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Constructing Stationary Gaussian Processes from Deterministic Processes With Random Initial Conditions

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Abstract

We consider a family of stationary Gaussian processes that includes the stationary Ornstein-Uhlenbeck process. We show that processes in this family can be attained as the limit of a sequence of deterministic processes with random initial conditions. Weak convergence in the supremum norm on finite time-intervals is shown. We also establish the convergence of a wide variety of long-term statistics. Our construction provides a rigorous example of how macroscopic stochastic dynamics can be derived from microscopic deterministic dynamics.

1 Introduction

In molecular dynamics, microscopic deterministic systems with many degree of freedom are often modeled by macroscopic stochastic systems with fewer degrees of freedom. For example, consider simulating a large molecule immersed in a bath of smaller particles. We may prefer not to determine the motion of all the smaller particles, but instead approximate their effect on the larger molecule by a stochastic forcing term. Thus, rather than solve a large system of ordinary differential equations for the motion of all of the particles, we solve a stochastic differential equation for the motion of only the larger molecule [4].

The goal of the present work is to provide a simple model for this type of situation. We consider a collection of non-interacting linear oscillators with random initial positions and momenta. Their motion is described by a system of linear ODEs. If we examine the sum of the positions of the oscillators as a function of time, we obtain a process that is a close approximation to a random process. Indeed, we will show that as the number of oscillators goes to infinity, this sum, suitable scaled, converges weakly in $C[0,T]$ to a continuous Gaussian process. In one choice of random frequencies, the limiting process will be the stationary Ornstein-Uhlenbeck process, the solution of a well-known linear SDE.

In [9] we proved the convergence of low-dimensional projections of sequences of non-stationary random ODEs to non-stationary stochastic processes. The present work is motivated that paper, but proves stronger results about long-term statistics and considers the situation where both the approximating and limiting processes are stationary. Physically, this choice corresponds to considering systems in their equilibrium state, which is more appropriate for many applications. Since our processes are stationary, results about long-term statistics follow almost immediately from the finite time interval results, as we shall see.

In Section 2 we will prove the weak convergence result in $C[0,T]$. In Section 3 we will discuss how our results can be used to prove the convergence of the long-term statistics of the deterministic processes to those of the limiting stochastic processes. In Section 4 we consider specific long-term statistics of interest, including autocovariances and transition probabilities.

The models in this paper are related to those used in [13], [9] and [11] to approximate the trajectories of stochastic differential equations with those of deterministic systems. Another approach to the problem is to use deterministic interacting particles systems: see [12] for an approximation to Brownian motion and [8] for an approximation to the OU process. Also, compare with [2] which studies systems driven by chaotic maps approximating white noise.

The study of approximation of Markov processes is a vast field with sophisticated techniques. See, for example, [6] and [10]. In particular, Kushner's perturbed test-function method [10] can be used to prove convergence on finite time-intervals and of long-term statistics for non-Markov process converging the Markov processes. However, given the complicated dependence structures of our deterministic processes, these methods do not easily apply here. Moreover, our example is explicit enough that we can use the theory of weak convergence in $C[0,T]$ in a straightforward fashion; see [3].

2 Weak Convergence in $C[0,T]$

We consider a collection of n oscillators, with equations of motion

$$\ddot{u}_m(t) = \omega_m^2 u_m(t), \quad m = 1, \dots n.$$

We choose the frequencies ω_m by generating an i.i.d. sequence of random variables $\{u_m\}$, $m \geq 1$, each distributed uniformly on $[0,1]$. Then we let $\omega_m = n^a u_m$ for $m = 1, \dots, n$, where a is a constant in $(0,1)$. Next, we generate an i.i.d. sequence $\{\eta_m\}$, $m \geq 1$ independently of $\{u_m\}$, each distributed as a standard

Gaussian random variable. We let oscillator m have an amplitude of $\eta_m f(n^a u_m)$, where f is a non-negative real-valued function on $[0, \infty)$ that satisfies

$$f(\omega) \leq B_1, \quad f(\omega) \leq B_2/\omega \tag{2.1}$$

for some constants B_1, B_2. The function f allows us to specify the spectral composition of the process.

With these choices, the motion of each oscillator is given by

$$u_m(t) = \eta_m f(n^a u_m) \sin(n^a u_m t + \psi_m), \quad m = 1, \ldots n,$$

for some choice of ψ_m. In order that the system of oscillators has a distribution of positions and velocities that is stationary in time, we choose $\{\psi_m\}$, $m \geq 1$ to be an i.i.d. sequence of variables uniformly distributed on $[0, 2\pi]$ independent of the $\{u_m\}, \{\eta_m\}$. Let $b = (1 - a)/2$. If we take the sum of the positions of the n oscillators and scale by n^{-b} we obtain the process

$$U_n(t) = \sum_{m=1}^{n} n^{-b} \eta_m f(n^a u_m) \sin(n^a u_m t + \psi_m). \tag{2.2}$$

We will show that with the above choice of b the approximating process converges to a Gaussian stochastic limit.

The interpretation of U_n as a random process is subtle but important. We imagine $\{u_m\}$, $\{\eta_m\}$ to be selected randomly and then held fixed. We allow n to vary without changing these two sequences. Then we generate the sequence $\{\psi_m\}$. So for each fixed $\{u_m\}$, $\{\eta_m\}$, and n, the process U_n is a random process for which the randomness only enters through $\{\psi_m\}$. Our theorems will all hold almost surely with respect to $\{u_m\}$, $\{\eta_m\}$. We use $\mathbb{E}_\psi$ to denote taking the expectation of a random variable with respect to $\{\psi_m\}$ while holding $\{\eta_m\}, \{u_m\}$ fixed. Conversely, $\mathbb{E}_{\eta,u}$ will be used to denote expectation with respect to the measure on $\{\eta_m\}, \{u_m\}$.

