IDENTIFICATION OF CONSUMER DEMAND AND PRODUCTION SYSTEMS WITH LIMITED DEPENDENT VARIABLES

by

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ABSTRACT

Some similarities between the simultaneous equation Tobit model and the consumer demand and production systems with limited dependent variables are addressed. Such systems have their distinctive features for parameter identification. We consider the identification of some consumer demand and production structures without price variations across samples. The linear expenditure system and translog indirect utility function for consumers demand and the translog profit function for production economics with binding non-negative constraints are analyzed. The presence of binding nonnegative constraints has effects on the identification of all the parameters in the non-homothetic translog indirect utility function and the translog profit function but not the linear expenditure system.

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1. Introduction

Consumer demand and production systems have been studied extensively in both the theoretical and empirical economics literature; see, for example, Theil [1975, 1976], Deaton and Muellbauer [1980] and Phlips [1983], among others. The traditional empirical studies with aggregated data do not consider the presence of binding nonnegative constraints. Several recent household budget surveys from both the developed and developing countries contain detailed information on the demand for disaggregated commodities; see, Deaton and Irish [1982], Wales and Woodland [1983], Pitt [1983a, 1983b] and Strauss [1983]. A common attribute of these data sets is the presence of nonconsumption of certain commodities for some households. Such samples contain limited dependent variables in that zero expenditures on some goods occur with finite positive probabilities.

As is well known, a system of demand equations is the result of utility maximization subject to a budget constraint. The specification of demand systems with limited dependent variables must also be compatible with demand theory. This issue has been addressed in the articles by Wales and Woodland [1983] and Lee and Pitt [1983, 1984]. The Wales and Woodland's approach is a direct random utility approach in which the Kuhn-Tucker inequalities determine the zero demanded quantities. Lee and Pitt [1983, 1984] provide a general framework based on the concept of virtual prices in which both the direct and dual utility approaches can be used.

In this article, we point out the similarity of the structures of consumer demand and production systems with the structures of the conventional Tobit model (Tobin [1958]) and the simultaneous equation Tobit model (Amemiya [1974]). The simultaneous equation Tobit model has some special identification features that differ from the conventional simultaneous equation models; an example has been pointed out in Lee [1976]. This motivates the investigation of the possibility of identification of some consumer demand and production systems when there are no price variations across samples in cross sectional data. Since parametric identification depends on the specification of functional forms of the systems, our analysis will focus on some specific systems. For the consumer demand analysis, the linear expenditure system and the demand system, derived from the indirect utility function, are analyzed. For the production analysis, the translog profit function will be analyzed. These systems are analyzed either because of their simple functional form or their flexibility and popularity in empirical studies.

2. Consumer Demand, Production Structures and Tobit Models

The econometric models of consumer demand and production structures with binding nonnegative constraints set out in Wales and Woodland [1983] and Lee and Pitt [1983], have structural differences but also similarities with the familiar Tobit model (Tobin [1958]) and the simultaneous equations Tobit model of Amemiya [1974].

As in traditional demand theory, the demand for goods is a function of prices p and income M;

$$(2.1) \qquad q_i = D_i(p, M) \qquad\qquad i = 1,\ldots,K$$

which are defined on $p > 0$ and $M > 0$, where K is the number of goods and p is a K dimensional price vector. These equations are derived from utility maximization subject to a budget constraint $py = M$. When $q_i \geq 0$ for all i, it corresponds to the solution of the traditional problem

$$(2.2) \qquad \max_y \{U(y) \mid py = M, \quad y \geq 0\}.$$

When some of the vector q in (2.1) are negative, the corresponding nonnegativity constraint problem (2.2) will have a boundary solution. Zero consumption of certain goods can occur only if the utility function does not possess the interior property.[1/] Therefore, we can assume that the utility function U is not only well-defined on the nonnegative commodity space, but can also be extended mathematically into regions with negative quantities such that the equations (2.1) are the unique solutions of the problem:

$$(2.3) \qquad \max_y \{U(y) \mid py = M\}.$$

The equations (2.1) are the solutions to the utility maximization problem without nonnegativity constraints (2.3), and are referred to as notional demand equations.

The notional demands q_i are latent variables which provide the underlying structure as in the Tobit model to determine the limited observations.

For a given vector of prices p, the observed demand quantities vector x is the (nonnegative) solution to (2.2). If some of the x_i are zero, the vector x may not necessarily equal $D(p, M) = (D_1(p, M),\ldots,D_K(p, M))'$. There are many possible patterns of consumed and non-consumed goods. Each pattern constitutes a single demand regime. In Lee and Pitt [1983], the concept of virtual prices in the quantity rationing literature (Rothbarth [1941] and Deaton [1981]) is used. Each observed demand vector x can be supported by a virtual price vector which provides justification of structural change in each regime. The virtual prices replace the actual prices of those goods which are not consumed. For example, the vector of virtual prices p^* corresponding to the regime with zero quantities demanded for the first L goods and positive quantities for the remaining K-L goods is characterized by $p_i^* = p_i$ for $i = L+1,\ldots,K$, and

$$(2.4) \qquad x_i = D_i(p_1^*,\ldots,p_L^*, p_{L+1},\ldots,p_K, M) \qquad i = L+1,\ldots,K$$

$$(2.5) \qquad 0 = D_i(p_1^*,\ldots,p_L^*, p_{L+1},\ldots,p_K, M) \qquad i = 1,\ldots,L.$$

The virtual prices $p_1^*,\ldots,p_L^*$ can be solved from (2.5) and are functions of the observed prices $p_{L+1},\ldots,p_K$ and income M, i.e.,

$$(2.6) \qquad p_i^* = \xi_i(\bar{p}, M) \qquad i = 1,\ldots,L,$$

where $\bar{p} = (p_{L+1},\ldots,p_K)'$. The observed positive demand quantities can be rewritten as

$$(2.7) \qquad x_i = D_i(\xi_1(\bar{p}, M),\ldots,\xi_L(\bar{p}, M), \bar{p}, M) \qquad i = L+1,\ldots,K.$$

The notional demand equations and virtual prices provide, also, the conditions to characterize the occurrence of the different consumption regimes. The regime, for which the demanded quantities of the first L goods are zero and the quantities of the remaining K-L goods are positive, is characterized by the conditions

$$
(2.8)\quad
\begin{aligned}
&D_1(p_1,\ \xi_{12}(p_1,\ \bar{p},\ M),\ldots,\xi_{1L}(p_1,\ \bar{p},\ M),\ \bar{p},\ M) \leq 0\\
&D_2(\xi_{21}(p_2,\ \bar{p},\ M),\ p_2,\ \xi_{23}(p_2,\ \bar{p},\ M),\ldots,\xi_{2L}(p_2,\ \bar{p},\ M),\ \bar{p},\ M) \leq 0\\
&\quad\vdots\\
&D_L(\xi_{L1}(p_L,\ \bar{p},\ M),\ldots,\xi_{LL-1}(p_L,\ \bar{p},\ M),\ p_L,\ \bar{p},\ M) \leq 0\\
&D_i(\xi_1(\bar{p},\ M),\ldots,\xi_L(\bar{p},\ M),\ \bar{p},\ M) > 0, \qquad i = L+1,\ldots,K
\end{aligned}
$$

where $\xi_j(\bar{p},\ M)$, $j = 1,\ldots,L$ are the solutions to (2.5) and the virtual prices $\xi_{12}(p_1,\ \bar{p},\ M),\ldots,\xi_{1L}(p_1,\ \bar{p},\ M)$ are the solutions to the equations

$$
(2.9)\qquad 0 = D_i(p_1,\ p_2^*,\ldots,p_L^*,\ \bar{p},\ M) \qquad i = 2,3,\ldots,L
$$

and the other ξ's are similarly defined. For econometric analysis, the notional demand equations are specified with a finite unknown parameter vector θ and stochastic components ε, i.e.,

$$
(2.10)\qquad q_i = D_i(p,\ M;\ \theta,\ \varepsilon) \qquad i = 1,\ldots,K.
$$

The stochastic components reflect random preferences or other unexplained factors in consumer decision, not observed by econometricians.

