A PROPOSAL FOR CALCULUS
IN THE SENIOR HIGH SCHOOL

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CHAPTER I

THE PRESENT SITUATION IN THE HIGH SCHOOL MATHEMATICS CURRICULUM

For several years leading mathematicians, teachers, and to some extent, professional educators have recognized that there exists some kind of deficiency in the mathematics curriculum of the high school. The exact nature of this deficiency is open to conjecture, but several causes have been suggested for this inadequacy.

I. PROPOSED CAUSES FOR THE INADEQUACY OF THE HIGH SCHOOL MATHEMATICS CURRICULUM

These complaints against high school mathematics have been launched from all sides, from without and from within. One of the surprising characteristics of these complaints is that there is little consistency among them. The critics place the blame for the allegedly poor curriculum first on one thing, then another.

The wide variance among these accusations is made clear when one looks at several of them. It is claimed, in particular, that (1) the present-day mathematics curriculum is too difficult, (2) the mathematics presented in the existing curriculum can not be applied to real-life situations, (3) the pupils have no interest in mathematics, (4) the mathematics teachers are ill-prepared, (5) the pupils are given poor mathematical backgrounds by the elementary teachers, and (6) the mathematics curriculum is out-dated and at least one hundred years behind the times.[1]

Perhaps these criticisms are valid; it is even conceivable that all the above conditions exist in the modern high school. It is not the purpose of the author to judge the validity of these charges, but to consider the one most fundamental failing of the mathematics curriculum in particular, and to propose an immediate measure to alleviate some of the deficiency.

II. THE FUNDAMENTAL FAILING OF THE HIGH SCHOOL MATHEMATICS CURRICULUM

It is contended by the author that the fundamental cause for the inadequacy of the mathematics curriculum is the subject matter taught. In other words, it is maintained that the mathematics curriculum is, indeed, out-dated and <u>at least</u> one hundred years behind the times.

What does this mean? Are not the concepts of geometry, algebra and trigonometry used in modern day living? Most assuredly so. But these mathematical methods have been

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[1]Commission on Mathematics of the College Entrance Examination Board, <u>Modernizing the Mathematics Curriculum</u> (Pamphlet. New York: College Entrance Examination Board, 1958), p. 2.

in existence for centuries. Geometry and trigonometry had their origins in the ancient Greek civilization. Other methods of computation taught in the high school have been developed through the ages, with the newest of these, logarithmic computation, being only three hundred years old.[2] There is validity in using these means for they have proven themselves through centuries of use. They are time-honored methods, and tradition says that we may use them.

What if this same criterion were applied to the field of chemistry? At one time, it was thought that there were only ninety-two elements which comprised all things, animate and inanimate. This concept is not as dated as the belief of the ancient Greeks that earth, air, fire and water were the components of all things, but in comparison to the chemistry of 1958, it is as ancient as geometry is to the modern mathematician. What would the modern chemistry class be like if the idea of only ninety-two elements still permeated the classroom? It would be difficult to imagine. Chemistry is being changed constantly, and these new changes are being incorporated into the chemistry course as rapidly as they happen, so as not to be behind the times. Chemistry is not a slave to the beliefs of antiquity.

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[2] Philip E. B. Jourdain, "The Nature of Mathematics," _The World of Mathematics_, James R. Newman, editor (New York: Simon and Schuster, 1956), p. 23.

New discoveries and concepts are being added to all scientific fields as they occur. With these refinements, teachers now can explain the theories of the atomic bomb and the effect of radiation upon genetic mutations. Mathematics plays an equally important role in these new scientific discoveries, but its place is disregarded in the high school.

Unfortunately, new developments in mathematics are not added to the curriculum. In the past one hundred years more has been developed in the fields of mathematics than was in the preceeding forty centuries. However, the high school curriculum hangs tenaciously to the older concepts of geometry, algebra and trigonometry and refuses to let these modern developments gain any hold. It is not that these new mathematical developments are difficult; they are fundamental to mathematical knowledge, and much more fundamental than the present high school courses. Rather, educators do not realize the need for them. How can they see this when some of them do not even recognize the practical applications and uses of elementary algebra?

