number of points on the Dwork K3 surface in $\mathbb{A}^4(\mathbb{F}_p)$. Then

$$\#X_\lambda^4(\mathbb{F}_p) = \frac{N_p^A(\lambda) - 1}{p-1}. \tag{5.1.5}$$

Letting $f(\overline{x}) = x_1^4 + x_2^4 + x_3^4 + x_4^4 - 4\lambda x_1x_2x_3x_4$ we can write

$$\begin{aligned} pN_p^A(\lambda) &= p^4 + \sum_{z\in\mathbb{F}_p^*}\sum_{x_1,x_2,x_3,x_4}\theta(zf(\overline{x})) \\ &= p^4 + \sum_{z\in\mathbb{F}_p^*}\sum_{x_i\neq 0}\theta(zf(\overline{x})) + \sum_{z\in\mathbb{F}_p^*}\sum_{\substack{x_i \\ some\ x_i=0}}\theta(zf(\overline{x})). \end{aligned}$$

We will call the first summand A and the second B. We first work to rewrite B. We can have 1, 2, 3, or 4 of the x_i's equal to zero, and there are 4, 6, 4, and 1 way, respectively, for this to occur. We will call these sums B_1, B_2, B_3, and B_4, respectively. $B_4 = p-1$ because θ is an additive character. We simplify the others using basic facts about characters and Gauss sums

$$\begin{aligned} B_1 &= 4\sum_{z\in\mathbb{F}_p^*}\sum_{x_i\neq 0}\theta(zx_1^4)\theta(zx_2^4)\theta(zx_3^4) \\ &= \frac{4}{(p-1)^3}\sum_{x_i,z\in\mathbb{F}_p^*}\sum_{a,b,c=0}^{p-2} g(T^{-a})T^{4a}(x_1)g(T^{-b})T^{4b}(x_2)g(T^{-c})T^{4c}(x_3)T^{a+b+c}(z) \\ &= \frac{4}{(p-1)^3}\sum_{a,b,c=0}^{p-2} g(T^{-a})g(T^{-b})g(T^{-c})\sum_{x_1}T^{4a}(x_1)\sum_{x_2}T^{4b}(x_2)\sum_{x_3}T^{4c}(x_3)\sum_{z}T^{a+b+c}(z). \end{aligned}$$

This sum is non-zero only when the following congruences hold:

$$4a, 4b, 4c \equiv 0 \pmod{p-1}, \text{ and } a+b+c \equiv 0 \pmod{p-1}.$$

Since $p \not\equiv 1 \pmod 4$, these congruences simultaneously hold only when 0 or 2 of a, b, c are $\frac{p-1}{2}$ and the remaining terms are 0. In this case, each character sum is $p-1$. Note that there are 3 ways to have two of a, b, c equal to zero. Thus,

$$\begin{aligned} B_1 &= 4(p-1)\left(g(T^0)^3 + 3g\left(T^{(p-1)/2}\right)^2 g(T^0)\right) \\ &= 4(p-1)\left(-1 + 3(-1p)(-1)\right) \\ &= 4(p-1)(3p-1). \end{aligned}$$

Similarly,

$$\begin{aligned} B_2 &= 6 \sum_{z \in \mathbb{F}_p^*} \sum_{x_i \neq 0} \theta(zx_1^4)\theta(zx_2^4) \\ &= \frac{6}{(p-1)^2} \sum_{a,b=0}^{p-2} g(T^{-a})g(T^{-b}) \sum_{x_1} T^{4a}(x_1) \sum_{x_2} T^{4b}(x_2) \sum_{z} T^{a+b}(z). \end{aligned}$$

This sum is non-zero only when the following congruences hold:

$$4a, 4b \equiv 0 \pmod{p-1}, \text{ and } a+b \equiv 0 \pmod{p-1}.$$

Since $p \not\equiv 1 \pmod 4$, these congruences simultaneously hold only when 0 or 2 of a, b are $\frac{p-1}{2}$. Thus,

$$\begin{aligned} B_2 &= 6(p-1)\left(g(T^0)^2 + g(T^{(p-1)/2})^2\right) \\ &= -6(p-1)(p-1). \end{aligned}$$

Finally,

$$\begin{aligned} B_3 &= 4 \sum_{z \in \mathbb{F}_p^*} \sum_{x_1 \neq 0} \theta(zx_1^4) \\ &= \frac{4}{p-1} \sum_{a=0}^{p-2} g(T^{-a}) \sum_{x_1} T^{4a}(x_1) \sum_{z} T^{a}(z). \end{aligned}$$

This sum is non-zero only when $a = 0$. Thus,

$$B_2 = 4(p-1)g(T^0) = -4(p-1).$$

Putting this all together gives

$$\begin{aligned} B &= B_1 + B_2 + B_3 + B_4 \\ &= 4(p-1)(3p-1) - 6(p-1)(p-1) - 4(p-1) + (p-1) \\ &= (p-1)(6p-1). \end{aligned}$$

Rewriting summand A requires more work and we will end up with $p-$adic hypergeometric functions.

$$
\begin{aligned}
A &= \sum_{z\in\mathbb{F}_p^*}\sum_{x_i\neq 0}\theta(zf(\overline{x}))\\
&= \sum_{z\in\mathbb{F}_p^*}\sum_{x_i\neq 0}\theta(zx_1^4)\theta(zx_2^4)\theta(zx_3^4)\theta(zx_4^4)\theta(-4\lambda x_1x_2x_3x_4)\\
&= \frac{1}{(p-1)^5}\sum_{i,j,k,l,m=0}^{p-2} g(T^{-i})g(T^{-j})g(T^{-k})g(T^{-l})g(T^{-m})T^m(-4\lambda)\sum T^{4i+m}(x_1)\\
&\qquad\times\sum T^{4j+m}(x_2)\sum T^{4k+m}(x_3)\sum T^{4l+m}(x_4)\sum T^{i+j+k+l+m}(z).
\end{aligned}
$$

We consider congruences that must hold for i,j,k,l,m as we did for summand B. This sum is non-zero only when the following congruences hold:

$$4i+m, 4j+m, 4k+m, 4l+m \equiv 0 \pmod{p-1},\ \text{and}\ i+j+k+l+m\equiv 0 \pmod{p-1}.$$

There are two cases. In the first case, which we will denote by A_1, i,j,k,l are equal in pairs. Here we need two equal to each other and the remaining two equal to that value plus $\frac{p-1}{2}$. For example, $i=j$ and $k=l=i+\frac{p-1}{2}$. In this case, $m\equiv -4i \pmod{p-1}$. There are 3 ways for this to occur. In the second case, $i=j=k=l$ and $m\equiv -4j \pmod{p-1}$. We will denote this case by A_2.

We now work to rewrite A_1. We start with

$$A_1 = 3\sum_{i=0}^{p-2} g(T^{-i})^2 g(T^{-(i+(p-1)/2})^2 g(T^{4i})T^{-4i}(-4\lambda).$$

We use the Hasse-Davenport relation with $\chi^{(p-1)/2}$ and $\psi=T^{-i}$ to get

$$
\begin{aligned}
g(T^{-(i+(p-1)/2})^2 &= \left(\frac{-g(T^{-2i})T^{2i}(2)g(T^0)g(T^{((p-1)/2})}{g(T^{-i})}\right)^2\\
&= \frac{-pg(T^{-2i})^2T^{2i}(4)}{g(T^{-i})^2}.
\end{aligned}
$$

Thus,

$$A_1 = -3p \sum_{i=0}^{p-2} g(T^{-2i})^2 g(T^{4i}) T^{2i}(4) T^{-4i}(-4\lambda)$$
$$= 6p - 3p \sum_{i \neq 0, (p-1)/2} g(T^{-2i})^2 g(T^{4i}) T^{2i}(4) T^{-4i}(-4\lambda)$$

We multiply the summand by $g(T^{2i})/g(T^{2i})$ to get

$$A_1 = 6p - 3p^2 \sum_{i \neq 0, (p-1)/2} \frac{g(T^{-2i}) g(T^{4i}) T^{2i}(4) T^{-4i}(-4\lambda)}{g(T^{2i})}.$$

As we have done in previous proofs, we let $T = \omega$, the Teichmüller character. Thus,

$$A_1 = 6p - 3p^2 \sum_{i \neq 0, (p-1)/2} \frac{g(\overline{\omega}^{2i}) g(\overline{\omega}^{-4i}) \overline{\omega}^{-2i}(4) \overline{\omega}^{4i}(4\lambda)}{g(\overline{\omega}^{-2i})}.$$

We use the Gross-Koblitz formula to write this in terms of the $p-$adic Gamma function.

$$A_1 = 6p + 3p^2 \sum_{i \neq 0, (p-1)/2} \frac{(-p)^a \Gamma_p\left(\left\langle \frac{2i}{p-1} \right\rangle\right) \Gamma_p\left(\left\langle \frac{-4i}{p-1} \right\rangle\right) \overline{\omega}^{4i}(2\lambda)}{\Gamma_p\left(\left\langle \frac{-2i}{p-1} \right\rangle\right)},$$

where

$$a = \left\langle \frac{2i}{p-1} \right\rangle + \left\langle \frac{-4i}{p-1} \right\rangle - \left\langle \frac{-2i}{p-1} \right\rangle$$
$$= \frac{2i}{p-1} - \left\lfloor \frac{2i}{p-1} \right\rfloor + \frac{-4i}{p-1} - \left\lfloor \frac{-4i}{p-1} \right\rfloor - \frac{-2i}{p-1} + \left\lfloor \frac{-2i}{p-1} \right\rfloor$$
$$= \left\lfloor \frac{-2i}{p-1} \right\rfloor - \left\lfloor \frac{2i}{p-1} \right\rfloor - \left\lfloor \frac{-4i}{p-1} \right\rfloor.$$

On page 232 of [38] we see that

$$\left\lfloor \frac{-2i}{p-1} \right\rfloor - \left\lfloor \frac{4i}{p-1} \right\rfloor = -\left\lfloor \frac{3}{4} - \frac{i}{p-1} \right\rfloor - \left\lfloor \frac{1}{4} - \frac{i}{p-1} \right\rfloor.$$

Furthermore,

$$\left\lfloor \frac{2i}{p-1} \right\rfloor = \begin{cases} 0 & \text{if } i < \frac{p-1}{2}, \\ 1 & \text{if } i \geq \frac{p-1}{2}, \end{cases}$$

so that $\left\lfloor \frac{2i}{p-1} \right\rfloor = \left\lfloor \frac{i}{p-1} \right\rfloor + \left\lfloor \frac{1}{2} + \frac{i}{p-1} \right\rfloor$. Thus,

$$a = -\left\lfloor \frac{3}{4} - \frac{i}{p-1} \right\rfloor - \left\lfloor \frac{1}{4} - \frac{i}{p-1} \right\rfloor - \left\lfloor \frac{i}{p-1} \right\rfloor - \left\lfloor \frac{1}{2} + \frac{i}{p-1} \right\rfloor .$$

We use Lemma 4.1 of [38] to rewrite the $p-$adic Gamma functions that appear in the summand. This yields the following expression for A_1.