We define the process U on $[0, \infty)$ by

$$U(t) := \frac{1}{\sqrt{2}} \int_0^\infty f(\omega) \sin(\omega t) dW_1(\omega) + \frac{1}{\sqrt{2}} \int_0^\infty f(\omega) \cos(\omega t) dW_2(\omega), \tag{2.3}$$

where W_1 and W_2 are independent Brownian motions. An application of Ito's Lemma shows that this process is defined for all t with the assumptions (2.1) on f. In fact, U is the unique stationary Gaussian stochastic process with covariance

$$\mathbb{E}[U(s)U(t)] = C(s,t) = \frac{1}{2} \int_0^\infty f^2(v) \cos(v(t-s)) dv.$$

Kolmogorov's criterion can be applied to the covariance to show that the process U is almost surely continuous. The case of primary interest to us is when

$$f(v) = \frac{2}{\sqrt{\pi}} \frac{\sqrt{\alpha}}{\sqrt{\alpha^2 + v^2}}, \tag{2.4}$$

for some $\alpha > 0$. In this case the covariance $C(s,t) = \exp(-\alpha|t-s|)$, so U is converges to an Ornstein-Uhlenbeck process, a stationary Gaussian Markov process.

Our main theorem shows that the process U_n converges weakly in $C[0,T]$ to the process U, almost surely with respect to $\{\eta_m\}, \{u_m\}$.

Theorem 2.1 *Let f be a non-negative real-valued function that satisfies (2.1) for some B_1, B_2, for all $\omega \geq 0$. Let $\{u_m\}$, $\{\eta_m\}$, $\{\psi_m\}$, $m \geq 1$ be mutually independent i.i.d. sequences with*

$$u_1 \sim \mathcal{U}[0,1], \;\; \eta_1 \sim \mathcal{N}(0,1), \;\; \psi_1 \sim \mathcal{U}[0,2\pi].$$

Let a, b be constants such that $a, b > 0$ and $a + 2b = 1$. For $t \geq 0$, let

$$U_n(t) = \sum_{m=1}^{n} V_{n,m}(t), \quad V_{n,m}(t) = n^{-b} \eta_m f(n^a u_m) \sin(n^a u_m t + \psi_m). \tag{2.5}$$

Almost surely with respect to $\{\eta_m\}$ and $\{u_m\}$, $U_n \Rightarrow U$ in $C[0,T]$ for all $T > 0$.

Remark: Another way of stating this result is that, if we restrict U_n and U to $[0,T]$ then , for any bounded continuous $g : C[0,T] \to \mathbb{R}$,

$$\mathbb{E}_\psi g(U_n) \to \mathbb{E} g(U)$$

as $n \to \infty$, for almost all $\{\eta_m\}$ and $\{u_m\}$. Thus the statistics of U_n converge to those of U on any finite interval.

In our proof we will use the following theorem from [7].

Theorem 2.2 *(From [7]) Suppose that $\{U_n\}$ is a collection of real-valued almost surely continuous stochastic processes in $C[0,\infty)$ and that:*

- *(i) the finite dimensional distributions of U_n weakly converge to those of an almost surely continuous process U as $n \to \infty$;*
- *(ii) there exist positive constants α, β, K such that for all n*

  $$\mathbb{E}|U_n(t+u) - U_n(t)|^\alpha \leq K|u|^{(1+\beta)}$$

  for all $t, u \in [0,\infty)$.

Then $U_n \Rightarrow U$ in $C[0,T]$ for all $T > 0$. □

Proof of Theorem 2.1 The proof is contained in two lemmas in the Appendix. In Lemma A.1 we show that criterion (i) is satisfied by U_n for almost all sequences $\{u_m\}$, $\{\eta_m\}$. In Lemma A.4 we prove criterion (ii) with $\alpha = 4$, $\beta = 1$, for almost all sequences $\{u_m\}$, $\{\eta_m\}$. This gives us the desired result. □

3 Applications to Long-Term Statistics

In the previous section we considered the convergence of the statistics of U_n to those of U on the finite interval. In this section we consider long-term statistics of U_n: in particular, long-term averages of functions of processes. Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function. We define the long-term average or empirical average of g to be

$$\langle g(U_n(t)) \rangle_\infty := \lim_{\tau\to\infty} \frac{1}{\tau} \int_0^\tau g(U_n(t))dt.$$

We can ask whether the long-term averages of a function for U_n converge to those for U, that is, whether

$$\langle g(U_n(t)) \rangle_\infty \to \langle g(U(t)) \rangle_\infty \tag{3.6}$$

as $n \to \infty$.

There is another way to denote such an average. Let $\pi_t : C[0,\infty) \to \mathbb{R}$ be the function that samples the process U_n at the time t, i.e. $\pi_t(U_n) = U_n(t)$. Then we can write

$$\langle g(U_n(t)) \rangle_\infty \equiv \langle g(\pi_t(U_n)) \rangle_\infty.$$

Thus the function g depends on U_n in a point-wise fashion.

This notation suggests the following generalization. For each interval $[t, t+T]$ we define the function

$$\pi_{[t,t+T]} : C[0,\infty) \to C[0,T]$$

by

$$\pi_{[t,t+T]}U(s) = U(t+s)$$

for $s \in [0,T]$. The effect of this function is to restrict U to a finite sub-interval, and then slide it back to the origin so that it is defined on $[0,T]$. Now given any function $g : C[0,T] \to \mathbb{R}$ we can ask if, as $n \to \infty$,

$$\langle g(\pi_{[t,t+T]}U_n) \rangle_\infty \to \langle g(\pi_{[t,t+T]}U) \rangle_\infty. \tag{3.7}$$

Thus we have generalized our above question about long-term averages to functions that depend on the process at more than one point in time.