III. THE LEADERS IN CURRICULUM REVISION

If this basic deficiency of the mathematics curriculum is ever to be relieved, it must be the mathematicians and the informed high school teachers that will be the driving force behind this revision. Naturally this revision must not be made hastily, nor should it be made for the sake of revision itself, but must be made only after determining that revision will produce a desired effect and that there is a definite need for the change.[3] Without these considerations, any revision is next to worthless.

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[3]Harold Spears, The Emerging High School Curriculum and Its Direction (New York: American Book Company, 1948), Chap. II.

CHAPTER II

PROPOSALS FOR MODERNIZING MATHEMATICS

If the fundamental deficiency of high school mathematics is recognized as the outmoded curriculum, and if this defection is kept in mind, what programs, if any, can be instituted to modernize mathematics?

I. TWO PROPOSALS

At least two definite proposals have been made with this inadequacy in mind. They are proposals that have received wide publicity, both for the progressiveness of their content and their unique methods of presentation. An examination of the fundamental recommendations of these two proposals may prove valuable.

Commission on Mathematics

Founded in 1955, the Commission on Mathematics of the College Entrance Examination Board has worked constantly on a set of proposals for the modernization of the high school mathematics curriculum. Other areas of interest to the Commission are (1) teacher education, (2) content and methodology of proposed new courses, (3) writing of sample text materials, and (4) the publishing of progress reports. In one of these reports, the following recommendations are explicitely given:

(1) That increased emphasis be placed upon the teaching of algebra as the study of mathematical structure in contrast to the development of manipulative skills alone, and that in a limited way the ideas of modern mathematics be introduced into this instruction.

(2) That the ideas of graphing commonly taught be extended into a development of the concepts of elementary analytic geometry.

(3) That increased emphasis be placed upon deductive reasoning in areas of mathematics other than geometry.

(4) That the traditional course in deductive solid geometry be abandoned, but that spacial concepts be developed in connection with those of the plane.

(5) That increased emphasis be placed upon the trigonometric functions and their properties as functions of real numbers, with a consequent lessened emphasis upon such computational trigonometry as solution of triangles by logarithms.

(6) That increased emphasis be placed upon probability and statistical inference as a type of thinking of the greatest importance in the contemporary world.

(7) That provision be made for the inclusion of the elementary calculus of polynomials in the high school program, but that a standard course in analytic geometry and calculus be considered as a college-level course, which if taught in high school should be regarded as a college course taught to able students for advanced placement.

(8) That a student who completes a full four-year program in secondary school mathematics should be prepared to take analytic geometry and calculus as his freshman college course.[4]

One has not too much difficulty in seeing the direction these proposals would take if enacted. They are not too

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[4]Commission on Mathematics of the College Entrance Examination Board, Commission on Mathematics (Pamphlet. New York: College Entrance Examination Board, 1957), pp. 4-5.

radical a diversion from the present curriculum, in that all present courses taught in the curriculum are included. The modernization in these recommendations comes with the inclusion of modern algebra in the ninth year, and the more advanced topics of analytic geometry, probability and statistical inference in advanced algebra and twelfth year mathematics. The incorporation of these "extra" topics necessitates the condensation of the present curriculum by eliminating most of the deductive study of solid geometry and the computational trigonometry. It should be noted that these recommendations imply that analytic geometry and calculus should not be taught as separate courses, but should be included only as a direct application of the function concept.

