

# REGULARITY CRITERION IN TERMS OF PRESSURE FOR THE NAVIER-STOKES EQUATIONS

Dongho Chae <sup>(a)</sup> and Jihoon Lee <sup>(b)</sup>

School of Mathematics <sup>(a)</sup>

University of Minnesota

127 Vincent Hall, 206 Church Street SE

Minneapolis, MN 55455, USA

and

Department of Mathematics <sup>(a),(b)</sup>

Seoul National University

Seoul 151-742, Korea

## Abstract

We obtain a regularity criterion of a Leray-Hopf weak solution of the Navier-Stokes equations in  $\mathbb{R}^3 \times [0, T]$  in terms of the integrability of pressure. We first establish a priori estimates for  $L^s(\mathbb{R}^3)$ ,  $s \geq 3$ , smooth solutions, and then use the standard continuation argument for the local smooth solutions. Our criterion, in particular, implies the previous results due to Serrin on the regularity criterion in terms of velocity fields.

Key words: Navier-Stokes equations, regularity criterion

## 1 Introduction

In this paper, we consider the initial value problem for the Navier-Stokes equations in  $\mathbb{R}^3 \times [0, T)$ ,

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u - \Delta u = -\nabla p \quad (1)$$

$$\operatorname{div} u = 0 \quad (2)$$

$$u(x, 0) = u_0(x), \quad (3)$$

where  $u = (u_1, u_2, u_3)$  with  $u = u(x, t)$  and  $p = p(x, t)$  denote unknown fluid velocity and scalar pressure, respectively, while  $u_0$  is a given initial velocity satisfying  $\operatorname{div} u_0 = 0$ . We denoted

$$(u \cdot \nabla)u = \sum_{j=1}^3 u_j \frac{\partial u}{\partial x_j}, \quad \Delta u = \sum_{j=1}^3 \frac{\partial^2 u}{\partial x_j^2}$$

in the above. For given  $u_0 \in L^2(\mathbb{R}^3)$  with  $\operatorname{div} u_0 = 0$  in the sense of distribution a global weak solution  $u$  to the Navier-Stokes equations, belonging to the space  $L^\infty(0, T; L^2(\mathbb{R}^3)) \cap L^2(0, T; H^1(\mathbb{R}^3))$  was constructed by Leray[10] and Hopf[6], which we call Leray-Hopf weak solution. Regularity of such Leray-Hopf weak solutions is one of the most outstanding open problems in mathematical fluid mechanics. We note here that there are partial regularity results by Schaffer and by Caffarelli-Kohn-Nirenberg(See[2] and references therein. See also for a recent simplification of the proof in [2] due to Lin[11].). On the other hand, pioneered by Serrin[13] (See also [13].), and extended and improved by Fabes-Jones-Riviere[4], Giga[5] and Struwe[14](For the recent progress on the problem of the marginal case see also [9], and references therein.), there exist regularity criteria of the Leray-Hopf weak solutions which state that if  $u$  is a Leray-Hopf weak solution belonging to  $L_T^{\alpha, \gamma} = L^\alpha(0, T; L^\gamma(\mathbb{R}^3))$  with exponents  $s$  and  $\gamma$  satisfying  $\frac{2}{\alpha} + \frac{3}{\gamma} \leq 1$ , then  $u$  is actually a solution of class  $C^\infty(\mathbb{R}^3 \times (0, T))$ . For the space  $L_T^{\alpha, \gamma}$  let us introduce the norm

$$\|u\|_{L_T^{\alpha, \gamma}} = \left( \int_0^T \|u(t)\|_\gamma^\alpha dt \right)^{\frac{1}{\alpha}},$$

where

$$\|u(t)\|_\gamma = \begin{cases} \left( \int_{\mathbb{R}^3} |u(x, t)|^\gamma dx \right)^{\frac{1}{\gamma}} & \text{if } 1 \leq \gamma < \infty \\ \operatorname{ess\,sup}_{x \in \mathbb{R}^3} |u(x, t)| & \text{if } \gamma = \infty \end{cases}.$$

