HOPF BIFURCATION ON A SQUARE LATTICE

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MARY SILBER* and EDGAR KNOBLOCH†

Abstract. A complete classification of the generic $D_4 \times_s T^2$–equivariant Hopf bifurcation problems is presented. This bifurcation arises naturally in the study of extended systems, invariant under the Euclidean group E(2), when a spatially uniform quiescent state loses stability to waves of wavenumber $k \neq 0$ and frequency $\omega \neq 0$. The $D_4 \times_s T^2$ symmetry group applies when periodic boundary conditions are imposed in two orthogonal horizontal directions. The center manifold theorem allows a reduction of the infinite dimensional problem to a bifurcation problem on C^4. In normal form, the vector field on C^4 commutes with an S^1 symmetry, which is interpreted as a time translation symmetry. The spatial and spatio–temporal symmetries of all possible solutions are classified in terms of isotropy subgroups of $D_4 \times_s T^2 \times S^1$. On a neighborhood of the Hopf bifurcation point, all small amplitude oscillatory solutions are found and their stabilities calculated relative to perturbations that preserve the spatial periodicity of the square lattice. There are five oscillatory solutions with maximal isotropy that bifurcate from the trivial solution; these are interpreted in terms of standing and travelling wave convection patterns. An unstable submaximal solution is also found to exist in an open region of parameter space. All possible bifurcation diagrams are given. The possibility of a primary bifurcation to a structurally stable heteroclinic cycle is explored.

1. Introduction.

Many pattern–forming systems produce approximately spatially periodic patterns. This observation has motivated a number of studies of pattern selection in systems with periodic boundary conditions. For doubly periodic patterns in a plane the resulting spatial domain may be identified with a two–torus. There are three doubly periodic lattices that tile a plane and for which the wavelength is the same in the two directions of periodicity. These are the rhombic, square and hexagonal lattices. The formulation of the problem on such a lattice introduces important symmetries into the problem. These are the symmetries of the unit cell D_n, $n = 2, 4, 6$ respectively, in addition to the two–torus of translations. The presence of these symmetries is responsible for the multiple solution branches that bifurcate from the trivial (homogeneous) solution, and can be exploited in analyzing their relative stability, *i.e.*, their stability properties with respect to perturbations preserving the lattice. This approach has met with considerable success in describing pattern formation arising from a steady state bifurcation [1–3]. Pattern formation arising from a Hopf bifurcation is less well studied. In this paper we focus attention on patterns, periodic on a square lattice, that form at a Hopf bifurcation. The spatial symmetry of the problem is described by the semi–direct product group $D_4 \times_s T^2$. Certain aspects of this problem have been considered previously. Swift [4] derived the form of the general equivariant vector field and determined the solutions of

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the third order truncation. However, he did not consider the stability properties of these solutions. Pismen [5] determined the stabilities of the standing wave solutions in the case of the third order truncation, but did not complete the stability analysis for the travelling wave solutions. Neither paper addresses the symmetry properties of the solutions. Additional work, complementary to this paper, has been done on Hopf bifurcation with O(2) symmetry [6–9], D_4 symmetry [10,11], and $D_6 \times_s T^2$ symmetry [12].

In this paper we present an essentially complete description of the generic Hopf bifurcation on a square lattice. We employ the techniques of equivariant bifurcation theory relying heavily on the symmetry properties of the possible solutions. In the next section we point out that the symmetry is responsible for the quadrupling of the eigenvalues at the bifurcation and state explicitly the assumptions that allow a formulation of the bifurcation problem on an eight–dimensional center manifold. We discuss the symmetries of this problem and observe that in Poincaré–Birkhoff normal form it will have an additional S^1 phase shift symmetry which we interpret as a time translation symmetry. In section 3 we determine the isotropy subgroup structure of the resulting symmetry group $D_4 \times_s T^2 \times S^1$. This analysis determines not only the possible solutions but also classifies them by their spatial and spatio–temporal symmetries. Five nontrivial solutions, guaranteed by the equivariant Hopf theorem [6], are found. In section 4, the general form of the equivariant vector field on the center manifold is derived. The oscillatory solutions that bifurcate from the trivial solution at the Hopf bifurcation are determined from the third order truncation of this vector field in section 5. It is found that in addition to the five solutions guaranteed by abstract theory a sixth solution with submaximal isotropy bifurcates in a primary bifurcation in an open region of parameter space. The stabilities of all six solutions are calculated in section 6, and all possible bifurcation diagrams depicted in section 7. In section 8 we describe the possible heteroclinic orbits connecting the various solutions and remark on the possible dynamics when all primary branches bifurcate supercritically but none are stable. In particular, we determine the region of the coefficient space where we expect a primary bifurcation to a structurally stable heteroclinic cycle. Finally, in section 9 we summarize our results and discuss possible applications.

2. Problem Formulation.

We consider the oscillatory instability of the trivial state of an isotropic homogeneous spatially extended system invariant under E(2), the Euclidean group of symmetries of the plane, *i.e.*, translations, rotations and reflections. Here a "trivial state" is any time–independent solution which is uniform in the horizontal and hence does not break the E(2) symmetry. It is assumed that the instability sets in with a unique *nonzero* wavenumber k_c and frequency ω_c leading to solutions of the linear problem that take the form of standing and travelling waves. Specifically, let $\mathcal{S}(\mu, k; R) = 0$ be the dispersion relation for linear modes of the form $e^{\mu t + i\mathbf{k}\cdot\mathbf{x}}g(y)$, where $g(y)$ is an appropriate vertical eigenfunction, $\mathbf{x} \in \mathbb{R}^2$ is a horizontal vector and $\mathcal{S}, \mu \in \mathbb{C}$, $k, R \in \mathbb{R}$, $k \equiv |\mathbf{k}|$. Here R is a control parameter of the system. The neutral stability curve $R(k)$ is determined from the real and imaginary parts of $\mathcal{S}(\mu = i\omega, k; R) = 0$. Its minimum defines k_c, $R_c = R(k_c)$ (see Figure 1). The critical frequency satisfies $\mathcal{S}(i\omega_c, k_c; R_c) = 0$. For the remainder of the paper, we let $\lambda \propto (R - R_c)$ be the bifurcation parameter and further assume

$$\left.\frac{\partial\, \mathrm{Re}(\mu)}{\partial \lambda}\right|_{k=k_c, \lambda=0} > 0. \tag{2.1}$$

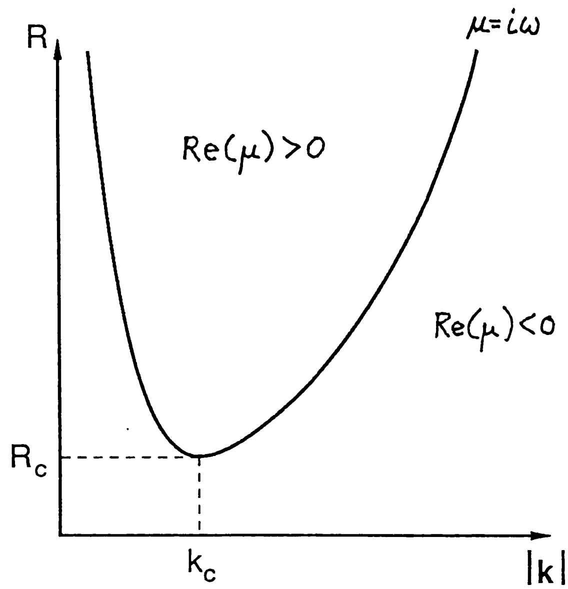


Figure 1: Hopf bifurcation curve $R(k)$. The trivial solution is unstable above the curve and linearly stable below it.

At $\lambda = 0$ there is an entire circle of wavevectors $\mathbf{k}$ with magnitude k_c and hence an infinite number of neutrally stable modes at the Hopf bifurcation point. In the presence of periodic boundary conditions in the horizontal this orientational degeneracy is absent. Specifically, we restrict the space of solutions to those that are periodic on a square lattice $\mathcal{L}_s$, where

$$\mathcal{L}_s = \{n_1\mathbf{a}_1 + n_2\mathbf{a}_2 :\ n_1, n_2 \in \mathbb{Z},\ \mathbf{a}_1 = \frac{2\pi}{k_c}\hat{\mathbf{x}}_1,\ \mathbf{a}_2 = \frac{2\pi}{k_c}\hat{\mathbf{x}}_2\}. \tag{2.2}$$

Here, $\hat{\mathbf{x}}_1$, $\hat{\mathbf{x}}_2$ are orthonormal vectors that span a horizontal plane. The symmetry of the problem thereby reduces to that of a compact subgroup of the Euclidean group; translational symmetry is identified with the two–torus $\mathrm{T}^2 = \mathbb{R}^2/\mathcal{L}_s$ since translations by vectors in $\mathcal{L}_s$ act trivially on the space of periodic solutions. The reflectional and rotational symmetries are those of the unit square cell, corresponding to the discrete group D_4. The symmetry group is thus the semi–direct product of these groups: $\Gamma_s = D_4 \times_s T^2$.

The periodic boundary conditions are essential to the analysis which follows. Specifically, at $\lambda = 0$ only a finite number of the linear modes have eigenvalues on the imaginary axis while the rest of the spectrum is bounded away from the imaginary axis. This allows us to invoke the center manifold theorem [13] and rigorously formulate the bifurcation problem in terms of ordinary differential equations for the amplitudes of the critical modes. The Γ_s–invariant subspace of neutrally stable modes at $\lambda = 0$ is

$$\begin{aligned}\mathcal{F}_s = \Big\{ &\big(v_1(t)e^{ik_cx_1} + v_2(t)e^{ik_cx_2} + w_1(t)e^{-ik_cx_1} + w_2(t)e^{-ik_cx_2}\big)g(y) : \\ &\mathbf{z} \equiv (v_1, v_2, w_1, w_2) \in \mathbb{C}^4 \Big\}.\end{aligned} \tag{2.3}$$

The linear part of the ordinary differential equation for the critical amplitudes is $\dot{\mathbf{z}} = \mu(\lambda)\mathbf{z}$, where $\mu(0) = i\omega_c$ and $\mathrm{Re}(\mu'(0)) > 0$. Hence, we interpret $|w_j|$ as the amplitude of a travelling wave in the positive $\hat{\mathbf{x}}_j$–direction and $|v_j|$ as the amplitude of a travelling wave in the negative $\hat{\mathbf{x}}_j$–direction, $j = 1, 2$.

The action of the symmetry group Γ_s on the amplitudes of the critical modes is inherited from its natural action on $\mathbb{R}^2$. From (2.3) it follows that a translation $(x_1, x_2) \to (x_1 + l_1, x_2 + l_2)$ is equivalent to a phase shift of the amplitudes

$$(\theta_1, \theta_2) : \begin{pmatrix} v_1 \\ v_2 \\ w_1 \\ w_2 \end{pmatrix} \to \begin{pmatrix} e^{i\theta_1} v_1 \\ e^{i\theta_2} v_2 \\ e^{-i\theta_1} w_1 \\ e^{-i\theta_2} w_2 \end{pmatrix}, \quad (\theta_1, \theta_2) \in T^2, \tag{2.4a}$$

where $\theta_j = k_c l_j, \;\; j = 1, 2$. The discrete group D_4 is generated by a reflection $\sigma_v : x_2 \to -x_2$ and a $\pi/2$ rotation $r_{\pi/2} : (x_1, x_2) \to (x_2, -x_1)$. As indicated in Figure 2 the corresponding action on $\mathbb{C}^4$ is

$$\sigma_v : \begin{pmatrix} v_1 \\ v_2 \\ w_1 \\ w_2 \end{pmatrix} \to \begin{pmatrix} v_1 \\ w_2 \\ w_1 \\ v_2 \end{pmatrix}, \qquad r_{\pi/2} : \begin{pmatrix} v_1 \\ v_2 \\ w_1 \\ w_2 \end{pmatrix} \to \begin{pmatrix} w_2 \\ v_1 \\ v_2 \\ w_1 \end{pmatrix}. \tag{2.4b}$$

This representation of $D_4 \times_s T^2$ is not irreducible; the group representations relevant to Hopf bifurcation problems are (generically) either (a) irreducible, but not absolutely irreducible, or (b) isomorphic to the direct sum of two copies of an absolutely irreducible representation

[6]. The group representation of Γ_s, generated by (2.4), is of type (b): it readily decomposes into two copies of an absolutely irreducible representation acting diagonally on

$$\begin{pmatrix} v_1 + \overline{w}_1 \\ v_2 + \overline{w}_2 \\ i(v_1 - \overline{w}_1) \\ i(v_2 - \overline{w}_2) \end{pmatrix}. \tag{2.5}$$

Here, absolutely irreducible means that the only 4×4 real matrices that commute with the group action on $\mathbb{R}^4$ are scalar multiples of the identity matrix. The 4–dimensional representation is relevant when the trivial solution loses stability via a steady state bifurcation. In this case, the action of $D_4 \times_s T^2$ is that of (2.4) on, say, the first two components of (2.5). This equivariant bifurcation problem was analyzed by Swift [4].

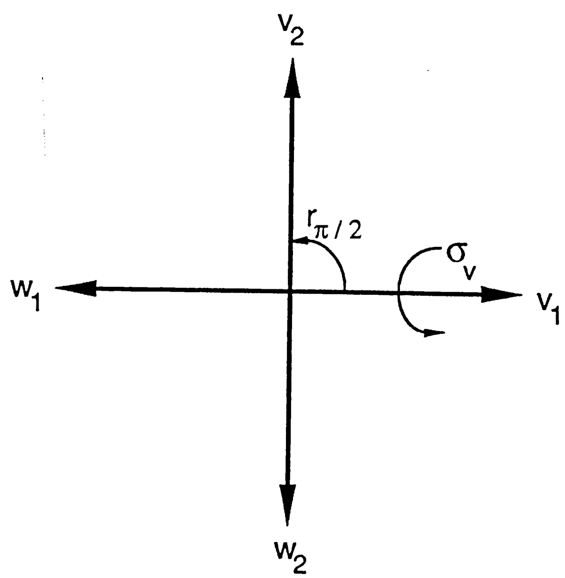


Figure 2: Action of D_4 on the amplitudes (v_1, v_2, w_1, w_2).

The vector field that commutes with the spatial symmetries (2.4) can be put into Poincaré–Birkhoff normal form. As with other Hopf bifurcation problems this procedure introduces an extra phase shift symmetry in time into the vector field. We denote this symmetry by S^1, and henceforth assume that the full spatio–temporal symmetry of the problem is specified by the group $\Gamma_s \times S^1$, where

$$\varphi : z \rightarrow e^{i\varphi} z, \quad \varphi \in S^1. \tag{2.6}$$

In the next section the isotropy subgroups of $\Gamma_s \times S^1$, together with their fixed point subspaces are determined. The isotropy subgroups determine the symmetries of all possible solutions, and the equivariant Hopf theorem [6] guarantees the existence of primary branches with two–dimensional fixed point subspaces. This abstract analysis provides much important information without explicit computation.

3. Isotropy Subgroups.

The symmetry of a solution $\mathbf{z}$ is characterized by the isotropy subgroup $\Sigma_{\mathbf{z}} \subset \Gamma_s \times \mathrm{S}^1$, which leaves it fixed, *i.e.*,

$$\Sigma_{\mathbf{z}} = \{\sigma \in \Gamma_s \times \mathrm{S}^1 : \ \sigma\mathbf{z} = \mathbf{z}\}. \tag{3.1}$$

Two solutions on the same group orbit have conjugate isotropy subgroups: if $\mathbf{z}$ is a solution with symmetry $\Sigma_{\mathbf{z}}$, then $\gamma\mathbf{z}$ has isotropy subgroup

$$\Sigma_{\gamma\mathbf{z}} = \gamma\Sigma_{\mathbf{z}}\gamma^{-1}. \tag{3.2}$$

We consider two solutions on the same group orbit to be equivalent. In this section, we determine all of the isotropy subgroups of $\Gamma_s \times \mathrm{S}^1$ up to conjugacy. In addition, we associate a fixed point subspace $\mathrm{Fix}(\Sigma)$ with every isotropy subgroup Σ, where

$$\mathrm{Fix}(\Sigma) = \{\mathbf{z} \in \mathbb{C}^4 : \ \sigma\mathbf{z} = \mathbf{z} \text{ for all } \sigma \in \Sigma\}. \tag{3.3}$$

In section 5, we will use the observation that $\mathrm{Fix}(\Sigma)$ is invariant under the dynamics [14] to determine the periodic solutions with symmetry Σ.

We denote the group elements of $\Gamma_s \times \mathrm{S}^1$ by $[\gamma, \varphi]$, where $\gamma \in \Gamma_s$ and $\varphi \in \mathrm{S}^1$. The elements of Γ_s may be denoted by $r^n_{\pi/2}\sigma^m_{\mathrm{v}}(\theta_1, \theta_2)$, with $n \in \{0,1,2,3\}$, $m \in \{0,1\}$, and $(\theta_1, \theta_2) \in \mathrm{T}^2$. Hence, the action of a general element of $\Gamma_s \times \mathrm{S}^1$ on $\mathbb{C}^4$ is

$$[r^n_{\pi/2}\sigma^m_{\mathrm{v}}(\theta_1,\theta_2),\varphi] : \begin{pmatrix} v_1 \\ v_2 \\ w_1 \\ w_2 \end{pmatrix} \to r^n_{\pi/2}\sigma^m_{\mathrm{v}} \begin{pmatrix} e^{i(\varphi+\theta_1)}v_1 \\ e^{i(\varphi+\theta_2)}v_2 \\ e^{i(\varphi-\theta_1)}w_1 \\ e^{i(\varphi-\theta_2)}w_2 \end{pmatrix}, \tag{3.4}$$

where the actions of σ_{v} and $r_{\pi/2}$ are given by (2.4b). The subgroup of $\Gamma_s \times \mathrm{S}^1$ generated by $\sigma_1, \ldots, \sigma_k \in \Gamma_s \times \mathrm{S}^1$ is denoted $\langle\sigma_1, \ldots, \sigma_k\rangle$. Note that $[(\pi,\pi),\pi]$ acts trivially on $\mathbb{C}^4$ and hence is contained in every isotropy subgroup.

The remainder of this section is devoted to compiling Table 1 which contains a representative isotropy subgroup for each conjugacy class, together with its fixed point subspace. We begin by noting that every isotropy subgroup is conjugate to one whose fixed point subspace is of the form

$$(v_1, v_2, w_1, w_2) = (r_1, r_2e^{i\psi}, r_3, r_4e^{i\psi}), \quad r_j \geq 0, \quad 0 \leq \psi < \pi. \tag{3.5a}$$

That three of the phases of a solution may be chosen is a consequence of the three continuous symmetries (*i.e.*, $\mathrm{T}^2 \times \mathrm{S}^1$) which act on $\mathbb{C}^4$ by phase shifts. If any amplitude r_j is zero then we may, without loss of generality, further assume $\psi = 0$. The action of the discrete symmetries of D_4 allow us to order the amplitudes r_j such that

$$r_1 \geq r_2 \geq r_4 \geq 0, \qquad r_1 \geq r_3 \geq 0. \tag{3.5b}$$

The trivial solution has the symmetry of the full group, whereas if all four amplitudes r_j are unequal, the isotropy subgroup is the trivial one: $Z_2^c = \langle[(\pi,\pi),\pi]\rangle$. We now consider the cases where $r_1 \neq 0$ and $r_i = r_j$ for some $i \neq j$; these fall into the following categories:

$$\begin{aligned} &(a)\ r_2 = r_3 = r_4 = 0 \\ &(b)\ r_3 = r_4 = 0, \quad r_2 \neq 0 \\ &(c)\ r_2 = r_4 = 0, \quad r_3 \neq 0 \\ &(d)\ r_1 = r_2 \neq 0, \quad r_3 \neq 0 \\ &(e)\ r_1 = r_3 \neq 0, \quad r_2 \neq 0. \end{aligned} \tag{3.6}$$

For cases (d) and (e) we do not assume the ordering of (3.5b).

The entries in Table 1 are verified by considering in turn each of the cases in (3.6):

(a) $\mathbf{z} = (r_1, 0, 0, 0)$ corresponds to travelling rolls (TR) and has isotropy subgroup $O(2) \times S^1 = \langle\sigma_v, (0,\theta_2)\rangle \times [(-\varphi, 0), \varphi]$. Here S^1 is a spatio–temporal symmetry; a time translation of the oscillation by approximately φ/ω_c is followed by a translation in the $\hat{x}_1$ direction by $-\varphi/k_c$.

(b) $\mathbf{z} = (r_1, r_2, 0, 0)$ has isotropy subgroup $S^1 = [(-\varphi, -\varphi), \varphi]$ if $r_1 \neq r_2$ (VIII in Table 1), and isotropy subgroup $Z_2 \times S^1$ if $r_1 = r_2$, where $Z_2 = \langle r_{\pi/2}\sigma_v\rangle$ (travelling squares (TS)).

(c) $\mathbf{z} = (r_1, 0, r_3, 0)$ has isotropy subgroup $Z_2^c \times O(2)$ if $r_1 \neq r_3$ (IX in Table 1). If $r_1 = r_3$ then the isotropy subgroup is enlarged by $Z_2 = \langle r_{\pi/2}^2\rangle$ (standing rolls (SR)).

(d) $\mathbf{z} = (r_1, r_1 e^{i\psi}, r_3, r_4 e^{i\psi})$ must have $r_3 = r_4$ for the isotropy subgroup to be nontrivial. If $r_3 = r_4 \neq r_1$, then the isotropy subgroup is nontrivial for those values of ψ satisfying

$$[r_{\pi/2}\sigma_v(\theta_1,\theta_2),\varphi]\begin{pmatrix} r_1 \\ r_1 e^{i\psi} \\ r_3 \\ r_3 e^{i\psi}\end{pmatrix} = \begin{pmatrix} r_1 e^{i(\varphi+\theta_2+\psi)} \\ r_1 e^{i(\varphi+\theta_1)} \\ r_3 e^{i(\varphi-\theta_2+\psi)} \\ r_3 e^{i(\varphi-\theta_1)}\end{pmatrix} = \begin{pmatrix} r_1 \\ r_1 e^{i\psi} \\ r_3 \\ r_3 e^{i\psi}\end{pmatrix}. \tag{3.7}$$

This equation has a solution $[(\theta_1,\theta_2),\varphi] \in T^2 \times S^1$ provided $\psi = 0, \pi/2$. For $\mathbf{z} = (r_1, r_1, r_3, r_3)$ the isotropy subgroup is $Z_2^c \times Z_2$, where $Z_2 = \langle r_{\pi/2}\sigma_v\rangle$ (X in Table 1). For $\mathbf{z} = (r_1, ir_1, r_3, ir_3)$ the isotropy subgroup is $Z_2^c \times \widetilde{Z}_2$, where $\widetilde{Z}_2 = \langle[r_{\pi/2}\sigma_v(0,\pi),\pi/2]\rangle$ (XI in Table 1). Here the tilde on $\widetilde{Z}_2$ indicates that it is a spatio–temporal symmetry of the solution.