The similarity of this system and the Tobit model can be first illustrated in the two goods case. The notional demand equations for the two goods case are

$$
q_1 = D_1(p_1,\ p_2,\ M;\ \theta,\ \varepsilon)
$$

and

$$
q_2 = D_2(p_1,\ p_2,\ M;\ \theta,\ \varepsilon).
$$

The two equations are functionally dependent because they satisfy the budget constraint $p_1q_1 + p_2q_2 = M$. There are three different regimes for the observed quantities $x = (x_1, x_2)$;

Regime 1: Both $x_1 > 0$, $x_2 > 0$. The positive demand equations are $x_1 = D_1(p_1, p_2, M; \theta, \varepsilon)$ and $x_2 = D_2(p_1, p_2, M; \theta, \varepsilon)$. The regime conditions are $D_1(p_1, p_2, M; \theta, \varepsilon) > 0$ and $D_2(p_1, p_2, M; \theta, \varepsilon) > 0$.

Regime 2: $x_1 = 0$ and $x_2 > 0$. The positive demand equation is $x_2 = D_2(\xi_1, p_2, M; \theta, \varepsilon) = M/p_2$ and the regime condition is $D_1(p_1, p_2, M; \theta, \varepsilon) \leq 0$.

Regime 3: $x_1 > 0$ and $x_2 = 0$. The positive demand equation is $x_1 = D_1(p_1, \xi_2, M; \theta, \varepsilon) = M/p_1$ and the regime condition is $D_2(p_1, p_2, M; \theta, \varepsilon) \leq 0$.

Because of the budget constraint, the condition $D_2(p_1, p_2, M; \theta, \varepsilon) \leq 0$ is equivalent to the condition $D_1(p_1, p_2, M; \theta, \varepsilon) \geq M/p_1$ and the condition $D_1(p_1, p_2, M; \theta, \varepsilon) \leq 0$ is equivalent to the condition $D_2(p_1, p_2, M; \theta, \varepsilon) \geq M/p_2$. The three regimes can be combined into a switching equation model and we have

$$
\begin{aligned}
x_1 &= M/p_1 && \text{if } D_1(p_1, p_2, M; \theta, \varepsilon) \geq M/p_1 \\
&= D(p_1, p_2, M; \theta, \varepsilon) && \text{if } M/p_1 > D_1(p_1, p_2, M; \theta, \varepsilon) > 0 \\
&= 0 && \text{if } D_1(p_1, p_2, M; \theta, \varepsilon) \leq 0.
\end{aligned}
\tag{2.9}
$$

The model (2.9) is, in effect, the two-limit Tobit model of Rosett and Nelson [1975]. If good 2 was always consumed in some positive amounts, we have a standard Tobit model. In the general case, consumer demand equations are more closely related to the multivariate and simultaneous equations Tobit models of Amemiya [1974]. As an example, consider the case where one of the commodities is always consumed, say, $x_K > 0$. Following Wales and Woodland [1983], the Kuhn-Tucker conditions for utility maximization are

$$p_K U_i(x;\ \theta,\ \varepsilon) - p_i U_K(x;\ \theta,\ \varepsilon) \leq 0, \qquad i = 1,\ldots,K-1$$

$$p_1 x_1 + \ldots + p_K x_K = M$$

and

$$x_i(p_K U_i(x;\ \theta,\ \varepsilon) - p_i U_k(x;\ \theta,\ \varepsilon)) = 0, \qquad i = 1,\ldots,K-1$$

After substituting $x_K = (M - \Sigma_{i=1}^{K-1} p_i x_i)/p_K$ into the remaining equations, these conditions can be rewritten as

$$F_i(x_1,\ldots,x_{K-1};\ \theta,\ \varepsilon) \geq 0, \qquad i = 1,\ldots,K-1$$

where $F_i(x_1,\ldots,x_{K-1};\ \theta,\ \varepsilon) = p_i U_K(x;\ \theta,\ \varepsilon) - p_K U_i(x;\ \theta,\ \varepsilon)$, and

$$x_i = 0 \quad \text{if} \quad F_i(x_1,\ldots,x_{i-1},\ 0,\ x_{i+1},\ldots,x_{K-1};\ \theta,\ \varepsilon) > 0;$$

$$F_i(x_1,\ldots,x_{K-1};\ \theta,\ \varepsilon) = 0, \quad \text{otherwise.}$$

This system, which relates the dependent variables $x_1,\ldots,x_{K-1}$ in a direct interactive way, is similar to the formulation of the simultaneous equation model with limited dependent variables in Amemiya [1974], even though the equations in the latter model are usually expressed in linear form and the former system may not have the conventional meaning of simultaneity. The strict quasi-concavity of the utility function implies a unique solution to the above system and therefore is analogous to the coherency condition requirement in Amemiya's model.

The problem of corner solutions can also occur in production economics with microeconomic data. It may be optimal for economic reasons for the firm to choose certain inputs but not others, or, in the case of a multiple output technology, nonproduced outputs. The approach for the consumer demand can be extended to the production analysis. The detail is referred to in Lee and Pitt [1983b]. The derived inputs and outputs equations are also related to the Tobit models. Consider, for example, the profit maximization of firms with production function $f(x_1,\dots,x_m)$ where x's are inputs, i.e.,

$$\max_x \; rf(x) - p'x$$
$$\text{subject to} \quad x \geq 0$$

where $x = (x_1,\dots,x_m)'$, r is the ouptut price and p is the vector of input prices. The Kuhn-Tucker conditions are

$$r\frac{\partial f(x)}{\partial x_i} - p_i \leq 0, \qquad i = 1,2,\dots,m$$

and

$$x_i = 0, \quad \text{if} \quad r\frac{\partial f(x)}{\partial x_i} - p_i < 0;$$

$$r\frac{\partial f(x)}{\partial x_i} - p_i = 0, \quad \text{otherwise}$$

With stochastic elements introduced into the system, it is related to the simultaneous Tobit models in Amemiya [1974].

Simultaneous equation models with limited dependent variables have special structures which differ from the traditional simultaneous equation models. They have also distinctive identification properties which have been illustrated in a simple two equation model in Lee [1976]. This motivates the investigation of the possibility of identification of the consumer demand and production systems with limited dependent variables and without price variations in cross sectional micro-economics data in the subsequent sections. It is hoped that the zero consumption of certain goods creates spillover effects to the consumption of other goods and helps identification of the parameters in the system.