Following the notion of scaling dimensions for various quantities in the Navier-Stokes equations in [2], the norm  $\|u\|_{L_T^{\alpha, \gamma}}$  has the scaling dimension zero, or negative if  $\frac{2}{\alpha} + \frac{3}{\gamma} \leq 1$ . On the other hand, if  $\frac{2}{\alpha} + \frac{3}{\gamma} \leq 2$ , both  $\|Du\|_{L_T^{\alpha, \gamma}}$  and  $\|p\|_{L_T^{\alpha, \gamma}}$  have scaling dimension zero, or negative. Related to this point, Beirão da Veiga[1] proposed another regularity criterion in terms of  $Du$ , which states that for Leray-Hopf weak solution  $u$ , the condition  $Du \in L_T^{\alpha, \gamma}$  with  $\frac{2}{\alpha} + \frac{3}{\gamma} \leq 2$  where  $\frac{3}{2} < \gamma < \infty$  implies that  $C^\infty(\mathbb{R}^3 \times (0, T))$ . We also remark that in [3] authors have improved Beirão da Veiga's regularity criterion by imposing that only the two components vorticity field belong to the same class. It is thus natural to consider the problem of obtaining the regularity condition in terms of  $p \in L_T^{\alpha, \gamma}$  with  $\frac{2}{\alpha} + \frac{3}{\gamma} < 2$  with appropriate restriction on the range of  $\alpha$  and  $\gamma$ . Actually, in [2] connection between the regularity of velocity fields and strong ingerability of pressure was suggested. We study that problem in this paper, and the followings are our main results. The first one is on the a priori estimates for smooth solutions.

**Theorem 1 ( a priori estimates )** Suppose  $s \geq 3$ ,  $\frac{3}{2} < \gamma \leq \infty$ , and  $1 < \alpha \leq \infty$  are given. Let us define  $\eta = \eta(\gamma, s, \alpha)$  by

$$\eta = \begin{cases} \max\{\gamma, s\} & \text{if } \frac{3s}{s+1} \leq \gamma < \infty \\ \alpha & \text{if } \frac{3}{2} < \gamma < \frac{3s}{s+1} \end{cases}.$$

Suppose  $u_0 \in L^s(\mathbb{R}^3)$  with  $\operatorname{div} u_0 = 0$ . Assume  $(u, p)$  is a smooth solution of the problem (1.1)-(1.3) in  $\mathbb{R}^3 \times (0, T)$ . If  $p \in L_T^{\alpha, \gamma}$  with  $2 - \frac{2}{\alpha} - \frac{3}{\gamma} \geq \frac{s-3}{\eta}$ , or  $\|p\|_{L_T^{\infty, \frac{3}{2}}}$  is sufficiently small, then  $u \in L^\infty(0, T; L^s) \cap L^s(0, T; L^{3s})$ . Moreover, the following estimates hold true.

(i) If  $\gamma = \infty$ , then

$$\sup_{0 \leq t < T} \|u(t)\|_s^s + \|u\|_{L_T^{s, 3s}}^s \leq C \|u_0\|_s^s \exp\left(C \|p\|_{L_T^{1, \infty}}\right), \quad (4)$$

where  $C = C(s)$ .

(ii) If  $\frac{3}{2} < \gamma < \infty$ , then

$$\sup_{0 \leq t < T} \|u(t)\|_s^s + \|u\|_{L_T^{s, 3s}}^s \leq C(\|u_0\|_s^s + \|p\|_{L_T^{\alpha, \gamma}}^\eta), \quad (5)$$

where  $C = C(s, \alpha, \gamma, T)$ .

(iii) If  $\gamma = \frac{3}{2}$  and  $\|p\|_{\infty, \frac{3}{2}}$  is sufficiently small, then

$$\sup_{0 \leq t < T} \|u(t)\|_s^s + \|u\|_{L_T^{s, 3s}}^s \leq C \|u_0\|_s^s, \quad (6)$$

where  $C = C(s, \|p\|_{\infty, \frac{3}{2}})$ .