For many functions g of interest, the limits (3.6) and (3.7) do hold, as we shall prove. First, we will show that long-term statistics of U_n are equivalent to statistics of U_n on a finite interval.

Theorem 3.1 *Let $g : C[0,T] \to \mathbb{R}$ be a continuous function. Let U_n be defined as in (2.2). Then for each n and almost surely with respect to $\{\eta_m\}$, $\{u_m\}$,*

$$\langle g(\pi_{[t,t+T]}U_n)\rangle_\infty = \mathbb{E}_\psi g(\pi_{[0,T]}U_n). \tag{3.8}$$

□

We will use a result of Weyl [14] as cited in [1]; here $\mathbb{T}^n$ denotes the n-dimensional torus, the numbers $\{\omega_m\}_{m=1}^n$ are *independent* if $\sum_{m=1}^n \omega_m k_m = 0$ for integers k_m implies $k_m \equiv 0$.

Lemma 3.2 *([1, page 286]) Suppose that $(\theta_1(t),\dots,\theta_n(t))$ is a vector valued function of time such that $\theta_m(t) = \omega_m t \mod 2\pi$. Suppose that $\omega_m, m = 1,\dots,n$ are independent. Then for any Riemann integrable $h : \mathbb{T}^n \to \mathbb{R}$*

$$\lim_{\tau\to\infty} \frac{1}{\tau}\int_0^\tau h(\theta_1(t),\dots,\theta_n(t))dt$$

is defined, and is equal to

$$\mathbb{E}_\Theta h(\theta_1,\dots,\theta_n) := (2\pi)^{-n}\int_0^{2\pi}\cdots\int_0^{2\pi} h(\theta_1,\dots,\theta_n)d\theta_1\dots d\theta_n. \quad \square$$

Proof of Theorem 3.1 Note that

$$\begin{aligned} g(\pi_{[t,t+T]}U_n) &= g\left(\left\{\sum_{m=1}^n n^{-b}\eta_m f(n^a u_m)\sin(n^a u_m(t+s)+\psi_m)\right\}_{s\in[0,T]}\right) \\ &= g\left(\left\{\sum_{m=1} n^{-b}\eta_m f(n^a u_m)\sin(\theta_m(t)+n^a u_m s+\psi_m)\right\}_{s\in[0,T]}\right) \\ &=: h(\theta_1(t),\dots,\theta_n(t)), \end{aligned}$$

where we define

$$\theta_m(t) = n^a u_m t \mod 2\pi.$$

Almost surely $\{n^a u_m\}_{m=1}^n$ are independent. Therefore we can apply Lemma 3.2. We obtain

$$\begin{aligned}
\langle g(\pi_{[t,t+T]}U_n)\rangle_\infty &= \lim_{\tau\to\infty}\frac{1}{\tau}\int_0^\tau h(\theta_1,\ldots,\theta_n)dt = \mathbb{E}_\Theta h(\theta_1,\ldots,\theta_n) \\
&= \mathbb{E}_\Theta g\left(\left\{\sum_{m=1}^n n^{-b}\eta_m f(n^a u_m)\sin(\theta_m + n^a u_m s + \psi_m)\right\}_{s\in[0,T]}\right) \\
&= \mathbb{E}_\Theta g\left(\left\{\sum_{m=1} n^{-b}\eta_m f(n^a u_m)\sin(\theta_m + n^a u_m s)\right\}_{s\in[0,T]}\right)
\end{aligned}$$

since the ψ_m are constant in this calculation. This latter expression is equal to $\mathbb{E}_\psi g(U_n)$ giving us (3.8). The result (3.9) follows from (3.8) and Theorem 2.1. $\square$

Theorem 2.1 together with Theorem 3.1 gives us the following corollary.

Corollary 3.3 *If g is bounded and continuous then*

$$\langle g(\pi_{[t,t+T]}U_n)\rangle_\infty \to \mathbb{E}g(\pi_{[0,T]}U), \tag{3.9}$$

as $n\to\infty$, for almost all $\{\eta_m\},\{u_m\}$. $\square$

If U is an ergodic stochastic process, then

$$\langle g(\pi_{[t,t+T]}U)\rangle_\infty = \mathbb{E}g(\pi_{[0,T]}U),$$

almost surely, for all bounded continuous g. So we have that

$$\langle g(\pi_{[t,t+T]}U_n)\rangle_\infty \to \langle g(\pi_{[t,t+T]}U)\rangle_\infty$$

as $n\to\infty$, for all bounded continuous g. This is the case when, for example

$$f(v) = \frac{2}{\sqrt{\pi}}\frac{\sqrt{\alpha}}{\sqrt{\alpha^2+v^2}}$$

and hence the limiting process U is an Ornstein-Uhlenbeck process.

4 Examples of Convergence of Long-Term Statistics

In many important cases the function g of interest is either unbounded or discontinuous and the result of Corollary 3.3 does not apply. We address some of these cases separately in the following examples.