$$\begin{aligned} A_1 &= 6p + 3p^2 \sum_{i \neq 0, (p-1)/2} (-p)^a \cdot \frac{\Gamma_p\left(\left\langle \frac{i}{p-1} \right\rangle\right) \Gamma_p\left(\left\langle \frac{1}{2} + \frac{i}{p-1} \right\rangle\right) \prod_{h=0}^{3} \Gamma_p\left(\left\langle \frac{1+h}{4} - \frac{i}{p-1} \right\rangle\right) \overline{\omega}^i(\lambda^4)}{\prod_{h=1}^{3} \Gamma_p\left(\frac{h}{4}\right) \cdot \Gamma_p\left(\left\langle \frac{1}{2} - \frac{i}{p-1} \right\rangle\right) \Gamma_p\left(\left\langle 1 - \frac{i}{p-1} \right\rangle\right)} \\ &= 6p + 3p^2 \sum_{i \neq 0, (p-1)/2} (-p)^a \cdot \frac{\Gamma_p\left(\left\langle \frac{i}{p-1} \right\rangle\right) \Gamma_p\left(\left\langle \frac{1}{2} + \frac{i}{p-1} \right\rangle\right)}{\Gamma_p\left(\frac{1}{4}\right) \Gamma_p\left(\frac{2}{4}\right) \Gamma_p\left(\frac{3}{4}\right)} \\ &\qquad \times \Gamma_p\left(\left\langle \frac{1}{4} - \frac{i}{p-1} \right\rangle\right) \Gamma_p\left(\left\langle \frac{3}{4} - \frac{i}{p-1} \right\rangle\right) \overline{\omega}^i(\lambda^4). \end{aligned}$$

If $i = 0$ or $(p-1)/2$, then

$$(-p)^a \cdot \frac{\Gamma_p\left(\left\langle \frac{i}{p-1} \right\rangle\right) \Gamma_p\left(\left\langle \frac{1}{2} + \frac{i}{p-1} \right\rangle\right) \Gamma_p\left(\left\langle \frac{1}{4} - \frac{i}{p-1} \right\rangle\right) \Gamma_p\left(\left\langle \frac{3}{4} - \frac{i}{p-1} \right\rangle\right) \overline{\omega}^i(\lambda^4)}{\Gamma_p\left(\frac{1}{4}\right) \Gamma_p\left(\frac{2}{4}\right) \Gamma_p\left(\frac{3}{4}\right)} = (-p)^0 = 1.$$

Thus,

$$\begin{aligned} A_1 = 6p - 6p^2 + 3p^2 \sum_{i=0}^{p-2} (-p)^a \cdot \frac{\Gamma_p\left(\left\langle \frac{1}{4} - \frac{i}{p-1} \right\rangle\right) \Gamma_p\left(\left\langle \frac{3}{4} - \frac{i}{p-1} \right\rangle\right)}{\Gamma_p\left(\left\langle \frac{-1}{4} \right\rangle\right) \Gamma_p\left(\left\langle \frac{-3}{4} \right\rangle\right)} \\ \times \frac{\Gamma_p\left(\left\langle \frac{i}{p-1} \right\rangle\right) \Gamma_p\left(\left\langle \frac{1}{2} + \frac{i}{p-1} \right\rangle\right)}{\Gamma_p(0) \Gamma_p\left(\left\langle \frac{1}{2} \right\rangle\right)} \overline{\omega}^i(\lambda^4). \end{aligned}$$

We recognize this sum as the following $p-$adic hypergeometric function expression

$$A_1 = 6p - 6p^2 - 3(p-1)p^2 {}_2G_2 \left[\begin{array}{cc|} 3/4 & 1/4 \\ 0 & 1/2 \end{array} \lambda^4 \right]_p .$$

We now work to rewrite A_2. In this case we have

$$A_2 = \sum_j g(T^{-j})^4 g(T^{4j}) T^{-4j}(-4\lambda).$$

As we have done in previous proofs, we let $T = \omega$, the Teichmüller character. Then

$$\begin{aligned} A_2 &= \sum_{j=0}^{p-2} g(\overline{\omega}^j)^4 g(\overline{\omega}^{-4j})\overline{\omega}^{4j}(4\lambda) \\ &= -1 + \sum_{j=1}^{p-2} g(\overline{\omega}^j)^4 g(\overline{\omega}^{-4j})\overline{\omega}^{4j}(4\lambda) \\ &= -1 + p\sum_{j=1}^{p-2} \frac{g(\overline{\omega}^j)^3 g(\overline{\omega}^{-4j})\overline{\omega}^j(-1)\overline{\omega}^{4j}(4\lambda)}{g(\overline{\omega}^{-j})}. \end{aligned}$$

We would like to rewrite the summand in terms of the p-adic Gamma function. To do this, we use the Gross-Koblitz formula.

$$A_2 = -1 - p\sum_{j=1}^{p-2}(-p)^k \frac{\Gamma_p\left(\langle\frac{j}{p-1}\rangle\right)^3 \Gamma_p\left(\langle\frac{-4j}{p-1}\rangle\right)}{\Gamma_p\left(\langle\frac{-j}{p-1}\rangle\right)}\overline{\omega}^j(-1)\overline{\omega}^{4j}(4\lambda),$$

where

$$\begin{aligned} k &= 3\left\langle\frac{j}{p-1}\right\rangle + \left\langle\frac{-4j}{p-1}\right\rangle - \left\langle\frac{-j}{p-1}\right\rangle \\ &= \frac{3j}{p-1} - 3\left\lfloor\frac{j}{p-1}\right\rfloor + \frac{-4j}{p-1} - \left\lfloor\frac{-4j}{p-1}\right\rfloor - \frac{-j}{p-1} + \left\lfloor\frac{-j}{p-1}\right\rfloor \\ &= -\left\lfloor\frac{-4j}{p-1}\right\rfloor - 1 \\ &= \begin{cases} 0 & \text{if } 0 < j \le \frac{p-1}{4}, \\ 1 & \text{if } \frac{p-1}{4} < j \le \frac{2(p-1)}{4}, \\ 2 & \text{if } \frac{2(p-1)}{4} < j \le \frac{3(p-1)}{4}, \\ 3 & \text{if } \frac{3(p-1)}{4} < j \le p-2. \end{cases} \end{aligned}$$

We use Lemma 4.1 in [38] to rewrite $\Gamma_p\left(\left\langle\frac{-j}{p-1}\right\rangle\right)$ and $\Gamma_p\left(\left\langle\frac{-4j}{p-1}\right\rangle\right)$. This leads to

$$\begin{aligned} A_2 &= -1 - p\sum_{j=1}^{p-2}(-p)^k \frac{\Gamma_p\left(\langle\frac{j}{p-1}\rangle\right)^3 \prod_{h=0}^{3}\Gamma_p\left(\left\langle\frac{1+h}{4} - \frac{j}{p-1}\right\rangle\right)}{\Gamma_p\left(\langle 1 - \frac{j}{p-1}\rangle\right)\prod_{h=1}^{3}\Gamma_p\left(\frac{h}{4}\right)}\overline{\omega}^j(-\lambda^4) \\ &= -1 - p\sum_{j=1}^{p-2}(-p)^k \frac{\Gamma_p\left(\left\langle\frac{j}{p-1}\right\rangle\right)^3}{\Gamma_p(0)^3} \cdot \prod_{i=1}^{3}\frac{\Gamma_p\left(\left\langle\frac{i}{4} - \frac{j}{p-1}\right\rangle\right)}{\Gamma_p\left(\frac{i}{4}\right)}\overline{\omega}^j(-\lambda^4). \end{aligned}$$

This expression can be written in terms of a p-adic hypergeometric function. By Definition 2.4.4, we have

$$ {}_3G_3\left[\begin{array}{ccc|c} 1/4 & 2/4 & 3/4 & \lambda^4 \\ 0 & 0 & 0 & \end{array}\right]_p := \frac{-1}{p-1}\sum_{j=0}^{p-2}\overline{\omega}^j(-\lambda^4)\prod_{i=1}^{3}\frac{\Gamma_p\left(\left\langle \frac{i}{4}-\frac{j}{p-1}\right\rangle\right)}{\Gamma_p\left(\frac{i}{4}\right)}\cdot\frac{\Gamma_p\left(\left\langle \frac{j}{p-1}\right\rangle\right)^3}{\Gamma_p(0)^3} $$

$$ \cdot(-p)^{-\lfloor\frac{1}{4}-\frac{j}{p-1}\rfloor-\lfloor\frac{2}{4}-\frac{j}{p-1}\rfloor-\lfloor\frac{3}{4}-\frac{j}{p-1}\rfloor-3\lfloor\frac{j}{p-1}\rfloor}. $$

Note that when $j=0$, the summand is simply $-\frac{1}{p-1}$. Our main task now is to determine what the power of $-p$ is for other values of j. First note that since $0\le j\le p-2$, we have $\lfloor\frac{j}{p-1}\rfloor=0$. For $i=1,2,3$ we have

$$ \left\lfloor\frac{i}{4}-\frac{j}{p-1}\right\rfloor=\begin{cases}0 & \text{if } \frac{j}{p-1}\le\frac{i}{4},\\ -1 & \text{if } \frac{j}{p-1}>\frac{i}{4}.\end{cases} $$

Thus, the exponent of $-p$ is

$$ -\lfloor\tfrac{1}{4}-\tfrac{j}{p-1}\rfloor-\lfloor\tfrac{2}{4}-\tfrac{j}{p-1}\rfloor-\lfloor\tfrac{3}{4}-\tfrac{j}{p-1}\rfloor-3\lfloor\tfrac{j}{p-1}\rfloor=\begin{cases}0 & \text{if } 0<j\le\frac{p-1}{4},\\ 1 & \text{if } \frac{p-1}{4}<j\le\frac{2(p-1)}{4},\\ 2 & \text{if } \frac{2(p-1)}{4}<j\le\frac{3(p-1)}{4},\\ 3 & \text{if } \frac{3(p-1)}{4}<j\le p-2.\end{cases} $$

Note how this coincides with the powers of $-p$ in the summand A_2. Thus,

$$ {}_3G_3\left[\begin{array}{ccc|c} 1/4 & 2/4 & 3/4 & \lambda^4 \\ 0 & 0 & 0 & \end{array}\right]_p = $$

$$ \frac{-1}{p-1}\left[1+\sum_{j=1}^{p-2}(-p)^k\frac{\Gamma_p\left(\left\langle\frac{j}{p-1}\right\rangle\right)^3}{\Gamma_p(0)^3}\prod_{i=1}^{3}\frac{\Gamma_p\left(\left\langle\frac{i}{4}-\frac{j}{p-1}\right\rangle\right)}{\Gamma_p\left(\frac{i}{4}\right)}\overline{\omega}^j(-\lambda^4)\right]. $$

This allows us to conclude that

$$ A_2=p(p-1){}_3G_3\left[\begin{array}{ccc|c} 1/4 & 2/4 & 3/4 & \lambda^4 \\ 0 & 0 & 0 & \end{array}\right]_p+p-1. $$