**Theorem 2 ( regularity criterion )** Let  $u_0 \in L^2(\mathbb{R}^3) \cap L^q(\mathbb{R}^3)$  for some  $q > 3$ , and  $\operatorname{div} u_0 = 0$  in the sense of distribution. Suppose  $u$  is a Leray-Hopf solution of (1)-(3) in  $[0, T)$ . If  $p \in L_T^{\alpha, \gamma}(\mathbb{R}^3)$  with  $\frac{2}{\alpha} + \frac{3}{\gamma} < 2$  and  $1 < \alpha \leq \infty$ ,  $\frac{3}{2} < \gamma < \infty$ , or  $p \in L_T^{1, \infty}$ , or else  $\|p\|_{L_T^{\infty, \frac{3}{2}}}$  is sufficiently small, then  $u$  is a regular solution in  $[0, T)$ .

In order to discuss implications of Theorem 2, and for later use in the proof of Theorem 1 we recall the well-known pressure-velocity relation in  $\mathbb{R}^3$ , given by

$$p = \sum_{i, j=1}^3 R_i R_j (u_i u_j),$$

where  $(R_i)_{i=1}^3$  are the Riesz transforms in  $\mathbb{R}^3$ . (This is obtained formally by taking divergence of (1), and solving the resulting equation for  $p$ .) The Calderon-Zygmund inequality implies, then

$$\|p\|_\gamma \leq C \|u\|_{2\gamma}^2, \quad 1 < \gamma < \infty, \quad (7)$$

and from this we immediately have

$$\|p\|_{L_T^{\alpha,\gamma}} \leq C \|u\|_{L_T^{2\alpha,2\gamma}}^2. \quad (8)$$

**Remark 1.** From the inequality (8) we find that the Serrin type of condition for  $u$ ,  $u \in L_T^{\alpha,\gamma}$  with  $\frac{2}{\alpha} + \frac{3}{\gamma} < 1$ , implies our condition for  $p$  in Theorem 1, and we obtain regularity for the weak solution  $u$  by applying our theorem, i.e. our result implies previous regularity criterion results in [13].

**Remark 2.** In [7], Kaniel\* obtained actually a regularity criterion in terms of pressure, which states that  $p \in L_T^{\infty,\gamma}$ ,  $\gamma > \frac{12}{5}$  implies regularity of a weak solution. This result is obviously implied by Theorem 2.

\*The authors would like to thank Professor G. Galdi for informing them of Kaniel's paper[7].

## 2 Proof of Main Theorems

We first establish the following Gronwall type of lemma.

**Lemma 3** *Let  $a(t)$  and  $b(t)$  be nonnegative functions on  $[0, \infty)$  and  $0 < \delta < 1$ . Suppose a nonnegative function  $y(t)$  satisfies the differential inequality*

$$y'(t) + b(t) \leq a(t)y^\delta(t) \quad \text{on } [0, \infty), \quad y(0) = y_0. \quad (9)$$

*Then, there exists a constant  $C = C(\delta)$  such that*

$$y(t) + \int_0^t b(s)ds \leq C \left\{ y_0 + \left( \int_0^t a(s)ds \right)^{\frac{1}{1-\delta}} \right\} \quad \forall t \in [0, \infty). \quad (10)$$

**Proof:** Solving the differential inequality  $y' \leq a(t)y^\delta$ , we obtain easily

$$\begin{aligned} y(t) &\leq \left\{ y_0^{1-\delta} + (1-\delta) \int_0^t a(s)ds \right\}^{\frac{1}{1-\delta}} \\ &\leq \left\{ y_0^{1-\delta} + \int_0^t a(s)ds \right\}^{\frac{1}{1-\delta}} \quad t \geq 0. \end{aligned} \quad (11)$$