3. Identification of The Linear Expenditure System for Consumer Demand

The linear expenditure system has been used extensively in empirical works (see, e.g. Stone [1954]). This system is attractive because of its linear structures in expenditures, even though it is restrictive in that the implied Engel curve is linear. The linear expenditure system is specified as

$$(3.1) \qquad p_i x_i = p_i \beta_i + \alpha_i (M - \Sigma_{j=1}^{K} \beta_j p_j) + \varepsilon_i \qquad i = 1,\ldots,K$$

where ε_i are additive disturbances and are assumed to be multivariate normal $N(0, \Sigma)$. The system implies that $\Sigma_{i=1}^{K} \alpha_i = 1$ and $\Sigma_{i=1}^{K} \varepsilon_i = 0$. Without price variations, the prices p_i, $i = 1,\ldots,K$, are (known) constants across samples. However, for cross sectional data, income M will, in general, be varied in the sample. Denote

$$(3.2) \qquad c_i = p_i \beta_i - \alpha_i \Sigma_{j=1}^{K} \beta_j p_j \qquad i = 1,\ldots,K$$

which are constants. It is noted that $\Sigma_{i=1}^{K} c_i = 0$. Equation (3.1) can be rewritten as

$$(3.1') \qquad p_i x_i = c_i + \alpha_i M + \varepsilon_i \qquad i = 1,\ldots,K$$

First, consider the interior regime where all the goods are consumed. The conditions that determine the occurrence of this regime are

$$\varepsilon_i > -(c_i + \alpha_i M) \qquad i = 1,\ldots,K$$

or equivalently,

$$\varepsilon_i > -(c_i + \alpha_i M) \qquad i = 1,\ldots,K-1$$

$$\Sigma_{j=1}^{K-1} \varepsilon_j < - \Sigma_{j=1}^{K-1} c_j + (1 - \Sigma_{j=1}^{K-1} \alpha_j)M$$

Conditional on the subsamples corresponding to this interior regime, the observed expenditure equations $p_1x_1,\ldots,p_{K-1}x_{K-1}$ will be a multivariate truncated normal Tobit model with a likelihood function consisting of products of truncated multivariate normal density functions (e.g., Hausman and Wise [1977]). The parameters c_i, α_i, $i = 1,\ldots,K$ and Σ can be identified from the truncated distribution, given that the incomes M are varying across samples. A detailed argument for the identification of such parameters is provided in the appendix. Given the identification of α and c, it remains to identify the K parameters β_i, $i = 1,\ldots,K$. Since $\Sigma_{i=1}^K c_i = 0$, the K equations in (3.2) are functionally dependent. There are more unknown parameters β_i than the number of independent equations, and hence the parameters β_i, $i = 1,\ldots,K$ cannot be identified from (3.2) without additional information.

Consider now the information contained in the regimes with only one nonconsumed good. Without loss of generality, consider the regime that $x_1 = 0$, $x_i > 0$ for $i = 2,\ldots,K$. The virtual price ξ_1 for good 1 at zero is

$$\xi_1 = -\frac{\alpha_1}{(1-\alpha_1)\beta_1}(M - \Sigma_{j=2}^K p_j\beta_j) - \frac{\varepsilon_1}{(1-\alpha_1)\beta_1} \tag{3.4}$$

and the remaining positive expenditure equations are

$$\begin{aligned} p_ix_i &= p_i\beta_i + \alpha_i(M - \xi_1\beta_1 - \Sigma_{j=2}^K p_j\beta_j) + \varepsilon_i \\ &= p_i\beta_i - \frac{\alpha_i}{1-\alpha_1}\Sigma_{j=2}^K p_j\beta_j + \frac{\alpha_i}{1-\alpha_1}M + \varepsilon_i + \frac{\alpha_i}{1-\alpha_1}\varepsilon_1 \qquad i = 2,\ldots,K. \end{aligned} \tag{3.5}$$

Denote the constants c_{1i} as

$$c_{1i} = p_i\beta_i - \frac{\alpha_i}{1-\alpha_1}\Sigma_{j=2}^K p_j\beta_j \qquad i = 2,\ldots,K. \tag{3.6}$$

We note that $\Sigma_{i=2}^K c_{1i} = 0$. The regime conditions for this regime are

$$\begin{aligned} &\varepsilon_1 < -(c_1 + \alpha_1 M) \\ &\varepsilon_i + \frac{\alpha_i}{1-\alpha_1}\varepsilon_1 > -\left(c_{1i} + \frac{\alpha_i}{1-\alpha_1}M\right), \qquad i = 2,\ldots,K. \end{aligned}$$

Conditional on the subsamples of this regime, the truncated multivariate density function will provide the identification of the parameters c_{1i}, $i = 2,\ldots,K$ given the identification of the α, Σ and c from the interior regime. The remaining question is whether the identified constants c_{1i}, $i = 2,\ldots,K$ in addition to c_i, $i = 1,\ldots,K$ can provide the identification for β. Unfortunately, the answer to this question is negative. This is so because c_{1i}, $i = 2,\ldots,K$ are linearly dependent on the constants c_i, $i = 1,\ldots,K$. This can be shown as follows. Consider the equations (3.2) and (3.6),

$$\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_{K-1} \\ c_{1i} \end{pmatrix} = A_i \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_K \end{pmatrix}$$

where

$$(3.7)\quad A_i = \begin{pmatrix} (1-\alpha_1)p_1 & -\alpha_1 p_2 & \cdots\cdots & -\alpha_1 p_{K-1} & -\alpha_1 p_K \\ -\alpha_2 p_1 & (1-\alpha_2)p_2 & & -\alpha_2 p_{K-1} & -\alpha_2 p_K \\ \vdots & \vdots & & \vdots & \vdots \\ -\alpha_{K-1}p_1 & -\alpha_{K-1}p_2 & \cdots\cdots & (1-\alpha_{K-1})p_{K-1} & -\alpha_{K-1}p_K \\ 0 & -\frac{\alpha_i}{1-\alpha_1}p_2 \;\cdots\; -\frac{\alpha_i}{1-\alpha_1}p_{i-1} & (1-\frac{\alpha_i}{1-\alpha_1})p_i \; -\frac{\alpha_i}{1-\alpha_1}p_{i+1} \;\cdots & -\frac{\alpha_i}{1-\alpha_1}p_{K-1} & -\frac{\alpha_i}{1-\alpha_1}p_K \end{pmatrix}$$

The matrix A_i can be rewritten as

$$A_i = B_1 + B_{2i} + B_{3i}$$

where

$$B_1 = \begin{bmatrix} p_1 & & & \\ & p_2 & & 0 \\ & & \ddots & \\ 0 & & & p_K \end{bmatrix}, \qquad B_{2i} = -\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{K-1} \\ \frac{\alpha_i}{1-\alpha_1} \end{pmatrix} [p_1 \; p_2 \; \cdots \; p_K]$$

and

$$B_{3i} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} [\, \frac{\alpha_i}{1-\alpha_1} p_1, \;\; 0,\ldots,0, \;\; p_i, \;\; 0,\ldots,0, \;\; -p_K].$$

The matrix $B_1 + B_{2i}$ is invertible and the inverse is

$$(B_1+B_{2i})^{-1} = \begin{bmatrix} p_1^{-1} & & 0 \\ & \ddots & \\ 0 & & p_K^{-1} \end{bmatrix} + \frac{1-\alpha_1}{\alpha_K(1-\alpha_1)-\alpha_i} \begin{bmatrix} \alpha_1 p_1^{-1} \\ \vdots \\ \alpha_{K-1} p_{K-1}^{-1} \\ \alpha_i (1-\alpha_1)^{-1} p_K^{-1} \end{bmatrix} [1 \;\; 1 \; \ldots . \; 1].$$

The determinant of A_i can be evaluated by the following formula (see, e.g. Dhrymes [1978], p.38),

$$|A_i| = \delta_i |B_1 + B_{2i}|$$

where the constant δ_i is

$$\begin{aligned} \delta_i &= 1 + (\, \frac{\alpha_i}{1-\alpha_1} p_1, \;\; 0,\ldots,0, \;\; p_i, \;\; 0,\ldots,0, \;\; -p_K)(B_1 + B_{2i})^{-1}(0,\ldots,0, \;\; 1)' \\ &= 1 - p_K p_K^{-1} + \frac{1-\alpha_1}{\alpha_K(1-\alpha_1)-\alpha_i} \left(\frac{\alpha_i}{1-\alpha_1} \alpha_1 + \alpha_i - \frac{\alpha_i}{1-\alpha_1} \right) \\ &= 0. \end{aligned}$$