In the case where $\mathbf{z} = (r_1, r_1 e^{i\psi}, r_1, r_1 e^{i\psi})$, the isotropy subgroup is $Z_2^c \times Z_2^2 = Z_2^c \times$

Name	Fixed Point Subspace	Isotropy Subgroup
I. Trivial Solution (T)	$v_1 = v_2 = w_1 = w_2 = 0$	$\Gamma_s \times S^1$
II. Travelling Rolls (TR)	$v_1 \neq 0,\ v_2 = w_1 = w_2 = 0$	$O(2) \times S^1 :\ O(2) = \langle \sigma_v, (0, \theta_2) \rangle,$ $S^1 = [(-\varphi, 0), \varphi]$
III. Travelling Squares (TS)	$v_1 = v_2 \neq 0,\ w_1 = w_2 = 0$	$Z_2 \times S^1 :\ Z_2 = \langle r_{\pi/2}\sigma_v \rangle,$ $S^1 = [(-\varphi, -\varphi), \varphi]$
IV. Standing Rolls (SR)	$v_1 = w_1 \neq 0,\ v_2 = w_2 = 0$	$Z_2^c \times Z_2 \times O(2) :\ Z_2 = \langle r_{\pi/2}^2 \rangle$
V. Standing Squares (SS)	$v_1 = v_2 = w_1 = w_2 \neq 0$	$Z_2^c \times D_4$
VI. Alternating Rolls (AR)	$v_1 = -iv_2 = w_1 = -iw_2 \neq 0$	$Z_2^c \times \widetilde{D}_4 :$ $\widetilde{D}_4 = \langle \sigma_v, [r_{\pi/2}(0, \pi), \pi/2] \rangle$
VII. Standing Cross-Rolls (SCR)	$v_1 = w_1 \neq 0,\ v_2 = w_2 \neq 0,$ $\lvert v_1 \rvert \neq \lvert v_2 \rvert$	$Z_2^c \times Z_2^2 :\ Z_2^2 = \langle \sigma_v, r_{\pi/2}^2 \rangle$
VIII.	$v_1 \neq 0,\ v_2 \neq 0,\ w_1 = w_2 = 0,$ $\lvert v_1 \rvert \neq \lvert v_2 \rvert$	$S^1 :\ S^1 = [(-\varphi, -\varphi), \varphi]$
IX.	$v_1 \neq 0,\ w_1 \neq 0,\ v_2 = w_2 = 0,$ $\lvert v_1 \rvert \neq \lvert w_1 \rvert$	$Z_2^c \times O(2)$
X.	$v_1 = v_2 \neq 0,\ w_1 = w_2 \neq 0,$ $\lvert v_1 \rvert \neq \lvert w_1 \rvert$	$Z_2^c \times Z_2 :\ Z_2 = \langle r_{\pi/2}\sigma_v \rangle$
XI.	$v_1 = -iv_2 \neq 0,\ w_1 = -iw_2 \neq 0,$ $\lvert v_1 \rvert \neq \lvert w_1 \rvert$	$Z_2^c \times \widetilde{Z}_2 :$ $\widetilde{Z}_2 = \langle [r_{\pi/2}\sigma_v(0, \pi), \pi/2] \rangle$
XII.	$v_1 = w_1 \neq 0,\ v_2 \neq 0,$ $\lvert v_2 \rvert \neq \lvert w_2 \rvert$	$Z_2^c \times Z_2 :\ Z_2 = \langle r_{\pi/2}^2 \sigma_v \rangle$
XIII.	$v_1 \neq 0,\ w_1 \neq 0,\ v_2 \neq 0,$ $\lvert v_2 \rvert \neq \lvert w_2 \rvert \neq \lvert v_1 \rvert \neq \lvert w_1 \rvert \neq \lvert w_2 \rvert$	$Z_2^c \equiv [(\pi, \pi), \pi]$

Table 1: All isotropy subgroups of $\Gamma_s \times S^1$, up to conjugacy, listed with their associated fixed point subspaces.

$\langle \sigma_v, r^2_{\pi/2} \rangle$ for $\psi \neq 0, \pi/2$. If $\psi = 0$ the isotropy subgroup is $Z_2^c \times D_4$ (standing squares (SS)), whereas if $\psi = \pi/2$ the isotropy subgroup is $Z_2^c \times \widetilde{D}_4$, where $\widetilde{D}_4 = \langle \sigma_v, [r_{\pi/2}(0,\pi), \pi/2] \rangle$ (alternating rolls (AR)).

(*e*) $z = (r_1, r_2 e^{i\psi}, r_1, r_4 e^{i\psi})$ has isotropy subgroup $Z_2^c \times Z_2 = Z_2^c \times \langle r^2_{\pi/2}\sigma_v \rangle$ if $r_2 \neq r_4$ (XII in Table 1). If $r_2 = r_4 \neq r_1$ then the isotropy subgroup is $Z_2^c \times Z_2^2$ where $Z_2^2 = \langle \sigma_v, r^2_{\pi/2} \rangle$ as in case (*d*) above (standing cross–rolls (SCR) in Table 1). The case $r_1 = r_2 = r_4$ is also examined in (*d*).

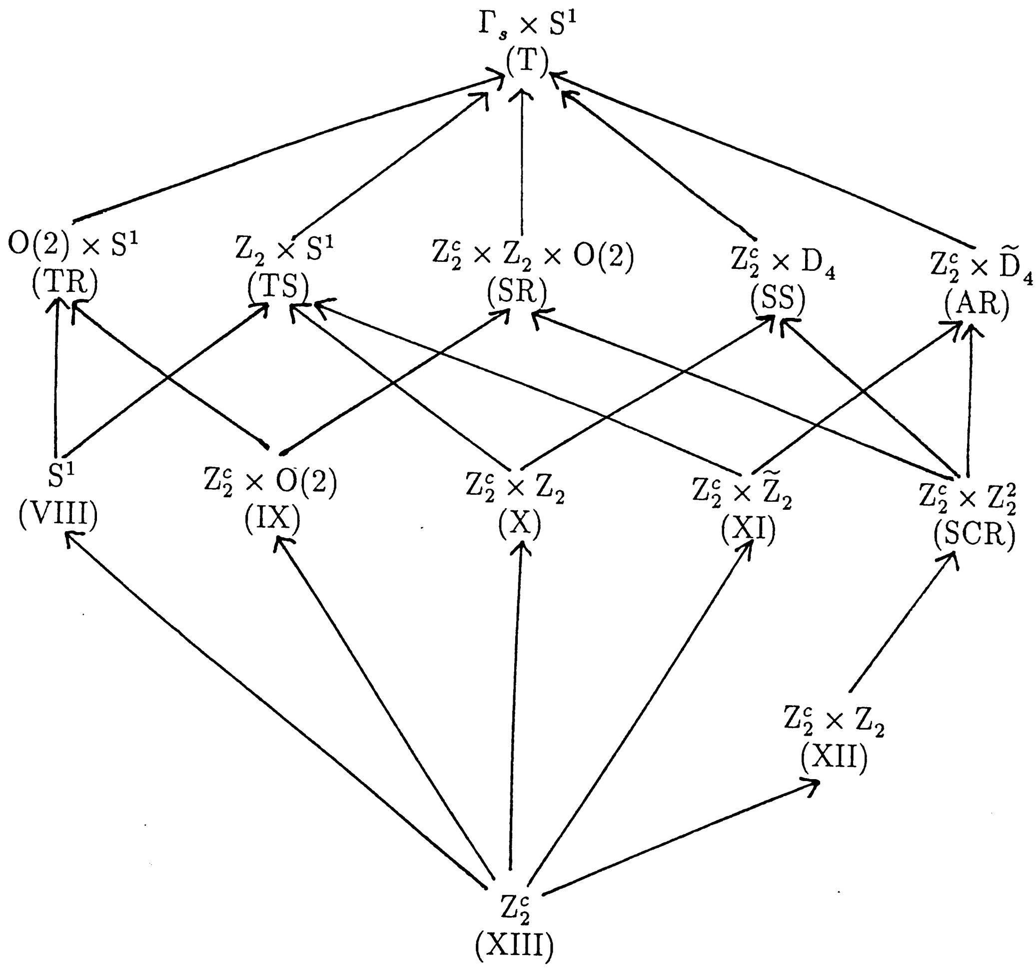
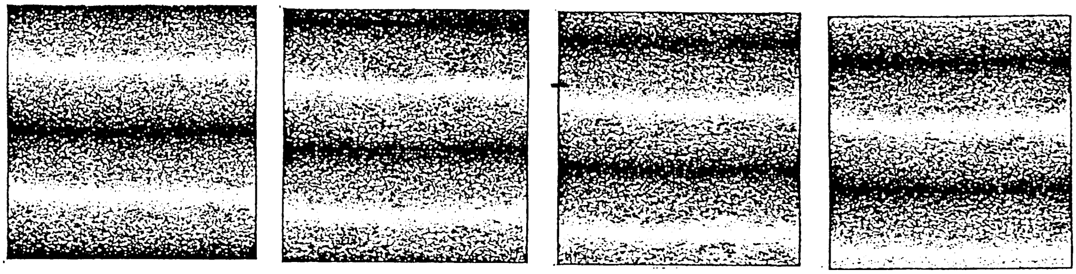


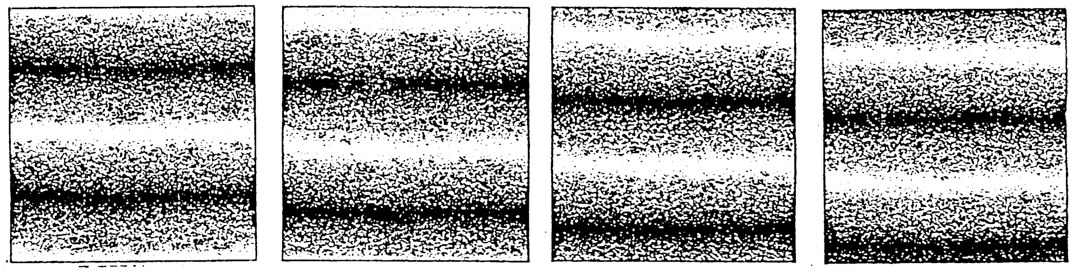
Figure 3: Lattice of isotropy subgroups of $\Gamma_s \times S^1$. Each entry represents the entire conjugacy class of the isotropy subgroup. Inclusion is indicated by an arrow.

The "lattice" of isotropy subgroups in Figure 3 summarizes the ways in which symmetry may be broken in this problem. The five maximal isotropy subgroups describe the symmetries of solutions II–VI in Table 1; each has a two–dimensional fixed point subspace. The equivariant Hopf theorem [6] guarantees that these five solutions bifurcate from the trivial solution at $\lambda = 0$. The planforms associated with these solutions are determined from

TRAVELLING ROLLS

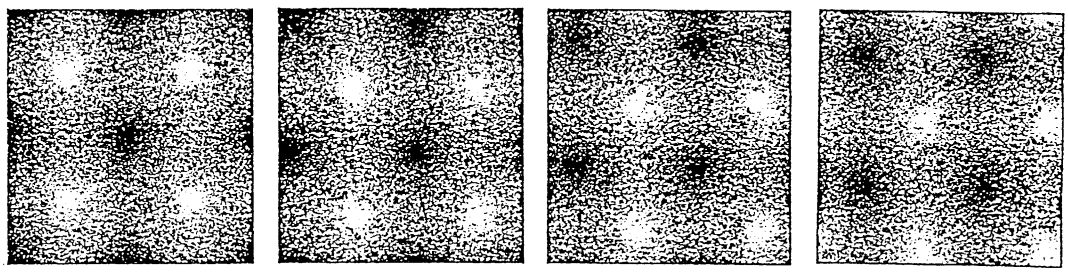


t = 0 t = 1/8 t = 1/4 t = 3/8

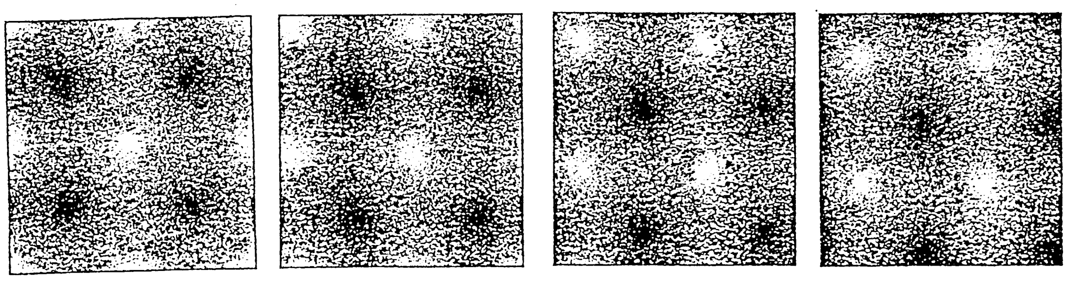


t = 1/2 t = 5/8 t = 3/4 t = 7/8

TRAVELLING SQUARES



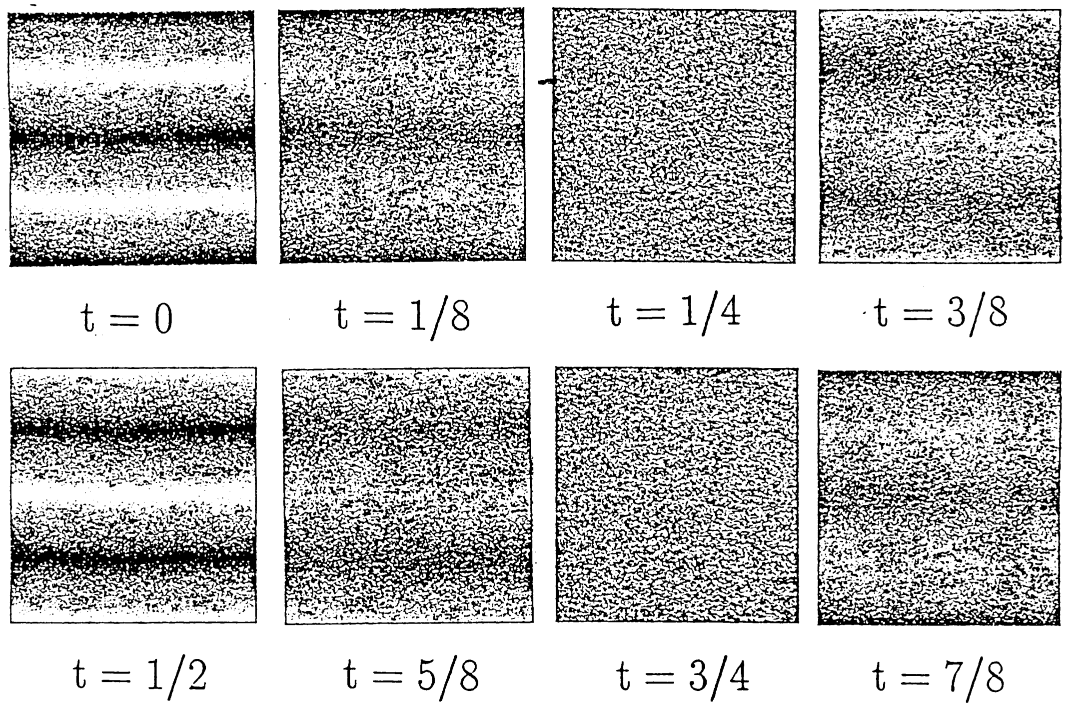
t = 0 t = 1/8 t = 1/4 t = 3/8



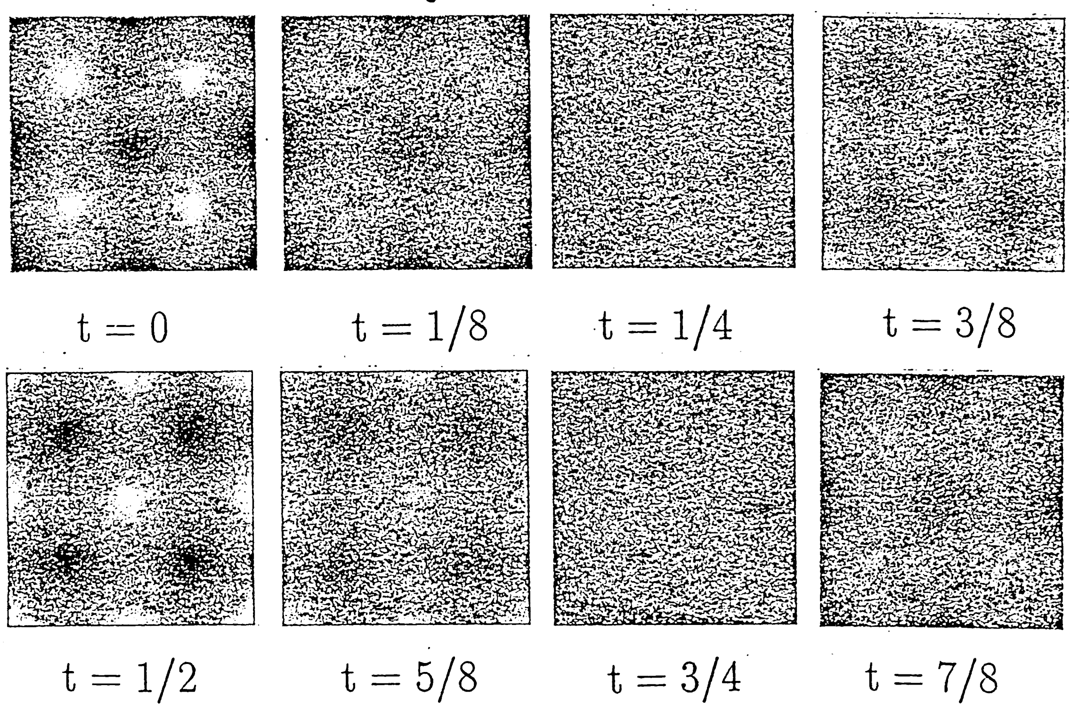
t = 1/2 t = 5/8 t = 3/4 t = 7/8

(Figure 4 continued on the next page)

STANDING ROLLS



STANDING SQUARES



(Figure 4 continued on the next page)

ALTERNATING ROLLS

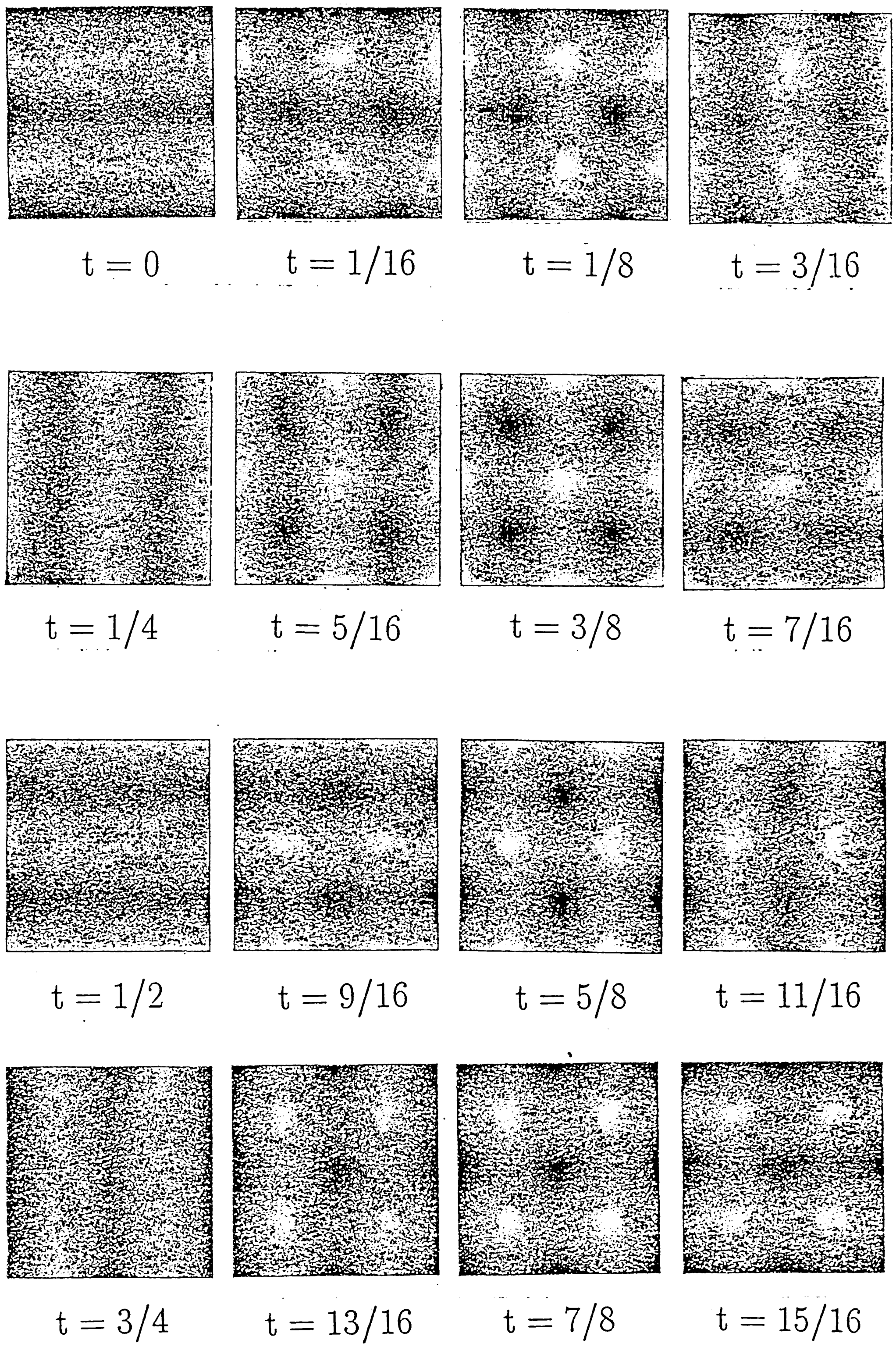


Figure 4: Planforms associated with the solutions guaranteed by the equivariant Hopf theorem. The square lattice periodicity is indicated by depicting four of the periodic cells. Note the symmetry between the solutions at $t = 0$ and half a cycle later at $t = 1/2$ (*i.e.*, the trivial symmetry Z_2^c).

the real part of the linear eigenfunctions in (2.3). The time evolution of each planform is depicted in Figure 4 as it would appear in a shadowgraph image of a convecting fluid layer. In this case dark regions correspond to hot rising fluid and light regions to cold descending fluid. Note that the solutions TR, TS and AR have nontrivial spatio–temporal symmetries, whereas the symmetries of SR and SS are purely spatial (excluding the trivial symmetry Z_2^c).