We finish the proof by combining these results to get an expression for $\#X_\lambda^4(\mathbb{F}_p)$. First note that

$$\begin{aligned}
A+B &= B + A_1 + A_2 \\
&= (p-1)(6p-1) + 6p - 6p^2 - 3(p-1)p^2{}_2G_2\left[\begin{array}{cc|c} 3/4 & 1/4 & \lambda^4 \\ 0 & 1/2 & \end{array}\right]_p \\
&\qquad + p(p-1){}_3G_3\left[\begin{array}{ccc|c} 1/4 & 2/4 & 3/4 & \lambda^4 \\ 0 & 0 & 0 & \end{array}\right]_p + p - 1 \\
&= -3(p-1)p^2{}_2G_2\left[\begin{array}{cc|c} 3/4 & 1/4 & \lambda^4 \\ 0 & 1/2 & \end{array}\right]_p + p(p-1){}_3G_3\left[\begin{array}{ccc|c} 1/4 & 2/4 & 3/4 & \lambda^4 \\ 0 & 0 & 0 & \end{array}\right]_p.
\end{aligned}$$

We now substitute this into Eq. 5.1.5.

$$\begin{aligned}
\#X_\lambda^4(\mathbb{F}_p) &= \frac{N_p^A(\lambda) - 1}{p-1} \\
&= \frac{\frac{1}{p}(p^4 + A + B) - 1}{p-1} \\
&= \frac{p^3-1}{p-1} - 3p\,{}_2G_2\left[\begin{array}{cc|c} 3/4 & 1/4 & \lambda^4 \\ 0 & 1/2 & \end{array}\right]_p + {}_3G_3\left[\begin{array}{ccc|c} 1/4 & 2/4 & 3/4 & \lambda^4 \\ 0 & 0 & 0 & \end{array}\right]_p
\end{aligned}$$

□

Proof of Theorem 1.0.3. We start with the point count result over $\mathbb{F}_p$ of Theorem 1.0.1

$$\begin{aligned}
\#X_\lambda^4(\mathbb{F}_p) &= p^2 + p + 12pT^t(-1)T^{2t}(1-\lambda^4) + 1 \\
&\qquad + p^2{}_3F_2\left(\begin{array}{ccc|c} T^t & T^{2t} & T^{3t} & \frac{1}{\lambda^4} \\ & \epsilon & \epsilon & \end{array}\right)_p + 3p^2\binom{T^{3t}}{T^t}{}_2F_1\left(\begin{array}{cc|c} T^{3t} & T^t & \frac{1}{\lambda^4} \\ & T^{2t} & \end{array}\right)_p
\end{aligned}$$

We use transformation properties from [38, 37] to rewrite the two finite field hypergeometric expressions in terms of McCarthy's $p-$adic hypergeometric function. In [37], McCarthy defines a new finite field hypergeometric function, normalized to satisfy transformation properties based on summation properties of Gauss sums. He also gives the relationship between McCarthy's hypergeometric function and Greene's hypergeometric function in [37, Prop. 2.5].

$$ {}_{n+1}F_n\left(\begin{array}{cccc} A_0, & A_1, & \dots, & A_n \\ & B_1, & \dots, & B_n \end{array}\middle| x\right)^M_p = \prod_{i=1}^{n}\binom{A_i}{B_i}^{-1} {}_{n+1}F_n\left(\begin{array}{cccc} A_0, & A_1, & \dots, & A_n \\ & B_1, & \dots, & B_n \end{array}\middle| x\right)_p . \tag{5.1.6} $$

Furthermore, in [38, Lemma 3.3] McCarthy gives the following relationship between his finite field hypergeometric function and his p-adic hypergeometric function. For a fixed odd prime p, let A_i, B_k be given by $\overline{\omega}^{a_i(p-1)}$ and $\overline{\omega}^{b_k(p-1)}$ respectively. Then

$$ {}_{n+1}F_n\left(\begin{array}{cccc} A_0, & A_1, & \dots, & A_n \\ & B_1, & \dots, & B_n \end{array}\middle| x\right)^M_p = {}_{n+1}G_{n+1}\left[\begin{array}{cccc} a_0 & a_1 & \dots & a_n \\ 0 & b_1 & \dots & b_n \end{array}\middle| x^{-1}\right]_p \tag{5.1.7} $$

We use these two properties to rewrite the hypergeometric function expressions in Theorem 1.0.1. We start with the ${}_3F_2$ term. Equation 2.12 in [19] states that for a character $A \in \widehat{\mathbb{F}_q^{\times}}$ we have

$$ \binom{A}{\epsilon} = -\frac{1}{p}. $$

We use this and Equation 5.1.6 above to write

$$ \begin{aligned} p^2 {}_3F_2\left(\begin{array}{ccc} T^t & T^{2t} & T^{3t} \\ & \epsilon & \epsilon \end{array}\middle| \frac{1}{\lambda^4}\right)_p &= p^2\binom{T^{2t}}{\epsilon}\binom{T^{3t}}{\epsilon} {}_3F_2\left(\begin{array}{ccc} T^t & T^{2t} & T^{3t} \\ & \epsilon & \epsilon \end{array}\middle| \frac{1}{\lambda^4}\right)^M_p \\ &= p^2\left(-\frac{1}{p}\right)\left(-\frac{1}{p}\right) {}_3F_2\left(\begin{array}{ccc} T^t & T^{2t} & T^{3t} \\ & \epsilon & \epsilon \end{array}\middle| \frac{1}{\lambda^4}\right)^M_p \\ &= {}_3F_2\left(\begin{array}{ccc} T^t & T^{2t} & T^{3t} \\ & \epsilon & \epsilon \end{array}\middle| \frac{1}{\lambda^4}\right)^M_p . \end{aligned} $$

Equation 5.1.7 then tells us that this is equal to

$$ {}_3G_3\left[\begin{array}{ccc} 1/4 & 2/4 & 3/4 \\ 0 & 0 & 0 \end{array}\middle| \lambda^4\right]_p . $$

We now work to rewrite the ${}_3F_2$ term. First note that

$$\binom{T^{3t}}{T^t}\binom{T^t}{T^{2t}} = \frac{g(T^{3t})g(T^{3t})T^t(-1)}{g(T^{2t}p} \cdot \frac{g(T^t)g(T^{2t})T^{2t}(-1)}{g(T^{3t}p} = \frac{g(T^{3t})g(T^t)T^t(-1)}{p^2} = \frac{1}{p}.$$

We use this and Equations 5.1.6 and 5.1.7 to write

$$3p^2\binom{T^{3t}}{T^t}{}_2F_1\left(\begin{matrix} T^{3t} & T^t \\ & T^{2t}\end{matrix}\,\middle|\,\frac{1}{\lambda^4}\right)_p = 3p^2\binom{T^{3t}}{T^t}\binom{T^t}{T^{2t}}{}_2F_1\left(\begin{matrix} T^{3t} & T^t \\ & T^{2t}\end{matrix}\,\middle|\,\frac{1}{\lambda^4}\right)_p^M$$
$$= 3p_2F_1\left(\begin{matrix} T^{3t} & T^t \\ & T^{2t}\end{matrix}\,\middle|\,\frac{1}{\lambda^4}\right)_p^M$$
$$= 3p_2G_2\left[\begin{matrix} 3/4 & 1/4 \\ 0 & 2/4\end{matrix}\,\middle|\,\lambda^4\right]_p.$$

We have rewritten each finite field hypergeometric expression as a p-adic hypergeometric expression, and so we have proved the desired result.

□

5.1.3 Dwork K3 Surface Period Integrals

In this section we give a formula for certain period integrals associated to Dwork K3 surfaces. The periods we are interested in are the fundamental periods described in Chapter 1, which are solutions to the following Picard-Fuchs equation.

Proposition 5.1.8. *[43, Sec. 3.2] The Picard-Fuchs equation for the Dwork K3 surface is*

$$\left(\vartheta^3 - z(\vartheta+1/4)(\vartheta+2/4)(\vartheta+3/4)\right)\pi = 0, \tag{5.1.8}$$

where $\vartheta = z\frac{d}{dz}$ *and* $z = \lambda^{-4}$.

The proposition below gives us a classical hypergeometric series expression for the period. The result is not original, but we prove it to review the procedure.

Proposition 5.1.9. *The solution to Eq. 5.1.8 that is bounded near $z = 0$ is given by*

$$\pi = {}_3F_2\left(\begin{array}{ccc|c} 1/4 & 2/4 & 3/4 & \frac{1}{\lambda^4} \\ & 1 & 1 & \end{array}\right). \tag{5.1.9}$$

Proof. Define $D := \vartheta^3 - z(\vartheta + 1/4)(\vartheta + 2/4)(\vartheta + 3/4)$. Let $f(z) = \sum_m a_m z^m$ be a solution to $Df = 0$, normalized so that $f(0) = 1$, and let $f' := \frac{df}{dz}$. We see that

$$\begin{aligned}(\vartheta^3)\, f &= \left(z\frac{d}{dz}\right)^3 f \\ &= \left(z\frac{d}{dz}\right)^2 (zf') \\ &= \left(z\frac{d}{dz}\right)(zf' + z^2 f'') \\ &= z^3 f''' + 3z^2 f'' + zf'.\end{aligned}$$

Furthermore,

$$\begin{aligned}(\vartheta + 1/4)(\vartheta + 2/4)(\vartheta + 3/4)f &= (\vartheta + 1/4)(\vartheta + 2/4)(zf' + \tfrac{3}{4}f) \\ &= (\vartheta + 1/4)(z^2 f'' + \tfrac{9}{4}zf' + \tfrac{3}{8}f) \\ &= z^3 f''' + \tfrac{9}{2}z^2 f'' + \tfrac{51}{16}zf' + \tfrac{3}{32}f.\end{aligned}$$

Hence,

$$\begin{aligned}Df &= z^3 f''' + 3z^2 f'' + zf' - z(z^3 f''' + \tfrac{9}{2}z^2 f'' + \tfrac{51}{16}zf' + \tfrac{3}{32}f) \\ &= z^3(1 - z)f''' + z^2(3 - \tfrac{9}{2}z)f'' + z(1 - \tfrac{51}{16}z)f' - \tfrac{3}{32}zf.\end{aligned}$$

Taking derivatives of f gives us

$$\begin{aligned} f' &= \sum_m (m+1)a_{m+1}z^m, \\ f'' &= \sum_m (m+2)(m+1)a_{m+2}z^m, \\ f''' &= \sum_m (m+3)(m+2)(m+1)a_{m+3}z^m. \end{aligned}$$

We substitute these into the equation $Df = 0$ to get

$$\sum_m \big(z^3(1-z)(m+3)(m+2)(m+1)a_{m+3} + z^2(3-\tfrac{9}{2}z)(m+2)(m+1)a_{m+2} \\ +z(1-\tfrac{51}{16}z)(m+1)a_{m+1} - \tfrac{3}{32}za_m\big)\, z^m = 0.$$

From the expression on the left side of the above equation we identify the coefficients of z^m to rewrite the equation as

$$\sum_m \big((m+1)^3 a_{m+1} - \tfrac{1}{64}(4m+1)(4m+2)(4m+3)a_m\big)\, z^m = 0$$

This gives the relationship

$$\begin{aligned} a_{m+1} &= \frac{(4m+1)(4m+2)(4m+3)}{64(m+1)^3} a_m \\ &= \frac{(m+1/4)(m+2/4)(m+3/4)}{(m+1)^3} a_m. \end{aligned}$$

Given that $a_0 = 1$ (so that $f(0) = 1$) we get that

$$a_m = \frac{\left(\frac{1}{4}\right)_m \left(\frac{2}{4}\right)_m \left(\frac{3}{4}\right)_m}{(1)_m (1)_m m!}.$$

We now use Eq. 2.2.1 to express the function f as the hypergeometric function

$$f = {}_3F_2\left(\begin{array}{ccc|c} 1/4 & 2/4 & 3/4 & \\ & 1 & 1 & z \end{array}\right). \tag{5.1.10}$$

We conclude the proof by setting $\pi = f$ and by recalling that $z = \frac{1}{\lambda^4}$.