Example: Empirical Covariances. A question of some interest is whether the empirical covariances of U_n converge to those of U. Let $g : C[0, s] \to \mathbb{R}$ be the function

$$g(U) := U(0)U(s)$$

for some $s \in [0, \infty)$. The empirical covariance of U_n is

$$\lim_{\tau\to\infty} \frac{1}{\tau} \int_0^\tau U_n(t)U_n(t+s)dt = \langle g(\pi_{[t,t+s]}U)\rangle_\infty.$$

Since g is continuous we can apply Theorem 3.1 to obtain that

$$\langle g(\pi_{[t,t+s]}U)\rangle_\infty = \mathbb{E}_\psi g(\pi_{[0,s]}U) = \mathbb{E}_\psi U_n(0)U_n(s).$$

Since g is not bounded, we cannot use Corollary 3.3. However, in proving Lemma A.1, we show that the covariances of U_n converge to those of U. So we obtain

$$\lim_{n\to\infty} \lim_{\tau\to\infty} \frac{1}{\tau} \int_0^\tau U_n(t)U_n(t+s)dt = \mathbb{E}U(0)U(s),$$

the covariance of U.

Example: Empirical Measures. Let I be a sub-interval of $\mathbb{R}$. We may be interested in the portion of time the approximating process spends in I. This is given by $\langle \mathbf{1}_I(U_n(t))\rangle_\infty$ or $\langle \mathbf{1}_I(\pi_t U_n)\rangle_\infty$, where $\mathbf{1}_I$ is the indicator function of I. We can use neither Corollary 3.3 nor Theorem 3.1, since the function $\mathbf{1}_I$ is not continuous. However, we can still use Lemma 3.2. For each $n \geq 1$ define $h : \mathbb{T}^n \to \mathbb{R}$ by

$$h(\theta_1, \dots, \theta_n) := \mathbf{1}_I(\sum_{m=1}^n n^{-b}\eta_m f(n^a u_m)\sin(\theta_m + n^a u_m s + \psi_m)).$$

Since there are at most two points where $\mathbf{1}_I$ is discontinuous, one can check that the set of points where h is discontinuous is of measure zero in $\mathbb{T}^n$. Thus, h is Riemann integrable. As in the proof of Theorem 3.1, we can apply Lemma 3.2 to obtain that

$$\langle \mathbf{1}_I(U_n(t))\rangle_\infty = \mathbb{E}_\psi \mathbf{1}_I(U_n(0)) = \mathbb{P}\{U_n(0) \in I\}.$$

Since from Theorem 2.1, $U_n(0) \Rightarrow U(0)$, and $\mathbb{P}\{U(0) \in \partial I\} = 0$, we have from standard results about weak convergence that this quantity converges to $\mathbb{P}\{U(0) \in I\}$. If U is ergodic we then have

$$\langle \mathbf{1}_I(U_n(t))\rangle_\infty \to \langle \mathbf{1}_I(U(t))\rangle_\infty$$

as $n \to \infty$ almost surely. So for any interval I, the long-term fraction of the time that the process U_n spends in I converges to the corresponding value for U, as $n \to \infty$. This shows the convergence of univariate empirical measures.

We can also consider the case of multivariate empirical measures. Let $s_1 < \ldots < s_k \in \mathbb{R}^+$ and $I_1, \ldots I_k$ be intervals in $\mathbb{R}$. Then the analogous result holds for $g = \mathbf{1}_{\{U(s_1)\in I_1,\ldots,U(s_k)\in I_k\}}$.

Example: Empirical Transition Probabilities. Let I_1 and I_2 be sub-intervals of $\mathbb{R}$ such that I_1 has positive length. For any n, the empirically computed transition probability from I_1 to I_2 in time s is

$$\frac{\langle \mathbf{1}_{U_n(t)\in I_1, U_n(t+s)\in I_2}\rangle_\infty}{\langle \mathbf{1}_{U_n(t)\in I_1}\rangle_\infty}. \tag{4.10}$$

To understand this, imagine that U_n is in I_1 at some point in time. Roughly, (4.10) is the probability that it will be in I_2 a time s later. From the previous example, the numerator weakly converges to $\mathbb{P}\{U(0) \in I_1, U(s) \in I_2\}$ and the denominator weakly converges to $\mathbb{P}\{U(0) \in I_1\} \neq 0$. So the quotient converges to $\mathbb{P}\{U(s) \in I_2 | U(0) \in I_1\}$ as $n \to \infty$. Once again, if U is ergodic, the quotient converges to the analogous empirical transition probability for U.

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A Appendix

Lemma A.1 *Let* $\{s_k\}_{k=1}^{l}$ *be given in* $[0, T]$. *For almost all* $\{\eta_m\}$, $\{u_m\}$

$$\{U_n(s_k)\}_{k=1}^{l} \Rightarrow \mathcal{N}(0, \Sigma)$$

where Σ *has entries*

$$\sigma_{i,j} := \frac{1}{2} \int_0^\infty f^2(v) \cos(v(s_i - s_j)) dv.$$

Here, $\mathcal{N}(0, \Sigma)$ *is the Multivariate Gaussian with zero mean and covariance matrix* Σ.

To prove Lemma A.1 we need the Multivariate Central Limit Theorem as stated in [5]. The result there applies to sequences of random variables with martingale dependence relations. Here is the result specialized to independent random variables.

Theorem A.2 *(From [5]) For each* n, *let* $X_{n,m}$, $1 \leq m \leq n$ *be independent random column vectors with* $\mathbb{E}X_{n,m} = 0$. *Let* Σ *be a* $k \times k$ *matrix. For a vector* X, *denote its norm by* $|X|$ *and its transpose by* X^T. *For an event* A, *let*

$$\mathbb{E}(X; A) := \mathbb{E}(X \mathbf{1}_A).$$

Suppose

(i) $\lim_{n\to\infty} \sum_{m=1}^{n} \mathbb{E}X_{n,m} X_{n,m}^T = \Sigma$ *;*

(ii) for all $\epsilon > 0$, $\lim_{n\to\infty} \sum_{m=1}^{n} \mathbb{E}(|X_{n,m}|^2; |X_{n,m}| > \epsilon) = 0$.