Finally, it is of interest to determine the normalizer $\mathcal{N}(\Sigma)$ of each isotropy subgroup $\Sigma \in \Gamma_s \times S^1$, where

$$\mathcal{N}(\Sigma) = \{\gamma \in \Gamma_s \times S^1 : \gamma^{-1}\Sigma\gamma = \Sigma\}. \tag{3.8}$$

The elements of $\mathcal{N}(\Sigma)$ map the fixed point subspace Fix(Σ) to itself, with the elements of $\mathcal{N}(\Sigma)/\Sigma$ acting nontrivially. Hence, a $\Gamma_s \times S^1$–equivariant vector field restricted to the invariant subspace Fix(Σ) is $\mathcal{N}(\Sigma)$–equivariant. For example, the maximal isotropy subgroups have $\mathcal{N}(\Sigma) = S^1$. In Table 2 we specify $\mathcal{N}(\Sigma)/\Sigma$ for the isotropy subgroups with four–dimensional fixed point subspaces (*i.e.*, entries VII–XI of Table 1). Note that $\mathcal{N}(\Sigma)/\Sigma \approx O(2) \times S^1$ for isotropy subgroups VIII–XI and hence that the Hopf bifurcation problem with $O(2) \times S^1$ symmetry applies in these invariant subspaces. This bifurcation problem has been extensively analyzed using singularity theory methods [8,9]. In particular, universal unfoldings of the normal forms for all degenerate bifurcation problems through codimension–two have been determined. The $D_4 \times S^1$–equivariant Hopf bifurcation problem, which applies in the SCR subspace, has been analyzed by Swift [11]. It should be noted that the results obtained in this subspace apply not only to the Hopf bifurcation problem with periodic boundary conditions (as posed in the previous section), but also to the same problem posed on a square domain with Neumann boundary conditions. Specifically, it is well known that, suitably extended, solutions obtained with Neumann boundary conditions also solve the corresponding problem with periodic boundary conditions. This extension can introduce additional symmetries into the problem (see, for example, Crawford *et al.* [15]). In the case of the square domain, these symmetries are precisely those listed for the SCR subspace in Table 2. Note that the "reflection symmetry" $(v_1, v_2) \to (v_1, -v_2)$ actually corresponds to a translation $(0, \pi) \in T^2$.

In the next section, we show by direct calculation that the form of the vector field restricted to any of the four–dimensional subspaces Fix(Σ) is exactly what we expect from considerations of the normalizer $\mathcal{N}(\Sigma)$ (*i.e.*, there are no additional restrictions placed on a $\mathcal{N}(\Sigma)$–equivariant vector field obtained in this way). This calculation is necessary in order to exclude the possibility that a degenerate $\mathcal{N}(\Sigma)$–equivariant bifurcation problem arises naturally when the equivariant vector field is restricted to Fix(Σ).

Representative Fixed Point Subspace	$\mathcal{N}(\Sigma)/\Sigma$
VII. (SCR) $(v_1, v_2, w_1, w_2) = (z_1, z_2, z_1, z_2) \in \mathbb{C}^2$	$D_4 \times S^1$ $r_{\pi/2} : (z_1, z_2) \to (z_2, z_1)$ $(0, \pi) : (z_1, z_2) \to (z_1, -z_2)$ $\varphi : (z_1, z_2) \to e^{i\varphi}(z_1, z_2)$ $\{r_{\pi/2}, (0, \pi)\} \in D_4; \quad \varphi \in S^1$
VIII. $(v_1, v_2, w_1, w_2) = (z_1, z_2, 0, 0) \in \mathbb{C}^2$	$O(2) \times S^1$ $(r_{\pi/2}\sigma_v) : (z_1, z_2) \to (z_2, z_1)$ $(\theta, -\theta) : (z_1, z_2) \to (e^{i\theta} z_1, e^{-i\theta} z_2)$ $\varphi : (z_1, z_2) \to e^{i\varphi}(z_1, z_2)$ $\{(r_{\pi/2}\sigma_v), (\theta, -\theta)\} \in O(2); \quad \varphi \in S^1$
IX. $(v_1, v_2, w_1, w_2) = (z_1, 0, z_2, 0) \in \mathbb{C}^2$	$O(2) \times S^1$ $r^2_{\pi/2} : (z_1, z_2) \to (z_2, z_1)$ $(\theta, 0) : (z_1, z_2) \to (e^{i\theta} z_1, e^{-i\theta} z_2)$ $\varphi : (z_1, z_2) \to e^{i\varphi}(z_1, z_2)$ $\{r^2_{\pi/2}, (\theta, 0)\} \in O(2); \quad \varphi \in S^1$
X. $(v_1, v_2, w_1, w_2) = (z_1, z_1, z_2, z_2) \in \mathbb{C}^2$	$O(2) \times S^1$ $r^2_{\pi/2} : (z_1, z_2) \to (z_2, z_1)$ $(\theta, \theta) : (z_1, z_2) \to (e^{i\theta} z_1, e^{-i\theta} z_2)$ $\varphi : (z_1, z_2) \to e^{i\varphi}(z_1, z_2)$ $\{r^2_{\pi/2}, (\theta, \theta)\} \in O(2); \quad \varphi \in S^1$
XI. $(v_1, v_2, w_1, w_2) = (z_1, iz_1, z_2, iz_2) \in \mathbb{C}^2$	$O(2) \times S^1$ $r^2_{\pi/2} : (z_1, z_2) \to (z_2, z_1)$ $(\theta, \theta) : (z_1, z_2) \to (e^{i\theta} z_1, e^{-i\theta} z_2)$ $\varphi : (z_1, z_2) \to e^{i\varphi}(z_1, z_2)$ $\{r^2_{\pi/2}, (\theta, \theta)\} \in O(2); \quad \varphi \in S^1$

Table 2: Action of $\mathcal{N}(\Sigma)$ on $\text{Fix}(\Sigma)$ for all four–dimensional fixed point subspaces.

4. Equivariant Vector Field.

The symmetry $\Gamma_s \times S^1$ severely restricts the form of the dynamical equation on the center manifold

$$\dot{\mathbf{z}} = \mathbf{f}(\mathbf{z}, \lambda); \quad \mathbf{f} : \mathbf{C}^4 \times \mathbf{R} \to \mathbf{C}^4. \tag{4.1a}$$

Specifically, $\mathbf{f}$ must be $\Gamma_s \times S^1$–equivariant:

$$\gamma \mathbf{f}(\mathbf{z}, \lambda) = \mathbf{f}(\gamma \mathbf{z}, \lambda) \quad \text{for all} \quad \gamma \in \Gamma_s \times S^1. \tag{4.1b}$$

We further assume that $\mathbf{f}$ is smooth (C^∞). The general form of such a vector field is

$$\begin{aligned}
\dot{v}_1 &= v_1 \; g(N_1, \delta_1, N_2, \Delta_2, Q) + \overline{w}_1 v_2 w_2 \; h(N_1, \delta_1, N_2, \Delta_2, \overline{Q}) \\
\dot{v}_2 &= v_2 \; g(N_2, \delta_2, N_1, \Delta_1, \overline{Q}) + \overline{w}_2 v_1 w_1 \; h(N_2, \delta_2, N_1, \Delta_1, Q) \\
\dot{w}_1 &= w_1 \; g(N_1, -\delta_1, N_2, \Delta_2, Q) + \overline{v}_1 v_2 w_2 \; h(N_1, -\delta_1, N_2, \Delta_2, \overline{Q}) \\
\dot{w}_2 &= w_2 \; g(N_2, -\delta_2, N_1, \Delta_1, \overline{Q}) + \overline{v}_2 v_1 w_1 h \; (N_2, -\delta_2, N_1, \Delta_1, Q),
\end{aligned} \tag{4.2}$$

where g and h are C^∞ complex–valued functions of $N_j \equiv |v_j|^2 + |w_j|^2$, $\delta_j \equiv |w_j|^2 - |v_j|^2$, $\Delta_j \equiv \delta_j^2$, and $Q \equiv v_1 \overline{v}_2 w_1 \overline{w}_2$. This section is devoted to the derivation of (4.2).

Consider a general term in the Taylor expansion of the first component of (4.1a), *i.e.*, of $\dot{v}_1 = f_1(v_1, v_2, w_1, w_2)$:

$$A_{\alpha\beta\gamma\delta\sigma\tau\mu\nu} \; v_1^\alpha \; v_2^\beta \; w_1^\gamma \; w_2^\delta \; \overline{v}_1^\sigma \; \overline{v}_2^\tau \; \overline{w}_1^\mu \; \overline{w}_2^\nu. \tag{4.3}$$

The coefficient $A_{\alpha\cdots\nu}$ is nonzero for those terms that commute with the action of $T^2 \times S^1$. For the T^2 action (2.4a) this implies $A_{\alpha\cdots\nu} \neq 0$ for

$$\begin{aligned}
(\alpha - \sigma) - (\gamma - \mu) &= 1 \\
(\beta - \tau) - (\delta - \nu) &= 0.
\end{aligned} \tag{4.4a}$$

Thus $(\alpha - \sigma) = 1 + (\gamma - \mu) \equiv m$ and $(\tau - \beta) = (\nu - \delta) \equiv n$, $\; m, n \in \mathbf{Z}$. The term (4.3) must also commute with the S^1 action (2.6), so that $A_{\alpha\cdots\nu} \neq 0$ for

$$\alpha + \beta + \gamma + \delta - \sigma - \tau - \mu - \nu = 1. \tag{4.4b}$$

It follows that $m = n + 1$. Thus the terms in the Taylor expansion of f_1 are

$$B_{n\alpha\beta\gamma\delta} \; v_1^{\alpha+n+1} \; v_2^\beta \; w_1^{\gamma+n} \; w_2^\delta \; \overline{v}_1^\alpha \; \overline{v}_2^{\beta+n} \; \overline{w}_1^\gamma \; \overline{w}_2^{\delta+n}. \tag{4.5}$$

Those terms with $n \geq 0$ may be expressed as

$$B_{n\alpha\beta\gamma\delta} \; |v_1|^{2\alpha} |v_2|^{2\beta} |w_1|^{2\gamma} |w_2|^{2\delta} Q^n v_1, \tag{4.6}$$

where $Q \equiv v_1\overline{v}_2 w_1\overline{w}_2$ is an invariant of the $T^2 \times S^1$ action. The terms with $n \leq -1$ have the form

$$B_{n\alpha\beta\gamma\delta}\ |v_1|^{2(\alpha+n+1)}|v_2|^{2(\beta+n)}|w_1|^{2(\gamma+n)}|w_2|^{2(\delta+n)}\overline{Q}^{-(n+1)}v_2w_2\overline{w}_1. \quad (4.7)$$

Thus the form of $f_1(v_1, v_2, w_1, w_2)$, which commutes with $T^2 \times S^1$, is

$$f_1 = v_1 G + v_2 w_2 \overline{w}_1 H, \quad (4.8)$$

where $G \equiv G(N_1, \delta_1, N_2, \delta_2, Q)$ and $H \equiv H(N_1, \delta_1, N_2, \delta_2, \overline{Q})$ are general invariant functions of $T^2 \times S^1$. In addition, G and H must be invariant under the action of $\sigma_v : (v_2, w_2) \to (w_2, v_2)$. Any smooth function $F(x, y)$ with the property $F(x, y) = F(y, x)$ may be expressed in terms of the invariants $N = x + y$ and $\Delta = (y - x)^2$ (*i.e.*, $F = \widehat{F}(N, \Delta)$). Hence we write $G = g(N_1, \delta_1, N_2, \Delta_2, Q)$ and $H = h(N_1, \delta_1, N_2, \Delta_2, \overline{Q})$. The remaining components of (4.2) are generated from (4.8) by the action of D_4 on a $\Gamma_s \times S^1$-equivariant vector field. Therefore

$$\begin{aligned} \dot{v}_2 &= f_1(v_2, w_1, w_2, v_1) \\ \dot{w}_1 &= f_1(w_1, w_2, v_1, v_2) \\ \dot{w}_2 &= f_1(w_2, v_1, v_2, w_1). \end{aligned} \quad (4.9)$$

As noted in the previous section, the $\Gamma_s \times S^1$-equivariant vector field (4.2) restricted to the four–dimensional fixed point subspaces of Table 1 commutes with either the action of $D_4 \times S^1$ or $O(2) \times S^1$. The general $D_4 \times S^1$-equivariant vector field is

$$\begin{aligned} \dot{v}_1 &= v_1\ g_{D4}(|v_1|^2, |v_2|^2, v_1^2\overline{v}_2^2) + \overline{v}_1 v_2^2\ h_{D4}(|v_1|^2, |v_2|^2, \overline{v}_1^2 v_2^2) \\ \dot{v}_2 &= v_2\ g_{D4}(|v_2|^2, |v_1|^2, v_2^2\overline{v}_1^2) + \overline{v}_2 v_1^2\ h_{D4}(|v_2|^2, |v_1|^2, \overline{v}_2^2 v_1^2). \end{aligned} \quad (4.10)$$

This vector field pertains to the standing cross–rolls subspace where $v_1 = w_1$, $v_2 = w_2$. The third order truncation of (4.10) was analyzed by Swift [11]. The general $O(2) \times S^1$-equivariant vector field is

$$\begin{aligned} \dot{z}_1 &= z_1 g_{O2}(|z_1|^2, |z_2|^2) \\ \dot{z}_2 &= z_2 g_{O2}(|z_2|^2, |z_1|^2). \end{aligned} \quad (4.11)$$

These equations pertain to the four–dimensional fixed point subspaces VIII–XI in Table 1 and have been extensively studied by a number of authors [6–9]. Note that the equations for the magnitudes of the complex amplitudes decouple from the phases so that we can study the two–dimensional D_4–equivariant system [4]:

$$\begin{aligned} \dot{r}_1 &= r_1\ g_r(r_1^2, r_2^2) \\ \dot{r}_2 &= r_2\ g_r(r_2^2, r_1^2), \end{aligned} \quad (4.12)$$

where $z_j = r_j e^{i\phi_j}$, $j = 1, 2$, and $g_r : \mathbf{R}^2 \to \mathbf{R}$.

5. Periodic Solutions.

In this section we let the functions g and h in (4.2) depend on a bifurcation parameter λ as discussed in section 2. We assume that the trivial solution is stable subcritically, and that the bifurcation parameter is scaled such that

$$g(0,\lambda) = \lambda + i\omega(\lambda) + \mathcal{O}(\lambda^2), \qquad \omega(0) = \omega_c. \tag{5.1}$$

We use a result of Golubitsky and Stewart [6] to determine the small amplitude periodic solutions of (4.2) that bifurcate from the trivial solution at $\lambda = 0$. Specifically, these authors show that the small amplitude periodic solutions with period $2\pi/(\omega_c + \tau)$ are in one–to–one correspondence with the solutions of

$$\phi(\mathbf{z},\lambda,\tau) \equiv \mathbf{f}(\mathbf{z},\lambda) - i(\omega_c + \tau)\mathbf{z} = \mathbf{0} \tag{5.2}$$

on a neighborhood of $(\mathbf{z},\lambda,\tau) = (\mathbf{0},0,0)$.

Consider the Taylor expansion of (5.2) about $(\mathbf{z},\lambda,\tau) = (\mathbf{0},0,0)$ through third order in $\mathbf{z}$:

$$\mathbf{\Phi} \equiv \begin{pmatrix} \nu v_1 + (a|w_1|^2 + bN_1 + cN_2)v_1 + d\,\overline{w}_1 v_2 w_2 \\ \nu v_2 + (a|w_2|^2 + bN_2 + cN_1)v_2 + d\,\overline{w}_2 v_1 w_1 \\ \nu w_1 + (a|v_1|^2 + bN_1 + cN_2)w_1 + d\,\overline{v}_1 v_2 w_2 \\ \nu w_2 + (a|v_2|^2 + bN_2 + cN_1)w_2 + d\,\overline{v}_2 v_1 w_1 \end{pmatrix} = \mathbf{0}, \tag{5.3}$$

where $\nu \equiv \lambda + i\sigma$, $\sigma \equiv \omega - (\omega_c + \tau)$. The cubic coefficients in (5.3) are complex and may be identified with the coefficients of the cubic terms in the Taylor expansion of (4.2) as follows

$$\begin{aligned} a &= 2g_2(\mathbf{0}), \quad & b &= g_1(\mathbf{0}) - g_2(\mathbf{0}), \\ c &= g_3(\mathbf{0}), \quad & d &= h(\mathbf{0}). \end{aligned} \tag{5.4}$$

Here $g_j(\mathbf{0})$ is the partial derivative of g with respect to its j^{th} argument evaluated at $(\mathbf{z},\lambda) = (\mathbf{0},0)$. Solutions of (5.3) with $Q \equiv v_1\overline{v}_2 w_1 \overline{w}_2 \neq 0$ satisfy

$$\hat{\mathbf{\Phi}} \equiv \begin{pmatrix} \nu r_1 + (ar_3^2 + bN_1 + cN_2)r_1 + de^{2i\psi}r_2r_3r_4 \\ \nu r_2 + (ar_4^2 + bN_2 + cN_1)r_2 + de^{-2i\psi}r_3r_4r_1 \\ \nu r_3 + (ar_1^2 + bN_1 + cN_2)r_3 + de^{2i\psi}r_4r_1r_2 \\ \nu r_4 + (ar_2^2 + bN_2 + cN_1)r_4 + de^{-2i\psi}r_1r_2r_3 \end{pmatrix} = \mathbf{0}, \tag{5.5}$$

where $r_1 = |v_1|$, $r_2 = |v_2|$, $r_3 = |w_1|$, $r_4 = |w_2|$ and $\psi = -\frac{1}{2}\arg(Q)$. The remainder of this section is devoted to compiling Table 3 which contains all nontrivial solutions of (5.3).

We begin by noting that the nontrivial solutions have either one, two, or four nonzero amplitudes. This follows from the fourth component of (5.3) which shows that $w_2 =$

Name	Branching Equations	Solution
Travelling Rolls (TR)	$\nu + b\|v_1\|^2 = 0$ $v_1 \neq 0,\ v_2 = w_1 = w_2 = 0$	$\|v_1\|^2 = -\lambda/b_r$ $\sigma = -b_i\|v_1\|^2$
Travelling Squares (TS)	$\nu + (b+c)\|v_1\|^2 = 0$ $v_1 = v_2 \neq 0,\ w_1 = w_2 = 0$	$\|v_1\|^2 = -\lambda/(b_r + c_r)$ $\sigma = -(b_i + c_i)\|v_1\|^2$
Standing Rolls (SR)	$\nu + (a+2b)\|v_1\|^2 = 0$ $v_1 = w_1 \neq 0,\ v_2 = w_2 = 0$	$\|v_1\|^2 = -\lambda/(a_r + 2b_r)$ $\sigma = -(a_i + 2b_i)\|v_1\|^2$
Standing Squares (SS)	$\nu + (a+2b+2c+d)\|v_1\|^2 = 0$ $v_1 = v_2 = w_1 = w_2 \neq 0$	$\|v_1\|^2 = -\lambda/(a_r + 2b_r + 2c_r + d_r)$ $\sigma = -(a_i + 2b_i + 2c_i + d_i)\|v_1\|^2$
Alternating Rolls (AR)	$\nu + (a+2b+2c-d)\|v_1\|^2 = 0$ $v_1 = -iv_2 = w_1 = -iw_2 \neq 0$	$\|v_1\|^2 = -\lambda/(a_r + 2b_r + 2c_r - d_r)$ $\sigma = -(a_i + 2b_i + 2c_i - d_i)\|v_1\|^2$
Standing Cross-Rolls (SCR)	$\nu + (a+2b)r_1^2 + (2c + de^{2i\psi})r_2^2 = 0$ $\nu + (a+2b)r_2^2 + (2c + de^{-2i\psi})r_1^2 = 0$ $r_1 = \|v_1\|,\ r_2 = \|v_2\|,\ 2\psi = \arg(\bar{v}_1^2 v_2^2)$ $v_1 = w_1 \neq 0,\ v_2 = w_2 \neq 0,\ r_1 \neq r_2$	$r_1^2 + r_2^2 = -2\lambda\ \mathrm{Im}(\bar{d}f)/[d_i(\|f\|^2 - \|d\|^2) + 2(a_r + 2b_r)\ \mathrm{Im}(\bar{d}f)]$ $\sigma = -[d_r(\|d\|^2 - \|f\|^2) + 2(a_i + 2b_i)\ \mathrm{Im}(\bar{d}f)](r_1^2 + r_2^2)/2\ \mathrm{Im}(\bar{d}f)$ $\cos(2\psi) = \mathrm{Re}(\bar{d}f)/\|d\|^2,\ \sin(2\psi) = \sqrt{1 - \cos^2(2\psi)}$ $(r_1/r_2)^2 = [\ \mathrm{Im}(\bar{d}f) - \|d\|^2\sin(2\psi)]/[\ \mathrm{Im}(\bar{d}f) + \|d\|^2\sin(2\psi)]$

Table 3: Solutions of (5.3). Subscripts r and i on the coefficients a, b, c, d refer to real and imaginary parts, $\nu \equiv \lambda + i\sigma$, and $f \equiv a + 2b - 2c$.