The University of Illinois

On the other hand, the program advanced by the University of Illinois, under the leadership of Max Beberman, a teacher at the University High School, and Herbert E. Vaughan, a mathematician on the university faculty, is so modern in its approach as to make the previously mentioned plan look almost antique. The cornerstone of the Illinois program is the approach to mathematics via abstract generalizations. The program is extended through the four years of the University High School and is being used experimentally in other schools in Illinois and Missouri. The general practicality of this program is not yet known, although the work done at the University High School by Beberman and his students has been most successful.[5]

II. A THIRD PROPOSAL

These proposals are conscientiously made, and are seen to be valid without too much difficulty. The author is wholeheartedly in favor of many of these suggestions, but believes that too little emphasis is placed upon the importance of calculus. It is contended that calculus can and should be introduced to the high school mathematics student at the earliest possible moment.

What are the reasons behind this proposal? At first glance, it would appear that the suggestion to include calculus in the high school mathematics curriculum in no way modernizes the curriculum, but rather makes it slightly more difficult. This is the case if one considers only that calculus was invented four hundred years ago, and is an ancient topic compared to set theory. But it does not stop there. Consider the extensive use made of calculus in many fields of modern knowledge[6]: sociology, psychology, physiology,

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[5]E. P. Rosenbaum, "The Teaching of Elementary Mathematics," Scientific American, 198: 64-73, May, 1958.

[6]Howard F. Fehr, "The Place of Analytics and Calculus in the Secondary School," The Mathematics Teacher, 27: 301, October, 1934; and Eugene W. Hellmich, The Mathematics in Certain Elementary Social Studies in Secondary Schools and Colleges (New York: Teachers College, Columbia University), pp. 6-7.

economics, not to mention the physical sciences. One realizes that in order to communicate in these fields, one must have a knowledge of the language. Calculus is a part of the language of these fields, particularly on the advanced levels. Thus the inclusion of calculus at the high school level will facilitate and hasten the acquisition of language, and, in turn, the understanding of a field of knowledge.

Another point to consider is that calculus can provide the eager student an outlet for his desire to learn. One of the fundamental aims of education is to give the student the fullest opportunity to learn.[7] While it is true that not all pupils are students, it is necessary that the student be provided for. At the same time, the pupil will be challenged by the fascination of calculus to put forth more effort into his studies.

Then there are two important reasons for including calculus in the high school curriculum:

(1) To bring the high school mathematics curriculum more in line with the modern progressive age in which we live.

(2) To help the student satisfy a desire for knowledge and give the pupil a greater appreciation of mathematics.

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[7]Harold Alberty, Reorganizing the High-School Curriculum (New York: The MacMillan Company, 1953), p. 45.

CHAPTER III

THE PRECEDENT FOR CALCULUS IN THE HIGH SCHOOL

I. THE DOWNWARD TREND

If we inspect the history of mathematical pedagogy, one fact especially should impress itself on our minds. All mathematical systems have shown a tendency to move downward in the educational echelon as they have become more widely known.[8] This has been the case of arithmetic, geometry, trigonometry, algebra and analytic geometry. When these fields were first developed, they were taught only to the intellectually elite. But as the general level of education rose, more and more of the younger students desired knowledge of these subjects. Thus trigonometry, as the others, which was once considered solely a university-level course has been introduced to the high school student, and seems more than likely to remain a high school level subject.

This practice has been followed in the United States, but stopped with the introduction of trigonometry. It seems rather inconsistent that the trend should stop here, rather

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[8] M. A. Nordgaard, "Introductory Calculus as a High School Subject," Selected Topics in the Teaching of Mathematics, Third Yearbook of the National Council of Teachers of Mathematics (New York: Teachers College, Columbia University, 1928), p. 67; and W. H. Tyler, "Calculus for Schools," The Mathematics Teacher, 15: 208-9, April, 1922.

than continue on to include calculus. This is not as damaging to mathematical education, however, as has been the almost complete disregard for one of the most fundamental of mathematical concepts: one that is essential to a thorough understanding of calculus -- the function concept.