□

Our final result for Dwork K3 surfaces relates the trace of Frobenius to the fundamental period associated to the surface. Recall that the Lefschetz fixed point formula describes the relationship between the number of points on a variety over a finite field $\mathbb{F}_q$ and the ℓ-adic étale cohomology $H^i(X) = H^i_{\acute{e}t}(\bar{X}, \mathbb{Q}_\ell)$, where the prime ℓ is coprime to q. For primes p of good reduction, the Lefschetz fixed point formula for K3 surfaces simplifies to

$$\begin{aligned}\#X_\lambda^4(\mathbb{F}_p) &= \sum_{i=0}^{4} \mathrm{trace}(\mathrm{Frob}_p, H^i(X_\lambda^4)) \\ &= 1 + \mathrm{tr}_p^2 + p^2,\end{aligned}$$

where $\mathrm{tr}_p^2 = \mathrm{trace}(\mathrm{Frob}_p, H^2(X_\lambda^4))$. We are interested in this particular trace because certain eigenvalues of the Frobenius map on $H^2(X)$, where X is a singular K3 surface, correspond to Fourier coefficients of weight 3 newforms [12].

Combining this formula with the point count formula of Theorem 1.0.1 yields

$$\begin{aligned}\mathrm{tr}_p^2 = p &+ 12pT^t(-1)T^{2t}(1-\lambda^4) \\ &+ p^2 {}_3F_2\left(\begin{matrix} T^t & T^{2t} & T^{3t} \\ & \epsilon & \epsilon \end{matrix} \,\middle|\, \frac{1}{\lambda^4}\right)_p + 3p^2\binom{T^{3t}}{T^t} {}_2F_1\left(\begin{matrix} T^{3t} & T^t \\ & T^{2t} \end{matrix} \,\middle|\, \frac{1}{\lambda^4}\right)_p.\end{aligned}$$

We will show that, modulo p, the only term that remains is the ${}_3F_2$ hypergeometric function. We have the following lemma for the ${}_2F_1$ term.

Lemma 5.1.10.

$$3p^2\binom{T^{3t}}{T^t} {}_2F_1\left(\begin{matrix} T^{3t} & T^t \\ & T^{2t} \end{matrix} \,\middle|\, x\right)_p \equiv 0 \pmod{p}.$$

Proof. Recall from the proof of Proposition 5.1.6 that

$$p^2\binom{T^{3t}}{T^t} {}_2F_1\left(\begin{matrix} T^{3t} & T^t \\ & T^{2t} \end{matrix} \,\middle|\, x\right)_p = g(T^{2t})g(T^{3t})^2 T^t(-1) {}_2F_1\left(\begin{matrix} T^{3t} & T^t \\ & T^{2t} \end{matrix} \,\middle|\, x\right)_p.$$

We rewrite the hypergeometric function as

$$
\begin{aligned}
{}_2F_1\left(\begin{array}{cc|c} T^{3t} & T^t & \\ & T^{2t} & x \end{array}\right)_p &= \frac{p}{p-1}\sum_{\chi}\binom{T^{3t}\chi}{\chi}\binom{T^t\chi}{T^{2t}\chi}\chi(x) \\
&= \frac{1}{p(p-1)}\sum_{\chi} J(T^{3t}\chi,\overline{\chi})J(T^t\chi,T^{2t}\overline{\chi})\chi(-x) \\
&= \frac{1}{p(p-1)}\sum_{\chi}\frac{g(T^{3t}\chi)g(\overline{\chi})}{g(T^{3t})}\frac{g(T^t\chi)g(T^{2t}\overline{\chi})}{g(T^{3t})}\chi(-x).
\end{aligned}
$$

As we have done in previous proofs, we let $T=\overline{\omega}$ and $\chi=\overline{\omega}^{-j}$, where ω is the Teichmüller character and rewrite our Gauss sum expression to get

$$
\begin{aligned}
&p^2\binom{T^{3t}}{T^t}{}_2F_1\left(\begin{array}{cc|c} T^{3t} & T^t & \\ & T^{2t} & x \end{array}\right)_p \\
&\quad= \frac{g(\overline{\omega}^{2t})g(\overline{\omega}^{3t})^2\overline{\omega}^t(-1)}{p(p-1)}\sum_{j=0}^{p-2}\frac{g(\overline{\omega}^{3t-j})g(\overline{\omega}^j)}{g(\overline{\omega}^{3t})}\frac{g(\overline{\omega}^{t-j})g(\overline{\omega}^{2t+j})}{g(\overline{\omega}^{3t})}\overline{\omega}^{-j}(-x) \\
&\quad= \frac{g(\overline{\omega}^{2t})\overline{\omega}^t(-1)}{p(p-1)}\sum_{j=0}^{p-2}g(\overline{\omega}^{3t-j})g(\overline{\omega}^j)g(\overline{\omega}^{t-j})g(\overline{\omega}^{2t+j})\overline{\omega}^{-j}(-x).
\end{aligned}
$$

Using the Gross-Koblitz formula from Theorem 2.4.3 allows us to rewrite this as

$$
\frac{-\pi^{2t}\overline{\omega}^t(-1)}{p(p-1)}\sum_{j=0}^{p-2}\pi^{6t}\Gamma_p\left(\frac{3t-j}{p-1}\right)\Gamma_p\left(\frac{j}{p-1}\right)\Gamma_p\left(\frac{t-j}{p-1}\right)\Gamma_p\left(\frac{2t+j}{p-1}\right)\Gamma_p\left(\frac{2t}{p-1}\right)\overline{\omega}^{-j}(-x).
$$

Since $\pi^{2t}\cdot\pi^{6t}=\pi^{8t}=p^2$, the above expression is equal to

$$
\frac{-p\overline{\omega}^t(-1)}{p-1}\sum_{j=0}^{p-2}\Gamma_p\left(\frac{3t-j}{p-1}\right)\Gamma_p\left(\frac{j}{p-1}\right)\Gamma_p\left(\frac{t-j}{p-1}\right)\Gamma_p\left(\frac{2t+j}{p-1}\right)\Gamma_p\left(\frac{2t}{p-1}\right)\overline{\omega}^{-j}(-x).
$$

Recall that $\omega(x)\equiv x \pmod p$ for all x in $\{0,\dots,p-1\}$ and $p-1\equiv -1 \pmod p$. Thus, the above sum is congruent modulo p to the expression

$$
p(-1)^t\sum_{j=0}^{p-2}\Gamma_p\left(\frac{3}{4}+j\right)\Gamma_p(-j)\Gamma_p\left(\frac{1}{4}+j\right)\Gamma_p\left(\frac{2}{4}-j\right)\Gamma_p\left(\frac{2}{4}\right)(-x)^j.
$$

This expression is congruent to 0 modulo p since $\Gamma_p : \mathbb{Z}_p \to \mathbb{Z}_p^*$. Hence,

$$p^2 \binom{T^{3t}}{T^t} {}_2F_1 \left(\begin{matrix} T^{3t} & T^t \\ & T^{2t} \end{matrix} \,\middle|\, x \right)_p \equiv 0 \pmod{p}.$$

□

We can now write that

$$\mathrm{tr}_p^2 \equiv p^2 {}_3F_2 \left(\begin{matrix} T^t & T^{2t} & T^{3t} \\ & \epsilon & \epsilon \end{matrix} \,\middle|\, \frac{1}{\lambda^4} \right)_p \pmod{p}.$$

Recall that in Corollary 3.2.2 we showed that

$${}_3F_2 \left(\begin{matrix} \frac{1}{4} & \frac{2}{4} & \frac{3}{4} \\ & 1 & 1 \end{matrix} \,\middle|\, x \right)_{\mathrm{tr}(p)} \equiv p^2 \, {}_3F_2 \left(\begin{matrix} T^t & T^{2t} & T^{3t} \\ & \epsilon & \epsilon \end{matrix} \,\middle|\, x \right)_p \pmod{p}.$$

This truncated series is equal to

$$\sum_{j=0}^{t} \frac{\left(\frac{1}{4}\right)_j \left(\frac{2}{4}\right)_j \left(\frac{3}{4}\right)_j}{j!^3} x^j.$$

Note that for the truncated series and the classical series we have the congruence

$$\sum_{j=0}^{t} \frac{\left(\frac{1}{4}\right)_j \left(\frac{2}{4}\right)_j \left(\frac{3}{4}\right)_j}{j!^3} x^j \equiv \sum_{j=0}^{\infty} \frac{\left(\frac{1}{4}\right)_j \left(\frac{2}{4}\right)_j \left(\frac{3}{4}\right)_j}{j!^3} x^j \pmod{p},$$

since the terms with $j > t$ are congruent to 0 modulo p. Thus, we have

$$\mathrm{tr}_p^2 \equiv {}_3F_2 \left(\begin{matrix} 1/4 & 2/4 & 3/4 \\ & 1 & 1 \end{matrix} \,\middle|\, \frac{1}{\lambda^4} \right) \pmod{p}.$$

The expression on the right side of the above congruence is the classical hypergeometric series that we used in Eq. 5.1.9 to express the fundamental period of the K3 surface, so we have proved the result.

□

5.2 Dwork Threefolds

We start this Section with another result for Gauss sum expressions. This is, in fact, just a specialization of Theorem 2.3.6 to the case where $n = 2$, so we omit the proof.

Proposition 5.2.1. *Let q be a prime power such that $q \equiv 1 \pmod{k}$ and let a, b, c, d be integer multiples of t, not all 0, such that $a+b+c+d \equiv 0 \pmod{q-1}$. Then, for $\psi \neq 0$,*

$$\sum_{j=0}^{q-2} g(T^{a+j})g(T^{b+j})g(T^{c-j})g(T^{d-j})T^j(\psi) = q^2(q-1)T^c(\psi)T^{b+d}(-1)\,{}_2F_1\left(\begin{matrix} T^{b+c} & T^{a+c} \\ & T^{c-d} \end{matrix} \,\middle|\, \psi\right)_q .$$

For our work with Dwork hypersurfaces, we will want $\psi = \lambda^k$. To mirror the main term of the hypergeometric point count formula, we would like our ${}_2F_1$ functions to be evaluated at $1/\lambda^k$. The following corollary writes the Gauss sum expression in this form.

