WAITING-LINE AUCTIONS

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Bidders compete for prizes, leases, or procurement contracts on the basis of monetary bids through a variety of price auctions. Yet many other allocation mechanisms also exist which resemble auctions because agents compete in some non-price dimension for a limited number of prizes. In particular, if commodities are distributed to consumers on a first-come-first-served basis, a high bid corresponds to arriving early and being first in line. This type of "waiting-line auction" is pervasive in countries with price controls but it is common in other countries as well, and its use can cause a large total amount of time to be spent waiting in line. Our purpose is to consider the cost and performance of several alternative waiting-line auction arrangements.

The problem examined here is not one in which a queue persists throughout a service period and individuals can decide whether to join the queue after estimating an expected waiting time, which is the same for all individuals. Instead, the good or service to be obtained will be awarded at a particular time and arrival in advance of that time determines one's ranking in the queue, a ranking that matters because the number of units to be distributed is less than the number of potential claimants. In this situation an individual's chosen arrival time will be influenced by the anticipated actions of others who are interested in the good or service, so the arrival time has some of the properties of a bid at auction. An early arrival increases the probability of obtaining a prize but it also increases the waiting time cost. Waiting-line allocation procedures are widely used for
rationing tickets to sporting events like the World Series or college basketball games, and such procedures also can be used to allocate theater tickets, theater seats for ticket holders, access to retail sales, work through labor exchanges, certain classes of airline tickets, seats in many travel modes or even in college classrooms, and many other opportunities.

Part I presents a simple model of the waiting-line auction, formulated as a noncooperative game with incomplete information. A fixed number of units of some commodity (the prizes) will be distributed on a first-come-first-served basis, one prize to each successful consumer, and there is a positive probability that some potential claimants will not receive a prize. Individuals are assumed to be risk neutral and to differ in their opportunity costs of time. They know how long they are willing to wait for a prize, but they do not know with certainty the willingness to wait of other potential claimants. A Nash equilibrium for this game is characterized in part II. Alternative waiting-line allocation procedures are analyzed in part III. In one procedure, individuals who arrive too late to receive a unit of the prize commodity are informed of this when they arrive, before they waste time waiting in line. Another alternative procedure is to allow individuals to take turns waiting in line for each other. These procedures would seem to conserve on waiting time but we show the equilibrium effect of instituting them is to cause individuals to arrive earlier, so that the equilibrium expected waiting time for each individual actually is unchanged. Limited possibilities for resale are also considered. Part IV contains a summary and conclusion.
I. A Model for the Waiting Line Auction

Waiting in line has been studied primarily as the result of a process in which individuals are served continuously, and all individuals can expect to spend the same amount of time waiting to be served. With such a uniform expected waiting time it has often been assumed that poorer individuals are more likely to join the queue, because the opportunity cost of the time spent waiting is lower for them. Yoram Barzel (1974) has shown that poorer individuals will not always line up first in a queue even when the opportunity cost of time is directly proportional to income. He considered an example in which the value of a prize for a particular consumer is an increasing function of that consumer's opportunity cost of time. If the elasticity of this value function is greater than one, an increase in the opportunity cost of time is associated with an even greater increase in the prize value, and individuals with relatively high time costs will be found in the queue. When all individuals have the same expected waiting time, the elasticity of this value function with respect to the opportunity cost of time determines whether the queue will contain individuals with relatively high or low time costs.

The question of when to join a waiting line is more complicated if all awards are made at the same time and individuals queue in advance of that time. In this case individuals' waiting times will presumably be correlated with their opportunity costs of time. In deciding when to arrive in advance of the award time, individuals must consider both their own opportunity costs of time and the likely arrival times of other prospective claimants. These considerations make the equilibrium analysis of the competition for prizes more complicated than when prizes are distributed continuously.
The Barzel analysis suggests that there are two cases to be considered: one where the time cost elasticity of the value function is less than one and low time cost individuals are more willing to wait, and the other where the value function elasticity is greater than one and high time cost individuals are more willing to wait. However, Barzel's analysis does not apply to the case where waiting times vary across individuals. Indeed, analyses of queuing have not yielded a process through which an individual, acting independently in a decentralized manner, would select a waiting time when that is variable. Moreover there is nothing to indicate that a decentralized equilibrium would then exist so there is no basis for estimating the aggregate waiting time cost of a particular waiting-line arrangement. An explicit equilibrium model of the behavior of prospective claimants acting independently with incomplete information would make it possible to investigate the structural conditions which determine how much time is required in waiting-line allocation procedures.