Substituting (11) into (9), and integrating the resulting inequality over  $[0, t]$ , we deduce

$$\begin{aligned} y(t) - y_0 + \int_0^t b(s)ds &\leq \int_0^t a(s)ds \left\{ y_0^{1-\delta} + \int_0^t a(s)ds \right\}^{\frac{\delta}{1-\delta}} \\ &\leq \left\{ y_0^{1-\delta} + \int_0^t a(s)ds \right\}^{\frac{1}{1-\delta}} \\ &\leq 2^{\frac{\delta}{1-\delta}} \left\{ y_0 + \left( \int_0^t a(s)ds \right)^{\frac{1}{1-\delta}} \right\}. \end{aligned}$$

Setting  $C(\delta) = 2^{\frac{\delta}{1-\delta}} + 1$ , we have (10). ■

**Proof of Theorem 1:** Taking  $L^2(\mathbb{R}^3)$ -inner product of (1) with  $su|u|^{s-2}$ , we obtain after integration by part

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^3} |u|^s dx + 2 \int_{\mathbb{R}^3} |\nabla |u|^{\frac{s}{2}}|^2 dx &= s \int_{\mathbb{R}^3} p(u \cdot \nabla) |u|^{s-2} dx \\ &\leq 2(s-2) \int_{\mathbb{R}^3} |p| |u|^{\frac{s}{2}-1} |\nabla |u|^{\frac{s}{2}}| dx = I, \end{aligned} \quad (12)$$

where we used the equality

$$\begin{aligned} \int_{\mathbb{R}^3} (\Delta u) \cdot u |u|^{s-2} dx &= - \int_{\mathbb{R}^3} |\nabla u|^2 |u|^{s-2} dx - \frac{s-2}{2} \int_{\mathbb{R}^3} |\nabla |u|^2| |u|^{s-2} dx \\ &= -\frac{s}{2} \int_{\mathbb{R}^3} |\nabla |u|^2| |u|^{s-2} dx = -\frac{2}{s} \int_{\mathbb{R}^3} |\nabla |u|^{\frac{s}{2}}|^2 dx. \end{aligned}$$

Below we estimate  $I$  of (12) case by case. Before doing that we recall the well-known  $L^p$ -interpolation inequality

$$\|u\|_r \leq \|u\|_p^\theta \|u\|_q^{1-\theta}, \quad (13)$$

where  $\frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q}$ ,  $0 \leq \theta \leq 1$ ,  $1 \leq p, q \leq \infty$ , and the Sobolev inequality,

$$\|u\|_{3s} = \| |u|^{\frac{s}{2}} \|_6^{\frac{2}{s}} \leq C \|\nabla |u|^{\frac{s}{2}} \|_2^{\frac{2}{s}} \quad (14)$$

In the all cases below it is assumed that  $1 < \alpha \leq \infty$ .

**Case 1:**  $\gamma = \infty$ .

$$\begin{aligned} I &\leq 2(s-2) \|p\|_\infty^{\frac{1}{2}} \|p\|_\infty^{\frac{1}{2}} \|u\|_s^{\frac{s-2}{2}} \|\nabla |u|^{\frac{s}{2}} \|_2 \quad (\text{By the Hölder's inequality}) \\ &\leq C \|p\|_\infty^{\frac{1}{2}} \|u\|_s^{\frac{s}{2}} \|\nabla |u|^{\frac{s}{2}} \|_2 \quad (\text{By the Calderon-Zygmund inequality, (7)}) \\ &\leq C \|p\|_\infty \|u\|_s^s + \|\nabla |u|^{\frac{s}{2}} \|_2^2. \quad (\text{By Young's inequality}) \end{aligned} \quad (15)$$

From (15) and (12) we obtain

$$\frac{d}{dt} \|u\|_s^s + \|\nabla |u|^{\frac{s}{2}} \|_2^2 \leq C \|p\|_\infty \|u\|_s^s. \quad (16)$$

Applying the standard Gronwall inequality, we deduce from (16)

$$\sup_{0 \leq t \leq T} \|u\|_s^s + \int_0^T \|\nabla |u|^{\frac{s}{2}} \|_2^2 dt \leq \|u_0\|_s^s \exp(C \|p\|_{L^1, \infty}) \quad (17)$$

From the Sobolev inequality, (14), for second term of the left hand side of (17) we have (4).