0, $v_1 v_2 w_1 \neq 0$, implies $d = 0$. Thus generically there are no solutions with exactly three nonzero amplitudes. To verify Table 3, consider the four cases

$$\begin{aligned} &(1)\ v_1 \neq 0, \quad v_2 = w_1 = w_2 = 0 \\ &(2)\ v_1 \neq 0, \quad v_2 \neq 0, \quad w_1 = w_2 = 0 \\ &(3)\ v_1 \neq 0, \quad w_1 \neq 0, \quad v_2 = w_2 = 0 \\ &(4)\ Q \neq 0. \end{aligned} \tag{5.6}$$

By restricting (5.3) to the appropriate invariant fixed point subspace it is straightforward to check that these equations admit only the solutions TR, TS, and SR for cases (1), (2), and (3), respectively. For case (4), we consider equation (5.5). Let $\hat{\Phi}_j$ denote the j^{th} component of $\hat{\Phi}$; then

$$\begin{aligned} r_1 \hat{\Phi}_1 - r_3 \hat{\Phi}_3 &= (\nu + bN_1 + cN_2)(r_1^2 - r_3^2) = 0 \\ r_2 \hat{\Phi}_2 - r_4 \hat{\Phi}_4 &= (\nu + bN_2 + cN_1)(r_2^2 - r_4^2) = 0. \end{aligned} \tag{5.7}$$

Thus case (4) divides into three distinct subcases:

$$\begin{aligned} &(4a)\ r_1^2 = r_3^2, \quad r_2^2 = r_4^2 \\ &(4b)\ \ \nu + bN_1 + cN_2 = \nu + bN_2 + cN_1 = 0 \\ &(4c)\ \ \nu + bN_1 + cN_2 = 0, \quad r_2^2 = r_4^2. \end{aligned} \tag{5.8}$$

The solutions SS, AR and SCR all satisfy the conditions of (4a). These are determined as follows. Consider the following equation evaluated at $r_1 = r_3$, $r_2 = r_4$,

$$\frac{\hat{\Phi}_1}{r_1} - \frac{\hat{\Phi}_2}{r_2} = (f - d\cos(2\psi))(r_1^2 - r_2^2) + id(r_1^2 + r_2^2)\sin(2\psi) = 0, \tag{5.9}$$

where $f \equiv a + 2b - 2c$. Solutions with $r_1^2 = r_2^2$ have $\sin(2\psi) = 0$. These correspond to SS and AR for $\psi = 0, \pi/2$, respectively. If, on the other hand, $r_1^2 \neq r_2^2$, then the real and imaginary parts of (5.9) may be solved for $\cos(2\psi)$ and $(r_1/r_2)^2$ to give

$$\begin{aligned} \cos(2\psi) &= \frac{\mathrm{Re}(\bar{d}f)}{|d|^2} \\ \left(\frac{r_1}{r_2}\right)^2 &= \frac{\mathrm{Im}(\bar{d}f) - |d|^2\sin(2\psi)}{\mathrm{Im}(\bar{d}f) + |d|^2\sin(2\psi)}. \end{aligned} \tag{5.10}$$

This is the solution SCR in Table 3. Without loss of generality we may assume that $\sin(2\psi) > 0$ since letting $\sin(2\psi) \to -\sin(2\psi)$ in (5.10) has the same effect as interchanging r_1 and r_2. Thus a representative solution on the group orbit of SCR has $\sin(2\psi) = \sqrt{1 - \cos^2(2\psi)}$. From the conditions $|\cos(2\psi)| < 1$ and $(r_1/r_2)^2 > 0$, we find that the solution (5.10) exists provided

$$|f|^2 > |d|^2 > |\ \mathrm{Re}(\bar{d}f)|. \tag{5.11}$$

It remains to show that cases (4b) and (4c) do not contain any new solutions to (5.5).

In case (4b), (5.5) holds for

$$\begin{aligned} ar_1r_3 + de^{2i\psi}r_2r_4 &= 0 \\ ar_2r_4 + de^{-2i\psi}r_1r_3 &= 0, \end{aligned} \tag{5.12}$$

which for $Q \neq 0$ implies $a^2 = d^2$. Generically, there are no solutions in case (4b) with $Q \neq 0$. In case (4c), (5.5) holds for

$$ar_1r_3 + de^{2i\psi}r_2^2 = 0 \tag{5.13a}$$

$$\nu + ar_2^2 + bN_2 + cN_1 + de^{-2i\psi}r_1r_3 = 0. \tag{5.13b}$$

Solving (5.13a) for $r_1r_3e^{-2i\psi}$ and solving (4c) in (5.8) for ν, we can write (5.13b) as

$$(c-b)N_1 = \left(c - b + \frac{d^2 - a^2}{2a}\right)N_2. \tag{5.14}$$

Here we replaced r_2^2 by $N_2/2$. The real and imaginary parts of (5.14) give two equations for the ratio N_1/N_2 so that generically there are no new solutions in case (4c).

In summary we have found that generically there are either five or six nontrivial periodic solution branches in the case of the cubic truncation of (4.2). The existence of the sixth solution depends on the values of the cubic coefficients in (4.2). The five solutions that always bifurcate from the trivial solution at $\lambda = 0$ are those guaranteed by the equivariant Hopf theorem as discussed in section 3. The corresponding horizontal planforms are depicted in Figure 4. The sixth solution which may exist is SCR. This solution is a superposition of standing rolls aligned with the $\hat{\mathbf{x}}_1$ and $\hat{\mathbf{x}}_2$ directions. The relative phases and amplitudes of these standing rolls depend on the cubic coefficients through equations (5.10). We note that the standing wave solutions, which satisfy $r_1 = r_3$, $r_2 = r_4$ (*i.e.*, SR, SS, AR, SCR), were found by Swift [11] who studied the Hopf bifurcation with D_4 symmetry. The TR and SR solutions were found in the study of Hopf bifurcations with O(2) symmetry [6]. The travelling wave solution TS, which exist because of the spatial translation symmetry T^2, is new.

6. Solution Stability.

In this section we determine the linear stability of the solutions in Table 3 with respect to perturbations periodic on the square lattice. Since the solutions are neutrally stable to perturbations that merely translate them along their group orbit, we will consider the stability of the group orbit of a solution to small perturbations transverse to the group orbit. If such perturbations decay exponentially, then the solutions are linearly orbitally stable.

Let $\mathbf{D}\phi|_{\mathbf{z}}$ denote the Jacobian matrix associated with (5.2) evaluated on a solution branch $(\mathbf{z}(\lambda), \tau(\lambda))$ of $\phi = \mathbf{0}$. A solution is linearly orbitally stable if all eigenvalues of

$\mathbf{D}\phi|_{\mathbf{z}}$, not forced to be zero by symmetry, have negative real parts. The corresponding periodic solution of (4.2) then has Floquet multipliers lying inside the unit circle (except for those forced to be one by symmetry) [6]. If the solution $\mathbf{z}$ has symmetry $\Sigma_{\mathbf{z}}$ then the group orbit has the dimension of $(\Gamma_s \times S^1)/\Sigma_{\mathbf{z}}$ and the number of zero eigenvalues of $\mathbf{D}\phi|_{\mathbf{z}}$ is

$$n_{\mathbf{z}} = 3 - \dim(\Sigma_{\mathbf{z}}). \tag{6.1}$$

Thus if the symmetry group $\Sigma_{\mathbf{z}}$ is discrete, there are three zero eigenvalues: two associated with the T^2 translation symmetries and one associated with the S^1 phase shift symmetry. This is the case for the standing wave solutions SS, AR and SCR. In contrast $\mathbf{D}\phi$ has two zero eigenvalues when evaluated on the TS and SR solution branches and one zero eigenvalue on the TR branch.

The form of the Jacobian matrix is restricted for solutions with nontrivial symmetry. In particular, $\mathbf{D}\phi|_{\mathbf{z}}$ commutes with all elements of the isotropy subgroup $\Sigma_{\mathbf{z}}$, *i.e.*,

$$\sigma \mathbf{D}\phi|_{\mathbf{z}} = \mathbf{D}\phi|_{\mathbf{z}}\sigma \quad \text{for all } \sigma \in \Sigma_{\mathbf{z}}. \tag{6.2}$$

This restriction greatly simplifies the calculation of $\mathbf{D}\phi|_{\mathbf{z}}$ and its eigenvalues, especially in the case where $\mathbf{z}$ lies in a two-dimensional fixed point subspace. For these five solutions (TR, TS, SR, SS, AR) it is possible to put the Jacobian matrix into block diagonal form with each block consisting of a 2×2 real matrix. This is most readily done by decomposing $\mathbb{C}^4$ into subspaces, each of which is invariant under a different representation of the isotropy subgroup $\Sigma_{\mathbf{z}}$. Specifically, we form the isotypic decomposition [14]

$$\mathbb{C}^4 = W_0 \oplus W_1 \oplus \cdots \oplus W_k, \tag{6.3}$$

where each isotypic component W_i may be further decomposed into subspaces U_i^j, each of which transforms according to the i^{th} irreducible representation of $\Sigma_{\mathbf{z}}$:

$$W_i = U_i^1 \oplus U_i^2 \oplus \cdots \oplus U_i^{n_i}. \tag{6.4}$$

Thus W_i contains all irreducible subspaces of $\mathbb{C}^4$ which are $\Sigma_{\mathbf{z}}$–isomorphic, meaning that there exists an isomorphism $A : U_i^j \rightarrow U_i^l$ such that

$$A(\sigma u) = \sigma(Au) \quad \text{for all } u \in U_i^j \text{ and } \sigma \in \Sigma_{\mathbf{z}}. \tag{6.5}$$

The isotypic components in (6.3) are invariant under the mapping $\mathbf{D}\phi|_{\mathbf{z}} : \mathbb{C}^4 \rightarrow \mathbb{C}^4$ [14], *i.e.*,

$$\mathbf{D}\phi|_{\mathbf{z}}(W_j) \subset W_j, \quad j = 1, \cdots, k. \tag{6.6}$$

Hence the Jacobian matrix is block diagonal in these coordinates. Table 4 gives the isotypic decomposition of $\mathbb{C}^4$ for each of the solutions in Table 3, together with the appropriate representation of $\Sigma_{\mathbf{z}}$ that acts on each isotypic component. Note that for each solution $W_0 = \text{Fix}(\Sigma_{\mathbf{z}})$ so that W_0 transforms according to the identity representation of $\Sigma_{\mathbf{z}}$. In addition the orbit of $\mathbf{z}$ under the action of S^1 (*i.e.*, $\mathbf{z} \to e^{i\varphi}\mathbf{z}$) is in W_0 so that there is a zero eigenvalue in $\mathbf{D}\phi|W_0$ for each solution $\mathbf{z}$. Here $\mathbf{D}\phi|W_0$ is $\mathbf{D}\phi|_{\mathbf{z}}$ restricted to the W_0 subspace. Table 5 gives the stabilities of the solutions to perturbations in each W_j for each solution with $W_0 \approx \mathbb{C}$.

Orbit Representative	Isotypic Components of $\mathbb{C}^4$	Representation of Isotropy Subgroup
Travelling Rolls (TR) $\mathbf{z} = (v_1, 0, 0, 0)$ $v_1 \neq 0$	$W_0^{TR} = \{(z_0, 0, 0, 0)\}$ $W_1^{TR} = \{(0, 0, z_1, 0)\}$ $W_2^{TR} = \{(0, z_1, 0, z_2)\}$	I S^1 $O(2) \times S^1$
Travelling Squares (TS) $\mathbf{z} = (v_1, v_1, 0, 0)$ $v_1 \neq 0$	$W_0^{TS} = \{(z_0, z_0, 0, 0)\}$ $W_1^{TS} = \{(z_1, -z_1, 0, 0)\}$ $W_2^{TS} = \{(0, 0, z_1, z_2)\}$	I $Z_2 = \langle\pi\rangle$ $Z_2 \times S^1 : Z_2 = \langle\kappa_0\rangle$
Standing Rolls (SR) $\mathbf{z} = (v_1, 0, v_1, 0)$ $v_1 \neq 0$	$W_0^{SR} = \{(z_0, 0, z_0, 0)\}$ $W_1^{SR} = \{(z_1, 0, -z_1, 0)\}$ $W_2^{SR} = \{(0, z_1, 0, z_2)\}$	I $Z_2 = \langle\pi\rangle$ $O(2)$
Standing Squares (SS) $\mathbf{z} = (v_1, v_1, v_1, v_1)$ $v_1 \neq 0$	$W_0^{SS} = \{(z_0, z_0, z_0, z_0)\}$ $W_1^{SS} = \{(z_1, -z_1, z_1, -z_1)\}$ $W_2^{SS} = \{(z_1, z_2, -z_1, -z_2)\}$	I $Z_2 = \langle\pi\rangle$ D_4
Alternating Rolls (AR) $\mathbf{z} = (v_1, iv_1, v_1, iv_1)$ $v_1 \neq 0$	$W_0^{AR} = \{(z_0, iz_0, z_0, iz_0)\}$ $W_1^{AR} = \{(z_1, -iz_1, z_1, -iz_1)\}$ $W_2^{AR} = \{(z_1, -iz_2, -z_1, iz_2)\}$	I $Z_2 = \langle\pi\rangle$ D_4
Standing Cross-Rolls (SCR) $\mathbf{z} = (v_1, v_2, v_1, v_2)$ $v_1 \neq 0,\ v_2 \neq 0,\ \lvert v_1\rvert \neq \lvert v_2\rvert$	$W_0^{SCR} = \{(z_1, z_2, z_1, z_2)\}$ $W_1^{SCR} = \{(z_1, z_2, -z_1, -z_2)\}$	I $Z_2^2 = \langle\kappa_1, \kappa_2\rangle$

Table 4: Isotypic decomposition of $\mathbb{C}^4$ for each isotropy subgroup associated with a solution in Table 3. Here, $\mathbf{z} = (v_1, v_2, w_1, w_2)$ and $S^1 : \mathbf{z} \to e^{i\varphi}\mathbf{z}$, $\pi : \mathbf{z} \to -\mathbf{z}$, $O(2) = \langle\kappa_0, \theta\rangle$, and $D_4 = \langle\kappa_0, \kappa_1\rangle$, where $\theta : (z_1, z_2) \to (e^{i\theta}z_1, e^{-i\theta}z_2)$, $\kappa_0 : (z_1, z_2) \to (z_2, z_1)$, $\kappa_1 : (z_1, z_2) \to (z_1, -z_2)$, and $\kappa_2 : (z_1, z_2) \to (-z_1, z_2)$.

For those solutions with a two–dimensional fixed point subspace W_0, the nonzero eigenvalue of $\mathbf{D}\phi|W_0$ is given by its trace. The eigenvector associated with this eigenvalue is in the plane of the limit cycle and transverse to it. The stability of the limit cycle to a perturbation in W_0, in this "radial direction", depends on the direction of bifurcation of the solution branch. For the third order truncation (5.3), the radial eigenvalue is -2λ, just as in the Hopf bifurcation without symmetry. This is readily checked by computing $\mathrm{Tr}(\mathbf{D}\phi|W_0)$. Consider, for example, travelling rolls for which $(v_1, v_2, w_1, w_2) = (z_0, 0, 0, 0) \in W_0^{TR}$. Treating z_0, $\bar{z}_0$ as independent perturbations in W_0 we obtain

$$(\mathbf{D}\phi|W_0^{TR})\begin{pmatrix} z_0 \\ \bar{z}_0 \end{pmatrix} = \begin{pmatrix} \frac{\partial\phi_1}{\partial v_1} & \frac{\partial\phi_1}{\partial \bar{v}_1} \\ \frac{\partial\bar{\phi}_1}{\partial v_1} & \frac{\partial\bar{\phi}_1}{\partial \bar{v}_1} \end{pmatrix}_{TR} \begin{pmatrix} z_0 \\ \bar{z}_0 \end{pmatrix}. \tag{6.7}$$

Then

$$\mathrm{Tr}(\mathbf{D}\phi|W_0^{TR}) = 2\ \mathrm{Re}\Big(\frac{\partial\phi_1}{\partial v_1}\Big)_{TR}, \tag{6.8}$$

where the TR subscript indicates that the Jacobian matrix is evaluated on the TR solution branch. We evaluate (6.8) using the third order truncation $\mathbf{\Phi}$ given by (5.3):

$$2\ \mathrm{Re}\Big(\frac{\partial\Phi_1}{\partial v_1}\Big)_{TR} = 2b_r|v_1|^2 = -2\lambda, \tag{6.9}$$

where the r subscript on b specifies the real part of b (similarly, $b_i \equiv \mathrm{Im}(b)$). Similar calculations for the solutions TS, SR, SS and AR give the same result. Specifically, $\mathrm{Tr}(\mathbf{D\Phi}|W_0) = -2\lambda$. The remaining calculations performed in compiling Table 5 follow below. Throughout, we give first a general expression for the eigenvalues associated with a solution in terms of partial derivatives of the general equivariant vector field (5.2). These expressions are then evaluated using the third order truncation (5.3).

Let $z_1, \bar{z}_1$ be independent perturbations in $W_1 \approx \mathbb{C}$, then $\mathbf{D}\phi|W_1$ has the form

$$(\mathbf{D}\phi|W_1)\begin{pmatrix} z_1 \\ \bar{z}_1 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}\begin{pmatrix} z_1 \\ \bar{z}_1 \end{pmatrix}, \qquad \alpha, \beta \in \mathbb{C}. \tag{6.10}$$

The eigenvalues of (6.10) are determined by computing the trace and determinant, where

$$\mathrm{Tr}(\mathbf{D}\phi|W_1) = 2\ \mathrm{Re}(\alpha), \qquad \det(\mathbf{D}\phi|W_1) = |\alpha|^2 - |\beta|^2. \tag{6.11}$$

It follows from (6.2) that the matrix in (6.10) commutes with the action of Σ_z on $(z_1, \bar{z}_1)$, $z_1, \bar{z}_1 \in W_1$. For example, the action of Σ_{TR} on W_1^{TR} is that of $\mathrm{S}^1 = [(-\varphi, 0), \varphi]$ on $(v_1, v_2, w_1, w_2) = (0, 0, z_1, 0)$, namely

$$\mathrm{S}^1 : W_1^{TR} \to W_1^{TR} : \quad \begin{pmatrix} z_1 \\ \bar{z}_1 \end{pmatrix} \mapsto \begin{pmatrix} e^{2i\varphi} & 0 \\ 0 & e^{-2i\varphi} \end{pmatrix}\begin{pmatrix} z_1 \\ \bar{z}_1 \end{pmatrix}. \tag{6.12}$$

The condition (6.2) is

$$\begin{pmatrix} e^{2i\varphi} & 0 \\ 0 & e^{-2i\varphi} \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \begin{pmatrix} e^{2i\varphi} & 0 \\ 0 & e^{-2i\varphi} \end{pmatrix}, \tag{6.13}$$

for all $\varphi \in \mathrm{S}^1$. It follows that $\beta = 0$. The real parts of the eigenvalues $\alpha, \bar{\alpha}$ are

$$\mathrm{Re}(\alpha)_{TR} = \mathrm{Re}\Big(\frac{\partial \phi_3}{\partial w_1}\Big)_{TR} = a_r |v_1|^2. \tag{6.14}$$

For TS, SR, SS and AR, the action of the appropriate isotropy subgroup on each W_1 is generated by a reflection $(z_1, \bar{z}_1) \to -(z_1, \bar{z}_1)$, where z_1 is given in each case by

$$\begin{aligned} &(\mathrm{TS}): \ (v_1, v_2, w_1, w_2) = (z_1, -z_1, 0, 0) \in W_1^{TS} \\ &(\mathrm{SR}): \ (v_1, v_2, w_1, w_2) = (z_1, 0, -z_1, 0) \in W_1^{SR} \\ &(\mathrm{SS}): \ (v_1, v_2, w_1, w_2) = (z_1, -z_1, z_1, -z_1) \in W_1^{SS} \\ &(\mathrm{AR}): \ (v_1, v_2, w_1, w_2) = (z_1, -iz_1, z_1, -iz_1) \in W_1^{AR}. \end{aligned} \tag{6.15}$$

This representation commutes trivially with $\mathbf{D}\phi|W_1$ since it is a scalar multiple of the identity. The stability of each solution to perturbations in W_1 is thus determined by calculating both the trace and determinant of $\mathbf{D}\phi|W_1$. However, for both the TS and SR solutions, there is a null eigenvector in the corresponding W_1 subspace. For travelling squares, the null eigenvector is in the direction of the group orbit of the solution $(v_1, v_1, 0, 0)$ under the action of $(\theta_1, -\theta_1) \in \mathrm{T}^2$. It is given by

$$\frac{d}{d\theta_1}\bigg|_{\theta_1=0} \begin{pmatrix} e^{i\theta_1} v_1 \\ e^{-i\theta_1} v_1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} iv_1 \\ -iv_1 \\ 0 \\ 0 \end{pmatrix} \in W_1^{TS}, \tag{6.16}$$

and corresponds to a perturbation that translates the travelling squares pattern along the diagonal $(x_1 = -x_2)$. For standing rolls, the null eigenvector is associated with a translation of the pattern in the $\hat{\mathbf{x}}_1$-direction. Hence it is determined from the group orbit of the solution $(v_1, 0, v_1, 0)$ under $(\theta_1, 0) \in \mathrm{T}^2$ by

$$\frac{d}{d\theta_1}\bigg|_{\theta_1=0} \begin{pmatrix} e^{i\theta_1} v_1 \\ 0 \\ e^{-i\theta_1} v_1 \\ 0 \end{pmatrix} = \begin{pmatrix} iv_1 \\ 0 \\ -iv_1 \\ 0 \end{pmatrix} \in W_1^{SR}. \tag{6.17}$$

The nonzero eigenvalues of $\mathbf{D}\phi|W_1^{TS}$ and $\mathbf{D}\phi|W_1^{SR}$ are then

$$\mathrm{Tr}(\mathbf{D}\phi|W_1^{TS}) = 2\ \mathrm{Re}(\alpha)_{TS} = 2\mathrm{Re}\Big(\frac{\partial \phi_1}{\partial v_1} - \frac{\partial \phi_1}{\partial v_2}\Big)_{TS} = 2(b_r - c_r)|v_1|^2 \tag{6.18a}$$

$$\mathrm{Tr}(\mathbf{D}\phi|W_1^{SR}) = 2\ \mathrm{Re}(\alpha)_{SR} = 2\mathrm{Re}\Big(\frac{\partial \phi_1}{\partial v_1} - \frac{\partial \phi_1}{\partial w_1}\Big)_{SR} = -2a_r|v_1|^2. \tag{6.18b}$$