Corollary 5.2.2. *The Gauss sum expression*

$$\sum_{j=0}^{q-2} g(T^{a+j})g(T^{b+j})g(T^{c-j})g(T^{d-j})T^j(\lambda^k)$$

is equal to both of the following hypergeometric functions.

1. $q^2(q-1)T^{a+c}(-1)\,{}_2F_1\left(\begin{matrix} T^{b+c} & T^{b+d} \\ & T^{b-a} \end{matrix} \,\middle|\, \frac{1}{\lambda^k}\right)_q .$

2. $q^2(q-1)T^{b+d}(-1)\,{}_2F_1\left(\begin{matrix} T^{a+d} & T^{a+c} \\ & T^{a-b} \end{matrix} \,\middle|\, \frac{1}{\lambda^k}\right)_q .$

Proof. We use part (ii) of Theorem 4.2 in [19] to rewrite the hypergeometric term from Theorem 5.2.1 in the following way.

$${}_2F_1\left(\begin{matrix} T^{b+c} & T^{a+c} \\ & T^{c-d} \end{matrix} \,\middle|\, \frac{1}{\lambda^k}\right)_q = T^{a+b+3c-d}(-1)T^{-(b+c)}(\lambda^k)\,{}_2F_1\left(\begin{matrix} T^{b+c} & T^{b+d} \\ & T^{b-a} \end{matrix} \,\middle|\, \frac{1}{\lambda^k}\right)_q .$$

Note that $T^{kt}(\lambda) = 1$. Thus,

$$q^2(q-1)T^{b+d}(-1)\,{}_2F_1\left(\begin{array}{cc|c} T^{b+c} & T^{a+c} & \lambda^k \\ & T^{c-d} & \end{array}\right)_q = q^2(q-1)T^{a+c}(-1)\,{}_2F_1\left(\begin{array}{cc|c} T^{b+c} & T^{b+d} & \frac{1}{\lambda^k} \\ & T^{b-a} & \end{array}\right)_q.$$

Furthermore, part (i) of Theorem 4.2 in [19] tells us that

$${}_2F_1\left(\begin{array}{cc|c} T^{b+c} & T^{b+d} & \frac{1}{\lambda^k} \\ & T^{b-a} & \end{array}\right)_q = T^{b-a}(1/\lambda^k)\,{}_2F_1\left(\begin{array}{cc|c} T^{a+d} & T^{a+c} & \frac{1}{\lambda^k} \\ & T^{a-b} & \end{array}\right)_q,$$

so that

$$q^2(q-1)T^{b+d}(-1)\,{}_2F_1\left(\begin{array}{cc|c} T^{b+c} & T^{a+c} & \lambda^k \\ & T^{c-d} & \end{array}\right)_q = q^2(q-1)T^{b+d}(-1)\,{}_2F_1\left(\begin{array}{cc|c} T^{a+d} & T^{a+c} & \frac{1}{\lambda^k} \\ & T^{a-b} & \end{array}\right)_q.$$

□

Remark. It's not clear if it is possible to work backwards from this to obtain a formula that starts with $S_{[w]}$ (from Koblitz's point count formula) instead of the simplified version that we work with here. For example, in the K3 surface case, $S_{[0022]}$ yields a ${}_2F_1$ term but $S_{[0112]}$ does not. It seems to depend on how many Gauss sum factors remain after using Hasse-Davenport to rewrite the expression. Thus, this theorem should apply only in the case when two factors remain in the numerator (and, hence, the denominator) after applying Hasse-Davenport to the denominator. This is the case for all but one of the Gauss sum terms that appear in the point count for the Dwork threefold.

One reason we are interested in rewriting Gauss sum expressions of this sort is to mirror work that has been done for period integrals of Dwork hypersurfaces. For example, in [5], Candelas, de la Ossa, and Rodriguez-Villegas express the periods of the Dwork threefold in terms of classical hypergeometric series. The fundamental period (associated to the mirror manifold) is expressed as a ${}_4F_3$ series and the remaining periods are ${}_2F_1$ series. We can use the above theorem to give a hypergeometric point count formula for the Dwork threefold that "matches" the periods.

Furthermore, in [5], the authors express the number of points modulo p for the threefold in terms of truncated hypergeometric series. As we did for Dwork K3 surfaces in Theorem 1.0.1, we will be able to give an exact point count formula using finite field hypergeometric functions.

Theorem 5.2.3. *Let $q = p^e$ be a prime power such that $q \equiv 1 \pmod 5$, $t = \frac{q-1}{5}$, and T be a generator for $\widehat{\mathbb{F}_q^{\times}}$. Then*

$$\#X_\lambda^5(\mathbb{F}_q) = \frac{q^4-1}{q-1} + 24q^2\delta(1-\lambda^5) + q^3{}_4F_3\left(\begin{matrix} T^t & T^{2t} & \dots & T^{4t} \\ & \epsilon & \dots & \epsilon \end{matrix} \middle| \frac{1}{\lambda^5}\right)_q$$

$$+ 20q^2{}_2F_1\left(\begin{matrix} T^{2t} & T^{3t} \\ & \epsilon \end{matrix} \middle| \frac{1}{\lambda^5}\right)_q + 20q^2{}_2F_1\left(\begin{matrix} T^{t} & T^{4t} \\ & \epsilon \end{matrix} \middle| \frac{1}{\lambda^5}\right)_q$$

$$+ 30q^2{}_2F_1\left(\begin{matrix} T^{t} & T^{3t} \\ & T^{4t} \end{matrix} \middle| \frac{1}{\lambda^5}\right)_q + 30q^2{}_2F_1\left(\begin{matrix} T^{t} & T^{2t} \\ & T^{3t} \end{matrix} \middle| \frac{1}{\lambda^5}\right)_q,$$

where $\delta(x) = 1$ if $x = 0$ and $\delta(x) = 0$ otherwise.

To prove this, we will start with Koblitz's point count formula in [29]. We then break this down into 6 sets of Gauss sum terms. One of these matches the form of Theorem 5.3.1 of Section 5.3, which we will prove is a ${}_4F_3$ hypergeometric function. Four of the remaining sets can be rewritten using Proposition 5.2.1. Note that McCarthy gave a p-adic hypergeometric point count formula in [36] for the Dwork threefold that holds for $\lambda = 1$. Our formula should match this when we use $\lambda^5 = 1$.

Meanwhile, the periods (that are holomorphic at $\lambda = 0$) of the Dwork threefold are given by the classical series

$$ {}_4F_3\left(\begin{matrix} 1/5 & 2/5 & 3/5 & 4/5 \\ & 1 & 1 & 1 \end{matrix}\,\middle|\, \frac{1}{\lambda^5}\right), $$

$$ {}_2F_1\left(\begin{matrix} 2/5 & 3/5 \\ & 1 \end{matrix}\,\middle|\, \frac{1}{\lambda^5}\right), \quad {}_2F_1\left(\begin{matrix} 1/5 & 4/5 \\ & 1 \end{matrix}\,\middle|\, \frac{1}{\lambda^5}\right), $$

$$ {}_2F_1\left(\begin{matrix} 1/5 & 3/5 \\ & 4/5 \end{matrix}\,\middle|\, \frac{1}{\lambda^5}\right), \quad {}_2F_1\left(\begin{matrix} 1/5 & 2/5 \\ & 3/5 \end{matrix}\,\middle|\, \frac{1}{\lambda^5}\right), $$

with multiplicities matching the coefficients in the point count formula. There are an additional 24 periods that appear only when $\lambda^5 = 1$, which corresponds to the term $24q^2\delta(1-\lambda^5)$ in the point count formula. Our work in Chapter 3 shows that the ${}_4F_3$ and the first two ${}_2F_1$ classical hypergeometric series and the corresponding finite field hypergeometric functions are congruent modulo p (when $q = p$), however we do no yet have a congruence or identity for the remaining terms.

It should be noted that Dwork hypersurface families are particularly nice to work with because of their large group of automorphisms. In general, one should expect many more terms in the point count formula and more periods for a Calabi-Yau manifold. The expected number comes from the Hodge structure, which in turn gives us information about the complex structure of the moduli space and the Betti numbers, which dictates the number of expected periods.

For example, as discussed in Section 3 of [5], the non-trivial Hodge numbers of the Dwork threefold are $h^{1,1} = 1$ and $h^{2,1} = 101$. This gives a Betti number of $B_3 =$

$2(1 + h^{2,1}) = 204$. Thus, we should expect there to be 204 periods of the holomorphic $(3, 0)$-form. However, the automorphism group reduces this number to 101. The amount of computation that needs to be done is further whittled down since many of the periods are equivalent modulo the Jacobian ideal. In fact, when grouped together in this way, the number of sets corresponding to a particular period is the same as the coefficient of the "matching" term in the point count formula.

This phenomenon holds true for Dwork K3 surfaces, too. Here, we have Betti number $B_2 = 22$, so we should expect there to be 22 periods of the holomorphic $(2, 0)$-form. However, there are in fact 16 periods and they fall into three distinct types (see Dwork's exposition in Chapter 6 of [10] for more on this). We note that the three types are expressible as classical hypergeometric series and that these series match the hypergeometric functions in the point count formula.

5.2.1 Proof of Theorem 5.2.3

We now prove Theorem 5.2.3. This work is very similar to our work with Dwork K3 surfaces, so we omit some details. We also use some results of McCarthy [36] that apply here.

Proof of Theorem 5.2.3. Let W be the set of all 5-tuples $w = (w_1, w_2, w_3, w_4, w_5)$ satisfying $0 \leq w_i < 5$ and $\sum w_i \equiv 0 \pmod 5$. We denote the points on the diagonal hypersurface

$$x_1^5 + \ldots + x_5^5 = 0$$

by $N_q(0) := \sum N_q(0,w)$, where

$$N_q(0,w) = \begin{cases} 0 & \text{if some but not all } w_i = 0, \\ \frac{q^4-1}{q-1} & \text{if all } w_i = 0, \\ -\frac{1}{q} J\left(T^{\frac{w_1}{5}}, T^{\frac{w_2}{5}}, \ldots, T^{\frac{w_5}{5}}\right) & \text{if all } w_i \neq 0. \end{cases}$$

Theorem 2 of [28] tells us that the number of points on the Dwork threefold is given by

$$\#X_\lambda^5(\mathbb{F}_q) = N_q(0) + \frac{1}{q-1} \sum \frac{\prod_{i=1}^5 g\left(T^{w_i t + j}\right)}{g(T^{5j})} T^{5j}(5\lambda)$$

where the sum is taken over $j \in \{0, \ldots, q-2\}$ and $w = (w_1, w_2, w_3, w_4, w_5)$ in $W/\sim$ (defined below).

We wish to simplify this formula. We start by considering the term $N_q(0)$.