Hence the matrix A_i is singular. It is easy to see from A_i that it is of rank $K-1$ in general.[2/] Therefore, c_{1i}, $i = 2,\ldots,K$ are linearly dependent on $c_1,\ldots,c_K$ and provide no additional identification information for the identification of β.[3/]

It remains to consider the information in the other regimes with more than one nonconsumed good. Without loss of generality, consider the regime where the first L goods are not consumed. The virtual prices $\xi_1,\ldots,\xi_L$ can be derived from the following equations:

$$(3.8) \qquad 0 = \xi_i\beta_i + \alpha_i(M - \Sigma_{j=L+1}^{K} p_j\beta_j - \Sigma_{j=1}^{L} \xi_j\beta_j) + \varepsilon_i, \qquad i = 1,\ldots,L$$

and the remaining positive expenditure equations are

$$(3.9) \qquad p_i x_i = p_i\beta_i + \alpha_i(M - \Sigma_{j=L+1}^{K} p_j\beta_j - \Sigma_{j=1}^{L} \xi_j\beta_j) + \varepsilon_i, \qquad i = L+1,\ldots,K.$$

Summing over the equations in (3.8), we have

$$\Sigma_{j=1}^{L}\xi_j\beta_j = -\frac{\Sigma_{j=1}^{L} \alpha_j}{1 - \Sigma_{j=1}^{L} \alpha_j}(M - \Sigma_{j=L+1}^{K} p_j\beta_j) - \frac{\Sigma_{j=1}^{L} \varepsilon_j}{1 - \Sigma_{j=1}^{L} \alpha_j}.$$

It follows that the positive expenditure equations can be rewritten as

$$(3.10) \qquad p_i x_i = c_{Li} + \frac{\alpha_i}{1 - \Sigma_{j=1}^{L} \alpha_j} M + \varepsilon_i + \frac{\alpha_i}{1 - \Sigma_{j=1}^{L} \alpha_j} \Sigma_{j=1}^{L} \varepsilon_j \qquad i = L+1,\ldots,K$$

where

$$(3.11) \qquad c_{Li} = p_i\beta_i - \frac{\alpha_i}{1 - \Sigma_{j=1}^{L} \alpha_j} \Sigma_{j=L+1}^{K} p_j\beta_j \qquad i = L+1,\ldots,K.$$

This regime will identify the constants c_{Li}, $i = L+1,\ldots,K$. Consider now the linear equations,

$$\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_{K-1} \\ c_{Li} \end{pmatrix} = A_{Li} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_K \end{pmatrix}$$

where

$$A_{Li} = C_1 + C_{2i} + C_{3i}$$

with

$$C_1 = \begin{bmatrix} p_1 & & & 0 \\ & p_2 & & \\ & & \ddots & \\ 0 & & & p_K \end{bmatrix}, \qquad C_{2i} = - \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{K-1} \\ \dfrac{\alpha_i}{1 - \Sigma_{j=1}^{L} \alpha_j} \end{bmatrix} [p_1 \; p_2 \; \cdots \; p_K]$$

and

$$C_{3i} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \left[\frac{\alpha_i}{1 - \Sigma_{j=1}^{L} \alpha_j} p_1, \ldots, \frac{\alpha_i}{1 - \Sigma_{j=1}^{L} \alpha_j} p_L, \; 0, \ldots, 0, \; p_i, \; 0, \ldots, 0, \; -p_K\right].$$

The inverse of $C_1 + C_{2i}$ is

$$(C_1 + C_{2i})^{-1} = \begin{bmatrix} p_1^{-1} & & 0 \\ & \ddots & \\ 0 & & p_K^{-1} \end{bmatrix} + \frac{1 - \Sigma_{j=1}^{L} \alpha_j}{\alpha_K (1 - \Sigma_{j=1}^{L} \alpha_j) - \alpha_i} \begin{bmatrix} \alpha_1 p_1^{-1} \\ \vdots \\ \alpha_{K-1} p_{K-1}^{-1} \\ \alpha_i (1 - \Sigma_{j=1}^{L} \alpha_j)^{-1} p_K^{-1} \end{bmatrix} [1 \; 1 \; \ldots \; 1].$$

The determinant of A_{Li} is

$$|A_{Li}| = \lambda_i \; |C_1 + C_{2i}|$$

where

$$\lambda_i = 1 + \left(\frac{\alpha_i}{1-\Sigma_{j=1}^{L}\alpha_j} p_1, \ldots, \frac{\alpha_i}{1-\Sigma_{j=1}^{L}\alpha_j} p_L, \;\; 0, \ldots, 0, p_i, 0, \ldots, 0, -p_K\right).$$

$$(C_1 + C_{2i})^{-1}(0, 0, \ldots, 0, 1)'$$

$$= 1 - p_K p_K^{-1} + \frac{1-\Sigma_{j=1}^{L}\alpha_j}{\alpha_K(1-\Sigma_{j=1}^{L}\alpha_j)-\alpha_i} \left(\frac{\alpha_i}{1-\Sigma_{j=1}^{L}\alpha_j}\Sigma_{j=1}^{L}\alpha_j + \alpha_i - \frac{\alpha_i}{1-\Sigma_{j=1}^{L}\alpha_j}\right)$$

$$= \frac{1}{\alpha_K(1-\Sigma_{j=1}^{L}\alpha_j)-\alpha_i} \left(\alpha_i \Sigma_{j=1}^{L}\alpha_j + \alpha_i(1-\Sigma_{j=1}^{L}\alpha_j) - \alpha_i\right)$$

$$= 0.$$

Hence the matrix A_{Li} is singular. The c_{Li}, $i = L+1, \ldots, K$ are linearly dependent on c_i, $i = 1, \ldots, K$. This regime does not provide additional information for the identification of the β's.

From the above analysis, we conclude that the presence of corner solutions and their spill over effects do not provide additional information on the identification of the linear expenditure system that can be provided without the corner solutions.

4. Identification of The Translog Indirect Utility Function

The translog indirect utility function for consumer demand analysis was introduced by Christensen, Jorgenson and Lau [1975]. The translog indirect utility function has a flexible functional form and the implied Engle curve is not necessarily linear. With stochastic elements which capture the differences in individuals' preferences, introduced in the additive terms, the translog indirect utility function is

$$(4.1) \qquad H(v;\theta,\varepsilon) = \Sigma_{i=1}^{K} \alpha_i \ln v_i + 1/2\,\Sigma_{i=1}^{K}\Sigma_{j=1}^{K} \beta_{ij} \ln v_i \ln v_j + \Sigma_{i=1}^{K} \varepsilon_i \ln v_i$$

where $v_i = p_i/M$ are income normalized prices, $\varepsilon = (\varepsilon_1,\ldots,\varepsilon_K)'$ is a K-dimensional vector of multivariate normal variables $N(0,\Sigma)$ and $\beta_{ij} = \beta_{ji}$ for all $i,j = 1,\ldots,K$. For normalization, $\Sigma_{i=1}^{K} \alpha_i = -1$ and $\Sigma_{i=1}^{K} \varepsilon_i = 0$. The notional share equations derived from Roy's identity are

$$(4.2) \qquad \begin{aligned} s_i &= \frac{\alpha_i + \Sigma_{j=1}^{K} \beta_{ij} \ln v_j + \varepsilon_i}{-1 + \Sigma_{\ell=1}^{K}\Sigma_{j=1}^{K} \beta_{\ell j} \ln v_j} \\ &= \frac{\alpha_i + \Sigma_{j=1}^{K} \beta_{ij} \ln p_j - \beta_{i\cdot} \ln M + \varepsilon_i}{-1 + \Sigma_{j=1}^{K} \beta_{\cdot j} \ln p_j - \beta_{\cdot\cdot} \ln M} \qquad i = 1,\ldots,K. \end{aligned}$$

where $s_i = p_i x_i/M$, $\beta_{i\cdot} = \Sigma_{j=1}^{K} \beta_{ij}$, $\beta_{\cdot\cdot} = \Sigma_{i=1}^{K}\Sigma_{j=1}^{K} \beta_{ij}$ and $\beta_{\cdot j} = \beta_{j\cdot}$ by symmetry. We consider the case where the prices are constants but the incomes M vary across samples. Denote

$$\delta_i = \alpha_i + \Sigma_{j=1}^{K} \beta_{ij} \ln p_j \qquad i = 1,\ldots,K.$$