Then $S_n = X_{n,1} + \cdots + X_{n,n} \Rightarrow \mathcal{N}(0, \Sigma)$ *as* $n \to \infty$. □

Proof of Lemma A.1 We fix $\{\eta_m\}$, $\{u_m\}$ for all $m \geq 1$. To apply Theorem A.2 we let the kth components of S_n, $X_{n,m} \in \mathbb{R}^n$ be

$$S_{n,k} = U_n(s_k), \quad X_{n,m,k} = V_{n,m}(s_k),$$

where U_n and $V_{n,m}$ are defined in (2.5). For each k, $V_{n,m}(s_k)$ is independent of the others and has mean zero. To check criterion (i) we compute

$$\begin{aligned}
\sigma_{j,k,n} &:= \sum_{m=1}^{n} \mathbb{E}_\psi X_{n,m,j} X_{n,m,k}^T \\
&= \sum_{m=1}^{n} \mathbb{E}_\psi V_{n,m}(s_j) V_{n,m}(s_k) \\
&= n^{-2b} \sum_{m=1}^{n} \eta_m^2 f^2(n^a u_m) \mathbb{E}_\psi \sin(n^a u_m s_j + \psi_m) \sin(n^a u_m s_k + \psi_m) \\
&= n^{-2b} \sum_{m=1}^{n} \eta_m^2 f^2(n^a u_m) \mathbb{E}_\psi \frac{1}{2} [\cos(n^a u_m (s_j - s_k)) - \cos(n^a u_m (s_j + s_k) + 2\psi_m)] \\
&= \frac{1}{2} n^{-2b} \sum_{m=1}^{n} \eta_m^2 f^2(n^a u_m) \cos(n^a u_m (s_j - s_k)).
\end{aligned}$$

We need to show that $\sigma_{j,k,n} \to \sigma_{j,k}$ almost surely with respect to $\{\eta_m\}, \{u_m\}$ as $n \to \infty$.

Taking expectations:

$$\mathbb{E}_{\eta,u} \sigma_{j,k,n} = \frac{1}{2} \int_0^{n^a} f^2(v) \cos(v(s_j - s_k)) dv.$$

Fixing j, k, we define

$$\delta_n := \sigma_{j,k,n} - \frac{1}{2} \int_0^{n^a} f^2(v) \cos(v(s_j - s_k)) dv.$$

The second term on the right converges to $\sigma_{j,k}$, so we just need to show that δ_n almost surely converges to zero.

Select an integer p such that $2pb > 1$. We will show that

$$\mathbb{E}\delta_n^{2p} \le K n^{-2pb}$$

for some positive K, ϵ. Then, by the monotone convergence theorem,

$$\mathbb{E} \sum_{n=1}^{\infty} \delta_n^{2p} = \sum_{n=1}^{\infty} \mathbb{E}\delta_n^{2p} \le \sum_{n=1}^{\infty} K n^{-2pb} < \infty. \tag{A.11}$$

Therefore $\sum_{n=1}^{\infty} \delta_n^{2p}$ will converge almost surely, and thus δ_n must converge to zero almost surely.

Note that

$$\begin{aligned}\delta_n &= \frac{1}{2}n^{-2b}\sum_{m=1}^{n}[\eta_m^2 f^2(n^a u_m)\cos(n^a u_m(s_j - s_k)) \\ &\qquad -\frac{1}{n^a}\int_0^{n^a} f^2(v)\cos(v(s_j - s_k))dv] \\ &= \sum_{m=1}^{n} y_{n,m}\end{aligned}$$

where

$$\begin{aligned}y_{n,m} &= \frac{1}{2}n^{-2b}[\eta_m^2 f^2(n^a u_m)\cos(n^a u_m(s_j - s_k)) \\ &\qquad -n^{-a}\int_0^{n^a} f^2(v)\cos(v(s_j - s_k))dv].\end{aligned}$$

Note that each $y_{m,n}$ has mean zero.

By Lemma A.3, to show that (A.11) holds, we just need to show

$$\mathbb{E}y_{m,n}^{2p} = \mathcal{O}(n^{-4pb-a}). \tag{A.12}$$

Note that

$$\begin{aligned}\mathbb{E}y_{m,n}^{2p} &\leq 2^{-2p}n^{-4pb}\left[2^{2p}\mathbb{E}\left(\eta^2 f^2(\omega)\right)^{2p} + 2^{2p}\left(n^{-a}\int_0^{n^a} f^2(v)dv\right)^{2p}\right] \\ &\leq C_1 n^{-4pb}n^{-a}\int_0^{n^a} f^{4p}(v)dv + C_2 n^{-4pb-2pa}\left(\int_0^{n^a} f^2(v)dv\right)^{2p}\end{aligned}$$

where C_1, C_2 are positive constants. From the assumptions on f, the integrals above converge as $n \to \infty$. Hence

$$\begin{aligned}\mathbb{E}y_{m,n}^{2p} &= \mathcal{O}(n^{-4pb-a}) + \mathcal{O}(n^{-4pb-2pa}) \\ &= \mathcal{O}(n^{-4pb-a})\end{aligned}$$

as required by (A.12) above. This establishes the first condition of the Lindberg-Feller theorem.