II. THE FUNCTION CONCEPT

One of the cries of the many reform movements has been that mathematics must be made functional. This terminology can be used two ways. First, mathematics must be taught in such a way that the student in a terminating course may be able to use that mathematical knowledge in his chosen vocation, whatever it may be. Also, the college-bound student must have acquired the prerequisite knowledge for the first mathematics course he enters at the university level. Thus mathematics must be functional in that the students must be able to use it. Secondly, mathematics must be taught in such a way that x and y become not only variable quantities whose values are to be determined in a particular situation, but more important become related changing quantities whose variation is to be studied. This idea of the relation and variation of changeable quantities is known as the function concept. This is the fundamental concept which will facilitate further study of mathematics in that a student will be learning more actual mathematics, and will be in a better position to apply his mathematics to his environment and future studies. The function concept is, in fact, the basis for all modern study of mathematics as pioneered by the French in the nineteenth century.[9]

III. THE USE OF GRAPHS

A graph is a pictorial representation of a relationship between two variables. The main purpose of the graph is to teach, in a simple manner, the idea of functional relationships. With the use of graphs, continuity and irregularity of functions are made plain. The idea of graphing is not to draw an exact picture of a function, but rather to sketch a general pæcture of a function. This application of the function is of great value in elementary mathematics where the students have difficulty grasping fundamental notions, and should play an important role in the movement of calculus into the high school curriculum.

The function is of primary importance in the study of mathematics. While this concept has had some use in high school mathematics, it should be the core of the mathematical education program.

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[9]Cf. p. 23.

CHAPTER IV

THE VITAL ISSUES

I. THE PRESENTATION OF CALCULUS

When the subject of teaching calculus in the high school is mentioned, some teacher will immediately recall his college calculus class, wherein the professor expounded the beauties of the extended law of the mean, or labored for a considerable time over the existence of an integral. While it is necessary that a college mathematics student know this material, and must be exposed to it through a rigorous presentation, most certainly one can not expect that a seventeen-year-old boy or girl will be able to cope with the theoretical refinements of calculus.

The Utilitarian Approach

The author does not advocate the study of the more difficult topics of calculus in the high school, but does propose to go further into the subject than suggested by the National Committee on Mathematical Requirements in 1923:[10]

> The calculus of the algebraic polynomial is so simple that a boy or girl who is capable of grasping the idea of

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[10] The National Committee on Mathematical Requirements, The Reorganization of Mathematics in Secondary School (New York: The Mathematical Association of America, Inc., 1923), p. 41.

limit, of slope, and of velocity, may in a brief time gain an outlook upon the field of mechanics and other exact sciences, and acquire a fair degree of facility in using one of the most powerful tools of mathematics, together with the capacity for solving a number of interesting problems.

The Theoretical Approach

The National Committee recommendation stresses the utility of mathematics, but gives little importance to the benefit derived from the study of calculus as pure mathematics. It should be noted that:

> The nature of mathematics is twofold. On one side it deals with quantitative relationships between material objects and thus becomes a tool in the world of business, economics or science. On the other side it deduces theorems from arbitrarily chosen postulates and seeks to carry these theorems to their logical conclusions, giving but little heed to their use in the world of practical affairs. From this point of view mathematics belongs to philosophy.[11]

The high school teacher tends to favor the use of mathematics as a tool. This is fine, but nothing else is accomplished. Mathematics first of all is a deductive science, then a tool. This point should not be lost. Calculus forms a bridge between the two aspects of mathematics that can not be gapped if one viewpoint is divorced from the other. The study of calculus "would open up the field of pure mathematics and would be readily appreciated as the

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[11]Vera Sanford, A Short History of Mathematics (Boston: Houghton Mifflin Company, 1930), p. 1.

connecting link between pure and applied mathematics."[12]

The author proposes to include an elementary study of the polynomial and algebraic function, making the study as meaningful as possible by using a graphical approach, while attempting to include a selected amount of elementary theory.