The interior regime will provide identification of some parameters. All goods are consumed in some positive amounts in this regime. The regime conditions are

(4.3) $\varepsilon_i < \beta_{i\cdot} \ln M - \delta_i$ $\qquad i = 1,2,\ldots,K.$

Equivalently, these conditions are

(4.4) $\varepsilon_i < \beta_{i\cdot} \ln M - \delta_i$ $\qquad i = 1,2,\ldots,K-1$

(4.5) $\Sigma_{j=1}^{K-1} \varepsilon_j > \delta_K - \beta_{K\cdot} \ln M$

Conditional on the subsamples in this regime, the truncated normal density function implied by the K-1 independent share equations in (4.2) will identify the covariance matrix Σ and the constants

(4.6) $\beta_{i\cdot} \;, \; \delta_i$ $\qquad i = 1,\ldots,K.$

It remains to consider the identification of the individual parameters α_i and β_{ij}, $i,j = 1,\ldots,K$.

Consider now the regime with only one nonconsumed good. Without loss of generality, consider the regime where the first good is not consumed. The virtual price of good one at zero quantity is

(4.7) $\ln \xi_1 = -(\alpha_1 + \Sigma_{j=2}^{K} \beta_{1j} \ln p_j - \beta_{1\cdot} \ln M + \varepsilon_1)/\beta_{11}$

The remaining positive shares are

$$s_i = \frac{\delta_i + \beta_{i1}(\ln \xi_1 - \ln p_1) - \beta_{i\cdot} \ln M + \varepsilon_i}{-1 + \Sigma_{j=1}^{K} \beta_{\cdot j} \ln p_j + \beta_{\cdot 1}(\ln \xi_1 - \ln p_1) - \beta_{\cdot\cdot} \ln M} \qquad i = 2,\ldots,K.$$

Since $\ln \xi_1 - \ln p_1 = -1/\beta_{11}(\delta_1 - \beta_{1\cdot} \ln M + \varepsilon_1)$, we have

(4.8) $$s_i = \frac{\delta_i - \beta_{i\cdot} \ln M - \frac{\beta_{i1}}{\beta_{11}}(\delta_1 - \beta_{1\cdot} \ln M) + \varepsilon_i - \frac{\beta_{i1}}{\beta_{11}} \varepsilon_1}{-1 + \Sigma_{j=1}^{K} \beta_{\cdot j} \ln p_j - \beta_{\cdot\cdot} \ln M - \frac{\beta_{\cdot 1}}{\beta_{11}} (\delta_1 - \beta_{1\cdot} \ln M) - \frac{\beta_{\cdot 1}}{\beta_{11}} \varepsilon_1} \qquad i = 2,\ldots,K.$$

It is necessary to make a distinction between the case $K = 2$ with the cases $K \geq 3$. For the case $K = 2$, this regime provides no additional identification since $s_2 = 1$ and the regime condition is simply $\varepsilon_1 \geq -(\delta_1 - \beta_1 . \ln M)$.[4/] For the cases $K \geq 3$, we have $0 < s_2 < 1$ and there is additional identification information. The regime conditions for the cases $K \geq 3$ are

$$\varepsilon_1 \geq -(\delta_1 - \beta_1 . \ln M)$$

(4.9)
$$\varepsilon_i - \frac{\beta_{i1}}{\beta_{11}} \varepsilon_1 < \frac{\beta_{i1}}{\beta_{11}} (\delta_1 - \beta_1 . \ln M) - (\delta_i - \beta_i . \ln M) \qquad i = 2,\ldots,K.$$

Given the identification of Σ and $\beta_i .$, δ_i, $i = 1,\ldots,K$, the truncated multivariate density function implied by the shares $s_2,\ldots,s_{K-1}$ will identify the ratios

(4.10)
$$\frac{\beta_{i1}}{\beta_{11}} \qquad i = 1,\ldots,K.$$

With similar arguments, for the case $K \geq 3$, the information in the regime that the j^{th} good is not consumed will provide the identification of the ratios

(4.11)
$$\frac{\beta_{ij}}{\beta_{jj}} \qquad i = 1,\ldots,K.$$

Given the identification of the ratios in (4.10), by the symmetric property $\beta_{ij} = \beta_{ji}$, the ratio

$$\frac{\beta_{jj}}{\beta_{11}} = \frac{\beta_{j1}}{\beta_{11}} \frac{\beta_{jj}}{\beta_{1j}}$$

is identifiable and therefore the identification of the ratios in (4.11) is equivalent to the identification of the ratios

(4.12)
$$\frac{\beta_{ij}}{\beta_{11}} \qquad i = 1,\ldots,K.$$

Hence we can conclude that, for $K \geq 3$, the information in all the K regimes with only one nonconsumed good will provide the identificaiton of the parameter ratios λ_{ij};

$$(4.13) \qquad \lambda_{ij} = \frac{\beta_{ij}}{\beta_{11}} \qquad\qquad i,j = 1,2,\ldots,K.$$

Since $\beta_{i\cdot} = \Sigma_{j=1}^{K} \beta_{ij}$, it follows that $\beta_{11} = 1/\beta_{i\cdot} \, \Sigma_{j=1}^{K} \lambda_{ij}$ if $\beta_{i\cdot} \neq 0$ for some i, and therefore β_{11} will be identifiable. It follows then that all the parameters β_{ij} and α_i, $i,j = 1,\ldots,K$ are identifiable. The condition $\beta_{i\cdot} \neq 0$ for some i holds if the translog indirect utility is nonhomothetic. Hence, we conclude that for the consumer demand model with more than two goods and a nonhomothetic translog indirect utility function, the presence of corner solutions for each good will provide enough information to identify all the unknown parameters in the indirect utility function.

The homothetic translog indirect utility function implies $\beta_{i\cdot} = 0$ for all $i = 1,\ldots,K$ and the expenditure shares are constants and not dependent on incomes. The above analysis shows that the interior regime and all the regimes with only one nonconsumed good will provide information for the identification of the covariance matrix Σ, δ_i and λ_{ij} for $i,j = 1,2,\ldots,K$. It remains to show whether other regimes will provide additional identification information. Consider the general regime with zero consumption on the first m goods. The joint virtual prices for the goods at zero are

$$(4.14) \qquad \begin{bmatrix} \ln \xi_1 \\ \ln \xi_2 \\ \vdots \\ \ln \xi_m \end{bmatrix} = - \begin{bmatrix} \beta_{11} & \beta_{12} & \cdots & \beta_{1m} \\ \beta_{21} & \beta_{22} & \cdots & \beta_{2m} \\ \vdots & \vdots & & \vdots \\ \beta_{m1} & \beta_{m2} & \cdots & \beta_{mm} \end{bmatrix}^{-1} \begin{bmatrix} \alpha_1 + \Sigma_{j=m+1}^{K} \beta_{1j} \ln p_j + \varepsilon_1 \\ \alpha_2 + \Sigma_{j=m+1}^{K} \beta_{2j} \ln p_j + \varepsilon_2 \\ \vdots \\ \alpha_m + \Sigma_{j=m+1}^{K} \beta_{mj} \ln p_j + \varepsilon_m \end{bmatrix} .$$