The stability of standing squares to perturbations in W_1^{SS} depends on both α and β in (6.10), where

$$\alpha = \left(\frac{\partial \phi_1}{\partial v_1} - \frac{\partial \phi_1}{\partial v_2} + \frac{\partial \phi_1}{\partial w_1} - \frac{\partial \phi_1}{\partial w_2}\right)_{SS} = (f - 3d)|v_1|^2 \tag{6.19a}$$

$$\beta = \left(\frac{\partial \phi_1}{\partial \bar{v}_1} - \frac{\partial \phi_1}{\partial \bar{v}_2} + \frac{\partial \phi_1}{\partial \bar{w}_1} - \frac{\partial \phi_1}{\partial \bar{w}_2}\right)_{SS} = (f + d)v_1^2 \tag{6.19b}$$

(recall $f \equiv a + 2b - 2c$). Combining (6.11) and (6.19) gives

$$\mathrm{Tr}(\mathbf{D}\phi|W_1^{SS}) = 2(f_r - 3d_r)|v_1|^2 \tag{6.20a}$$

$$\det(\mathbf{D}\phi|W_1^{SS}) = 8(|d|^2 - \mathrm{Re}(\bar{d}f))|v_1|^4. \tag{6.20b}$$

The eigenvalues of $\mathbf{D}\phi|_{AR}$ may be inferred from those of $\mathbf{D}\phi|_{SS}$. This is because of a parameter symmetry [11] in (5.3). Specifically, (5.3) is equivariant under the transformation

$$(v_1, v_2, w_1, w_2;\ d) \to (v_1, iv_2, w_1, iw_2;\ -d). \tag{6.21}$$

Thus the AR solution is obtained from the SS solution by letting $d \to -d$ in (5.3), *i.e.*,

$$(v_1, v_1, v_1, v_1; d) \to (v_1, iv_1, v_1, iv_1; -d). \tag{6.22}$$

Combining (6.20) and (6.22) gives

$$\mathrm{Tr}(\mathbf{D}\phi|W_1^{AR}) = 2(f_r + 3d_r)|v_1|^2 \tag{6.23a}$$

$$\det(\mathbf{D}\phi|W_1^{AR}) = 8(|d|^2 + \mathrm{Re}(\bar{d}f))|v_1|^4. \tag{6.23b}$$

It remains to determine the stability of the solutions relative to perturbations in $W_2 \approx \mathbb{C}^2$. Since $z_1, \bar{z}_1, z_2, \bar{z}_2$ are independent perturbations in W_2, $\mathbf{D}\phi|W_2$ has the form

$$(\mathrm{D}\phi|W_2)\begin{pmatrix} z_1 \\ \bar{z}_1 \\ z_2 \\ \bar{z}_2 \end{pmatrix} = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}\begin{pmatrix} z_1 \\ \bar{z}_1 \\ z_2 \\ \bar{z}_2 \end{pmatrix}, \tag{6.24a}$$

$$A_j = \begin{pmatrix} \alpha_j & \beta_j \\ \bar{\beta}_j & \bar{\alpha}_j \end{pmatrix}, \quad \alpha_j, \beta_j \in \mathbb{C}, \quad j = 1,2,3,4. \tag{6.24b}$$

Let $(z_1, z_2) \in \mathbb{C}^2$ be defined for each solution in Table 5 by

$$(TR),(SR): \ (v_1, v_2, w_1, w_2) = (0, z_1, 0, z_2) \tag{6.25a}$$

$$(TS): \ (v_1, v_2, w_1, w_2) = (0, 0, z_1, z_2) \tag{6.25b}$$

$$(SS),(AR): \ (v_1, v_2, w_1, w_2) = (z_1, z_2, -z_1, -z_2). \tag{6.25c}$$

The action of $\sigma_v \in O(2) \subset \Sigma_{TR}, \Sigma_{SR}$ on $(z_1, z_2) \in W_2^{TR}, W_2^{SR}$ is the same as the action of $(r_{\pi/2}\sigma_v) \in \Sigma_{TS}, \Sigma_{SS}$ on $(z_1, z_2) \in W_2^{TS}, W_2^{SS}$. Specifically, σ_v and $(r_{\pi/2}\sigma_v)$ act on the appropriate W_2 subspaces by reflection $\kappa_0 : (z_1, z_2) \to (z_2, z_1)$. It follows that $\mathbf{D}\phi|W_2$ commutes with

$$\begin{pmatrix} 0 & \mathrm{I} \\ \mathrm{I} & 0 \end{pmatrix}, \tag{6.26}$$

where I and 0 are the 2×2 identity and zero matrices, respectively. Hence, (6.24a) has the form

$$\mathbf{D}\phi|W_2 = \begin{pmatrix} A_1 & A_2 \\ A_2 & A_1 \end{pmatrix}. \tag{6.27}$$

Note that this matrix is similar to

$$\begin{pmatrix} A_1 + A_2 & 0 \\ 0 & A_1 - A_2 \end{pmatrix} = S(\mathbf{D}\phi|W_2)S^{-1}, \tag{6.28a}$$

where

$$S = S^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathrm{I} & \mathrm{I} \\ \mathrm{I} & -\mathrm{I} \end{pmatrix}. \tag{6.28b}$$

In addition to the reflection symmetry, $O(2) \subset \Sigma_{TR}, \Sigma_{SR}$ contains a translational symmetry $\theta : (z_1, z_2) \to (e^{i\theta}z_1, e^{-i\theta}z_2)$. Thus $\mathbf{D}\phi|W_2^{TR}$ and $\mathbf{D}\phi|W_2^{SR}$ commute with

$$\begin{pmatrix} R_\theta & 0 \\ 0 & R_{-\theta} \end{pmatrix}, \qquad R_\theta = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}. \tag{6.29}$$

Therefore A_1, A_2 in (6.27) satisfy

$$A_1 R_\theta = R_\theta A_1, \qquad A_2 R_\theta = R_{-\theta} A_2, \tag{6.30}$$

and hence have the form

$$A_1 = \begin{pmatrix} \alpha & 0 \\ 0 & \bar{\alpha} \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & \beta \\ \bar{\beta} & 0 \end{pmatrix}. \tag{6.31}$$

The eigenvalues of $\mathbf{D}\phi|W_2^{SR}$ are those of

$$(A_1 \pm A_2)_{SR} = \begin{pmatrix} \alpha & \pm\beta \\ \pm\bar{\beta} & \bar{\alpha} \end{pmatrix}_{SR}. \tag{6.32}$$

The stability of SR to perturbations in W_2^{SR} is therefore determined by

$$\mathrm{Tr}(A_1 \pm A_2)_{SR} = 2\ \mathrm{Re}(\alpha)_{SR}, \quad \det(A_1 \pm A_2)_{SR} = (|\alpha|^2 - |\beta|^2)_{SR}, \tag{6.33}$$

where

$$(\alpha)_{SR} = \left(\frac{\partial \phi_2}{\partial v_2}\right)_{SR} = -f|v_1|^2, \quad (\beta)_{SR} = \left(\frac{\partial \phi_2}{\partial \bar{w}_2}\right)_{SR} = dv_1^2. \tag{6.34}$$

The eigenvalues of $\mathbf{D}\phi|W_2^{SR}$ are degenerate with

$$\mathrm{Tr}(A_1 \pm A_2)_{SR} = -2f_r|v_1|^2 \tag{6.35a}$$

$$\det(A_1 \pm A_2)_{SR} = (|f|^2 - |d|^2)|v_1|^4. \tag{6.35b}$$

The W_2^{TR} subspace has one additional symmetry; it is invariant under the S^1 action $[(-\varphi,0),\varphi]$ that phase shifts (z_1,z_2). Thus $\mathbf{D}\phi|W_2^{TR}$ commutes with

$$\begin{pmatrix} R_\varphi & 0 \\ 0 & R_\varphi \end{pmatrix}. \tag{6.36}$$

where R_φ is defined as in (6.29). In this case conditions (6.30) are supplemented by

$$A_2 R_\varphi = R_\varphi A_2. \tag{6.37}$$

Then $\beta = 0$ in (6.31) and $\mathbf{D}\phi|W_2^{TR}$ is diagonal with repeated eigenvalues $\alpha, \bar{\alpha}$, where

$$(\alpha)_{TR} = \left(\frac{\partial \phi_2}{\partial v_2}\right)_{TR} = (c-b)|v_1|^2. \tag{6.38}$$

The W_2^{TS} subspace is invariant under the action of (6.36), but not under the action of (6.29). It follows that A_1, A_2 in (6.27) have the form

$$A_j^{TS} = \begin{pmatrix} \alpha_j & 0 \\ 0 & \bar{\alpha}_j \end{pmatrix}_{TS}, \quad j = 1,2. \tag{6.39}$$

Thus the eigenvalues are those of

$$(A_1 \pm A_2)_{TS} = \begin{pmatrix} \alpha_1 \pm \alpha_2 & 0 \\ 0 & \bar{\alpha}_1 \pm \bar{\alpha}_2 \end{pmatrix}_{TS}, \tag{6.40}$$

where

$$(\alpha_1)_{TS} = \left(\frac{\partial \phi_3}{\partial w_1}\right)_{TS} = a|v_1|^2, \quad (\alpha_2)_{TS} = \left(\frac{\partial \phi_3}{\partial w_2}\right)_{TS} = d|v_1|^2. \tag{6.41}$$

Hence the real parts of the eigenvalues of $\mathbf{D}\phi|W_2^{TS}$ are

$$\mathrm{Re}(\alpha_1 \pm \alpha_2)_{TS} = (a_r \pm d_r)|v_1|^2. \tag{6.42}$$

Finally, $\mathbf{D}\phi|W_2^{SS}$ commutes with a reflection $(z_1,z_2) \to (z_1,-z_2)$ so that $A_2 = 0$ in (6.27) and the eigenvalues of $\mathbf{D}\phi|W_2^{SS}$ have multiplicity two. One degenerate pair of eigenvalues is zero since the null eigenvectors associated with the group orbit of the SS solution under the T^2 action reside in W_2^{SS}. These eigenvectors are

$$\left.\frac{d}{d\theta_1}\right|_{\theta_1=0} \begin{pmatrix} e^{i\theta_1}v_1 \\ v_1 \\ e^{-i\theta_1}v_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} iv_1 \\ 0 \\ -iv_1 \\ 0 \end{pmatrix}, \quad \left.\frac{d}{d\theta_2}\right|_{\theta_2=0} \begin{pmatrix} v_1 \\ e^{i\theta_2}v_1 \\ v_1 \\ e^{-i\theta_2}v_1 \end{pmatrix} = \begin{pmatrix} 0 \\ iv_1 \\ 0 \\ -iv_1 \end{pmatrix}. \tag{6.43}$$

Solution	Null Eigenvectors	Eigenvalues	Stability
Travelling Rolls	$(iv_1, 0, 0, 0) \in W_0^{TR}$	$W_0^{TR} : \{2\ \mathrm{Re}(\frac{\partial\phi_1}{\partial v_1}),\ 0\}$	$\mathrm{sgn}(b_r)$
(TR)		$W_1^{TR} : \{\mu_1, \bar{\mu}_1\};\ \ \mathrm{Re}(\mu_1) = \mathrm{Re}(\frac{\partial\phi_3}{\partial w_1})$	$\mathrm{sgn}(a_r)$
$\mathbf{z} = (v_1, 0, 0, 0)$		$W_2^{TR} : \{\mu_2, \mu_2, \bar{\mu}_2, \bar{\mu}_2\};\ \ \mathrm{Re}(\mu_2) = \mathrm{Re}(\frac{\partial\phi_2}{\partial v_2})$	$\mathrm{sgn}(a_r - f_r)$
Travelling Squares	$(iv_1, iv_1, 0, 0) \in W_0^{TS}$	$W_0^{TS} : \{2\ \mathrm{Re}(\frac{\partial\phi_1}{\partial v_1} + \frac{\partial\phi_1}{\partial v_2}),\ 0\}$	$\mathrm{sgn}(a_r + 4b_r - f_r)$
(TS)	$(iv_1, -iv_1, 0, 0) \in W_1^{TS}$	$W_1^{TS} : \{2\ \mathrm{Re}(\frac{\partial\phi_1}{\partial v_1} - \frac{\partial\phi_1}{\partial v_2}),\ 0\}$	$\mathrm{sgn}(f_r - a_r)$
$\mathbf{z} = (v_1, v_1, 0, 0)$		$W_2^{TS} : \{\mu_1, \mu_2, \bar{\mu}_1, \bar{\mu}_2\};\ \ \mathrm{Re}(\mu_1) = \mathrm{Re}(\frac{\partial\phi_3}{\partial w_1} + \frac{\partial\phi_3}{\partial w_2}),$ $\mathrm{Re}(\mu_2) = \mathrm{Re}(\frac{\partial\phi_3}{\partial w_1} - \frac{\partial\phi_3}{\partial w_2})$	$\mathrm{sgn}(a_r + d_r)$ $\mathrm{sgn}(a_r - d_r)$
Standing Rolls	$(iv_1, 0, iv_1, 0) \in W_0^{SR}$	$W_0^{SR} : \{2\ \mathrm{Re}(\frac{\partial\phi_1}{\partial v_1} + \frac{\partial\phi_1}{\partial w_1}),\ 0\}$	$\mathrm{sgn}(a_r + 2b_r)$
(SR)	$(iv_1, 0, -iv_1, 0) \in W_1^{SR}$	$W_1^{SR} : \{2\ \mathrm{Re}(\frac{\partial\phi_1}{\partial v_1} - \frac{\partial\phi_1}{\partial w_1}),\ 0\}$	$\mathrm{sgn}(-a_r)$
$\mathbf{z} = (v_1, 0, v_1, 0)$		$W_2^{SR} : \{\mu_1, \mu_1, \mu_2, \mu_2\};\ \ \mu_1 + \mu_2 = 2\ \mathrm{Re}(\frac{\partial\phi_2}{\partial v_2}),$ $\mu_1\mu_2 = \left\lvert\frac{\partial\phi_2}{\partial v_2}\right\rvert^2 - \left\lvert\frac{\partial\phi_2}{\partial \bar{w}_2}\right\rvert^2$	$\mathrm{sgn}(-f_r)$ $\mathrm{sgn}(\lvert d\rvert^2 - \lvert f\rvert^2)$

(Table 5 continued on the next page)

Solution	Null Eigenvectors	Eigenvalues	Stability
Standing Squares	$(iv_1, iv_1, iv_1, iv_1) \in W_0^{SS}$	$W_0^{SS} : \{2\ \mathrm{Re}(\frac{\partial\phi_1}{\partial v_1} + \frac{\partial\phi_1}{\partial v_2} + \frac{\partial\phi_1}{\partial w_1} + \frac{\partial\phi_1}{\partial w_2}),\ 0\}$	$\mathrm{sgn}(2a_r + 4b_r + d_r - f_r)$
(SS)		$W_1^{SS} : \{\mu_1, \mu_2\}; \mu_1 + \mu_2 = 2\ \mathrm{Re}(\alpha)$,	$\mathrm{sgn}(f_r - 3d_r)$
		$\mu_1\mu_2 = \|\alpha\|^2 - \|\beta\|^2$;	$\mathrm{sgn}(\mathrm{Re}(\bar{d}f) - \|d\|^2)$
$\mathbf{z} = (v_1, v_1, v_1, v_1)$		$\alpha = (\frac{\partial\phi_1}{\partial v_1} - \frac{\partial\phi_1}{\partial v_2} + \frac{\partial\phi_1}{\partial w_1} - \frac{\partial\phi_1}{\partial w_2})$	
		$\beta = (\frac{\partial\phi_1}{\partial \bar{v}_1} - \frac{\partial\phi_1}{\partial \bar{v}_2} + \frac{\partial\phi_1}{\partial \bar{w}_1} - \frac{\partial\phi_1}{\partial \bar{w}_2})$	
	$(iv_1, 0, -iv_1, 0)$, $(0, iv_1, 0, -iv_1) \in W_2^{SS}$	$W_2^{SS} : \{\mu, \mu, 0, 0\};\ \mu = 2\ \mathrm{Re}(\frac{\partial\phi_1}{\partial v_1} - \frac{\partial\phi_1}{\partial w_1})$	$\mathrm{sgn}(-d_r - a_r)$
Alternating Rolls	$(iv_1, -v_1, iv_1, -v_1) \in W_0^{AR}$	$W_0^{AR} : \{2\ \mathrm{Re}(\frac{\partial\phi_1}{\partial v_1} + i\frac{\partial\phi_1}{\partial v_2} + \frac{\partial\phi_1}{\partial w_1} + i\frac{\partial\phi_1}{\partial w_2}),\ 0\}$	$\mathrm{sgn}(2a_r + 4b_r + d_r - f_r)$
(AR)		$W_1^{AR} : \{\mu_1, \mu_2\}; \mu_1 + \mu_2 = 2\ \mathrm{Re}(\alpha)$,	$\mathrm{sgn}(f_r + 3d_r)$
		$\mu_1\mu_2 = \|\alpha\|^2 - \|\beta\|^2$;	$\mathrm{sgn}(-\mathrm{Re}(\bar{d}f) - \|d\|^2)$
$\mathbf{z} = (v_1, iv_1, v_1, iv_1)$		$\alpha = (\frac{\partial\phi_1}{\partial v_1} - i\frac{\partial\phi_1}{\partial v_2} + \frac{\partial\phi_1}{\partial w_1} - i\frac{\partial\phi_1}{\partial w_2})$	
		$\beta = (\frac{\partial\phi_1}{\partial \bar{v}_1} + i\frac{\partial\phi_1}{\partial \bar{v}_2} + \frac{\partial\phi_1}{\partial \bar{w}_1} + i\frac{\partial\phi_1}{\partial \bar{w}_2})$	
	$(iv_1, 0, -iv_1, 0)$, $(0, -v_1, 0, v_1) \in W_2^{AR}$	$W_2^{AR} : \{\mu, \mu, 0, 0\};\ \mu = 2\ \mathrm{Re}(\frac{\partial\phi_1}{\partial v_1} - \frac{\partial\phi_1}{\partial w_1})$	$\mathrm{sgn}(d_r - a_r)$

Table 5: Eigenvalues of the Jacobian matrix $\mathbf{D}\phi$ evaluated on the solution branches guaranteed by the equivariant Hopf theorem. A solution to the third order truncation (5.3) is stable if the signed quantities in the fourth column are all negative. The SCR solution, not listed above, is unstable. (Recall $f \equiv a + 2b - 2c$.)

Hence the stability of the SS solution to perturbations in W_2^{SS} is determined by $\mathrm{Tr}(\mathbf{D}\phi|W_2^{SS}) = 4\,\mathrm{Re}(\alpha)_{SS}$, where

$$(\alpha)_{SS} = \left(\frac{\partial \phi_1}{\partial v_1} - \frac{\partial \phi_1}{\partial w_1}\right)_{SS} = -(d+a)|v_1|^2. \tag{6.44}$$

It follows from the parameter symmetry (6.22) that the stability of the AR solution to perturbations in W_2^{AR} depends on

$$\mathrm{Tr}(\mathbf{D}\phi|W_2^{AR}) = 4(d_r - a_r)|v_1|^2. \tag{6.45}$$

So far we have considered the stability of those solutions guaranteed by the equivariant Hopf theorem, *i.e.*, those with $W_0 \approx \mathbf{C}$. However, for coefficient values in (5.3) satisfying

$$|f|^2 > |d|^2 > |\mathrm{Re}(\bar{d}f)|, \tag{6.46}$$

there is an additional solution (SCR) for which $W_0^{SCR} \approx \mathbf{C}^2$. The SCR solution was determined to be unstable in the W_0^{SCR} subspace by Swift [11]. His stability argument hinges on the observation that the SCR existence conditions (6.46) are the same as the conditions for (6.20b), (6.23b) and (6.35b) to be positive. Specifically, consider the bifurcations that occur as d and f are varied through a point in the (d, f)-parameter space satisfying one of the following degeneracy conditions:

$$\begin{aligned} |d|^2 &= \mathrm{Re}(\bar{d}f) \\ |d|^2 &= -\mathrm{Re}(\bar{d}f) \\ |d|^2 &= |f|^2. \end{aligned} \tag{6.47}$$

These are surfaces in the (d, f)-parameter space where pitchfork bifurcations occur along the SS, AR and SR solution branches, respectively. Two SCR solutions (both on the same group orbit) are created at the pitchfork bifurcations. Swift invokes the exchange of stability at pitchfork bifurcations to infer the instability of the SCR solution branch. We now calculate the eigenvalues of $\mathbf{D}\Phi|_{SCR}$ directly for the cubic truncation (5.3).

The isotropy subgroup Σ_{SCR} acts trivially on the W_0^{SCR} subspace and hence does not restrict the form of $\mathbf{D}\phi|W_0^{SCR}$. However, two of the four eigenvalues are known since they are associated with eigenvectors in the phase and radial directions of the limit cycle. Thus the eigenvalues of $\mathbf{D}\Phi|W_0^{SCR}$ are $0, -2\lambda, \mu_1, \mu_2$ and the characteristic polynomial associated with $\mathbf{D}\Phi|W_0^{SCR}$ may be formally factored

$$\mu^4 + a_1\mu^3 + a_2\mu^2 + a_3\mu = \mu(\mu + 2\lambda)(\mu - \mu_1)(\mu - \mu_2) = 0. \tag{6.48}$$

The sum and product of μ_1 and μ_2 may then be expressed in terms of λ, a_1, a_2 as follows

$$\mu_1 + \mu_2 = 2\lambda - a_1, \tag{6.49a}$$

$$\mu_1\mu_2 = 2\lambda(2\lambda - a_1) + a_2. \tag{6.49b}$$

These quantities were computed using the computer algebra program, Macsyma:

$$\mu_1 + \mu_2 = -\left[\frac{3d_i(|f|^2 - |d|^2)}{\mathrm{Im}(\bar{d}f)} + 2f_r\right](|v_1|^2 + |v_2|^2) \tag{6.50a}$$

$$\mu_1\mu_2 = \frac{8(|d|^2 - \mathrm{Re}(\bar{d}f))(|d|^2 + \mathrm{Re}(\bar{d}f))(|f|^2 - |d|^2)|v_1|^4}{|d|^2(|f|^2 - |d|^2) - 4(\ \mathrm{Im}(\bar{d}f))^2|v_1|^2/(|v_1|^2 + |v_2|^2)}\ . \tag{6.50b}$$

It follows from the inequalities (6.46) that the numerator of (6.50b) is positive. The denominator is negative, as demonstrated by the following chain of inequalities

$$\begin{aligned}\text{denominator} &< |d|^2(|f|^2 - |d|^2) - 2(\mathrm{Im}(\bar{d}f))^2 \\ &= -[|d|^2(|f|^2 + |d|^2) - 2(\mathrm{Re}(\bar{d}f))^2] \\ &< -2\left[|d|^4 - (\mathrm{Re}(\bar{d}f))^2\right] < 0,\end{aligned} \tag{6.51}$$

where, without loss of generality, we have assumed $(|v_1|^2 + |v_2|^2) < 2|v_1|^2$ in (6.50b). This explicit calculation confirms Swift's result that SCR are unstable.