We now show that the second condition holds: with probability one with respect to $\{\eta_m\}, \{u_m\}$ and for all $\epsilon > 0$

$$\lim_{n\to\infty} \sum_{m=1}^{n} \mathbb{E}_\psi \left(\sum_{k=1}^{l} V_{n,m}^2(s_k); \sum_{k=1}^{l} V_{n,m}^2(s_k) > \epsilon^2 \right) = 0.$$

Since f is bounded by a constant B_1 we have

$$V_{n,m}^2(s_k) \leq n^{-2b}\eta_m^2 B_1^2,$$

for all k.

Defining $\tilde{\epsilon} = \epsilon/lB_1^2$, we can bound each term of the summation as follows:

$$\begin{aligned} \mathbb{E}_\psi(\sum_{k=1}^{l} V_{n,m}^2(s_k); \sum_{k=1}^{l} V_{n,m}^2(s_k) > \epsilon^2) &\leq ln^{-2b}\eta_m^2 B_1^2 \mathbb{E}_\psi(1; n^{-2b}\eta_m^2 > \tilde{\epsilon}) \\ &= ln^{-2b}\eta_m^2 B_1^2 \mathbf{1}_{\{n^{-2b}\eta_m^2 > \tilde{\epsilon}\}} \end{aligned}$$

So if we define $z_m = n^{-2b}\eta_m^2 B_1^2 \mathbf{1}_{\{n^{-2b}\eta_m^2 > \tilde{\epsilon}\}}$, and $\zeta_n = \sum_{m=1}^{n} z_m$, we just need to show that $\lim_{n\to\infty} \zeta_n = 0$, almost surely with respect to $\{\eta_m\}_{m\geq 1}$. To do this it suffices to show that $\mathbb{E}_{\eta,u}\zeta_n^2 = \mathcal{O}(n^{-2})$. Now

$$\begin{aligned} \mathbb{E}_{\eta,u}\zeta_n^2 &= \mathbb{E}_{\eta,u}\left(\sum_{m=1}^{n} z_m\right)^2 = \mathbb{E}_{\eta,u}\left(\sum_{m=1}^{n}(z_m - \mathbb{E}_{\eta,u}z_m) + n\mathbb{E}_{\eta,u}z_m\right)^2 \\ &\leq 2\mathbb{E}_{\eta,u}\sum_{m=1}^{n}(z_m - \mathbb{E}_{\eta,u}z_m)^2 + 2n^2(\mathbb{E}_{\eta,u}z_m)^2 \\ &\leq 2\sum_{m=1}^{n}[\mathbb{E}_{\eta,u}z_m^2] + 2n^2[\mathbb{E}_{\eta,u}z_m]^2 \\ &= 2nn^{-4b}B_1^4\mathbb{E}_{\eta,u}[\eta_m^4\mathbf{1}_{\{\eta_m^2 > n^{2b}\tilde{\epsilon}\}}] + 2n^2n^{-4b}B_1^4\mathbb{E}_{\eta,u}[\eta_m^2\mathbf{1}_{\{\eta_m^2 > n^{2b}\tilde{\epsilon}\}}]^2 \\ &\leq Cn^{2-4b}\mathbb{E}_{\eta,u}[\eta_m^4\mathbf{1}_{\{\eta_m^2 > n^{2b}\tilde{\epsilon}\}}]. \end{aligned}$$

But

$$\mathbb{E}[\eta_m^4\mathbf{1}_{\{\eta_m^2 > n^2b\tilde{\epsilon}\}}] \leq \frac{1}{2n^{4-4b}}\mathbb{E}\eta_m^8 + \frac{n^{4-4b}}{2}\mathbb{E}\mathbf{1}_{\{\eta_m^2 > n^2b\tilde{\epsilon}\}}.$$

By noting that the distribution of η_m has exponentially decaying tails, the required $\mathcal{O}(n^{-2})$ bounds follows. $\Box$

Lemma A.3 *For $n \geq 1$, $m = 1, \ldots, n$, let $y_{n,m}$ be i.i.d. random variables with $\mathbb{E}y_{n,m} = 0$ and*

$$\mathbb{E}y_{m,n}^{2p} = \mathcal{O}(n^{-4pb-a})$$

for all m, n. Let

$$\delta_n = \sum_{m=1}^{n} y_{m,n}$$

for $n \geq 1$. Then

$$\mathbb{E}\delta_n^{2p} = \mathcal{O}(n^{-2bp}).$$

Proof For each n, $\{y_{n,m}\}, m = 1, \ldots, n$ is a sequence of i.i.d. random variables of mean zero. This means that when we take an even power of δ_n most of the terms have expectation 0. We will bound the sum of the others. Let y be a random variable distributed as $y_{n,m}$. Then we can write $\mathbb{E}\delta_n^{2p}$, p a positive integer, as

$$\begin{aligned} & K_1 \sum_{1 \leq i \leq n} \mathbb{E}y^{2p} \\ + \quad & K_2 \sum_{1 \leq i_1 \leq i_2 \leq n} \quad \sum_{p_1 + p_2 = p} \mathbb{E}\left[y^{2p_1}\right] \mathbb{E}\left[y^{2p_2}\right] \\ + \quad & \cdots \\ + \quad & K_j \sum_{1 \leq i_1 \leq \ldots \leq i_j \leq n} \quad \sum_{p_1 + \ldots + p_j = p} \mathbb{E}\left[y^{2p_1}\right] \cdots \mathbb{E}\left[y^{2p_j}\right] \\ + \quad & \cdots \\ + \quad & K_p \sum_{1 \leq i_1 \leq \ldots \leq i_p \leq n} \left[\mathbb{E}y^2\right]^p . \end{aligned}$$

Here the summations over the indices $i_k, k = 1, \ldots, j$ denotes the summation over all ordered sequences of j integers between 1 and n, of which there are $\mathcal{O}(n^j)$. The summation over indices $p_k, k = 1, \ldots, j$ is over all sets of j positive integers that add up to p, of which the number is independent of n. The K_j depend on p but not on n.