**Case 2:**  $s \leq \gamma < \infty$  and  $2 - \frac{2}{\alpha} - \frac{3}{\gamma} \geq \frac{s-3}{\gamma}$ .

$$\begin{aligned}
I &\leq 2(s-2)\|p\|_s \|u\|_s^{\frac{s-2}{2}} \|\nabla|u|^{\frac{s}{2}}\|_2 \quad (\text{By the Hölder's inequality}) \\
&\leq 2(s-2)\|p\|_{\gamma}^{\frac{\gamma}{2\gamma-s}} \|p\|_{\frac{s}{2}}^{\frac{\gamma-s}{2}} \|u\|_{\gamma}^{\frac{s-2}{2}} \|\nabla|u|^{\frac{s}{2}}\|_2 \quad (\text{By the interpolation inequality, (13)}) \\
&\leq C\|p\|_{\gamma}^{\frac{\gamma}{2\gamma-s}} \|u\|_s^{\frac{2\gamma s-2s-s^2}{4\gamma-2s}} \|\nabla|u|^{\frac{s}{2}}\|_2 \quad (\text{By the Calderon-Zygmund's estimate, (7)}) \\
&\leq C\|p\|_{\gamma}^{\frac{2\gamma}{2\gamma-s}} \|u\|_s^{s-\frac{2s}{2\gamma-s}} + \|\nabla|u|^{\frac{s}{2}}\|_2^2. \quad (\text{By Young's inequality}) \tag{18}
\end{aligned}$$

Combining this with (12), the following inequality holds.

$$\frac{d}{dt} \|u\|_s^s + \|\nabla|u|^{\frac{s}{2}}\|_2^2 \leq C\|p\|_{\gamma}^{\frac{2\gamma}{2\gamma-s}} \|u\|_s^{s-\frac{2s}{2\gamma-s}}. \tag{19}$$

Applying (10) with  $\delta = 1 - \frac{2}{2\gamma-s}$ , and, using Hölder's inequality, we obtain

$$\begin{aligned}
\sup_{0 \leq t \leq T} \|u\|_s^s + \int_0^T \|\nabla|u|^{\frac{s}{2}}\|_2^2 dt &\leq C \left\{ \|u_0\|_s^s + \left( \int_0^T \|p(t)\|_{\gamma}^{\frac{2\gamma}{2\gamma-s}} dt \right)^{\frac{2\gamma-s}{2}} \right\} \\
&\leq C(T) (\|u_0\|_s^s + \|p\|_{L_T^{\alpha,\gamma}}^{\gamma}) \tag{20}
\end{aligned}$$

Applying the Sobolev inequality, (14), to estimate (20) in the left hand direction, we obtain (5).

**Case 3:**  $\frac{3s}{s+1} \leq \gamma < s$  and  $2 - \frac{2}{\alpha} - \frac{3}{\gamma} \geq \frac{s-3}{s}$ .

$$\begin{aligned}
I &\leq 2(s-2)\|p\|_{\gamma} \|u\|_s^{\frac{s-2}{2}} \|\nabla|u|^{\frac{s}{2}}\|_2 \quad (\text{By Hölder's inequality}) \\
&\leq 2(s-2)\|p\|_{\gamma} \|u\|_s^{\frac{\gamma s-3s+\gamma}{2\gamma}} \|u\|_{3s}^{\frac{3s-3\gamma}{2\gamma}} \|\nabla|u|^{\frac{s}{2}}\|_2 \quad (\text{By the interpolation inequality, (13)}) \\
&\leq C\|p\|_{\gamma} \|u\|_s \|\nabla|u|^{\frac{s}{2}}\|_2^{\frac{3s+\gamma s-3\gamma}{\gamma s}} \quad (\text{By the Sobolev inequality, (14)}) \\
&\leq C\|p\|_{\gamma}^{\frac{2\gamma s}{\gamma s-3s+3\gamma}} \|u\|_s^{s-\frac{2\gamma s}{\gamma s-3s+3\gamma}} + \|\nabla|u|^{\frac{s}{2}}\|_2^2. \quad (\text{By Young's inequality}) \tag{21}
\end{aligned}$$