We now calculate the remaining eigenvalues of $\mathbf{D\Phi}|_{SCR}$. Let $(z_1, z_2) \in W_1^{SCR}$ be defined by $(v_1, v_2, w_1, w_2) = (z_1, z_2, -z_1, -z_2)$; then Σ_{SCR} acts on (z_1, z_2) by the reflections κ_1, κ_2, where $\kappa_1 : (z_1, z_2) \to (z_1, -z_2)$ and $\kappa_2 : (z_1, z_2) \to (-z_1, z_2)$. The matrix $\mathbf{D}\phi|W_1^{SCR}$, of the form (6.24), commutes with matrices

$$\begin{pmatrix} \epsilon \mathrm{I} & 0 \\ 0 & \delta \mathrm{I} \end{pmatrix}, \tag{6.52}$$

where ϵ, δ take on values of ± 1. It follows that

$$\mathbf{D}\phi|W_1^{SCR} = \begin{pmatrix} A_1 & 0 \\ 0 & A_4 \end{pmatrix}, \quad A_j = \begin{pmatrix} \alpha_j & \beta_j \\ \bar{\beta}_j & \bar{\alpha}_j \end{pmatrix}, \quad j = 1, 4. \tag{6.53}$$

As in the case of $\mathbf{D}\phi|W_2^{SS}$, there are two zero eigenvalues in $\mathbf{D}\phi|W_1^{SCR}$ associated with translations of the pattern under T^2. The null eigenvectors, determined by (6.43), are $z_1 = (iv_1, 0)$, $z_2 = (0, iv_2)$. Hence A_1 and A_4 each contain a zero eigenvalue. The stability of SCR to perturbations in W_1^{SCR} is determined by

$$\mathrm{Tr}(A_j)_{SCR} = 2\mathrm{Re}(\alpha_j)_{SCR}, \quad j = 1, 4, \tag{6.54}$$

where

$$(\alpha_1)_{SCR} = \left(\frac{\partial \phi_1}{\partial v_1} - \frac{\partial \phi_1}{\partial w_1}\right)_{SCR} = \nu + bN_1 + cN_2 \tag{6.55a}$$

$$(\alpha_4)_{SCR} = \left(\frac{\partial \phi_2}{\partial v_2} - \frac{\partial \phi_2}{\partial w_2}\right)_{SCR} = \nu + bN_2 + cN_1 \quad . \tag{6.55b}$$

Substituting for N_1, N_2 on the SCR solution branch we obtain the following two nonzero eigenvalues of $\mathbf{D\Phi}|W_1^{SCR}$:

$$-\left[f_r + a_r + \frac{d_i(|f|^2 - |d|^2) \pm (f_r - a_r)|d|^2 \sin(2\psi)}{\mathrm{Im}(\bar{d}f)}\right](|v_1|^2 + |v_2|^2), \tag{6.56}$$

where $\sin(2\psi)$ is given in Table 3.

7. Bifurcation Diagrams.

The previous section determined that the solution stabilities depend on the following six nonlinear coefficients

$$a_r, b_r, d_r, f_r, d_i, f_i, \tag{7.1}$$

where f has been chosen in favor of $c = \frac{1}{2}(a + 2b - f)$. In this section we neglect the unstable SCR solution in our classification of the bifurcation problems; the stability results for the remaining five solutions, summarized in Table 5, suggest the following nondegeneracy conditions on (5.3):

$$b_r \neq 0; \qquad a_r \neq 0,\ f_r,\ \pm d_r,\ f_r - 4b_r,\ -2b_r,\ -2b_r + \frac{1}{2}(f_r \pm d_r); \tag{7.2a}$$

$$|f|^2 \neq |d|^2 \neq \pm\ \mathrm{Re}(\bar{d}f); \tag{7.2b}$$

$$f_r \neq 0, \pm 3d_r. \tag{7.2c}$$

The condition $f_r = 0$ defines a degeneracy in (5.3) only if $|f|^2 > |d|^2$. In this case the eigenvalues μ_1, μ_2 of $\mathbf{D\phi}|W_2^{SR}$ are purely imaginary with $\mu_1 = -\mu_2$. Similarly, the nondegeneracy conditions $f_r \neq 3d_r, -3d_r$ are only necessary if $|d|^2 > \mathrm{Re}(\bar{d}f), -\mathrm{Re}(\bar{d}f)$, respectively.

The inequalities (7.2) divide the six–dimensional parameter space (7.1) into regions characterized by distinct bifurcation diagrams. The dependence of the bifurcation diagrams on f_i, d_i is summarized in Figure 5 where d is plotted in the complex f plane. For the remainder of this section we restrict our analysis to the case $d_r > 0$. The parameter symmetry (6.21) allows us to infer the results for $d_r < 0$ from those obtained for positive d_r. It remains to determine the bifurcation diagrams in the various regions of the (a_r, b_r, f_r)–parameter space defined by the inequalities (7.2a). First, however, we review the bifurcation results for the four–dimensional invariant subspaces (*i.e.*, entries VII-XI in Table 1).

As noted in section 3, equation (5.3) restricted to subspace VII is $D_4 \times S^1$–equivariant; it has the form

$$\begin{aligned} 0 &= \nu v_1 + (a + 2b - f)(|v_1|^2 + |v_2|^2)v_1 + f|v_1|^2 v_1 + d v_2^2 \bar{v}_1 \\ 0 &= \nu v_2 + (a + 2b - f)(|v_1|^2 + |v_2|^2)v_2 + f|v_2|^2 v_2 + d v_1^2 \bar{v}_2. \end{aligned} \tag{7.3}$$

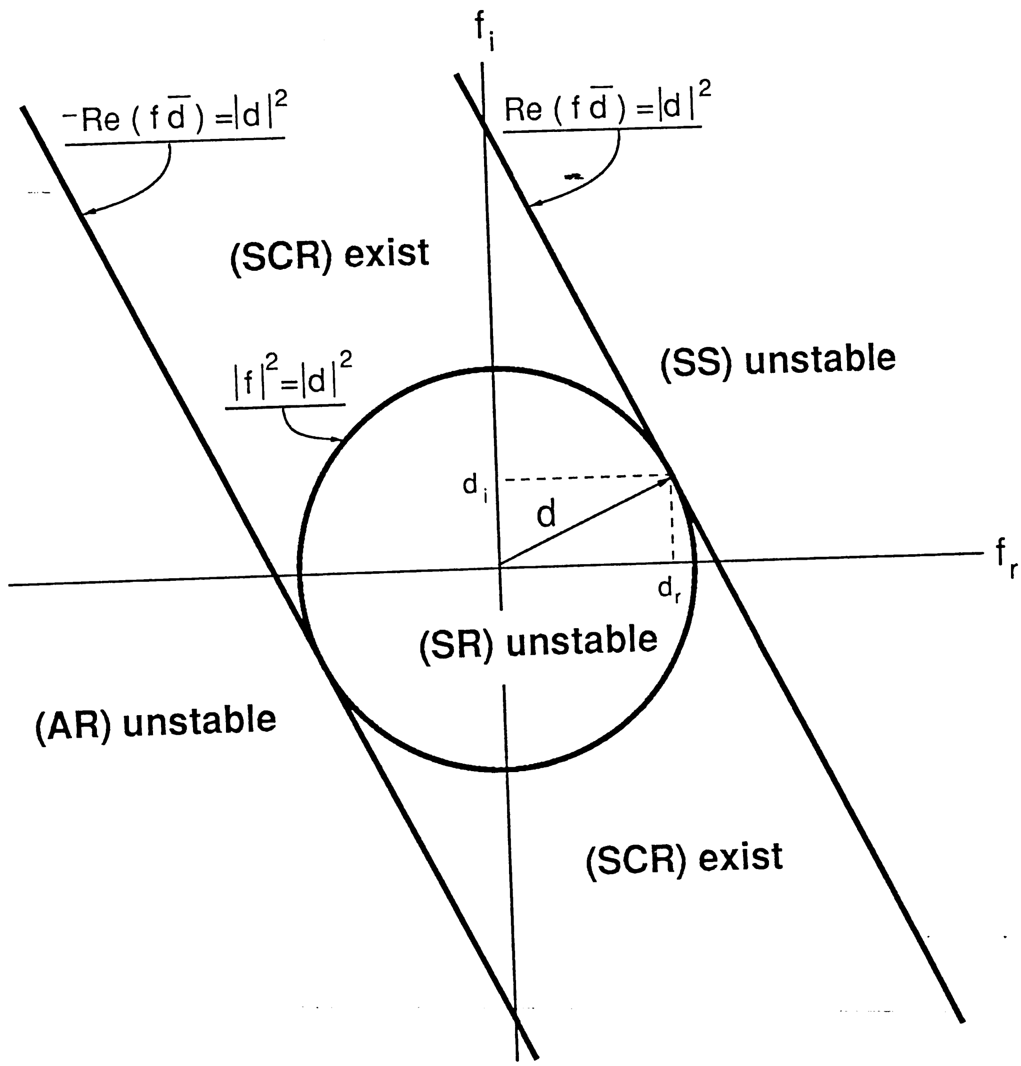


Figure 5: The complex coefficient d is plotted in the complex f plane. The nondegeneracy conditions (7.2b) divide this plane into five regions, each labeled by its distinguishing feature.

The nontrivial solutions of (7.3) are SR, SS, AR and SCR. The bifurcation diagrams for the SR, SS and AR solutions are given in Figure 6 for the various regions in the $(a_r + 2b_r, f_r)$–plane. Here, stability is determined relative to perturbations in subspace VII only. The results for $d_r < 0$ are obtained by interchanging the SS and AR labels on the bifurcation diagrams in Figure 6. Note that if any two solution branches are subcritical, then all three are unstable. If, on the other hand, all three branches are supercritical then it is possible for two of the three solutions to be stable.

In each of the subspaces VIII-XI, the restricted form of (5.3) on $\mathbb{C}^2$ is $O(2) \times S^1$–equivariant [6]:

$$\begin{aligned} 0 &= \nu z_1 + \alpha |z_2|^2 z_1 + \beta(|z_1|^2 + |z_2|^2) z_1 \\ 0 &= \nu z_2 + \alpha |z_1|^2 z_2 + \beta(|z_1|^2 + |z_2|^2) z_2. \end{aligned} \tag{7.4}$$

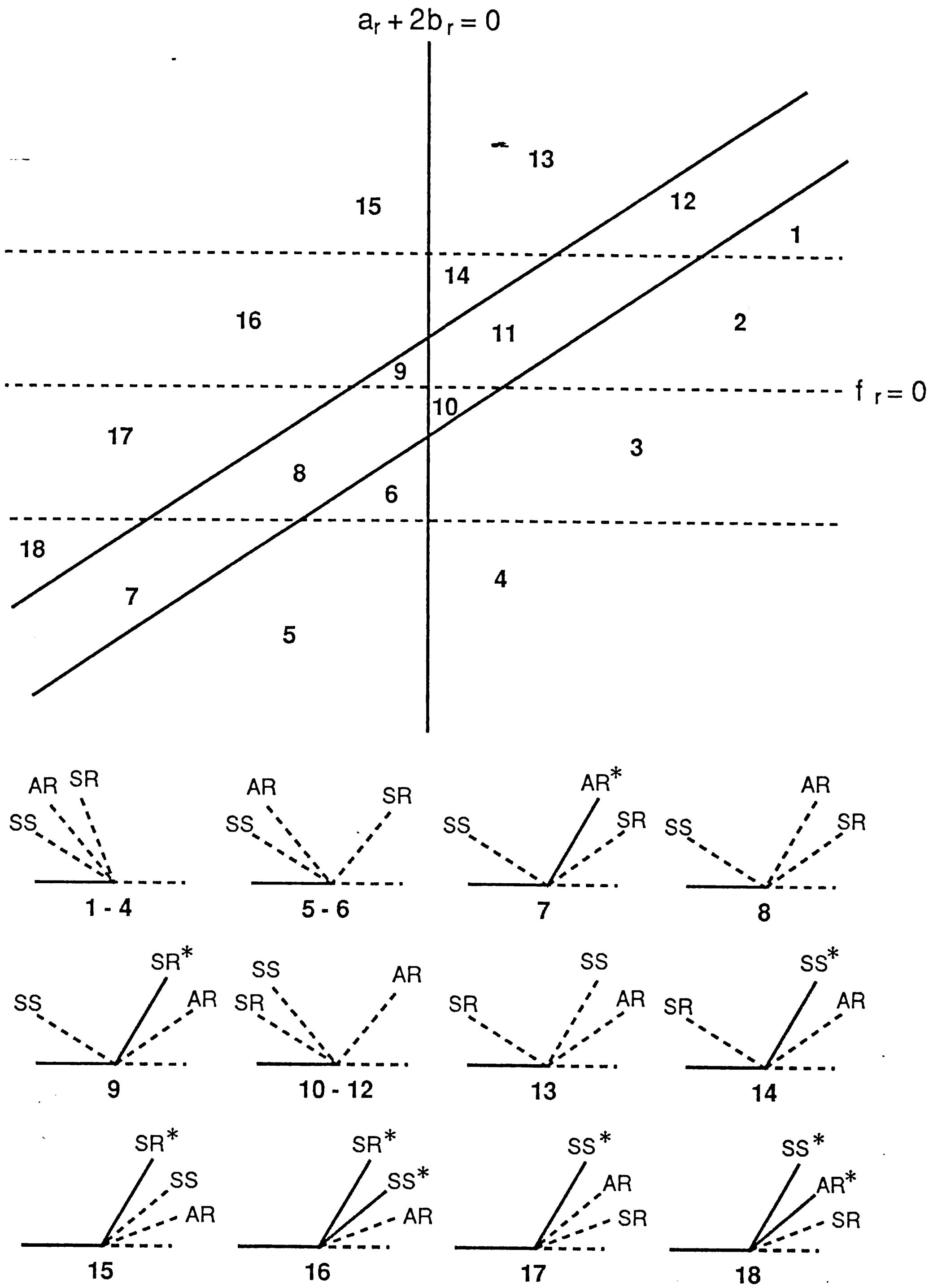


Figure 6: Bifurcation diagrams associated with (7.3). The solid lines in the $(a_r + 2b_r, f_r)$–plane are $(a_r + 2b_r) = 0, (f_r \pm |d_r|)/2$, the dashed lines are $f_r = 0, \pm 3|d_r|$. (See the text for a discussion of when the dashed lines represent degeneracies.) The bifurcation diagrams give the direction of bifurcation and stability of the solution branches for each region of the $(a_r + 2b_r, f_r)$–plane. Dashed bifurcation curves indicate unstable solutions. An asterisk on a solution indicates that it is possibly stable (stability depends on where f lies in Figure 5). The bifurcation diagrams are plotted for $d_r > 0$.

Subspace	α	β	S	R
VIII. $(z_1, z_2) = (v_1, v_2)$ $w_1 = w_2 = 0$	$\frac{1}{2}(a - f)$	b	TS	TR
IX. $(z_1, z_2) = (v_1, w_1)$ $v_2 = w_2 = 0$	a	b	SR	TR
X. $(z_1, z_2) = (v_1, w_1)$ $v_1 = v_2,\ w_1 = w_2$	$a + d$	$\frac{1}{2}(a + 4b - f)$	SS	TS
XI. $(z_1, z_2) = (v_1, w_1)$ $v_1 = -iv_2,\ w_1 = -iw_2$	$a - d$	$\frac{1}{2}(a + 4b - f)$	AR	TS*

Table 6: Correspondence between the coefficients and solution branches of (7.4) and those of (5.3) restricted to the fixed point subspaces VIII-XI. Here, TS* is a solution on the group orbit of TS.

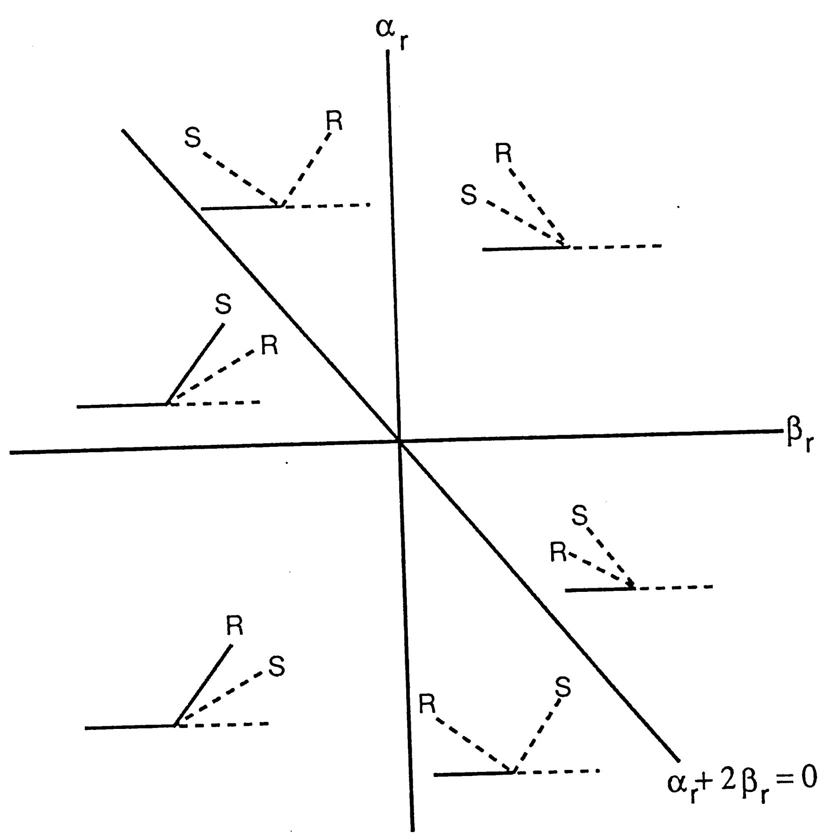


Figure 7: Bifurcation diagrams in (α_r, β_r)–plane.

There are two nontrivial solution types in this subspace:

$$\begin{aligned}(\mathrm{S}):&\ |z_1| = |z_2| \neq 0\\ (\mathrm{R}):&\ |z_1| \neq 0,\ z_2 = 0.\end{aligned} \tag{7.5}$$

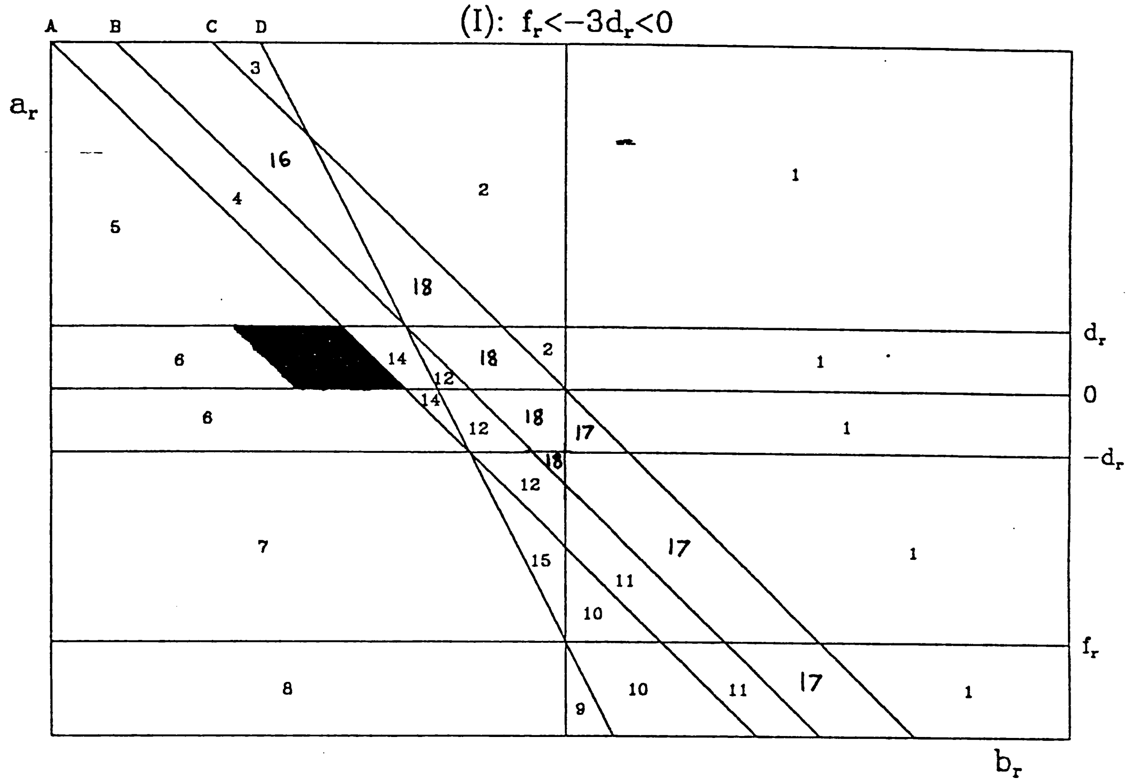
The stabilities of these solutions depend on α_r, β_r. In Table 6 the correspondence between (α, β) and the cubic coefficients of (5.3) is given for each of the subspaces VIII–XI, together with the solutions corresponding to S and R in (7.5). In Figure 7 the bifurcation diagrams are given in the (α_r, β_r)–plane. Note that both solutions are unstable if either is subcritical and that one and only one is stable if both branches bifurcate supercritically. Hence, for TS to be stable, we need the TS, TR, SS and AR branches all to bifurcate supercritically. In addition, if the TS branch is stable, then the TR, SS, and AR solutions are unstable. By similar reasoning, the TR branch is unstable if TR, SR or TS is subcritical. Note that the travelling wave solutions TS and TR cannot both be stable.