We can now use the bound on $y_{m,n}$ in the expression for $\mathbb{E}\delta_n^{2p}$, recalling that

$1 - a = 2b$.

$$\begin{aligned}\mathbb{E}\delta_n^{2p} &\leq \sum_{j=1}^{p} \mathcal{O}(n^j)\mathcal{O}(n^{-4pb-ja}) \\ &= \sum_{j=1}^{p} \mathcal{O}(n^{-4pb+j(1-a)}) \\ &= \mathcal{O}(n^{-4pb+p(1-a)}) = \mathcal{O}(n^{-2bp})\end{aligned}$$

as required. □

We now establish weak convergence by proving the required tightness result.

Lemma A.4 *For almost all* $\{\eta_m\}, \{u_m\}$ *there is a constant* K *such that*

$$\mathbb{E}_\psi |U_n(s+r) - U_n(s)|^4 \leq Kr^2$$

for all $s, r \in [0, T]$. *Here* K *depends on* η_m *and* u_m *but not on* n *or* r.

Proof First note that

$$\begin{aligned}&U_n(s+r) - U_n(s) \\ &= \sum_{m=1}^{n} V_{n,m}(s+r) - V_{n,m}(s) \\ &= \sum_{m=1}^{n} n^{-b}\eta_m f(n^a u_m)\{\sin(n^a u_m(s+r) + \psi_m) - \sin(n^a u_m s + \psi_m)\} \\ &= \sum_{m=1}^{n} n^{-b}\eta_m f(n^a u_m) 2\sin(\frac{n^a u_m r}{2})\cos(\frac{n^a u_m(2s+r)}{2} + \psi_m).\end{aligned}$$

We take the fourth power and then take the expectation. Noting that

$$\frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} \cos(\alpha+\psi_1)^2\cos(\beta+\psi_2)^2 d\psi_1 d\psi_2 = \frac{1}{4}$$

and

$$\frac{1}{2\pi}\int_0^{2\pi} \cos(\alpha+\psi)^4 d\phi = \frac{3}{8}$$

and that all of the other terms have zero expectation we get

$$
\begin{aligned}
&\mathbb{E}_\phi |U_n(s+r) - U_n(s)|^4 \\
&= 6 \sum_{j<k} n^{-4b} \eta_j^2 \eta_k^2 f(n^a u_j)^2 f(n^a u_k)^2 16 \sin(n^a u_j r/2)^2 \sin(n^a u_k r/2)^2 \frac{1}{4} \\
&\quad + \sum_{m=1}^{n} n^{-4b} \eta_m^4 f(n^a u_m)^4 16 \sin(n^a u_m r/2)^4 \frac{3}{8} \\
&\le 48 \left[\frac{1}{2} n^{-2b} \sum_{m=1}^{n} \eta_m^2 f^2(n^a u_m) \sin(n^a u_m r/2)^2 \right]^2
\end{aligned}
$$

Dividing this last bracketed term by r we see that we need to show

$$
\sup_{n, r \in [0,T]} n^{-2b} \sum_{m=1}^{n} \eta_m^2 f^2(n^a u_m) \sin(n^a u_m r/2)^2 r^{-1}
$$

is finite almost surely. We break the expression up into small parts that are easier to work with. Recalling that $f(\omega) \le B_2/\omega$,

$$
\begin{aligned}
&n^{-2b} \sum_{m=1}^{n} \eta_m^2 f^2(n^a u_m) \sin(n^a u_m r/2)^2 r^{-1} \\
&\quad \le C n^{-2b} \sum_{m=1}^{n} \eta_m^2 \min(r^2, u_m^{-2} n^{-2a}) r^{-1} \\
&\quad = C n^{-2b} \sum_{m=1}^{n} \eta_m^2 \mathbf{1}_{\{u_m^{-1} n^{-a} \le r\}} u_m^{-2} n^{-2a} r^{-1} + n^{-2b} \sum_{m=1}^{n} \eta_m^2 \mathbf{1}_{\{u_m^{-1} n^{-a} > r\}} r.
\end{aligned}
$$

Considering the first term we get

$$
\begin{aligned}
n^{-2b} \sum_{m=1}^{n} \eta_m^2 \mathbf{1}_{\{u_m^{-1} n^{-a} \le r\}} u_m^{-2} n^{-2a} r^{-1} &\le n^{-2b} \sum_{m=1}^{n} \eta_m^2 u_m^{-1} n^{-a} \\
&= n^{-1} \sum_{m=1}^{n} \eta_m^2 u_m^{-1}
\end{aligned}
$$