To construct a waiting-line model for the case of varying waiting times, we consider waiting in line to be like a bid at auction. Suppose there are \( m \) prizes to be given away, at most one per person. The opportunity costs of time of \( n \) prospective recipients (\( n > m \)) are denoted by \( w_1, w_2, \ldots, w_n \). We initially consider the case in which all individuals know \( m \) and \( n \); this assumption will be relaxed in part III. The prizes are to be awarded at a known time, on a first-come-first-served basis, so each individual must anticipate having to wait in line and must decide how early to arrive. Let \( t_i \) denote the amount of time the \( i \text{th} \) individual arrives in advance of the award time; the lowest possible value of \( t_i \) is zero. We shall refer to \( t_i \) as the \( i \text{th} \) individual's
"arrival time." A high value of $t_i$ increases the probability of obtaining a prize but it also increases the cost of waiting in line, which is $w_i t_i$. for individual $i$.

An individual's money value of a prize is assumed to be a function of the individual's opportunity cost of time. This money value also depends on any money price charged for the good, if that is not zero. The value function is denoted by $v(w)$ and is assumed to be continuously differentiable and positive valued. Barzel's analysis indicates that the elasticity of the value function may determine whether individuals with high or low time costs line up first, and this elasticity will be denoted by $\eta$: $\eta \equiv [v'(w)w]/v(w)$.6

Each individual who participates considers the payoffs for two possible outcomes: receiving a prize or not. The payoff if the individual decides not to participate at all is assumed to be zero. Not participating may be attractive for some because there is a non-negative entry cost which is incurred if an individual decides to travel to the line. The entry cost is assumed to be a fixed amount of time $k$, valued at time cost $w$. The time $k$ may be thought of as time required to reach the queue, and that time is more costly for those with higher time costs. Thus individual $i$ who waits for $t_i$ units of time and receives a prize will have a monetary payoff of $v(w_i) - w_i t_i - kw_i$.

Let $\pi^W_i$ denote the ratio of this monetary payoff to the individual's opportunity cost of time $w_i$:

$$\pi^W_i = \frac{a_i - t_i - k}{w_i}, \quad (1)$$

where $a_i = \frac{v(w_i)}{w_i}$. Thus, as it is expressed in time units, $a_i$ is the
"time value" of a prize for individual $i$, and $\pi^W_i$ is the payoff in time units for a successful claimant (a "winner") who waited $t_i$ units of time. Similarly, if individuals who do not receive prizes must wait to find out that they are unsuccessful, then the "loser's" payoff $\pi^L_i$ expressed in units of time is:

$$\pi^L_i = -t_i - k.$$  \hspace{1cm} (2)

Of course the monetary payoffs for winning and losing can be obtained by multiplying $\pi^W_i$ and $\pi^L_i$ by $w_i$. But by expressing payoffs in time units instead, through the ratios $\pi^W_i$ and $\pi^L_i$, equilibria for the full range of value function elasticities can be studied without technical complications that arise if, say, monetary payoffs are used.

The $i^{th}$ individual's optimal choice of arrival time $t_i$ will depend on the individual's time value of the prize $a_i$ and on how early other prospective recipients are expected to arrive. Of course each individual's expectations about others' arrival times will be affected by the individual's or willingness to wait, subjective beliefs regarding other individuals' time values for the prize. We assume that individuals know their own time values for the prize, and that all individuals have identical subjective beliefs about the possible time values of prizes for rival claimants. Specifically, each person believes that the time values of prizes for the $n-1$ rivals are independent realizations of a continuous random variable with a distribution function (d.f.) denoted by $G(a)$ and a density function (p.d.f.) denoted by $g(a)$, where the letter "$a$" represents the time value of a prize. Recall that $a_i = \frac{v(w_i)}{w_i}$, so the distribution of the time values of prizes is determined by the $v(w_i)$ function and by the underlying population distribution
of opportunity costs of time. Thus \( G(a) \) can be thought of as being the
d.f. of the distribution of time values of prizes in the underlying popu-
lation of potential recipients. It is further assumed that \( g(a) > 0 \) on
a finite open interval. The \( a_i \) and \( w_i \) are positively related if the
elasticity of the value function is greater than one, and they are nega-
tively related if the elasticity is less than one.