Finally, we summarize the bifurcation results in the (a_r, b_r, f_r)–coefficient space for all five nontrivial, maximally symmetric solutions (TR, TS, SR, SS, AR). All possible bifurcation diagrams are obtained by considering six slices through the (a_r, b_r, f_r)–space at fixed f_r satisfying the inequalities:

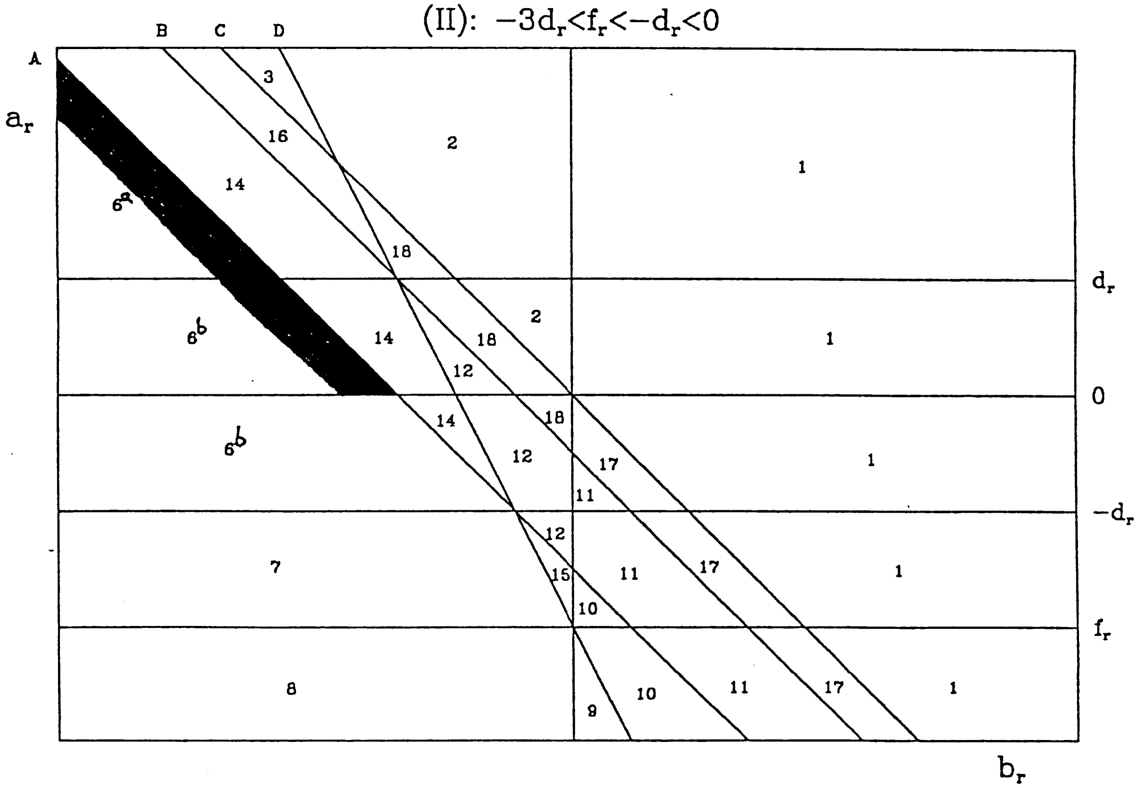
$$\begin{array}{ll} (I) & f_r < -3d_r < 0\\ (II) & -3d_r < f_r < -d_r < 0\\ (III) & -d_r < f_r < 0\\ (IV) & 0 < f_r < d_r\\ (V) & 0 < d_r < f_r < 3d_r\\ (VI) & 0 < 3d_r < f_r. \end{array} \tag{7.6}$$

The (a_r, b_r)–planes are divided into 33 regions each by the nine degeneracy lines defined by (7.2a) (see Figure 8). The bifurcation diagrams appropriate to the various regions are given in Figure 9. Note that it is possible to have two stable solutions bifurcate from the trivial solution when all five branches are supercritical (regions 5, 20 and 27 of Figure 9). It is also possible for all five branches to bifurcate supercritically with none being stable (regions 6 and 35 of Figure 9). The dynamics associated with these regions of the coefficient space has not been fully determined. However, in the next section we indicate why we expect the behavior to be complex and explore the possibility of a primary bifurcation to a structurally stable heteroclinic cycle connecting three of the periodic solutions found in section 5. The structural stability of the cycle is a consequence of symmetry (see, for example, Guckenheimer and Holmes [16]).

(I): $f_r<-3d_r<0$

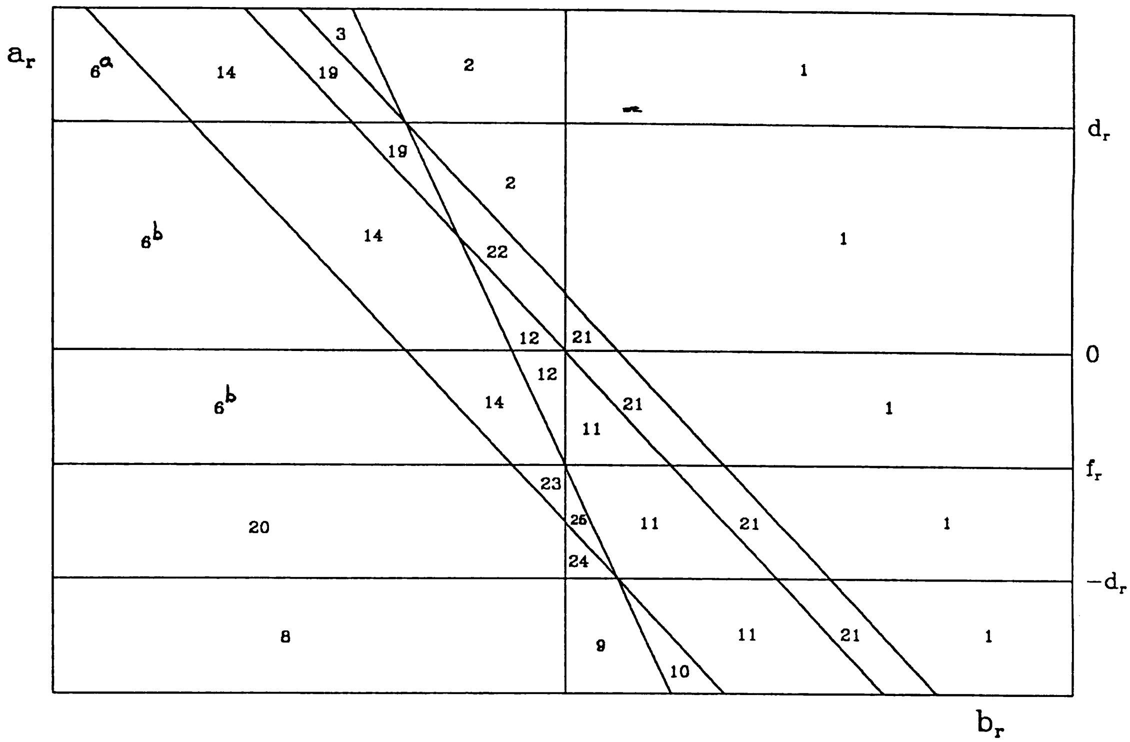


(II): $-3d_r<f_r<-d_r<0$

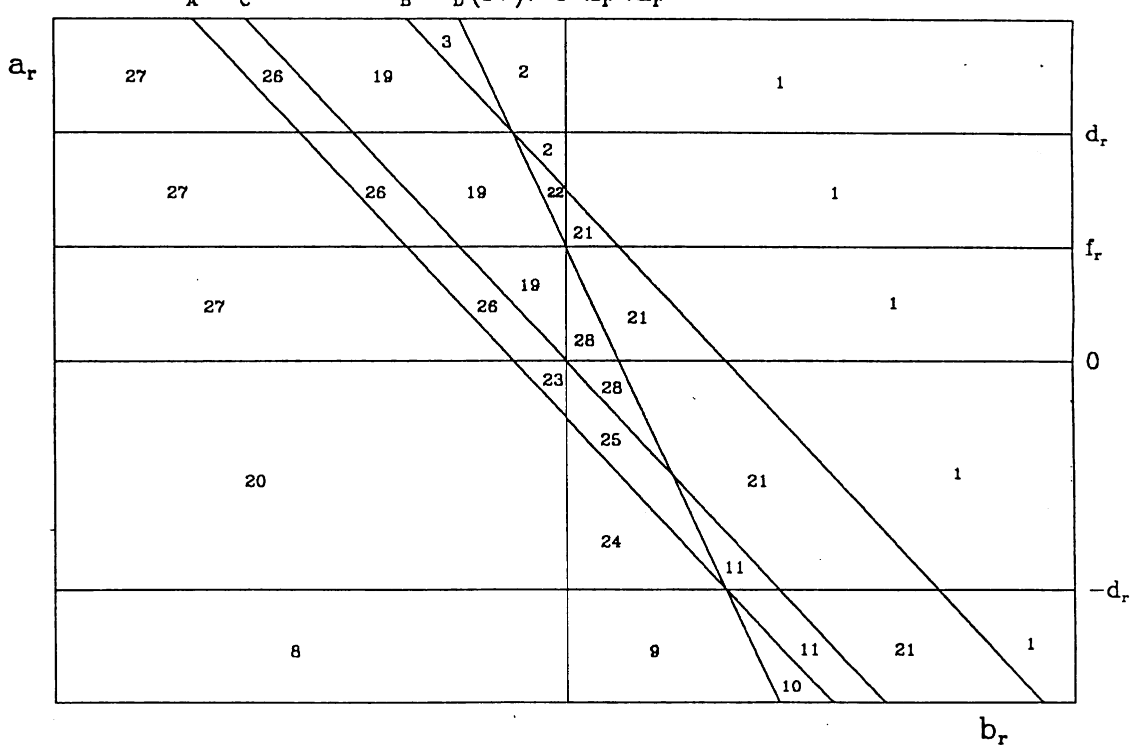


(Figure 8 continued on the next page)

(III): $-d_r<f_r<0$



(IV): $0<f_r<d_r$



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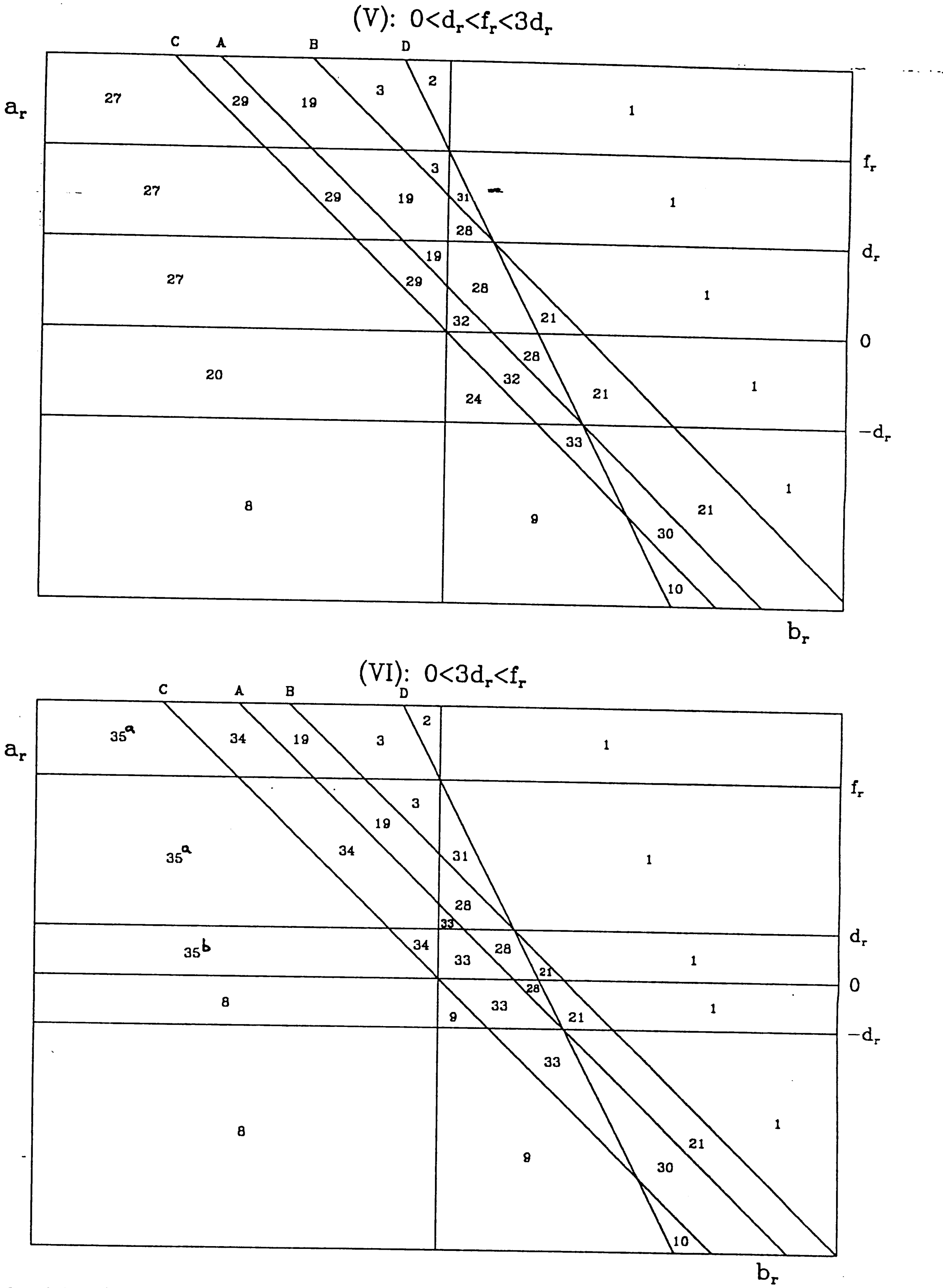

Figure 8: The (a_r, b_r)–plane partitioned by the nondegeneracy conditions (7.2a) for d_r positive, and f_r satisfying the inequalities (7.6) as labeled. The labeled lines are (A) $a_r = -2b_r + (f_r - d_r)/2$, (B) $a_r = -2b_r + (f_r + d_r)/2$, (C) $a_r = -2b_r$, and (D) $a_r = -4b_r + f_r$. The bifurcation diagrams are given in Figure 9. The darkened regions and those regions labeled with a superscript a or b are discussed in section 8.

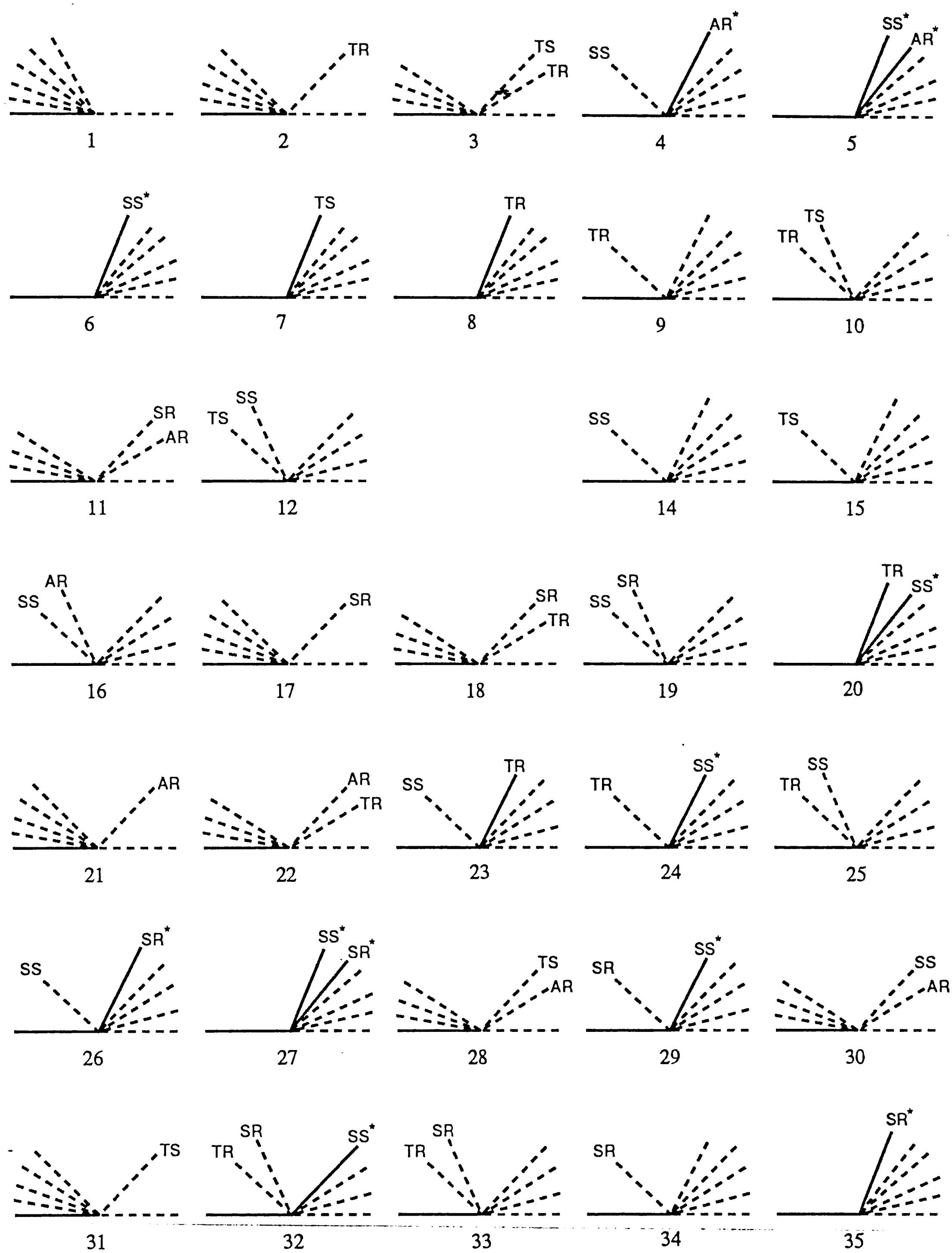

Figure 9: Bifurcation diagrams indicating direction of bifurcation and stability for the numbered regions in Figure 8. Solid lines correspond to stable or possibly stable solutions; as in Figure 6, an asterisk means that the solution stability depends on where f lies in Figure 5.

8. Heteroclinic Cycles.

In this section we investigate some of the complicated dynamics expected in certain regions of the coefficient space associated with the cubic truncation of (4.2). We begin by indicating why we expect a nontrivial attractor to be associated with regions of the coefficient space where all branches are supercritical and unstable (regions 6 and 35 in Figure 9). In this analysis we also assume that one of the following inequalities holds:

$$a_r < 0 \qquad \text{or} \qquad a_r + 2b_r - f_r < 0. \tag{8.1}$$

The second of these conditions is satisfied for regions labeled 35 in Figure 8 and for those labeled 6 in cross–section (*III*) of Figure 8 (*i.e.*, when $f_r > -|d_r|$). The inequalities (8.1) are not satisfied in the blackened portions of the regons labeled 6 in Figure 8 (*I*), (*II*). Note that for SS (SR) to be unstable in region 6 (35), we are necessarily in a part of the coefficient space where the SCR solution does not exist (*cf.* Figure 5).

Let the radius of a hypersphere in $\mathbb{R}^8$ be denoted by $\mathcal{R} \geq 0$, where

$$\mathcal{R}^2 \equiv |v_1|^2 + |v_2|^2 + |w_1|^2 + |w_2|^2. \tag{8.2}$$

We wish to show that $\dot{\mathcal{R}} < 0$ for sufficiently large $\mathcal{R}$ in the regions specified above, in which case trajectories starting near the origin cannot escape to infinity. We call the resulting attractor nontrivial since it is neither the fixed point at the origin nor one of the periodic orbits created at the Hopf bifurcation, these being unstable in the regions of interest. From the cubic truncation of (4.2), it follows that

$$\dot{\mathcal{R}} = \lambda + \frac{1}{\mathcal{R}}\big[2a_r(|v_1|^2|w_1|^2 + |v_2|^2|w_2|^2) + b_r\mathcal{R}^4 + (a_r - f_r)N_1N_2 + 4d_r\ \mathrm{Re}(Q)\big] \tag{8.3a}$$

$$\leq \lambda + \frac{1}{\mathcal{R}}\big[2a_r(|v_1|^2|w_1|^2 + |v_2|^2|w_2|^2) + b_r\mathcal{R}^4 + (a_r - f_r)N_1N_2 + 4|d_rQ|\big], \tag{8.3b}$$

for $\mathcal{R} \neq 0$. We introduce the following nonlinear coordinates:

$$\mathcal{Q} \equiv |Q| \geq 0, \tag{8.4a}$$

$$\mathcal{T} \equiv (|v_1||v_2| - |w_1||w_2|)^2 + (|v_1||w_2| - |w_1||v_2|)^2 \geq 0, \tag{8.4b}$$

$$\mathcal{U} \equiv (|v_1||w_1| - |v_2||w_2|)^2 \geq 0, \tag{8.4c}$$

$$\mathcal{V} \equiv \big[(|v_1| - |w_1|)^2 - (|v_2| - |w_2|)^2\big]\big[(|v_1| + |w_1|)^2 - (|v_2| + |w_2|)^2\big]. \tag{8.4d}$$

These quantities are of the same order (in the amplitudes $|v_j|$, $|w_j|$) as $\mathcal{R}^4$. Note that $\mathcal{T}$ vanishes on all solution branches except for TS, $\mathcal{U}$ vanishes on all but SR, $\mathcal{V}$ vanishes on all but TR, and $\mathcal{Q}$ vanishes on all but SS and AR. The inequality (8.3b), expressed in terms of the coordinates (8.4), is

$$\dot{\mathcal{R}} \leq \lambda + \frac{1}{\mathcal{R}}\left[b_r\mathcal{V} + (a_r + 4b_r - f_r)\mathcal{T} + 2(a_r + 2b_r)\mathcal{U} + 4(2a_r + 4b_r - f_r + |d_r|)\mathcal{Q}\right] \quad (8.5a)$$

$$= \lambda + \frac{1}{\mathcal{R}}\left[b_r(\mathcal{V} + 4\mathcal{U}) + (a_r + 4b_r - f_r)\mathcal{T} + 2a_r\mathcal{U} + 4(2a_r + 4b_r - f_r + |d_r|)\mathcal{Q}\right] \quad (8.5b)$$

$$= \lambda + \frac{1}{\mathcal{R}}\Big[b_r(\mathcal{V} + 2\mathcal{T}) + (a_r + 2b_r - f_r)\mathcal{T} + 2(a_r + 2b_r)\mathcal{U} + 4(2a_r + 4b_r - f_r + |d_r|)\mathcal{Q}\Big]. \quad (8.5c)$$

A consequence of our choice of coordinates (8.4) is that the coefficients of the nonconstant terms on the right–hand–side of (8.5a) determine the direction of branching of the solutions (*e.g.*, TS is supercritical when the coefficient of $\mathcal{T}$ is negative). For the case of interest here, all branches are supercritical and hence the coefficients are all negative:

$$b_r < 0, \quad a_r + 4b_r - f_r < 0, \quad a_r + 2b_r < 0, \quad 2a_r + 4b_r - f_r + |d_r| < 0. \qquad (8.6)$$

That $\dot{\mathcal{R}}$ is negative for sufficiently large $\mathcal{R}$, and for coefficient values satisfying (8.1) and (8.6), follows from the observation that $\mathcal{V} + 4\mathcal{U}$ in (8.5b) and $\mathcal{V} + 2\mathcal{T}$ in (8.5c) are nonnegative:

$$\mathcal{V} + 4\mathcal{U} = (|v_1|^2 + |w_1|^2 - |v_2|^2 - |w_2|^2)^2 \geq 0 \qquad (8.7a)$$

$$\mathcal{V} + 2\mathcal{T} = (|v_1|^2 - |w_1|^2)^2 + (|v_2|^2 - |w_2|^2)^2 \geq 0. \qquad (8.7b)$$

Hence each of the nonconstant terms on the right–hand–side of (8.5b) is negative if the first of the inequalities (8.1) is satisfied. (Similarly, each is negative in (8.5c) if the second inequality holds.) Since these terms dominate at large radius, $\mathcal{R}$ cannot grow without bound. We now investigate the possibility of a heteroclinic cycle being associated with certain of these regions of the coefficient space.