By the Strong Law of Large Numbers, this series is convergent as n goes to infinity for almost all $\{\eta\}$, $\{u\}$. Therefore, it is bounded almost surely. Thus we only need to show

$$
\sup_{n, r \in [0,T]} n^{-2b} \sum_{m=1}^{n} \eta_m^2 \mathbf{1}_{\{u_m^{-1} n^{-a} > r\}} r
$$

is bounded almost surely. This is more difficult. Note that if $r < n^{-a}$ then the term takes the very simple form

$$n^{-2b}\sum_{m=1}^{n}\eta_m^2 n^{-a} = n^{-1}\sum_{m=1}^{n}\eta_m^2$$

which is bounded almost surely by the SLLN as before. Moreover, we can simplify the $\sup$ with respect to r in $[n^{-a}, T]$. Let us define

$$r_k := \frac{T}{2^k}$$

for $k \geq 1$. Then for any $r \in [r_k, r_{k-1}]$

$$\begin{aligned} n^{-2b}\sum_{m=1}^{n}\eta_m^2 \mathbf{1}_{\{u_m^{-1}n^{-a}>r\}} r &\leq n^{-2b}\sum_{m=1}^{n}\eta_m^2 \mathbf{1}_{\{u_m^{-1}n^{-a}>r_k\}} r_{k-1} \\ &= 2n^{-2b}\sum_{m=1}^{n}\eta_m^2 \mathbf{1}_{\{u_m^{-1}n^{-a}>r_k\}} r_k. \end{aligned}$$

So

$$\sup_{r\in[n^{-a},T]} n^{-2b}\sum_{m=1}^{n}\eta_m^2 \mathbf{1}_{\{u_m^{-1}n^{-a}>r\}} r \leq \sup_k 2n^{-2b}\sum_{m=1}^{n}\eta_m^2 \mathbf{1}_{\{u_m^{-1}n^{-a}>r_k\}} r_k.$$

where k runs from 1 to $\lceil \log_2 T n^a \rceil$. Thus there are only $\mathcal{O}(\log n)$ terms with respect k.

Putting this all together, we have that

$$\begin{aligned} &\sup_{n,r\in[0,]} n^{-2b}\sum_{m=1}^{n}\eta_m^2 \mathbf{1}_{\left\{u_m^{-1}n^{-a}>r\right\}} r \\ &\leq \sup_{n,r\in[0,n^{-a}]} n^{-1}\sum_{m=1}^{n}\eta_m^2 + \sup_{n,r\in[n^{-a},T]} n^{-2b}\sum_{m=1}^{n}\eta_m^2 \mathbf{1}_{\left\{u_m^{-1}n^{-a}>r\right\}} r \\ &\leq C + \sup_{n,k=1,\ldots,\lceil T\log_2 n^a\rceil} 2n^{-2b}\sum_{m=1}^{n}\eta_m^2 \mathbf{1}_{\left\{u_m^{-1}n^{-a}>r_k\right\}} r_k. \end{aligned}$$

This reduces the problem to that of determining the supremum of countably many terms. So let us define

$$x_{n,k} := n^{-2b}\sum_{m=1}^{n}\eta_m^2 \mathbf{1}_{\left\{u_m^{-1}n^{-a}>r_k\right\}} r_k.$$

and define

$$\mu_{n,k} := \mathbb{E}x_{n,k} = n^{-2b}\sum_{m=1}^{n}\eta_m^2 n^{-a},$$

which does not depend on k, and

$$\mu_{\sup} := \sup_{n,k}\mu_{n,k}$$

which is finite by the SLLN.

We will show that $\sup_{n,k} x_{n,k}$ is finite almost surely by showing that $\mathbb{P}\{\sup_{n,k} x_{n,k} > K + \mu_{sup}\}$ converges to 0 as $K \to \infty$. First note that

$$\begin{aligned}\mathbb{P}\{\sup_{n,k} x_{n,k} > K + \mu_{sup}\} &\leq \sum_{n,k}\mathbb{P}\{x_{n,k} > K + \mu_{sup}\}\\ &\leq \sum_{n,k}\mathbb{P}\{x_{n,k} - \mu_{n,k} > K\}\end{aligned}$$

To obtain a bound on the summand of the last expression, we need a bound on some moment of $\delta_n := x_{n,k} - \mu_{n,k}$.

Choose p so that $4pb > 1$. Then

$$\begin{aligned}\delta_{n,k} &= x_{n,k} - \mu_{n,k}\\ &= n^{-2b}\sum_{m=1}^{n}\left[\eta_m^2\mathbf{1}_{\{u_m^{-1}n^{-a} > r_k\}}r_k - n^{-a}\right]\\ &= \sum_{m=1}^{n} y_{n,m}\end{aligned}$$

where

$$y_{n,m} := n^{-2b}\left[\eta_m^2\mathbf{1}_{\{u_m^{-1}n^{-a} > r_k\}}r_k - n^{-a}\right].$$

We can apply Lemma A.3. We just need to show that $\mathbb{E}y_{n,m}^{2p} = \mathcal{O}(n^{-4pb-a})$. Indeed

$$\begin{aligned}\mathbb{E}y_{n,m}^{2p} &\leq n^{-4pb}[2p\mathbb{E}(\eta^2\mathbf{1}_{\{u_m^{-1}n^{-a} > r_k\}}r_k^2)^{2p} + 2p(n^{-2a})^{2p}]\\ &= \mathcal{O}(n^{-4pb}n^{-a}r_k^{2p-1}) + \mathcal{O}(n^{-4pb}n^{-4ap})\\ &= \mathcal{O}(n^{-4pb-a}).\end{aligned}$$

So we have

$$\mathbb{E}\delta_{n,k}^{2p} = \mathcal{O}(n^{-2bp}).$$

This gives us a bound in probability on the our quantity of interest

$$\mathbb{P}\{|\delta_{n,k}| > K\} \leq Cn^{-2bp}/K^{2p}.$$

So

$$\begin{aligned}
\mathbb{P}\{\sup_{n,k} x_{n,k} > K + \mu_{sup}\} &\leq C \sum_{n,k} n^{-2bp}/K^{2p} \\
&\leq C \sum_{n} \lceil \log_2 Tn^a \rceil n^{-2bp}/K^{2p} \\
&\leq D/K^{2p}
\end{aligned}$$

as required. □