The assumed information structure is such that each individual's un-
certainty about rivals' time values of prizes is represented by the same
probability distribution. With symmetric information and symmetric
payoff functions, the essential difference among individuals is in their
time values for prizes, so we will only consider symmetric equilibria in
which an individual's equilibrium arrival time, measured from the time when
prizes are awarded, is a function of that individual's time value of a prize.
More precisely, a Nash equilibrium is characterized by (i) a condition which
determines which individuals decide to participate and incur the entry cost,
and (ii) a common arrival strategy function \( o(a) \) which determines the
equilibrium arrival times for all participants as a function of their own
time values for a prize.

II. Equilibrium in the Waiting Line Auction

The payoff structure in (1) and (2) is similar to that of a sealed bid
auction in which prizes are sold to the highest bidders at their own bid
prices. The value of the prize is \( a_i \) in (1), while the "bid" is \( t_i \),
and both bids and prize values are in units of time. The loser's payoff
in (2) has the somewhat unusual property that losers pay an amount equal to
their bids. This is a special case of the auction models analyzed in Holt
(1979) and Milgrom and Weber (1980). By drawing on these auction models we shall describe in this section Nash equilibrium behavior in the waiting-line auction.

In equilibrium, one would expect that individuals with higher time values for the prize will arrive earlier. This relationship, denoted by \( \sigma(a) \) with \( \sigma' > 0 \), will determine equilibrium arrival times: \( t_i = \sigma(a_i) \) for any individual \( (i = 1, \ldots, n) \) who decides to enter the queue. This \( \sigma(a) \) function will be called the equilibrium strategy function. The approach taken in this section is to assume that there is a positive valued, strictly increasing, and differentiable equilibrium strategy function and then to show by construction what this function must be.

Since the individuals with the \( m \) earliest arrival times will receive prizes, a typical individual, say individual \( i \), will win if \( t_i \) exceeds the \( m \)th largest of the other individuals' chosen arrival times. Because \( \sigma(a) \) is strictly increasing in the equilibrium to be determined, one can also say that individual \( i \) will win if \( a_i \) exceeds the \( m \)th largest of the \( n-1 \) rival time values for the prize. Let \( F(a) \) and \( f(a) \) denote the d.f. and the p.d.f. respectively for the order statistic of rank \( m \) among \( n-1 \) independent drawings from the population distribution of time values for the prize. It is straightforward, but unnecessary at this point, to compute the d.f. and the p.d.f. of this order statistic from \( m, n, \) and \( G(a) \). By definition, \( F(a) \) is the probability that the \( m \)th largest of the \( n-1 \) rival time values is less than or equal to \( a \). Thus in a symmetric equilibrium with a strictly increasing \( \sigma(a) \) function, \( F(a_i) \) is the probability that the \( i \)th individual will obtain a prize. (We ignore ties, which occur with probability zero.)
The non-negative time required to reach the waiting line, \( k \), may cause some individuals who have very low time values for the prize to decide against waiting in line. Indeed, we can identify a time value \( a^* \) so low that an individual \( i \) with \( a_i = a^* \) would be indifferent between arriving at the moment of the award and not participating at all. This value \( a^* \) has the property that the person entering the queue with a zero waiting time has the same expected payoff (zero) as that attainable by not participating at all. Then it follows from the payoff structure in (1) and (2) and the probability interpretation of \( F(a) \) that \( a^* \) is determined by

\[
F(a^*)a^* = k. \tag{3}
\]

Here \( F(a^*) \) is the probability that the \( m^{th} \) largest among rivals' time values for prizes is less than or equal to \( a^* \). Thus in a symmetric equilibrium, \( F(a^*) \) is the probability that an individual with a time value for the prize of \( a^* \) can arrive at the award time and still receive a prize. For such an individual, the expected time value of the prize benefit, \( F(a^*)a^* \), just equals the fixed time cost \( k \) of entering the auction without waiting. Thus condition (3) can be thought of as a "dissipation of expected rent at the margin" condition. If there is an \( a^* \) that satisfies equation (3) it will be unique, because \( \frac{d}{da}(F(a)a) > 0 \).