The remaining minus share equations for goods $m+1,\ldots,K$ are

$$(4.15) \qquad -s_i = \alpha_i + (\beta_{i1} \; \beta_{i2} \; \cdots \; \beta_{im})(\ln \xi_1 \ldots \ln \xi_m)' + \Sigma_{j=m+1}^{K} \beta_{ij} \ln p_j + \varepsilon_i$$

$$= \delta_i - (\lambda_{i1}\ \lambda_{i2}\ \cdots\ \lambda_{im})\Lambda^{-1}(\delta_1\ \cdots\ \delta_m)' + \varepsilon_i$$
$$- (\lambda_{i1}\ \lambda_{i2}\ \cdots\ \lambda_{im})\Lambda^{-1}(\varepsilon_1\ \varepsilon_2\ \cdots\ \varepsilon_m)' \qquad i = m+1,\ldots,K$$

where

$$\Lambda = \begin{bmatrix} 1 & \lambda_{12} \cdots & \lambda_{1m} \\ \lambda_{21} & \lambda_{22} \cdots & \lambda_{2m} \\ \vdots & \vdots & \vdots \\ \lambda_{m1} & \lambda_{m2} \cdots & \lambda_{mm} \end{bmatrix}.$$

The first regime condition for the occurance of this regime is that, conditional on goods $2,\ldots,m$ being zeros, the notional demand for good one is negative (see (2.8)). The joint virtual prices for goods $2,\ldots,m$ are

$$(4.16) \quad \begin{bmatrix} \ln \eta_2 \\ \vdots \\ \ln \eta_m \end{bmatrix} = - \begin{bmatrix} \beta_{22} & \beta_{23} & \cdots & \beta_{2m} \\ \vdots & \vdots & & \vdots \\ \beta_{m2} & \beta_{m3} & \cdots & \beta_{mm} \end{bmatrix}^{-1} \begin{bmatrix} \alpha_2 + \beta_{21} \ln p_1 + \Sigma_{j=m+1}^{K} \beta_{2j} \ln p_j + \varepsilon_2 \\ \vdots \\ \alpha_m + \beta_{m1} \ln p_1 + \Sigma_{j=m+1}^{K} \beta_{mj} \ln p_j + \varepsilon_m \end{bmatrix}.$$

The first regime condition is

$$(4.17) \quad \alpha_1 + \Sigma_{i=1}^{m} \beta_{1i} \ln \eta_i + \Sigma_{j=m+1}^{K} \beta_{1j} \ln p_j + \varepsilon_1 \geq 0.$$

Equivalently, after the substitution of (4.16) into (4.17), it becomes

$$(4.16) \quad \delta_1 - (\lambda_{12}\ \lambda_{13}\ \cdots\ \lambda_{1m}) \begin{bmatrix} \lambda_{22} & \lambda_{23} & \cdots & \lambda_{2m} \\ \vdots & \vdots & & \vdots \\ \lambda_{m2} & \lambda_{m3} & \cdots & \lambda_{mm} \end{bmatrix}^{-1} \begin{bmatrix} \delta_2 + \varepsilon_2 \\ \vdots \\ \delta_m + \varepsilon_m \end{bmatrix} + \varepsilon_1 \geq 0.$$

Similarly, the remaining regime conditions can be derived. A common feature of these conditions and the share equations in (4.15) is that the parameters involved are in the form of λ_{ij} and δ_i. Hence it provides no additional identification for parameters that have not been identified. Hence, we conclude that for a homothetic indirect translog utility funtion, the presence of corner solutions and their spill over effects does not provide enough information to identify all the parameters in the utility function. This is so because income variations do not have effects on the expenditure shares.

Even though not all the parameters are identifiable for the homothetic translog indirect utility function, the presence of corner solutions does provide additional identification information as compared to the situation with no corner solutions. With no corner solutions, only the covariance matrix Σ and the constants δ_i, $i = 1,\ldots,K$ can be identified without price variations. With corner solutions, it is possible to identify the ratios β_{ij}/β_{11}, $i,j = 1,\ldots,K$ in addiiton to the parameters Σ and δ's. Indeed if one extra piece of information were available to identify one of the β's, all the parameters would be identifiable. For example, if one of the prices does vary across samples, say, p_ℓ, the parameters $\beta_{1\ell}$, $\beta_{2\ell},\ldots,\beta_{K\ell}$ will be identifiable from the interior regime and it follows from the above analysis that all the parameters will be identifiable. This will be the case if the model is on the study of joint labor supply and demand for commodities since the wage rate is varying across individuals.

5. Identification of The Translog Profit Function

The above identification analysis can be extended to the analysis of models of production economics with limited dependent variables. The observed values of profit will provide an additional piece of information for identification. Consider the translog profit function

$$(5.1) \qquad \ln \Pi = \alpha_{00} + \Sigma_{i=1}^{K} \alpha_{i0} \ln p_i + 1/2\, \Sigma_{i=1}^{K} \Sigma_{j=1}^{K} \alpha_{ij} \ln p_i \ln p_j + \Sigma_{i=1}^{K} \varepsilon_i \ln p_i + \varepsilon_0$$

where K is the total number of inputs and outputs, $\alpha_{ij} = \alpha_{ji}$ for all $i,j = 1,\ldots K$ and $\varepsilon = (\varepsilon_0, \varepsilon_1,\ldots,\varepsilon_K)'$ is assumed to be multivariate normal $N(0, \Omega)$. Linear homogeneity of the profit function in prices implies the constraints $\Sigma_{i=1}^{K} \alpha_{i0} = 1$, $\Sigma_{j=1}^{K} \alpha_{ij} = 0$ and $\Sigma_{i=1}^{K} \varepsilon_i = 0$. This specification is quite general in that it allows for single output as well as multiple output technologies. We assume that profits are not zero in the samples so that the revenue shares and expense shares of profit are well-defined. The Hotelling and McFadden lemma implies that the notional (negative) expense shares and the revenue shares of profit from (5.1) are

$$(5.2) \qquad s_i = \alpha_{i0} + \Sigma_{j=1}^{K} \alpha_{ij} \ln p_j + \varepsilon_i \qquad\qquad i = 1,\ldots,K.$$

The goods are assumed to be arranged in a way such that the first m goods correspond to inputs and the remaining $K - m$ goods correspond to outputs. For single output technology, the last good will be the output commodity. For practical reasons, it is interesting to consider only the cases $K \geq 3$. Denote

$$(5.3) \qquad c_i = \alpha_{i0} + \Sigma_{j=1}^{K} \alpha_{ij} \ln p_j \qquad\qquad i = 1,\ldots,K.$$

All the inputs will be used with some positive amounts and all the outputs will be produced in some positive amounts in the interior regime. The regime conditions are

$$(5.4) \qquad -c_i > \varepsilon_i, \qquad\qquad i = 1,\ldots,m;$$

$$(5.5)\qquad -c_\ell < \varepsilon_\ell, \qquad\qquad \ell = m+1,\ldots,K.$$

The observed shares are

$$(5.6)\qquad s_i = c_i + \varepsilon_i \qquad\qquad i = 1,\ldots,K.$$

and the corresponding profit function is

$$(5.7)\qquad \ln \Pi = \alpha_{00} + 1/2\,\Sigma_{i=1}^K \alpha_{i0} \ln p_i + 1/2\,\Sigma_{i=1}^K (\alpha_{i0} + \Sigma_{j=1}^K \alpha_{ij} \ln p_j) \ln p_i + \Sigma_{i=1}^K \varepsilon_i \ln p_i + \varepsilon_0$$

$$= c_0 + 1/2\,\Sigma_{i=1}^K c_i \ln p_i + \Sigma_{i=1}^K \varepsilon_i \ln p_i + \varepsilon_0$$

where $c_0 = \alpha_{00} + 1/2\,\Sigma_{i=1}^K \alpha_{i0} \ln p_i$. The truncated multivariate normal density function of the $K - 1$ share equations $s_1,\ldots,s_{K-1}$ and the observed log profit function will identify the parameters c_0, c_i, $i = 1,\ldots,K$ and the covariance matrix Ω.