II. FOUR VITAL CONSIDERATIONS

Other considerations than method and content must be made. Assuming that the problems of method and content can be solved, there are certain vital questions that must be answered before the study of calculus in the high school could even be considered.

(1) Will essential and vital parts of the existing curriculum have to be sacrificed in order to make room for the study of calculus?

(2) Is the teaching force of sufficient size and preparation to permit the inclusion of calculus?

(3) Will it be necessary to re-arrange the present mathematics curriculum?

(4) Does a high school student have the mentality and experience to profit by such a course?

There are other questions that may be asked concerning calculus in the high school, but these four are the vital

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[12]Susie B. Farmer, "The Place and Teaching of Calculus in Secondary Schools," The Mathematics Teacher, 20: 187, April, 1927.

ones. The less important questions must remain unanswered, and be left to the high school administration and the mathematics teacher.

The Elective Course in Calculus

The answer to the first question is given when it is realized that the proposed course in calculus is elective, so that it will not crowd out any of the existing courses. However, a full semester or so given to the study of calculus necessitates one move. That will be to accelerate the present mathematics curriculum by casting out "dead wood" and eliminating duplication so that the remaining material may be given in about five semesters. This will not crowd any of the non-mathematical parts from the curriculum, and it will give the student with mathematical interest an opportunity to take up this enlightening branch.

The Calculus Instructor

As for the instructor, nothing could be more disastrous to a new course than to have it taught by incompetent teachers. Most mathematics teachers have studied calculus in college, and perhaps these can undertake the teaching of the elements of calculus to high school pupils. There is a danger in this assumption, however, for the meager knowledge of calculus required of prospective mathematics teachers by most teacher-training institutions does not acquaint one with much more than the mechanical operations of the topic, most of which can be forgotten too easily. With this in mind it is almost imperative that the high school calculus teacher have studied more calculus than in normally included in the requirements for graduation; and this study should come very shortly before the teacher attempts to give a course in calculus in the high school. Nor should a teacher out of sympathy with this proposal be pressed to teach it, no matter how thorough his education. In a small school, it may be inadvisable to give the course. In such cases, topics can be offered as part of the higher algebra and trigonometry courses, or perhaps it may be offered in alternate years.

The Revised Curriculum

As to the third question: In itself, this new course will not make a re-arrangement of the mathematics curriculum necessary. One may simply add this extra course for high school juniors or seniors. The revision in mathematics, as has been said before, will rise, not out of the inclusion of calculus, but from the recognition that much of the high school program is extraneous and can be easily omitted from formal instruction.[13] Particularly with more junior high schools including the study of informal geometry, the feeling is that one year is sufficient for a study of both plane and

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[13]Cf. p. 7.

solid geometry. Algebra and trigonometry are still necessary, but is there justification for arbitrarily parcelling these courses into two-semester and one-semester packages, respectively? In these courses, considerable "weeding out" could take place leaving at least one semester, and in most cases more than one semester, for the study of calculus and other advanced material.

<u>The Ability of the High School Student</u>

In regard to the fourth question we must first displace the fear that a course in elementary calculus is any more difficult than many topics now given in the high school course. Particularly is brought to mind the topic of trigonometric identities and equations, the genuine backbone of trigonometry. As taught in high schools, this topic brings no immediate response from the students. They are not taught to understand the importance and utility of these relations and must force themselves to memorize these mathematical aids because they are to be tested on them. What a far cry this is from the informal study of maxima and minima by studying the graph of a function, or by relating the sign of the derivative to the straight line motion of an automobile. Here the students can "see" the theory or can use their experience to benefit from the topic. The benefits of calculus are made clearer if we inspect the results of an experiment at the University of Iowa High School.