From (21) and (12), we find the following inequality holds.

$$\frac{d}{dt} \|u\|_s^s + \|\nabla|u|^{\frac{s}{2}}\|_2^2 \leq C\|p\|_{\gamma}^{\frac{2\gamma s}{\gamma s-3s+3\gamma}} \|u\|_s^{s(1-\frac{2\gamma}{\gamma s-3s+3\gamma})}, \tag{22}$$

Applying (10) with  $\delta = 1 - \frac{2\gamma}{\gamma s-3s+3\gamma}$ , we have

$$\begin{aligned}
\sup_{0 \leq t < T} \|u(t)\|_s^s + \int_0^T \|\nabla|u|^{\frac{s}{2}}\|_2^2 dt &\leq C \left\{ \|u_0\|_s^s + \left( \int_0^T \|p(t)\|_{\gamma}^{\frac{2\gamma s}{\gamma s-3s+3\gamma}} dt \right)^{\frac{\gamma s-3s+3\gamma}{2\gamma}} \right\} \\
&\leq C(T) (\|u_0\|_s^s + \|p\|_{L_T^{\alpha,\gamma}}^s) \tag{23}
\end{aligned}$$

Applying the Sobolev inequality to estimate (23) as previously, we obtain (5).

**Case 4:**  $\frac{3}{2} < \gamma < \frac{3s}{s+1}$  and  $2 - \frac{2}{\alpha} - \frac{3}{\gamma} \geq \frac{s-3}{\alpha}$ .

$$\begin{aligned}
I &\leq 2(s-2) \|p\|_{\frac{3s}{s+1}} \|u\|_{\frac{3s}{2}}^{\frac{s-2}{2}} \|\nabla|u|^{\frac{s}{2}}\|_2 \quad (\text{By Hölder's inequality}) \\
&\leq 2(s-2) \|p\|_{\gamma}^{\frac{(s-1)\gamma}{3s-2\gamma}} \|p\|_{\frac{3s}{2}}^{\frac{3s-\gamma-s\gamma}{3s-2\gamma}} \|u\|_{\frac{3s}{2}}^{\frac{s-2}{2}} \|\nabla|u|^{\frac{s}{2}}\|_2 \quad (\text{By the interpolation inequality, (13)}) \\
&\leq C \|p\|_{\gamma}^{\frac{(s-1)\gamma}{3s-2\gamma}} \|u\|_{\frac{3s}{2}}^{\frac{3s^2-6s\gamma+6s}{6s-4\gamma}} \|\nabla|u|^{\frac{s}{2}}\|_2 \quad (\text{By Calderon-Zygmund's estimate, (7)}) \\
&\leq C \|p\|_{\gamma}^{\frac{(s-1)\gamma}{3s-2\gamma}} \|\nabla|u|^{\frac{s}{2}}\|_2^{\frac{6s-8\gamma+6}{3s-2\gamma}} \quad (\text{By Sobolev's inequality, (14)}) \\
&\leq C \|p\|_{\gamma}^{\frac{(s-1)\gamma}{2\gamma-3}} + \|\nabla|u|^{\frac{s}{2}}\|_2^2. \quad (\text{By Young's inequality}) \tag{24}
\end{aligned}$$

From (24) and (12), we have

$$\frac{d}{dt} \|u\|_s^s + \|\nabla|u|^{\frac{s}{2}}\|_2^2 \leq C \|p\|_{\gamma}^{\frac{(s-1)\gamma}{2\gamma-3}}. \tag{25}$$

By integrating the both sides of (25) over  $[0, T]$ , the following inequality is immediate.

$$\sup_{0 \leq t < T} \|u(t)\|_s^s + \int_0^T \|\nabla|u|^{\frac{s}{2}}\|_2^2 dt \leq \|u_0\|_s^s + C \int_0^T \|p\|_{\gamma}^{\frac{(s-1)\gamma}{2\gamma-3}} dt. \tag{26}$$

Applying Hölder's inequality in the right hand direction, and the Sobolev inequality in the left hand direction of (26) as previously, we obtain (5).