Consider now the regimes where only one of the inputs in the multiple inputs technologies is not used or only one of the outputs in the multiple outputs technologies is not produced. Without loss of generality, consider the regime where the first input is not used. The virtual price vector for this input in log scale is

$$(5.8)\qquad \ln \xi_1 = -1/\alpha_{11}\,(\alpha_{10} + \Sigma_{j=2}^K \alpha_{1j} \ln p_j + \varepsilon_1)$$

and the nonzero shares are

$$(5.9)\qquad s_i = \alpha_{i0} + \alpha_{i1} \ln \xi_1 + \Sigma_{j=2}^K \alpha_{ij} \ln p_j + \varepsilon_i \qquad\qquad i = 2,\ldots,K$$

$$= c_i - \frac{\alpha_{i1}}{\alpha_{11}} c_1 + \varepsilon_i - \frac{\alpha_{i1}}{\alpha_{11}} \varepsilon_1$$

The first regime condition is

$$(5.10) \qquad \varepsilon_1 \geq -c_1$$

and the remaining conditions are

$$(5.11) \qquad \frac{\alpha_{i1}}{\alpha_{11}} c_1 - c_i > \varepsilon_i - \frac{\alpha_{i1}}{\alpha_{11}} \varepsilon_1, \qquad i = 2,\ldots,m;$$

$$(5.12) \qquad \frac{\alpha_{\ell 1}}{\alpha_{11}} c_1 - c_\ell < \varepsilon_\ell - \frac{\alpha_{\ell 1}}{\ell_{11}} \varepsilon_1, \qquad \ell = m+1,\ldots,K.$$

The truncated multivariate normal density function implied by the shares $s_2,\ldots,s_K$ will provide identification of the ratios

$$(5.13) \qquad \frac{\alpha_{i1}}{\alpha_{11}} \qquad i = 2,\ldots,K$$

given the identification of Ω and c_i, $i = 1,\ldots,K$ from the interior regime. The corresponding profit function will also provide additional information for identification. The profit function for this regime is

$$(5.14) \qquad \begin{aligned} &\ln \Pi \\ &= c_0 + 1/2\,\alpha_{10}(\ln \xi_1 - \ln p_1) + 1/2(c_1 + \alpha_{11}(\ln \xi_1 - \ln p_1))\ln \xi_1 \\ &\quad + 1/2\,\Sigma_{i=2}^K (c_i + \alpha_{i1}(\ln \xi_1 - \ln p_1))\ln p_i + \Sigma_{i=1}^K \varepsilon_i \ln p_i + \varepsilon_0 + \varepsilon_1(\ln \xi_1 - \ln p_1) \\ &= c_0 + 1/2\,\Sigma_{i=1}^K c_i \ln p_i + (c_1 + \varepsilon_1)(\ln \xi_1 - \ln p_1) + \Sigma_{i=1}^K \varepsilon_i \ln p_i + \varepsilon_0 + 1/2\,\alpha_{11}(\ln \xi_1 - \ln p_1) \end{aligned}$$

Since $\ln \xi_1 - \ln p_1 = -1/\alpha_{11}\ (c_1 + \varepsilon_1)$, it follows that

$$(5.15) \qquad \ln \Pi = c_0 + 1/2\,\Sigma_{i=1}^K c_i \ln p_i + \Sigma_{i=1}^K \varepsilon_i \ln p_i + \varepsilon_0 - \frac{1}{2\alpha_{11}}(c_1 + \varepsilon_1)^2.$$

Given the identification of the parameters c_0, c_i, $i = 1,\ldots,K$, Ω and α_{i1}/α_{11}, $i = 2,\ldots,K$, the truncated density function of $\ln \Pi$ of this regime will provide the identification of α_{11}. It follows that all the parameters α_{i1}, $i = 1,2,\ldots,K$, are identifiable. Furthermore, $\alpha_{10} = c_1 - \Sigma_{j=1}^K \alpha_{1j} \ln p_j = c_1 - \Sigma_{j=1}^K \alpha_{j1} \ln p_j$ is also identifiable.

With similar arguments, the information in the regime with the j^{th} good being zero will provide the identification of

$$\alpha_{j0}, \quad \alpha_{ij}, \qquad\qquad i = 1,2,\ldots,K.$$

We note that the identification of all the parameters α_{i0}, α_{ij}, $i,j = 1,\ldots,K$ requires that there are $K-1$ different regimes, in each of them one of the goods is zero, instead of K such regimes. This is so because the constraints $\Sigma_{i=1}^{K} \alpha_{i0} = 1$ and $\Sigma_{j=1}^{K} \alpha_{ji} = 0$ will identify the remaining parameters. This observation is important because for the single output technology, the output will be, in general, produced for all the firms in the samples and only different inputs may not be utilized. Finally $\alpha_{00} = c_0 - 1/2\,\Sigma_{i=1}^{K} \alpha_{i0} \ln p_i$ is identifiable given the identification of c_0 and α_{i0}, $i = 1,\ldots,K$. Hence, we conclude that all the parameters in the translog profit function can be identified.

6. Conclusion

Consumer demand and production systems with binding nonnegative constraints for the analysis of microeconomic data are related to the multivariate and simultaneous equation Tobit models. These systems have distinctive identification properties. The identification of the parameters in the linear expenditure system, the translog indirect utility function and the translog profit function are analyzed when there are no price variations across samples. The zero consumption of certain goods will create spillover effects to the consumption of other goods. The spillover effects are captured in the differences between the virtual prices which support the demanded quantities and the market prices. These create structural changes across different consumption or production regimes. The information in each regime will identify some sets of parameters. If each of the commodities is not consumed by some individuals in the samples, all the parameters in the nonhomothetic indirect utility function can be identified for the cases with more than two commodities. For the two goods model with binding nonnegative constraints, it is simply a single equation Tobit or two limit Tobit model and the presence of limited dependent variables does not help identification. On the other hand, not all the parameters in the homothetic indirect utility function can be identified because the expenditure shares are constant and are not functions of incomes. Similar phenomenon occur for the linear expenditure system which implies linear Engel curves. The presence of limited dependent variables does not help the identification of the linear expenditure system without price variation. Finally, all the parameters in the translog profit function for production analysis are identifiable because of the observability of the profit function.