Lemma 5.2.4. $N_q(0) = \frac{q^4-1}{q-1} + 50 \sum_{i=1}^4 g(T^{it})^2 g(T^{3it}) + \frac{1}{q} \sum_{i=1}^4 g(T^{it})^5$.

Proof. McCarthy proves this result in [36] (see Equation 3.3). □

Define the equivalence relation $\sim$ on W by $w \sim w'$ if $w - w'$ is a multiple of $(1,1,1,1,1)$. Up to permutation, McCarthy shows that there are six cosets (up to permutation) in $W/\sim$:

$$(0,0,0,0,0)^1, (0,1,2,3,4)^{24}, (0,0,0,1,4)^{20}, (0,0,0,2,3)^{20}, (0,0,1,1,3)^{30}, (0,0,1,2,2)^{30}.$$

Let

$$S_{[w]} = \frac{1}{q-1} \sum_{j=0}^{q-2} \frac{\prod_{i=1}^5 g\left(T^{w_i t + j}\right)}{g\left(T^{5j}\right)} T^{5j}(5\lambda) \tag{5.2.1}$$

denote the summands corresponding to all $w' \in [w]$. Our main tool for simplifying terms of this form is the Hasse-Davenport relation for Gauss sums.

In the lemmas that follow, we give explicit formulas for each $S_{[w]}$.

Lemma 5.2.5. *Let* $w = (0,0,0,0,0)$. *Then*

$$S_{[w]} = -\frac{1}{q}\sum_{i=1}^{4} g\left(T^{it}\right)^5 + q^2 {}_4F_3 \left(\begin{matrix} T^t & T^{2t} & T^{3t} & T^{4t} \\ & \epsilon & \epsilon & \epsilon \end{matrix} \,\middle|\, \frac{1}{\lambda^5} \right)_q$$

Proof. This is proved by specializing Theorem 8.1 of [18] or Theorem 5.3.1 of Section 5.3 of this thesis to the case where $d = 5$.

□

Lemma 5.2.6. *Let* $w = (0,1,2,3,4)$. *Then*

$$S_{[w]} = 24q^2\delta(1-\lambda^5),$$

where $\delta(x) = 1$ *if* $x = 0$ *and* $\delta(x) = 0$ *otherwise.*

Proof. When $w = (0,1,2,3,4)$, we have

$$S_{[w]} = \frac{24}{q-1}\sum_{j=0}^{q-2} \frac{g(T^j)g(T^{t+j})g(T^{2t+j})g(T^{3t+j})g(T^{4t+j})}{g(T^{5j})} T^{5j}(5\lambda).$$

We use Hasse-Davenport (see Corollary 2.3 of [18]) and properties of Gauss sums to rewrite this.

$$\begin{aligned} S_{[w]} &= \frac{24}{q-1}\sum_{j=0}^{q-2}\prod_{i=1}^{4} g(T^{it})T^{5j}(\lambda) \\ &= \frac{24q^2}{q-1}\sum_{j=0}^{q-2} T^{5j}(\lambda) \\ &= 24q^2\delta(1-\lambda^5), \end{aligned}$$

since

$$\sum_{j=0}^{q-2} T^j(\lambda^5) = \begin{cases} q-1 & \text{if } \lambda^5 = 1, \\ 0 & \text{otherwise.} \end{cases}$$

□

We now work to rewrite the remaining terms. Unlike in our work with Dwork K3 surfaces, these terms break down in a similar manner. Thus, we will carefully show our work for $w = (0,0,0,1,4)$ and state the remaining results.

Lemma 5.2.7. *Let* $w = (0,0,0,1,4)$. *Then*

$$S_{[w]} = -20g(T^{2t})^2 g(T^t) - 20g(T^{3t})^2 g(T^{4t}) + 20q^2 {}_2F_1\left(\begin{array}{cc|c} T^{2t} & T^{3t} & \frac{1}{\lambda^5} \\ & \epsilon & \end{array}\right)_q .$$

Proof. When $w = (0,0,0,1,4)$, we have

$$S_{[w]} = \frac{20}{q-1}\sum_{j=0}^{q-2} \frac{g(T^j)^3 g(T^{t+j}) g(T^{4t+j})}{g(T^{5j})} T^{5j}(5\lambda).$$

We use Hasse-Davenport to write

$$S_{[w]} = \frac{20q^2}{q-1}\sum_{j=0}^{q-2} \frac{g(T^j)^2}{g(T^{3t+j})g(T^{2t+j})} T^{5j}(\lambda).$$

Note that when $j = 2t$ or $3t$, we have $g(T^0)$ in the denominator. We separate these two cases from the summand and evaluate them to get

$$\frac{g(T^{2t})^2}{g(T^{3t+2t})g(T^{2t+2t})} T^{10t}(\lambda) = -\frac{1}{q} g(T^{2t})^2 g(T^t),$$
$$\frac{g(T^{3t})^2}{g(T^{3t+3t})g(T^{2t+3t})} T^{15t}(\lambda) = -\frac{1}{q} g(T^{3t})^2 g(T^{4t}).$$

For the remaining values of j we use the relationship $g(\chi)g(\overline{\chi}) = \chi(-1)q$, noting that $T^t(-1) = T^{5t}(-1) = 1$.

$$\sum_{j=0, j\neq 2t,3t}^{q-2} \frac{g(T^j)^2}{g(T^{3t+j})g(T^{2t+j})} T^{5j}(\lambda) = \frac{1}{q^2} \sum_{j=0, j\neq 2t,3t}^{q-2} g(T^j)^2 g(T^{2t-j}) g(T^{3t-j}) T^{5j}(\lambda).$$

Note that for $j = 2t, 3t$ we have

$$g(T^{2t})^2 g(T^{2t-2t}) g(T^{3t-2t}) T^{10t}(\lambda) = -g(T^{2t})^2 g(T^t),$$
$$g(T^{3t})^2 g(T^{2t-3t}) g(T^{3t-3t}) T^{15t}(\lambda) = -g(T^{3t})^2 g(T^{4t}).$$

Finally, we use Corollary 5.2.2 to rewrite the main Gauss sum term.

$$\sum_{j=0}^{q-2} g(T^j)^2 g(T^{2t-j}) g(T^{3t-j}) T^{5j}(\lambda) = q^2(q-1) {}_2F_1 \left(\begin{matrix} T^{2t} & T^{3t} \\ & \epsilon \end{matrix} \,\middle|\, \frac{1}{\lambda^5} \right)_q .$$

Hence,

$$\begin{aligned} S_{[w]} &= \frac{20q^2}{q-1} \Bigg(-\frac{1}{q} g(T^{2t})^2 g(T^t) - \frac{1}{q} g(T^{3t})^2 g(T^{4t}) + \frac{1}{q^2} g(T^{2t})^2 g(T^t) + \frac{1}{q^2} g(T^{3t})^2 g(T^{4t}) \\ &\qquad + (q-1) {}_2F_1 \left(\begin{matrix} T^{2t} & T^{3t} \\ & \epsilon \end{matrix} \,\middle|\, \frac{1}{\lambda^5} \right)_q \Bigg) \\ &= \frac{20}{q-1} g(T^{2t})^2 g(T^t)(1-q) + \frac{20}{q-1} g(T^{3t})^2 g(T^{4t})(1-q) + 20q^2 {}_2F_1 \left(\begin{matrix} T^{2t} & T^{3t} \\ & \epsilon \end{matrix} \,\middle|\, \frac{1}{\lambda^5} \right)_q \\ &= -20 g(T^{2t})^2 g(T^t) - 20 g(T^{3t})^2 g(T^{4t}) + 20q^2 {}_2F_1 \left(\begin{matrix} T^{2t} & T^{3t} \\ & \epsilon \end{matrix} \,\middle|\, \frac{1}{\lambda^5} \right)_q \end{aligned}$$

□

Similarly, we have

$$\begin{aligned} S_{[0,0,0,2,3]} &= -20 g(T^t)^2 g(T^{3t}) - 20 g(T^{4t})^2 g(T^{2t}) + 20q^2 {}_2F_1 \left(\begin{matrix} T^{t} & T^{4t} \\ & \epsilon \end{matrix} \,\middle|\, \frac{1}{\lambda^5} \right)_q , \\ S_{[0,0,1,1,3]} &= -30 g(T^{3t})^2 g(T^{4t}) - 30 g(T^{2t})^2 g(T^{t}) + 30q^2 {}_2F_1 \left(\begin{matrix} T^{t} & T^{3t} \\ & T^{4t} \end{matrix} \,\middle|\, \frac{1}{\lambda^5} \right)_q , \\ S_{[0,0,1,2,2]} &= -30 g(T^t)^2 g(T^{3t}) - 30 g(T^{4t})^2 g(T^{2t}) + 30q^2 {}_2F_1 \left(\begin{matrix} T^{t} & T^{2t} \\ & T^{3t} \end{matrix} \,\middle|\, \frac{1}{\lambda^5} \right)_q . \end{aligned}$$

We now combine all of these terms to get the complete point count formula for the Dwork threefold. Note that the extra Gauss sum terms from $N_q(0)$ will cancel with the extra Gauss sum terms from the $S_{[w]}$ terms. Hence,

$$\#X_\lambda^5(\mathbb{F}_q) = \frac{q^4-1}{q-1} + 24q^2\delta(1-\lambda^5) + q^3{}_4F_3\left(\begin{array}{cccc|c} T^t & T^{2t} & \dots & T^{4t} & \frac{1}{\lambda^5} \\ & \epsilon & \dots & \epsilon & \end{array}\right)_q$$

$$+ 20q^2{}_2F_1\left(\begin{array}{cc|c} T^{2t} & T^{3t} & \frac{1}{\lambda^5} \\ & \epsilon & \end{array}\right)_q + 20q^2{}_2F_1\left(\begin{array}{cc|c} T^{t} & T^{4t} & \frac{1}{\lambda^5} \\ & \epsilon & \end{array}\right)_q$$

$$+ 30q^2{}_2F_1\left(\begin{array}{cc|c} T^{t} & T^{3t} & \frac{1}{\lambda^5} \\ & T^{4t} & \end{array}\right)_q + 30q^2{}_2F_1\left(\begin{array}{cc|c} T^{t} & T^{2t} & \frac{1}{\lambda^5} \\ & T^{3t} & \end{array}\right)_q.$$

□

5.3 Higher Dimensional Dwork Hypersurfaces

We have made partial progress on similar point count and period results for higher dimensional Dwork hypersurfaces

$$X_\lambda^d: \quad x_1^d + x_2^d + \ldots + x_d^d = d\lambda x_1 x_2 \cdots x_d.$$

5.3.1 Point Count for Prime Powers $q \equiv 1 \pmod d$

We begin with a point count formula that holds for prime powers $q \equiv 1 \pmod d$. This is a partial breakdown of Koblitz's point count formula of Section 5.1 and demonstrates that we should expect to be able to develop hypergeometric point count formulas for a large class of varieties.