The symmetric Nash equilibrium in this waiting line auction game is completely characterized by the cutoff prize value \( a^* \) determined in (3) and by the equilibrium arrival strategy function \( \sigma(a) \). Holt (1979) has derived the equilibrium strategy function for a general auction model which includes the payoff structure in (1) and (2) as a special case. It is a
direct implication of the payoff functions defined in (1) and (2) and of equation (9) in Holt (1979) that the derivative of the equilibrium strategy function must satisfy:

\[ \sigma'(a) = af(a) \]  \hspace{1cm} (4)

for \( a > a^* \). The derivation of (4) is omitted for the sake of brevity, but we shall prove that the resulting \( \sigma(a) \) function determines a Nash equilibrium.

In order to find a specific \( \sigma(a) \) function it is necessary to have an initial condition, or starting point, for the differential equation in (4). Because a marginal individual with a time value for the prize of \( a^* \) would be indifferent between participating with a zero waiting time and not participating at all, it follows that a person with a time value for the prize which is slightly greater than \( a^* \) would be willing to participate but would not wait very long. This suggests that the initial condition is:

\[ \lim_{a \to a^*} \sigma(a) = 0 . \]  \hspace{1cm} (5)

This initial condition is a direct implication of Theorem 3 in Holt (1979) for the payoff structure in (1) and (2). One can easily verify that the \( \sigma(a) \) function in equation (6) below satisfies the differential equation in (4) and the initial condition in (5):^{11}

\[ \sigma(a) = \int_{a^*}^{a} y f(y) \, dy \]  \hspace{1cm} (6)
for \( a > a^* \). Appendix A contains a proof that if the \( i^{th} \) individual's \( n - 1 \) rivals select their arrival times \( t_j = \sigma(a_j) \) for \( a_j > a^* \), \( j \neq i \), then the \( i^{th} \) individual's expected payoff is globally maximized by choosing \( t_i = \sigma(a_i) \). Thus the \( \sigma(a) \) function in (6) determines a Nash equilibrium. Finally, it follows from (4) and (6) that the equilibrium \( \sigma(a) \) function in (6) is positive valued, strictly increasing, and differentiable as specified at the beginning of this section.

Equation (6) implies that individuals with relatively high time values for prizes will arrive relatively early. Recall that the time value of a prize \( a_i \) for individual \( i \) is \( \frac{v(w_i)}{w_i} \), so \( a_i \) and \( w_i \) are positively related when \( n > 1 \) and negatively related when \( n < 1 \). Therefore, individuals with relatively high (low) opportunity costs of time will choose relatively early (late) arrival times if \( n > 1 \), and they will choose relatively late (early) arrival times if \( n < 1 \).

Barzel's value function elasticity rule determines whether the equilibrium relationship between opportunity costs of time and arrival times is positive or negative. But with bidding behavior based on the time values for the prize rather than, say, the opportunity costs of time, the equilibrium strategy function \( \sigma(a) \) will always be strictly increasing.

III. Alternative Waiting-Line Allocation Procedures

In this section, we compare the transactions costs associated with several alternative waiting-line allocation procedures. To facilitate these comparisons, let us introduce parameters that allow adjustments to waiting time. Specifically, let time payoffs for winning and losing now be represented as
\[ \pi_i^W = a_i - \tau t_i - k \] (7)

\[ \pi_i^L = -\lambda t_i - k \] (8)

for \( i = 1, 2, \ldots, n \). In part II we considered only the case in which \( \tau = \lambda = 1 \). Suppose that two persons who arrive in succession can hold one another's position, thereby sharing the waiting time. If the individual not in the line can make full use of the time thus released, then such sharing reduces waiting time by 50%. That case can be represented by payoffs (7) and (8) with \( \tau = \lambda = \frac{1}{2} \). Or if individuals who arrive too late to receive a prize are immediately informed, so they do not have to spend time in line, the payoff structure would be as in (7) and (8) with \( \tau = 1 \) and \( \lambda = 0 \). We always assume \( 0 < \tau \leq 1 \) and \( 0 < \lambda \leq \tau \). Note these arrangements can make the actual waiting time of individual \( i \) less than the arrival time \( t_i \).