**Case 5:**  $\gamma = \frac{3}{2}$ .

Setting  $\gamma = \frac{3}{2}$  in the 4-th inequality from above of (24), we have

$$I \leq C_1 \|p\|_{\frac{3}{2}}^{\frac{1}{2}} \|\nabla|u|^{\frac{s}{2}}\|_2^2. \tag{27}$$

Then we see from (12),

$$\frac{d}{dt} \|u(t)\|_s^s + 2 \|\nabla|u|^{\frac{s}{2}}\|_2^2 \leq C_1 \left( \sup_{0 \leq t < T} \|p(t)\|_{\frac{3}{2}}^{\frac{1}{2}} \right) \|\nabla|u|^{\frac{s}{2}}\|_2^2. \tag{28}$$

If  $C_1 \left( \sup_{0 \leq t < T} \|p(t)\|_{\frac{3}{2}}^{\frac{1}{2}} \right) \leq 1$ , then

$$\sup_{0 \leq t < T} \|u(t)\|_s^s + \int_0^T \|\nabla|u|^{\frac{s}{2}}\|_2^2 dt \leq \|u_0\|_s^s. \tag{29}$$

Applying the Sobolev inequality (14) to (29), we obtain (6). Proof of Theorem 1 is complete. ■

In order to prove Theorem 2 we recall the following result due to Y. Giga.

**Theorem 4 (Giga, [5], pp. 202)** *Suppose  $u_0 \in L^s(\mathbb{R}^3)$ ,  $s \geq 3$ , then there exists  $T_0$  and a unique classical solution  $u \in BC([0, T_0]; L^s(\mathbb{R}^3))$ . Moreover, let  $(0, T_*)$  be the maximal interval such that  $u$  solves (1)-(3) in  $C((0, T_*); L^s(\mathbb{R}^3))$ ,  $s > 3$ . Then*

$$\|u(\tau)\|_s \geq \frac{C}{(T_* - \tau)^{\frac{s-3}{2s}}} \quad (30)$$

with constant  $C$  independent of  $T_*$  and  $s$ .

**Proof of Theorem 2:** We consider here the case  $p \in L_T^{\alpha, \gamma}$  with  $\frac{2}{\alpha} + \frac{3}{\gamma} < 2$ ,  $1 < \alpha < \infty$ ,  $\frac{3}{2} < \gamma \leq \infty$ . The other cases of which  $p \in L_T^{1, \infty}$ , or else  $\|p\|_{L_T^{\infty, \frac{3}{2}}}$  is sufficiently small are similar. Since  $u_0 \in L^2(\mathbb{R}^3) \cap L^q(\mathbb{R}^3)$  for some  $q > 3$ ,  $u_0 \in L^s(\mathbb{R}^3)$  for any  $s \in (3, q)$ . We now fix  $s \in (3, q)$  so that

$$\left(2 - \frac{2}{\alpha} - \frac{3}{\gamma}\right) \min\{\alpha, \gamma, 3\} \geq s - 3.$$

Then,  $2 - \frac{2}{\alpha} - \frac{3}{\gamma} \geq \frac{s-3}{\eta}$  holds, and, by the apriori estimate in Theorem 1 combined with the standard continuation argument, which is possible thanks to the estimate (30) of Theorem 4, we can continue our local smooth solution corresponding to initial data  $u_0 \in L^s(\mathbb{R}^3)$  to obtain  $u \in BC([0, T]; L^s(\mathbb{R}^3)) \cap C^\infty(\mathbb{R}^3 \times (0, T))$ . This completes the proof of Theorem 2. ■

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