Theorem 5.3.1. *Let $q \equiv 1 \pmod d$, $t = \frac{q-1}{d}$, and T be a generator for $\widehat{\mathbb{F}_q^\times}$. In addition, let W be the set of all d-tuples $w = (w_1, \ldots, w_d)$ satisfying $0 \leq w_i < d$ and $\sum w_i \equiv 0$*

(mod d). *Then the number of points over $\mathbb{F}_q$ on the Dwork hypersurface is given by*

$$\#X_\lambda^d(\mathbb{F}_q) = \frac{q^{d-1}-1}{q-1} + q^{d-2}{}_{d-1}F_{d-2}\left(\begin{matrix} T^t & T^{2t} & \dots & T^{(d-1)t} \\ & \epsilon & \dots & \epsilon \end{matrix}\,\middle|\, \frac{1}{\lambda^d}\right)_q$$

$$+\frac{1}{q}\sum_{\overline{w}\in W^{**}}\prod_i g(T^{w_it}) + \frac{1}{q-1}\sum_{\overline{w}\neq\overline{0}}\sum_{j=0}^{p-2}\frac{\prod_{i=1}^d g\left(T^{w_it+j}\right)}{g(T^{dj})}T^{dj}(d\lambda),$$

where $\overline{w} = (w_1, \dots, w_d) \in W/\sim$, and W^{**} is the subset of W with $w_i \neq 0$ and w_i not all equal.

Remark. Note that in the case where $d = 4$, i.e. for Dwork K3 surfaces, the point count formula had a main ${}_3F_2$ hypergeometric term, a ${}_2F_1$ term, and a sum of multiples of q. It seems likely that we will have a similar breakdown of terms in higher dimensional cases.

Remark. Theorem 5.3.1 gives a general formula for the number of points on a Dwork hypersurface. The formula has a mysterious Gauss sum term that has not been broken down into something more manageable. The reason for this is that it's not clear yet what the structures of the sets W^* and $W/\sim$ are. The possible number and form of elements in this group becomes more complicated as the dimension of the hypersurface increases. For example, the size of $W/\sim$ for the 4-dimensional Dwork hypersurface X_λ^6 is 1296. This value was computed in Sage [9] and we verified the method by testing the known $d = 4, 5$ cases. Note that many of these will be grouped together by permutations as in the case of $d = 4$ and $d = 5$, but it is unclear if there's a general algorithm for doing this.

Proof of Theorem 5.3.1. This proof follows our work in proving Theorem 1.0.1. Let W be the set of all d-tuples $w = (w_1, \dots, w_d)$ satisfying $0 \leq w_i < d$ and $\sum w_i \equiv 0 \pmod d$. We denote the points on the diagonal hypersurface

$$x_1^d + \ldots + x_d^d = 0$$

by $N_q(0) := \sum N_q(0,w)$, where

$$N_q(0,w) = \begin{cases} 0 & \text{if some but not all } w_i = 0, \\ \frac{q^{d-1}-1}{q-1} & \text{if all } w_i = 0, \\ -\frac{1}{q} J\left(T^{\frac{w_1}{d}}, \ldots, T^{\frac{w_d}{d}}\right) & \text{if all } w_i \neq 0. \end{cases}$$

Koblitz's formula in this general case is as follows.

$$\#X_\lambda^d(\mathbb{F}_q) = N_q(0) + \frac{1}{q-1} \sum \frac{\prod_{i=1}^d g\left(T^{w_i t + j}\right)}{g(T^{dj})} T^{dj}(d\lambda)$$

where the sum is taken over $j \in \{0, \ldots, q-2\}$ and $w \in W$.

Letting W^* be set of all $d-$tuples where no $w_i = 0$, we can write

$$N_q(0,w) = \frac{q^{d-1}-1}{q-1} + \frac{1}{q} \sum_{w \in W^*} \prod_i g(T^{w_i t}).$$

As in the Section 5.1 we consider cosets of W with respect to the equivalence relation $\sim$ on W defined by $w \sim w'$ if $w - w'$ is a multiple of $(1, \ldots, 1)$. In the case where $d = 4$ we had three cosets and their permutations. For general d, we should expect many more cosets. Regardless of the value of d, however, one of the cosets will be the zero element $w = (0, \ldots, 0)^1$. We show that the summand associated to this coset can be expressed as a finite field hypergeometric function.

When $w = (0, \ldots, 0)$ we have

$$S_{[0]} = \frac{1}{q-1} \sum_{j=0}^{q-2} \frac{g\left(T^j\right)^d}{g\left(T^{dj}\right)} T^{dj}(d\lambda)$$

If $t \mid j$, then

$$\frac{g\left(T^j\right)^d}{g\left(T^{dj}\right)} T^{dj}(d\lambda) = -g\left(T^j\right)^d.$$

Thus,

$$\begin{aligned}S_{[0]} &= -\frac{1}{q-1}\sum_{i=1}^{d-1} g\left(T^{it}\right)^d + \frac{1}{q-1}\sum_{j=0, t\nmid j}^{q-2} \frac{g\left(T^j\right)^d}{g\left(T^{dj}\right)} T^{dj}(d\lambda) \\ &= -\frac{1}{q-1}\sum_{i=1}^{d-1} g\left(T^{it}\right)^d + \frac{1}{q-1}\sum_{j=0, t\nmid j}^{q-2} \frac{g\left(T^j\right)^d g\left(T^{-dj}\right)}{T^{dj}(-1)q} T^{dj}(d\lambda) \\ &= -\frac{1}{q-1}\sum_{i=1}^{d-1} g\left(T^{it}\right)^d + \frac{1}{q(q-1)}\sum_{j=0, t\nmid j}^{q-2} g\left(T^j\right)^d g\left(T^{-dj}\right) T^{dj}(-d\lambda).\end{aligned}$$

Note that if $t \mid j$ then

$$g\left(T^j\right)^d g\left(T^{-dj}\right) T^{dj}(-d\lambda) = -g\left(T^j\right)^d.$$

Hence,

$$\begin{aligned}S_{[0]} &= -\frac{1}{q-1}\sum_{i=1}^{d-1} g\left(T^{it}\right)^d + \frac{1}{q(q-1)}\sum_{i=1}^{d-1} g(T^{it})^d + \frac{1}{q(q-1)}\sum_{j=0}^{q-2} g\left(T^j\right)^d g\left(T^{-dj}\right) T^{dj}(-d\lambda) \\ &= -\frac{1}{q}\sum_{i=1}^{d-1} g\left(T^{it}\right)^d + \frac{1}{q(q-1)}\sum_{j=0}^{q-2} g\left(T^j\right)^d g\left(T^{-dj}\right) T^{dj}(-d\lambda).\end{aligned}$$

We would like to express this is as a finite field hypergeometric function. Recall that the ${}_{d-1}F_{d-2}$ hypergeometric function is given by

$$\begin{aligned}&{}_{d-1}F_{d-2}\left(\begin{matrix} T^t & T^{2t} & \dots & T^{(d-1)t} \\ & \epsilon & \dots & \epsilon \end{matrix}\ \middle|\ \frac{1}{\lambda^d}\right)_q \\ &\qquad = \frac{q}{q-1}\sum_{\chi} \binom{T^t\chi}{\chi}\binom{T^{2t}\chi}{\epsilon\chi}\cdots\binom{T^{(d-1)t}\chi}{\epsilon\chi}\chi(1/\lambda^d).\end{aligned}$$

We rewrite this so that it is in terms of Gauss sums

$$\begin{aligned}&= \frac{q}{q-1}\sum_{\chi}\left(\frac{\chi(-1)}{q}\right)^{d-1} J(T^t\chi,\overline{\chi})\cdots J(T^{(d-1)t}\chi,\overline{\chi})\overline{\chi}(\lambda^d) \\ &= \frac{1}{q^{d-2}(q-1)}\sum_{\chi}\chi(-1)^{d-1}\frac{\prod_{i=1}^{d-1} g(T^{it}\chi)}{\prod_{i=1}^{d-1} g(T^{it})} g(\overline{\chi})^{d-1}\overline{\chi}(\lambda^d)\end{aligned}$$

Use the Hasse-Davenport formula of Theorem 2.3.1 to get

$$
\begin{aligned}
{}_{d-1}F_{d-2}\left(\begin{matrix} T^{t} & T^{2t} & \dots & T^{(d-1)t} \\ & \epsilon & \dots & \epsilon \end{matrix} \;\middle|\; \frac{1}{\lambda^d}\right)_q &= \frac{1}{q^{d-2}(q-1)}\sum_{\chi}\chi(-1)^{d-1}\frac{g(\chi^d)g(\overline{\chi})^{d-1}}{\chi^d(d)g(\chi)}\overline{\chi}(\lambda^d) \\
&= \frac{1}{q^{d-1}(q-1)}\sum_{\chi} g(\chi^d)g(\overline{\chi})^d\overline{\chi}^d(-d\lambda) \\
&= \frac{1}{q^{d-1}(q-1)}\sum_{j=0}^{q-2} g(T^{-dj})g(T^j)^d T^{dj}(-d\lambda).
\end{aligned}
$$

Thus,

$$
S_{[0]} = -\frac{1}{q}\sum_{i=1}^{d-1} g\left(T^{it}\right)^d + q^{d-2}{}_{d-1}F_{d-2}\left(\begin{matrix} T^{t} & T^{2t} & \dots & T^{(d-1)t} \\ & \epsilon & \dots & \epsilon \end{matrix} \;\middle|\; \frac{1}{\lambda^d}\right)_q .
$$

We now combine our results to get

$$
\begin{aligned}
\#X_\lambda^d(\mathbb{F}_q) = {}& \frac{q^{d-1}-1}{q-1} + q^{d-2}{}_{d-1}F_{d-2}\left(\begin{matrix} T^{t} & T^{2t} & \dots & T^{(d-1)t} \\ & \epsilon & \dots & \epsilon \end{matrix} \;\middle|\; \frac{1}{\lambda^d}\right)_q \\
& + \frac{1}{q}\sum_{w\in W^*}\prod_i g(T^{w_i t}) - \frac{1}{q}\sum_{i=1}^{d-1} g\left(T^{it}\right)^d + \frac{1}{q-1}\sum_{\overline{w}\neq\overline{0}}\sum_{j=0}^{p-2}\frac{\prod_{i=1}^d g\left(T^{w_i t+j}\right)}{g(T^{dj})}T^{dj}(d\lambda) \\
= {}& \frac{q^{d-1}-1}{q-1} + q^{d-2}{}_{d-1}F_{d-2}\left(\begin{matrix} T^{t} & T^{2t} & \dots & T^{(d-1)t} \\ & \epsilon & \dots & \epsilon \end{matrix} \;\middle|\; \frac{1}{\lambda^d}\right)_q \\
& + \frac{1}{q}\sum_{w\in W^{**}}\prod_i g(T^{w_i t}) + \frac{1}{q-1}\sum_{\overline{w}\neq\overline{0}}\sum_{j=0}^{p-2}\frac{\prod_{i=1}^d g\left(T^{w_i t+j}\right)}{g(T^{dj})}T^{dj}(d\lambda),
\end{aligned}
$$

where W^{**} is the set of $d-$tuples with $w_i \neq 0$ and w_i not all equal. □

5.3.2 Point Count for Primes $p \not\equiv 1 \pmod d$

In this section we discuss a conjecture for a point count formula that is written in terms of McCarthy's p-adic hypergeometric function for primes $p \not\equiv 1 \pmod d$.