When the payoff functions in (1) and (2) are replaced by (7) and (8), it is shown in Appendix A that the following \( \sigma(a) \) function is a Nash Equilibrium:

\[ \sigma(a) = \frac{1}{\frac{1}{\tau F(a) + \lambda(1 - F(a))}} \int_{a^*}^{a} y f(y) \, dy \] (9)

for \( a > a^* \). The \( a^* \) cutoff is still determined in equation (3). Note that (9) reduces to (6) when \( \tau = \lambda = 1 \).

Thus for a participant with a time value of the prize of \( a \), \( F(a) \) is the probability of winning after a wait of \( \tau \sigma(a) \), and \( 1 - F(a) \) is
the probability of losing after a wait of \( \lambda\sigma(a) \). The expected waiting
time, denoted \( T^e(a) \), will not necessarily equal \( \sigma(a) \) now because

\[
T^e(a) = \tau\sigma(a)F(a) + \lambda\sigma(a)[1 - F(a)] .
\]  

(10)

By comparing (10) with (9) it is clear that in equilibrium the expected
waiting time is

\[
T^e(a) = \int_{a}^{a^*} y f(y) \, dy ,
\]  

(11)

for \( a > a^* \). The equilibrium expected payoff (in time units) for a person
with a prize value of \( a \), facing the payoff structure in (7) and (8), can
now be written as

\[
aF(a) - T^e(a) - k .
\]  

(12)

The expected money payoff for an individual is the expected payoff in (12)
multiplied by the individual's opportunity cost of time.

We next evaluate alternative procedures through variations in the \( \tau \)
and \( \lambda \) parameters. We also examine the possibility of resale and the possi-
bility that \( m \) and \( n \) -- the numbers of prizes and participants -- are random.

A. Effect of Waiting by Losers

When losers and winners both wait in line until prizes are awarded,
the payoff functions are as in (1) and (2), which are the same as (7) and
(8) with \( \tau = \lambda = 1 \). If losers need not wait in line, however, \( \tau = 1 \)
and \( \lambda = 0 \); the value \( \tau = 1 \) includes the waiting cost in the payoff
for winning but the \( \lambda = 0 \) value eliminates the waiting cost from the
payoff for losing. The question of interest is whether waiting by losers affects arrival strategies, and if it does whether consumers are better off under either of the procedures represented by $\lambda = 1$ or $\lambda = 0$.

It is obvious that a change in $\lambda$ will not affect the $a^*$ cutoff determined in equation (3) but that such a change will affect the equilibrium arrival times determined by (9). The equilibrium effect of a reduction from $\lambda = 1$ to $\lambda = 0$ is to increase the waiting time for all participants regardless of their prize values. This is not surprising; individuals will tend to arrive earlier when the waiting cost penalty for losing is eliminated. However, observe that the expected waiting time, $T_e(a)$ in (11), is unaffected by $\lambda$. Thus for participants the time saving from not waiting in the event of a loss is exactly offset by the increased waiting time in the event of a win. That is why the equilibrium expected payoff in (12) for every possible prize value is independent of $\lambda$. The implication is that in this model no participant would benefit, in an expected payoff sense, from the elimination of a requirement that losers wait.

The elimination of waiting by losers might appear "fair" in an ex post sense. The waiting costs are borne entirely by those who receive prizes in each auction, so the variance of expected gain should be lower. Although the expected waiting time for any given time in (11) is the same whether losers wait or not, so over repeated auctions the average waiting burden for any one person should be the same, risk averse individuals might prefer the arrangement that requires no waiting by losers when it lowers the variance of expected gain. Risk aversion can therefore matter when alternative arrangements are considered, but its analysis is beyond our present scope.
B. **Effect of Shared Waiting**

The possibility that one person could hold a place in line for one or more others would seem to reduce wasteful queuing. Individuals will respond to any such reduction in the cost of waiting as they did when losers did not have to wait, however, by increasing their waiting time. Specifically, suppose that each person can hold the place in line for an adjacent participant and that adjacent participants will take turns waiting in line, so \( T = \lambda = \frac{1}{2} \). More generally, if each person can take turns with \( s \) others, \( T = \lambda = \frac{1}{s+1} \).