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Appendix: Identification of Truncated Multivariate Normal Distribution of Share Equations

Consider the share equations without price variation

(A.1) $s_i = c_i + \varepsilon_i \qquad i = 1,\ldots,K$

where c_i, $i = 1,\ldots,K$, are constants, $\Sigma_{i=1}^{K} s_i = 1$, $\Sigma_{i=1}^{K} \varepsilon_i = 0$ and $(\varepsilon_1,\ldots,\varepsilon_K)$ is multivariate normal distribution $N(0, \Omega)$. The regime that will be considered is the interior regime in which all the goods are purchased, i.e., $s_i > 0$ for all i, $i = 1,\ldots,K$. Since the sum of the shares is unity and ε_K is linearly dependent on the remaining shares $\varepsilon_1,\ldots,\varepsilon_{K-1}$, the density function of the shares can be written in terms of $K-1$ shares as

(A.2) $$f(y) = (2\pi)^{-\frac{K-1}{2}} |\Sigma|^{-\frac{1}{2}} \exp\{ -\frac{1}{2}(y-\mu)'\Sigma^{-1}(y-\mu) \}$$

where $y = (s_1,\ldots,s_{K-1})'$, $\mu = (c_1,\ldots,c_{K-1})'$ and Σ is the covariance matrix of y which is the submatrix of Ω with the last row and column deleted. The occurrence of the interior regime is determined by the conditions $s_1 > 0,\ldots,s_{K-1} > 0$ and $\Sigma_{i=1}^{K-1} s_i < 1$. The occurance probability of this regime is therefore

(A.3) $$P = \int_0^1 \int_0^{1-y_1} \cdots \int_0^{1-\Sigma_{i=1}^{K-2} y_i} f(y)dy_{K-1}\cdots dy_1$$

and the conditional density function is $f(y)/P$. The identification problem is to identify the parameters c_i, $i = 1,\ldots,K-1$ and the covariance matrix Σ from the sample information in this regime. In the following paragraph, we will show that all the unknown parameters can be identified from the moments of the conditional density function $f(y)/P$.

The normal density function (A.2) satisfies the following differential equation

(A.4) $$\frac{\partial}{\partial y_{K-1}} f(y) = -(\sigma^{K-1})'(y-\mu)f(y)$$

where σ^{K-1} is the $K-1^{th}$ column of Σ. Consider the equation

(A.5) $$y_j^r y_{K-1}^m (1-\Sigma_{i=1}^{K-1} y_i)^\ell \frac{\partial}{\partial y_{K-1}} f(y) = -(\sigma^{K-1})'(y-\mu)\, y_j^r y_{K-1}^m (1-\Sigma_{i=1}^{K-1} y_i)^\ell f(y)$$

for $r \geq 0$, $m \geq 1$, $\ell \geq 1$ and $j \neq K-1$. By the integration by parts,

$$\int_0^{1-\Sigma_{i=1}^{K-2} y_i} y_{K-1}^m (1-\Sigma_{i=1}^{K-1} y_i)^\ell \frac{\partial}{\partial y_{K-1}} f(y) dy_{K-1}$$

$$= y_{K-1}^m (1-\Sigma_{i=1}^{K-1} y_i)^\ell f(y) \Big|_{y_{K-1}=0}^{1-\Sigma_{i=1}^{K-2} y_i}$$

$$- \int_0^{1-\Sigma_{i=1}^{K-2} y_i} [m\, y_{K-1}^{m-1} (1-\Sigma_{i=1}^{K-1} y_i)^\ell - \ell y_{K-1}^m (1-\Sigma_{i=1}^{K-1} y_i)^{\ell-1}]\, f(y) dy_{K-1}$$

$$= -m \int_0^{1-\Sigma_{i=1}^{K-2} y_i} y_{K-1}^{m-1} (1-\Sigma_{i=1}^{K-1} y_i)^\ell f(y) dy_{K-1} + \ell \int_0^{1-\Sigma_{i=1}^{K-2} y_i} y_{K-1}^m (1-\Sigma_{i=1}^{K-1} y_i)^{\ell-1} f(y) dy_{K-1}$$

and hence (A.5) implies

$$\ell\, E(y_j^r\, y_{K-1}^m\, y_K^{\ell-1}) - m\, E(y_j^r\, y_{K-1}^{m-1}\, y_K^\ell)$$

(A.6) $$= E(y_j^r\, y_{K-1}^m\, y_K^\ell)(\sigma^{K-1})'\mu - E(y_j^r\, y_{K-1}^m\, y_K^\ell\, y')\, \sigma^{K-1}$$

where $y_K = 1-\Sigma_{i=1}^{K-1} y_i$ $(= s_k)$ and the expectations are taken with respect to the truncated density $f(y)/P$. As every good can be named the $K-1^{th}$ commodity, the equation (A.6) can be generalized as

(A.7) $$\ell\, E(y_j^r\, y_i^m\, y_K^{\ell-1}) - m\, E(y_j^r\, y_i^{m-1}\, y_K^\ell)$$

$$= E(y_j^r\, y_i^m\, y_K^\ell)(\sigma^i)'\mu - E(y_j^r\, y_i^m\, y_K^\ell\, y')\, \sigma^i$$

for $j \neq i$, $i,j = 1,\ldots,K-1$, $r \geq 0$, $m \geq 1$ and $\ell \geq 1$. Denote

(A.8) $$\alpha_i = (\sigma^i)'\mu \qquad\qquad i = 1,\ldots,K-1.$$

When $r = 0$, $\ell = m = 1$, equation (A.7) implies that

$$E(y_i) - E(y_K) = E(y_i y_K)\alpha_i - E(y_i y_K y')\sigma^i \qquad i = 1,\ldots,K-1 \tag{A.9}$$

and therefore

$$\alpha_i = (E(y_i) - E(y_K) + E(y_i y_K y')\sigma^i)/E(y_i y_K), \qquad i = 1,\ldots,K-1. \tag{A.10}$$

For $r = 0$, $\ell = 1$ and $m = 2$, we have

$$E(y_i^2) - 2E(y_i y_K) = E(y_i^2 y_K)\alpha_i - E(y_i^2 y_K y')\sigma^i, \qquad i = 1,\ldots,K-1. \tag{A.11}$$

On the other hand, when $r = \ell = m = 1$,

$$E(y_j y_i) - E(y_j y_K) = E(y_j y_i y_K)\alpha_i - E(y_j y_i y_K y')\sigma^i, \quad i \neq j;\ i,j = 1,\ldots,K-1. \tag{A.12}$$

Let $e_j = (0,\ldots,0,1,0,\ldots,0)'$ be the $K-1$ unit vector with one at the j^{th} entry. Equations (A.11) and (A.12) can be combined into

$$E(y_i y) - E(y_K(y+y_i e_i)) = E(y_K y_i y)\alpha_i - E(y_K y_i y y')\sigma^i \qquad i = 1,\ldots,K-1 \tag{A.13}$$

Substituting α_i in (A.10) into (A.13), σ^i can be solved as

$$\begin{aligned}\sigma^i = &\{E(y_K y_i y y') - E(y_K y_i y)E(y_K y_i y')/E(y_K y_i)\}^{-1}. \\ &\{E(y_K(y+y_i e_i)) - E(y_i y) + (E(y_i) - E(y_K))E(y_K y_i y)/E(y_K y_i)\}\end{aligned} \qquad i = 1,\ldots,K-1 \tag{A.14}$$

which is a function of some moments of this regime. Similarly, substituting (A.14) into (A.10), α_i is also a function of some moments. As these moments can be estimated by the sample moments from this regime, all the parameters α_i and σ^i, $i=1,\ldots,K-1$ are identifiable from the sample information in this regime. From (A.8), $\mu = \Sigma\alpha$ where $\alpha = (\alpha_1,\ldots,\alpha_{K-1})'$ is identified. Hence μ and Σ are identifiable.

The above analysis can be generalized to the cases with regressors. For those cases, moments involving y and the regressors will be used. The analysis can obviously be modified for models with expenditure equations instead of share equations.

Footnotes

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1). This refers to the property that the utility for a commodity combination in which one or more quantities is zero is lower than for any combination in which all quantities are positive. This property implies that the demand for all goods will be positive for all positive price vectors.

2) The submaxtix of A_i with the last row and column deleted is nonsingular.

3) For example, $c_{1K} = \delta_1 c_1 + \ldots + \delta_{K-1} c_{K-1}$ where $\delta_1 = \frac{\alpha_K}{1-\alpha_1} - 1$ and $\delta_i = -1$, $i = 2,\ldots,K-1$.

4) When $K = 2$, the model is a single equation Tobit or two limit Tobit model.

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