Conjecture 5.3.2. *Let d be an odd prime and p a prime number such that $p \not\equiv 1 \pmod{d}$. The number of points over $\mathbb{F}_p$ on the Dwork hypersurface is given by*

$$\#X_\lambda^d(\mathbb{F}_p) = \frac{p^{d-1}-1}{p-1} - {}_{d-1}G_{d-1}\left[\begin{array}{cccc|} 1/d & 2/d & \dots & (d-1)/d \\ 0 & 0 & \dots & 0 \end{array}\ \lambda^d\right]_p$$

Remark. Currently our conjecture is somewhat limited, only applying to Dwork hypersurfaces with d a prime. When d is not prime we have found that the number of terms to consider and simplify grows rather large. This is because we have to consider various congruences as we did in the proofs of Theorems 1.0.2 and 5.3.1, and the number of solutions to these is unwieldy when d is not prime.

After the initial submission of this Conjecture to arXiv [18], Barman, Rahman, and Saikia have demonstrated the validity of this conjecture in [2].

Partial proof of Conjecture 5.3.2. The start of the proof mirrors the work of McCarthy in [38]. Let $N_p^A(\lambda)$ denote the number of points on the Dwork hypersurface in $\mathbb{A}^d(\mathbb{F}_p)$. Then

$$N_{\mathbb{F}_p}(\lambda) = \frac{N_p^A(\lambda)-1}{p-1}. \tag{5.3.1}$$

Letting $f(\overline{x}) = x_1^d + x_2^d + \dots x_d^d - d\lambda x_1 x_2 \cdots x_d$ we can write

$$\begin{aligned} pN_p^A(\lambda) &= p^d + \sum_{z\in\mathbb{F}_p^*}\sum_{x_i}\theta(zf(\overline{x})) \\ &= p^d + \sum_{z\in\mathbb{F}_p^*}\sum_{x_i\neq 0}\theta(zf(\overline{x})) + \sum_{z\in\mathbb{F}_p^*}\sum_{\substack{x_i \\ \textit{some } x_i=0}}\theta(zf(\overline{x})). \end{aligned}$$

We first work to rewrite the second summand. We start by letting $f'(\overline{x}) = x_1^d + x_2^d + \dots x_d^d$ and N_p' be the number of solutions to $f'(\overline{x}) = 0$. Note that $x \to x^d$ is an automorphism when $p \not\equiv 1 \pmod{d}$ because d is prime, so $N_p' = p^{d-1}$. Furthermore,

$$pN_p' = p^d + \sum_{z\in\mathbb{F}_p^*}\sum_{x_i\neq 0}\theta(zf'(\overline{x})) + \sum_{z\in\mathbb{F}_p^*}\sum_{\substack{x_i \\ \textit{some } x_i=0}}\theta(zf'(\overline{x})).$$

Thus,

$$\sum_{z\in\mathbb{F}_p^*}\sum_{x_i\neq 0}\theta(zf'(\overline{x})) = -\sum_{z\in\mathbb{F}_p^*}\sum_{\substack{x_i \\ some\ x_i=0}}\theta(zf'(\overline{x})).$$

Noting that

$$\sum_{z\in\mathbb{F}_p^*}\sum_{\substack{x_i \\ some\ x_i=0}}\theta(zf'(\overline{x})) = \sum_{z\in\mathbb{F}_p^*}\sum_{\substack{x_i \\ some\ x_i=0}}\theta(zf(\overline{x})),$$

we can write

$$pN_p^A(\lambda) = p^d + \sum_{z\in\mathbb{F}_p^*}\sum_{x_i\neq 0}\theta(zf(\overline{x})) - \sum_{z\in\mathbb{F}_p^*}\sum_{x_i\neq 0}\theta(zf'(\overline{x})).$$

We will call the first summand A and the second B. We can simplify B using basic facts about characters and Gauss sums

$$\begin{aligned}
B &= \sum_{z\in\mathbb{F}_p^*}\sum_{x_i\neq 0}\theta(zf'(\overline{x})) \\
&= \sum_{z\in\mathbb{F}_p^*}\sum_{x_i\neq 0}\theta(zx_1^d)\theta(zx_2^d)\cdots\theta(zx_d^d) \\
&= \frac{1}{(p-1)^d}\sum_{a_i=0}^{p-2} g(T^{-a_1})\cdots g(T^{-a_d}) \\
&\qquad\qquad \times \sum_{x_1} T^{da_1}(x_1)\cdots\sum_{x_d} T^{da_d}(x_d)\sum_{z} T^{a_1+\ldots a_d}(z).
\end{aligned}$$

This sum is non-zero only when the following congruences hold:

$$da_1,\ldots,da_d \equiv 0 \pmod{p-1}, \text{ and } \sum a_i \equiv 0 \pmod{p-1}.$$

Since $p \not\equiv 1 \pmod d$ and d is prime, these congruences simultaneously hold only when $a_1,\ldots,a_d = 0$. Hence, $B = -(p-1)$.

We now work to rewrite A.

$$A = \frac{1}{(p-1)^{d+1}} \sum_{a_i=0}^{p-2} g(T^{-a_1}) \cdots g(T^{-a_{d+1}}) T^{d+1}(-d\lambda)$$
$$\times \sum_{x_1} T^{da_1+a_{d+1}}(x_1) \cdots \sum_{x_d} T^{da_d+a_{d+1}}(x_d) \sum_{z} T^{a_1+\ldots a_{d+1}}(z).$$

We consider congruences that must hold for the a_i. This sum is non-zero only when the following congruences hold:

$$da_1 + a_{d+1}, \ldots, da_d + a_{d+1} \equiv 0 \pmod{p-1}, \text{ and } a_1 + \ldots + a_{d+1} \equiv 0 \pmod{p-1}.$$

We first consider having the a_i not all equal. Here we would have $a_i = \frac{j_i(p-1)}{d}$, where $0 \leq j_i \leq d-1$, $\sum j_i \equiv 0 \pmod{d}$, and the a_i are not all identical. However, since d is prime, this is not possible. Thus, we must have all of the a_i being equal. Thus we have

$$A = \sum_{j=0}^{p-2} g(T^{-j})^d g(T^{dj}) T^{-dj}(-d\lambda).$$

We expect that this term can be expressed as a p-adic hypergeometric function of the form that we saw in Theorem 1.0.2. Our conjecture is that we have

$$A = -p(p-1)_{d-1}G_{d-1}\left[\begin{array}{cccc|} 1/d & 2/d & \ldots & (d-1)/d \\ 0 & 0 & \ldots & 0 \end{array}\ \lambda^d\right]_p$$

plus a term to cancel with B. Assuming this is true, we now write a formula for the point count.

$$\begin{aligned}
\#X_\lambda^d(\mathbb{F}_p) &= \frac{\frac{1}{p}(p^d + A - B) - 1}{p-1} \\
&= \frac{p^{d-1}-1}{p-1} + \frac{\frac{1}{p}(A-B)}{p-1} \\
&= \frac{p^{d-1}-1}{p-1} - {}_{d-1}G_{d-1}\left[\begin{array}{cccc|} 1/d & 2/d & \ldots & (d-1)/d \\ 0 & 0 & \ldots & 0 \end{array}\ \lambda^d\right]_p.
\end{aligned}$$

□

5.3.3 Dwork Hypersurface Period Calculation

Proposition 5.3.3. *[43, Section 3.2] The Picard-Fuchs equation for the Dwork hypersurface*

$$X_\lambda^d: \quad x_1^d + x_2^d + \ldots + x_d^d = d\lambda x_1 x_2 \cdots x_d$$

is given by

$$\left(\vartheta^{d-1} - z(\vartheta + \tfrac{1}{d}) \cdots (\vartheta + \tfrac{d-1}{d})\right) \pi = 0 \tag{5.3.2}$$

where $\vartheta = z\frac{d}{dz}$ *and* $z = \lambda^{-d}$.

The following is adapted from results in Section 46 of Rainville's text [44].

Proposition 5.3.4. *The solution to Eq. 5.3.2 in Proposition 5.3.3 that is bounded near* $z = 0$ *is given by*

$$\pi = {}_{d-1}F_{d-2}\left(\begin{matrix} \frac{1}{d} & \frac{2}{d} & \cdots & \frac{d-1}{d} \\ & 1 & \cdots & 1 \end{matrix} \;\middle|\; \frac{1}{\lambda^d}\right). \tag{5.3.3}$$

We saw in Theorem 3.2.1 that this is congruent (up to a sign) modulo p to the matching finite field hypergeometric function that appears in the point count. This leads us to a conjecture that extends the congruence we saw in Theorem 1.0.4

Conjecture 5.3.5. *For the Dwork hypersurface*

$$X_\lambda^d: \quad x_1^d + x_2^d + \ldots + x_d^d = d\lambda x_1 x_2 \ldots x_d,$$

we have that the trace of Frobenius over $\mathbb{F}_p$ *on* $H^{d-2}(X_\lambda^d)$ *and the fundamental period associated to the surface are congruent modulo* p *when* $p \equiv 1 \pmod{d}$.

The conjecture here is that, as in the Dwork K3 surface case, either the remaining Gauss sum terms in the point count formula of Theorem 5.3.1 are congruent to 0 modulo p or that these terms are canceled out in the trace of Frobenius – point count relationship. This relationship becomes more complicated for higher dimensional varieties.

For example, consider the family of Dwork threefolds

$$X_\lambda^5: \quad x_1^5 + x_2^5 + \ldots + x_5^5 = 5\lambda x_1 x_2 \cdots x_5.$$

The trace of Frobenius over $\mathbb{F}_p$ on $H^3(X_\lambda^5)$ when $p \equiv 1 \pmod 5$ is given by

$$tr_p^5 = p^3 + 25p^2 - 100p + 1 - \#X_\lambda^5(\mathbb{F}_p).$$

See [39, Sections 1.5, 3.1] for a proof of this. In the case where $\lambda = 1$ and $p \equiv 1 \pmod 5$, Conjecture 5.3.5 follows from Theorem 3.2.1 and the point count work of McCarthy in [36]. More generally, for $\lambda \neq 1$, the formula for $\#X_\lambda^5(\mathbb{F}_p)$ has a main ${}_4F_3$ hypergeometric term and several terms made up of products of Gauss sums.

A result relating the trace of Frobenius and the periods is expected for algebraic curves because of Manin's work in [34]. However, there is not a result of this sort that holds generally for higher dimensional algebraic varieties. We expect that it should be the case that the period and trace of Frobenius over $\mathbb{F}_p$ are congruent for a large class of varieties. In particular it would seem possible to show, at least by comparing explicit formulas, that the trace and the period are congruent when these expressions are both hypergeometric. Better yet, given that we expect there to be a congruence between these two quantities, it seems possible that it is exactly the varieties whose periods are solutions to hypergeometric differential equations that have a finite field hypergeometric point count.

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