It is apparent from (9) that such reductions in \( T \) and \( \lambda \) will cause all participants to arrive earlier. It follows from (11) that the earlier arrivals exactly offset the reduced waiting requirement so that the expected waiting time for each participant is unchanged. Thus the equilibrium expected waiting time for each participant is unchanged. Thus the equilibrium expected payoff in (12) is unchanged for each time value by the introduction of sharing arrangements. Again, changes in the values of \( \lambda \) and \( T \) will affect the variance of expected gains, so it is possible that risk averse individuals would prefer some sharing procedures over others even though expected gains are unchanged.

C. **Effect of Resale Possibility**

When the elasticity of the valuation function is greater than one then those with the highest opportunity costs of time are willing to wait in line the longest, and this suggests there will be opportunities for beneficial exchange. In particular, if resale of prizes is allowed then persons with lower time costs can wait in line and profit by reselling the prizes to those with higher opportunity costs of time, because the latter
also value prizes the most. This possibility of resale can be handled by reinterpretation of the waiting-line auction model we have discussed. The resale opportunity can make effective valuation elasticities less than one for those with lower time costs, thus motivating their participation in the auction as middlemen.

When resale is allowed and is profitable it will alter the prize valuation of those with low opportunity costs of time, effectively making \( n < 1 \) to motivate their participation in the lottery as middlemen. To see this, suppose it is legal for an individual to resell any prize obtained in the waiting-line auction. And assume there is a competitive resale market, so there will be a market clearing price denoted by \( \tilde{p} \). If this resale price is known, our previous analysis can be applied with

\[
a_i = \max \left\{ \frac{v(w_i)}{w_i}, \frac{\tilde{p}}{w_i} \right\}.
\]

(13)

Any individual with \( v(w_i) > \tilde{p} \) is a potential buyer, and any individual with \( v(w_i) < \tilde{p} \) is a potential middleman. For these middlemen, 
\[
a_i = \frac{\tilde{p}}{w_i} \quad \text{and} \quad n = (d\tilde{p}/w)(w/v(w)) = 0, \quad \text{which is less than one.}
\]

So individuals with lower opportunity costs of time will line up earlier and resell to those with high opportunity costs of time. Note that this analysis can easily be modified if one person is allowed to receive more than one prize, simply by multiplying \( \frac{\tilde{p}}{w_i} \) by the number of prizes allowed before substituting it for \( v(w_i) \). The receipt of many prizes by one person cannot be allowed to upset the competitive market for prizes, however, since we rely on the competitive price \( \tilde{p} \) to value prizes.
The formulation of the waiting-line auction in this paper is not convenient for deriving propositions about the effects of resale because changes in \( \bar{p} \) will alter the probability distribution of the time values of the prizes (the \( a_i \)). Given empirical estimates of model parameters it would be possible to compute the equilibrium expected payoffs with and without resale, however, and thereby to estimate the welfare consequences of prohibiting resale.

D. Effects of Random Numbers of Prizes and Participants

Although the number of prizes is often known with certainty, as in the case of tickets to theatrical or athletic events, it also can be uncertain. For example, recent accounts of the meat scarcity in Poland describe individuals lining up at their butcher shops in advance of the morning meat delivery without knowing the amount of meat that will be available. It is straightforward to relax the assumption that \( m \) and \( n \) are known with certainty because all probabilities that depend on \( m \) and \( n \) enter the expected payoff expressions in a multiplicative manner. For example, suppose the probability that the number of prizes is an integer, \( j \), is known by all to be \( \rho_j \). The shortage assumption, that the number of prizes is no greater than the number of participants, now becomes \( \sum_j \rho_j = 1 \) where \( \Sigma_j \) denotes a sum from \( j = 1 \) to \( j = n \).

For each possible value of \( j \) there is a corresponding probability distribution, with d.f. and p.d.f. denoted by \( F_j(\cdot) \) and \( f_j(\cdot) \) respectively, which determines the probability of winning a prize in a symmetric equilibrium. Therefore \( F_j(\cdot) \) is the d.f. of the \( j^{th} \) largest of \( n - 1 \) drawings from the population distribution of time costs.
As before, individuals with time costs below some cutoff $a^*$ will not participate, so $F_j(a^*)$ is the probability that an individual could arrive at the precise time of the awards and receive one of $j$ prizes without waiting. As before, the expected payoff for a person with a time value of the prize of $a^*$ who arrives at the award time must be zero, so $\sum_j \rho_j F_j(a^*) a^* - k = 0$. This condition is equivalent to (3) if $F(a) \equiv \sum_j \rho_j F_j(a)$ [and therefore $f(a) = \sum_j \rho_j f_j(a)$].