Given the isotropy subgroup lattice in Figure 3, it is straightforward to ennumerate all possible heteroclinic cycles. In particular, it is clear that the cycles must contain the TS solution; these possibilities are depicted in Figure 10. Note that in specifying a heteroclinic cycle we identify solutions with conjugate isotropy subgroups, *i.e.*, solutions on the same group orbit. The observation that the vector field (4.2) restricted to any of the fixed point subspaces VIII–XI is $\mathrm{O}(2) \times \mathrm{S}^1$–equivariant may be used to determine the conditions for existence of an orbit connecting the periodic solutions residing in these subspaces. In particular, one can analyze the two–dimensional real vector field (4.12) to determine that generically there is a saddle–sink connection whenever the branches bifurcate supercritically (see, for example, Melbourne *et al.* [17]). The problem of proving that there exists a heteroclinic cycle is thereby reduced to that of showing there is an orbit connecting the appropriate solutions in the SCR subspace. In order to investigate this possibility, we recall certain results of

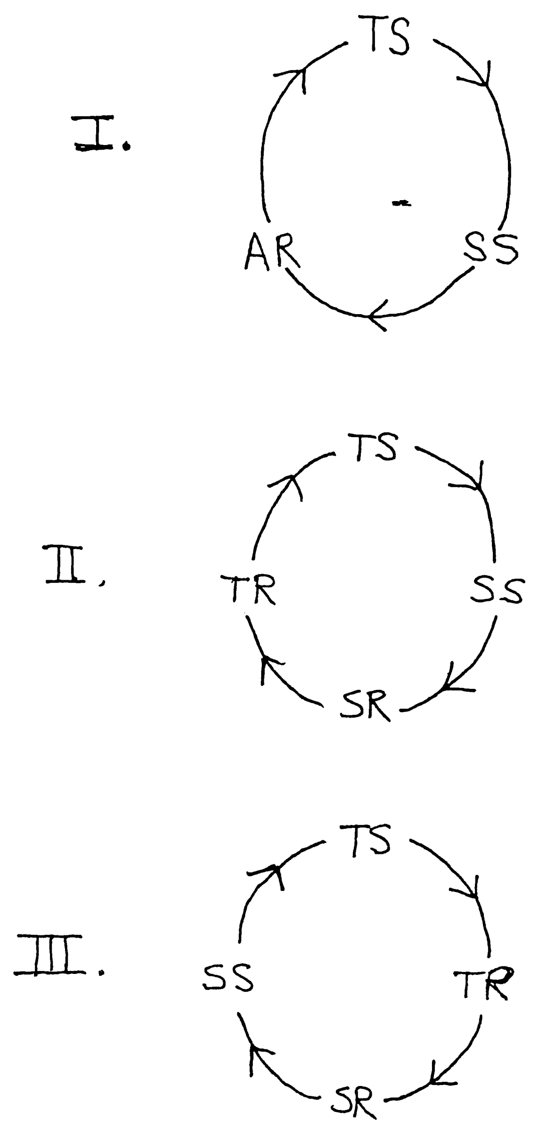

Figure 10: Possible heteroclinic cycles. Three additional possibilities are obtained by interchanging AR and SS. These may be treated using the parameter symmetry.

Swift on the $D_4 \times S^1$–equivariant Hopf bifurcation problem [11], which applies in the SCR subspace. In particular, Swift was able to project the dynamics associated with the third order truncation on to the surface of a sphere and in this way to obtain a two–dimensional flow. Specifically, the following coordinate transformation of $(v_1, v_2) \in \mathbb{C}^2$ was introduced:

$$\begin{aligned} v_1 &= \sqrt{r}\cos(\theta/2)e^{i(\phi+\psi)/2} \\ v_2 &= \sqrt{r}\sin(\theta/2)e^{i(-\phi+\psi)/2}. \end{aligned} \tag{8.8}$$

When this coordinate transformation is applied to the cubic truncation of (4.2), restricted to the SCR subspace, we obtain

$$\begin{aligned}\dot{r} &= r\{2\lambda + r[2(a_r + 2b_r - f_r) + f_r(1+\cos^2\theta) + d_r \sin^2\theta \cos 2\phi]\}\\ \dot{\theta} &= r\sin\theta[\cos\theta(-f_r + d_r\cos 2\phi) - d_i \sin 2\phi]\\ \dot{\phi} &= r[\cos\theta(f_i - d_i\cos 2\phi) - d_r \sin 2\phi].\end{aligned} \tag{8.9}$$

As noted by Swift, the radial dependence in the $(\dot{\theta}, \dot{\phi})$ equation may be removed by rescaling time $t \to t/r$. In this case, the (θ', ϕ') equation decouples from the r' equation (where the prime indicates differentiation with respect to the scaled time). The flow may then be studied on S^2. The fixed points of the "associated spherical system" may be identified with the SR, SS, and AR solutions (see Figure 11a).

Before investigating the possibility of heteroclinic cycle I in Figure 10 we rule out possibilities II and III. This is accomplished by noting that there does not exist a connecting orbit between the solutions in the SCR subspace whenever the other connections are present. For example, it follows from Table 6 and Figure 7 that there are saddle–sink connections SR→TR, TR→TS, and TS→SS in subspaces IX, VIII, and X, respectively, if and only if

$$\begin{gathered} a_r < 0, \qquad b_r < 0, \qquad a_r - f_r > 0, \qquad a_r + 4b_r - f_r < 0,\\ a_r + d_r > 0, \qquad a_r + 2b_r + \frac{1}{2}(d_r - f_r) < 0. \end{gathered} \tag{8.10a}$$

These inequalities confine us to regions of the cubic coefficient space where bifurcation diagram 6 of Figure 9 applies (*cf.* Figure 8). In order for there to be a connecting orbit SS→SR in the fixed point subspace SCR, the SS solution must be unstable, which is the case only if

$$\mathrm{Re}(f\bar{d}) > |d|^2. \tag{8.10b}$$

However, in this parameter regime SR is unstable, in the SCR subspace, in directions transverse to the "radial direction". Specifically, in the associated spherical system, the SS solution is a saddle and the SR solution is a source. Thus there cannot be a connecting orbit from the SS solution to the SR solution; heteroclinic cycle II does not appear (generically) in a primary bifurcation. A similar argument rules out the possibility of a primary bifurcation to heteroclinic cycle III.

In order for heteroclinic cycle I to exist, the SS solution must have an unstable direction in the SCR subspace and the AR solution must have at least one stable eigenvalue (in addition to the radial eigenvalue). The possibility of a heteroclinic cycle with a source→sink connection in the associated spherical system may be excluded since it is not consistent with AR→TS→SS connections. We will also not pursue the possibility of heteroclinic cycle I having a source→saddle connection in the associated spherical system since we do not expect

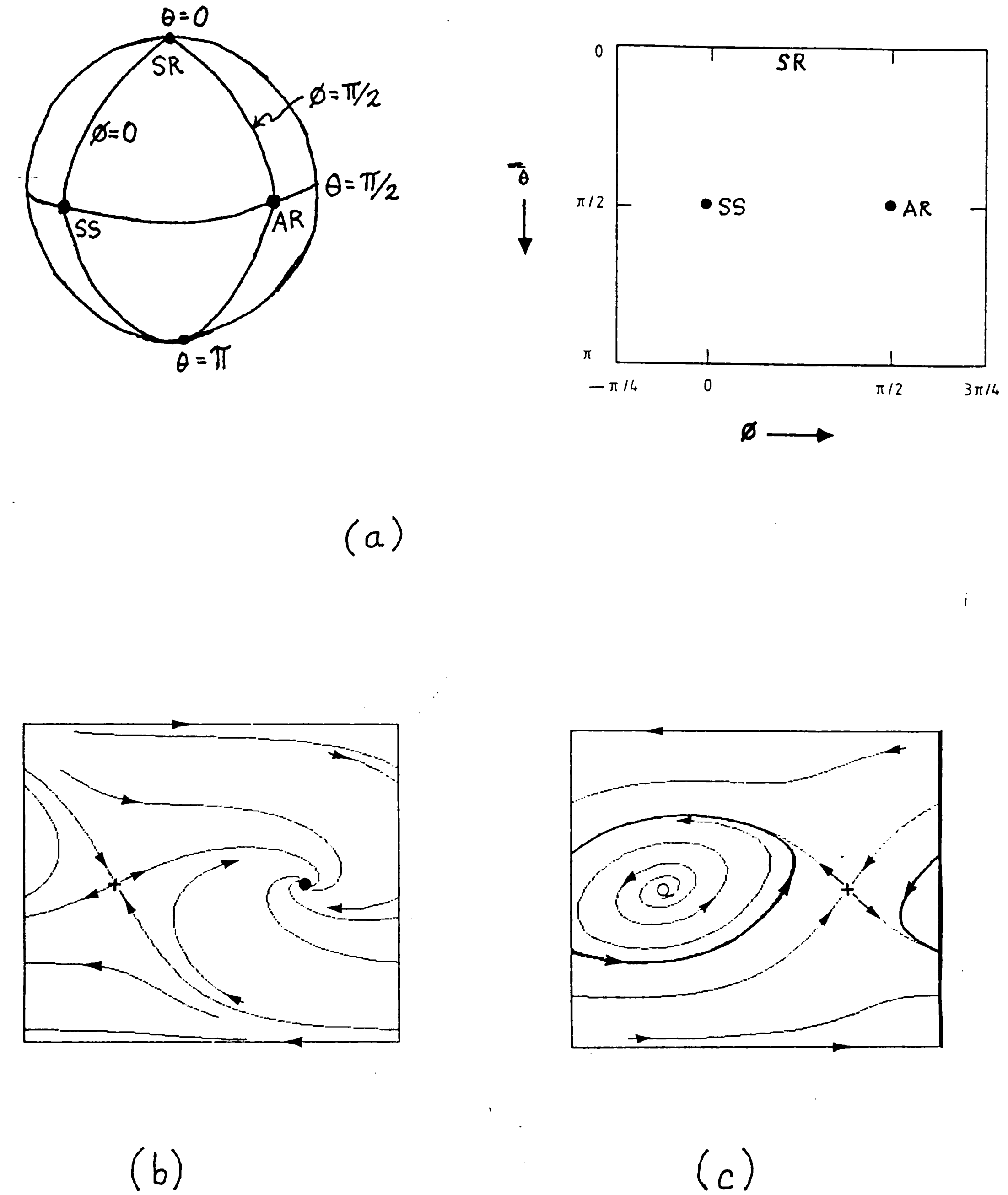

Figure 11: (a) Coordinate system for associated spherical system, shown in perspective, and in the (ϕ, θ)–plane. The SR, SS, and AR solutions appear as pairs of fixed–points (one on the group orbit of the other). (b) Phase portrait showing a saddle–sink connection in the associated spherical system (reproduced from [11]). (c) Phase portrait showing limit cycle in associated spherical system. This corresponds to a quasiperiodic solution in the SCR subspace (also from [11]).

such a cycle to be asymptotically stable. The final possibility of a saddle→sink connecting orbit in the associated spherical system exists for coefficient values in cross-section (I) of Figure 8. The condition that there be connections AR→TS→SS confines our analysis further to those regions of (I) labeled 6. (Note that for SS to be a saddle inequality (8.10b) must hold.) In the associated spherical system the SR solution is then identified with a source at the north pole of the sphere (the source at the south pole is on its group orbit), the SS (AR) solution is a saddle (sink) located on the equator of the sphere at $\phi = 0$ ($\phi = \pi/2$). In Figure 11b, we reproduce a phase portrait from Swift's paper in which there appears to be a connecting orbit from the SS solution to the AR solution. This connection is certainly expected for d_r and d_i sufficiently small. This is seen by noting that at the degenerate point $d = 0$ there is an invariant circle of equilibria on the equator of the spherical system. Specifically, at $d = 0$ the associated spherical system is

$$\begin{aligned} \theta' &= -f_r \sin\theta \cos\theta \\ \phi' &= f_i \cos\theta. \end{aligned} \tag{8.11}$$

When d is perturbed from zero we expect the invariant set to persist. In this case it consists of four equilibria (two each of the SS and AR solution types) and the heteroclinic orbits linking them.

So far we have indicated just one possibility for the dynamics associated with regions of the coefficient space where all branches are supercritical and unstable. This applied to the regions labeled 6 in cross–section (I) of Figure 8. We close this section by suggesting possibilities for the dynamics associated with the other regions of the coefficient space where all branches are supercritical and unstable. The simplest scenario applies if the SCR subspace is attracting in which case the results of Swift on the D_4–equivariant Hopf bifurcation apply. In particular, Swift showed that there are open regions of the coefficient space where there exists a quasiperiodic solution [11]. Specifically, the Poincaré–Bendixson theorem was used to show that there exists a pair of limit cycles in the associated spherical system (one on the group orbit of the other). He further conjectured that there is at most one such pair of limit cycles. With $d_r > 0$, we expect the limit cycles to exist and be stable in the SCR subspace in the following regions of the coefficient space:

$$(II): \quad \mathrm{Re}(|d|^2 - f\bar{d}) < 0, \quad (\mathrm{SS},\mathrm{AR},\mathrm{SR}) = (\mathrm{Sa},\mathrm{QP},\mathrm{So}) \tag{8.12a}$$

$$(III): \quad \mathrm{Re}(|d|^2 - f\bar{d}) < 0, \quad (\mathrm{SS},\mathrm{AR},\mathrm{SR}) = (\mathrm{Sa},\mathrm{So},\mathrm{QP}) \tag{8.12b}$$

$$(VI): \quad |f|^2 - |d|^2 < 0, \quad (\mathrm{SS},\mathrm{AR},\mathrm{SR}) = (\mathrm{QP},\mathrm{So},\mathrm{Sa}). \tag{8.12c}$$

(The parameter symmetry may be used to infer the results for $d_r < 0$.) Here the roman numerals refer to the cross–section of Figure 8 and the notation $(\mathrm{SS},\mathrm{AR},\mathrm{SR}) = (\mathrm{Sa},\mathrm{QP},\mathrm{So})$ specifies that in the associated spherical system the SS solution is a saddle, the SR solution is a source, and that the quasiperiodic solution appears as a limit cycle surrounding the source AR (*cf.* Figure 11c). We now focus on the regions of cross–sections (II), (III) and (VI) where all branches are supercritical and unstable (regions 6 and 35); a superscript

a indicates that the SCR subspace is attracting on neighborhoods of each of the AR, SS and SR solutions (*i.e.*, all eigenvalues associated with eigendirections transverse to the SCR subspace and its group orbit have negative real part). In these cases it is plausible that the SCR *subspace* is attracting and that the quasiperiodic solution is asymptotically stable. In the regions labeled with a superscript b the SCR subspace is not attracting and the dynamics associated with these regions may be quite complicated. In particular, it may be possible to get heteroclinic cycles involving not only the periodic solutions created at the Hopf bifurcation, but also the quasiperiodic solution in the SCR subspace. For example, with $a_r < 0$ in region 6^b of cross-section (III) there are heteroclinic connections SR→TR→TS→SS, but as already noted there is no connection SS→SR. However, in this region of coefficient space, in the associated spherical system the SS solution is a saddle and the SR solution is surrounded by a (presumably isolated) stable limit cycle. Hence one might expect a heteroclinic cycle involving the quasiperiodic solution rather than the SR solution (*i.e.*, QP→TR→TS→SS→QP).

9. Conclusion.

In this paper we have employed the techniques of equivariant bifurcation theory to analyze the generic Hopf bifurcation problem with $D_4 \times_s T^2$ symmetry. The problem is motivated by the Hopf bifurcation of a trivial (homogeneous) equilibrium in a continuous translation–invariant system. In the presence of periodic boundary conditions the problem has the spatial symmetry $G \times_s T^2$, where G is the symmetry of the unit cell of the resulting lattice and the two–torus T^2 describes translation in two independent directions. We have focused on the case $G = D_4$, *i.e.*, on patterns periodic on a square lattice. For this problem the center manifold is eight–dimensional, and the appropriate action of $D_4 \times_s T^2$ is described by (2.4). The resulting $D_4 \times_s T^2$–equivariant amplitude equations were assumed to be in Poincaré–Birkhoff normal form and hence to commute with an additional S^1 phase shift symmetry. All isotropy subgroups were determined, thereby classifying the possible solutions by their symmetries. The S^1 symmetry of the normal form introduced the possibility of patterns characterized by spatio–temporal symmetries, as well as by purely spatial symmetries. The equivariant Hopf theorem [6] was then invoked to determine five nontrivial periodic solutions that generically bifurcate from the trivial solution at the Hopf bifurcation. These solutions, depicted in Figure 4, correspond to two travelling wave patterns and three standing wave patterns.

The general form of a smooth $D_4 \times_s T^2 \times S^1$–equivariant vector field on $\mathbb{C}^4$ was derived in section 4. *All* small amplitude periodic solutions of the third order truncation of this vector field were determined on a neighborhood of the Hopf bifurcation. In addition to the five solutions guaranteed by the equivariant Hopf theorem, a sixth solution was found in an open region of the coefficient space. This solution resides in a four–dimensional invariant subspace and has submaximal symmetry. It is unstable to perturbations in the fixed point subspace of its isotropy subgroup.

In section 6 the linear orbital stabilities of the maximally symmetric solutions were determined with respect to perturbations periodic on the square lattice. Full advantage was taken of the symmetries of the solutions in calculating the eigenvalues of the associated 8×8 Jacobian matrices. This was done explicitly by forming the isotypic decomposition of $\mathbb{C}^4$ for each isotropy subgroup. General expressions were given for the eigenvalues of the Jacobian matrix evaluated on the solution branches with maximal symmetry. In the generic bifurcation problem, solution stabilities depend on six of the eight (real) cubic coefficients. All possible bifurcation diagrams were determined within this coefficient space. This paper thus provides an exhausting classification of the generic possibilities for Hopf bifurcation on a square lattice. In section 8 attention was drawn to the interesting dynamics expected to take place when all the primary branches bifurcate supercritically but none are stable. In particular, we presented evidence for the existence of a primary bifurcation to a structurally stable heteroclinic cycle in a particular region of the coefficient space. We also indicated in which regions of the coefficient space there exists a quasiperiodic solution. The existence of heteroclinic cycles involving quasiperiodic solutions is currently under investigation. We hope in the future to report on these more complicated cycles, as well as on the asymptotic stability of the heteroclinic cycles and quasiperiodic solutions.

The theory described in the present paper can be applied to any large aspect ratio system undergoing a Hopf bifurcation from a trivial solution. As noted in section 3, aspects of the theory also apply to small aspect ratio systems defined on square domains when Neumann boundary conditions are used. Convection in a binary fluid mixture provides us with a typical example where our bifurcation analysis applies. If the separation ratio characterizing such a mixture is sufficiently negative the trivial state, corresponding to pure conduction, loses stability at a Hopf bifurcation as the Rayleigh number is raised. With periodic boundary conditions in one direction it is well known that travelling rolls (TR) are generally selected over standing rolls (SR). These conclusions are based on explicit computation of the coefficients a_r and b_r in (5.3) with both idealized [18] and experimentally realistic [19] boundary conditions, and accords with both numerical and experimental observations. The remaining four coefficients necessary to make predictions for the selection of two-dimensional patterns have not, however, been computed. The present paper should serve as a guide to efficiently performing such calculations. In an interesting paper Pismen [20] observed that if in addition to no-mass-flux boundary conditions at the top and bottom nearly thermally insulating boundary conditions are employed the wavelength of the initial instability will be long. In this case a simple equation for the two-dimensional planform of the pattern can be derived. This equation takes the form (5.3) and the pattern selection problem can be solved. In one dimension Pismen shows that TR are selected; in two dimensions the solution AR is stable.

These results are a special case of our more general analysis. Other applications along these lines can be easily imagined. In particular we mention here the application of the analysis of the Hopf bifurcation with $D_6 \times_s T^2$ symmetry [12] to the convective instability of two superposed fluids heated from below [21].

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678 **Christophe Golé**, Monotone maps of $T^n \times R^n$ and their periodic orbits