It follows from these observations that if the distribution of number of prizes is known, the equilibrium arrival times can still be determined by equations (3) and (9) (and all calculations in Appendix A are unaffected by the reinterpretation of $F(a)$). Therefore the conclusions in the previous sections are unaffected by randomness in the number of prizes. The analysis of randomness in the number of participants can be handled in precisely the same manner.

IV. Summary

If the price and quantity of a commodity are fixed and inflexible, its available stock may have to be rationed in some nonprice way. This rationing is often done on a first-come-first-served basis with a limit on the amount of the commodity which can be claimed by any recipient. The rationing problem considered in this paper is modeled as an auction in which the prizes are units of the commodity to be distributed at a particular time, and lining up in advance of that time improves one's probability of winning a prize. Choosing an arrival time is then comparable to bidding in a sealed tender auction. Using standard methods
of analyzing such auctions, we analyze noncooperative equilibrium behavior under rather strong risk neutrality and symmetry assumptions. We consider the effects of alternative auction arrangements, as when losers are informed at the time of their arrival that they cannot win or when waiting can be shared among participants. Changes in the auction arrangements we consider do not affect expected waiting times; economies achieved in waiting are simply offset by earlier arrivals. The model can also be used to analyze situations in which resale of prizes is permitted and in which the numbers of prizes or prospective recipients are random.
Appendix A. Verification of the Nash Equilibrium Requirement

In this appendix we prove that if all but one of the individuals choose arrival times according to the strategy function (9), then the remaining individual's expected payoff is strictly maximized by also choosing an arrival time determined by (9).

Equation (11) contains a formula for the equilibrium expected waiting time which will be useful:

\[ T^e(a) = \int_{a^*}^{a} y f(y) \, dy \]  \hspace{1cm} (A1)

The resulting equilibrium expected payoff (in time units) was computed from (12). Let this equilibrium expected payoff by denoted by \( E(\pi(\sigma(a); a)) \), so

\[ E(\pi(\sigma(a); a)) = aF(a) - \int_{a^*}^{a} y f(y) \, dy - k. \]  \hspace{1cm} (A2)

While all rivals continue to use the strategy function in (9), suppose that one individual with a time value of the prize of \( a \) considers a positive arrival time \( T \neq \sigma(a) \). Recall that \( F(a) \) is the probability that an equilibrium arrival time \( \sigma(a) \) will result in a win, so the probability that \( T \) will result in a win is \( F(\sigma^{-1}(T)) \), where \( \sigma^{-1}(\cdot) \) is the inverse function for \( \sigma(\cdot) \). Thus the expected payoff (in time units) for this deviant decision, which will be denoted by \( E(\pi(T; a)) \), is:
By the definition of an inverse function, \( \sigma(\sigma^{-1}(T)) = T \). Let \( x \equiv \sigma^{-1}(T) \), so the expected waiting time in the curly braces in (A3) can be written as

\[
T^e(x) = \tau \sigma(x) F(x) + \lambda \sigma(x) [1 - F(x)].
\] (A4)

It follows immediately from a comparison of (A4) and (A1) that the expected waiting time for the deviation arrival time \( T \) is

\[
\int_{a^*}^{x} y f(y) \, dy.
\] (A5)

This result and the definition of \( x \) allow us to express the expected payoff in (A3) as

\[
E(\pi(T; a)) = a F(\sigma^{-1}(T)) - \int_{a^*}^{\sigma^{-1}(T)} y f(y) \, dy - k.
\] (A6)

The final step is to compare the individual's equilibrium expected payoff \( E(\pi(\sigma(a); a)) \) with the deviation expected payoff \( E(\pi(T; a)) \). There are two cases to be considered. First suppose that the arrival time \( T \) is less than the equilibrium arrival time \( \sigma(a) \), or equivalently that \( a > \sigma^{-1}(T) \). In this case,

\[
E(\pi(\sigma(a); a)) - E(\pi(T; a)) = a [F(a) - F(\sigma^{-1}(T))] - \int_{\sigma^{-1}(T)}^{a} y f(y) \, dy
\]

\[
= a \int_{\sigma^{-1}(T)}^{a} [1 - \frac{y}{a}] f(y) \, dy.
\] (A7)
The integrand on the right side of (A7) is positive for all $y < a$, so the integral is positive as required for a Nash equilibrium. The second case is that in which $a < \sigma^{-1}(T)$. It is straightforward to show that the equation analogous to (A7) is:

$$E\{\pi(\sigma(a); a)\} - E\{\pi(T; a)\} = a \int_{a}^{\sigma^{-1}(T)} \left[ \frac{Y}{a} - 1 \right] f(y) \, dy.$$  \hspace{1cm} (A8)

Note that the integrand on the right side of (A8) is positive for $y > a$, so the integral is positive as required for a Nash equilibrium.
Footnotes

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1 Time is used to acquire goods in virtually every economy, and Becker (1965) has described its allocation by consumers for this purpose. Consequent decisions by suppliers in markets were considered by Visscher (1975). The deliberate use of queues to achieve income distribution goals has been described by Nichols, Smolensky, and Tideman (1971).

2 See Harsanyi (1967, 1968) for a general discussion of games with incomplete information. Vickrey (1961) first analyzed a sealed tender price auction as a noncooperative game with incomplete information. See Holt (1980) for other references on price auction games. Finally, it is interesting to note that biologists have analyzed game theoretic models of animal conflicts which resemble auctions. John Riley has shown us that the payoff structure in the waiting-line auctions we consider is similar to that of the "war of attrition" game in Smith and Parker (1976).

3 Of course it may be misleading to associate low time cost people with the poor and high time cost people with the rich. The opportunity cost of time need not change monotonically with income if constraints such as the standardized work week prevent individuals from adjusting marginally the hours they work. See Sherman and Willett (1972) for analysis of this case.
Barzel's argument is quite simple. He assumes that there is a constant elasticity "demand function" for the prize commodity: \( q = \alpha p^\beta Y \gamma \) where \( q \) is quantity, \( p \) is price, \( Y \) is a consumer's income, and the Greek letters are parameters which are the same for all individuals. The inverse of this function is: \( p = \alpha^{-1/\beta} q^{1/\beta} Y^{-\gamma/\beta} \). If individuals' opportunity costs of time, denoted by \( w \), are proportional to income, then the inverse demand function can be expressed: \( p = A w^{-\gamma/\beta} \), where \( A \) is a constant which depends on \( \alpha, \beta, Y, q, \) and the constant of time cost proportionality. (We find it natural to think of \( p \) as being the reservation price of a prize consisting of \( q \) units of the prize commodity, so the demand function being considered is really an "all or nothing" demand function.) Then if all individuals must wait an equal amount of time, say \( \bar{t} \), the net value of a prize for an individual is \( A w^{-\gamma/\beta} - \bar{t} w \). Of course, the sign of the correlation between the net value of a prize and income is the same as the sign of \( -\gamma/\beta - 1 \). Thus the sign of \( -\gamma/\beta - 1 \) determines whether the queue will contain those with high or low time costs, and the waiting time \( \bar{t} \) determines the cutoff.

The argument in footnote 4 would not apply if waiting times were correlated with opportunity costs of time in equilibrium.

For the specific demand function examined in Barzel and discussed in footnote 4, \( \eta = -\gamma/\beta \).

Other arrangements, as when losers are informed on arrival and need not wait, are examined in Part III below.

This condition for winning is true even if some individuals with very low time values for the prize do not participate at all.

The relevant density formula can be found in most mathematical statistics books. For example, see Hogg and Craig (1978), p. 159, eq. (2).
If the a* which solves (3) exceeds the highest time value of a prize which can be found in the population, then there is no demand for the product. We rule out this case.

Alternatively, to derive (6) from (4) and (5) integrate both sides of (4) from a* to any time value which exceeds a* and use the initial condition, (5), to obtain the σ(a) function in (6).

This type of neutrality result has been called a "revenue equivalence theorem" in the auction literature. See Milgrom and Weber (1980).

The distribution of time values can still be known by participants even when they depend on p̅. Each person may believe the opportunity costs of time for n - 1 rivals are realizations of a continuous random variable from a known distribution function, and that information together with p̅ will yield a distribution function for a.
References