It was found that the students showed keen interest and were impressed by the power of calculus. They were particularly struck by the ease with which physics problems could be worked and were in agreement that calculus was the most useful and helpful mathematics tool they had come in contact with.[14]

What do we find in other countries? In Austria, Germany, France and Sweden certain schools introduce analytic geometry in the eleventh year, and in some of the advanced German and Austrian schools, differential calculus is developed and applied to physics in that year.[15] In the twelfth year the schools of Denmark, Germany, Austria, Belgium, Sweden, Switzerland, France and Russia and others offer differential and integral calculus.

No one believes that the mind of an American youth does not equal that of his European cousin. His school year may be shorter, his curriculum less intense, his teachers less well-equipped, but he certainly has the mental power to do in the eleventh and twelfth years what the European youth does in those same years. The European youth in his twelfth year is on the par of an American college sophomore in re-

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[14]Nordgaard, "Introductory Calculus as a High School Subject," p. 95.

[15]James E. Russell, German Higher Schools (New York: Longmans, Green and Company, 1916), p. 326.

spect to scholastic achievements thanks to better trained teachers and an intense well-planned curriculum, but that does not affect his native ability. Many of the high schools in Europe are highly selective, and thus do not have the great cross-section of pupils as do American schools. However, the calculus in Germany and France is required of all pupils.[16] The calculus we propose is elective, designed for the better mathematics student. These are exactly the students that enroll in the present elective eleventh- and twelfth-year mathematics courses, thus we are, in effect, designing the proposed calculus course for the present students of high school mathematics.

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[16] Ibid., p. 324; and F. E. Farrington, French Secondary Schools (New York: Longmans, Green and Company, 1910), p. 269.

CHAPTER V

THE DEVELOPMENT OF CALCULUS IN THE SECONDARY SCHOOL

For a more thorough understanding of the case for calculus in the high school, perhaps a look into the historical development of this subject in the secondary school is in order.

I. THE HISTORICAL DEVELOPMENT

Europe

Europe was the cardle of modern mathematics, and as such, was first to realize the benefit that the study of calculus in the high school can bring. Let us look at only a few of the countries that have pioneered this movement.

France.[17] In France the secondary schools have been giving work in calculus for over one hundred and fifty years under algèbre and analyse. For a long time, France has been giving a classical and scientific course in the upper years of her secondary schools, and has encouraged a degree of specialization in the twelfth year that in most countries is permitted only in the university. The direction of the secondary mathematics curriculum has been given by the exacting

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[17]Farrington, op. cit., pp. 257-287.

mathematical requirements for entrance to French scientific schools. School authority has been highly centralized and is directed from Paris. This possibly accounts for France being the first country in the world to include work in calculus as a regular and required part of the curriculum in her secondary schools.

In the last quarter of the nineteenth century a movement grew which would result in calculus being studied by all students, and not by just the superior. Correlation of mathematics with the sciences, the unification of the different branches of mathematics, and a psychological rather than a logical approach to the different topics were being agitated. Above all the function concept was to be made the core of the new mathematics instruction. It was from this idea that the question of teaching calculus to the non-specializing, non-pre-professional students had its beginning, in France as well as in other countries.

Since French schools teach all their mathematics as a unified course, they found an immediate use for calculus, especially in their work with algebra, and this work has been incorporated in the algebra texts. Even in branches that are not too close in union with calculus, it has been found useful, both in application and in theory. Scholastic standards are high, and the number of successful mathematics students is equally high.

Germany.[18] In Germany, the situation proved to be different. There was no strong centrally directed force prescribing the curriculum, although as far back as 1816 the curriculum for secondary schools included analytic geometry and calculus. This did not prove to be of any benefit, however, for entrance requirements to the university were much lower than the advanced courses provided in the secondary schools. The universities were hostile to the teaching of calculus in the high school and instruction in it was actually forbidden.

Not until the time the French began to "push" the function concept did the Germans begin to make extensive use of calculus in the secondary schools, but only in a disguised form by making use of the functional properties of conics. Gradually the function concept worked its way down from the higher classes to the lower classes. At the turn of the twentieth century Felix Klein began to take the leading role in the reform of mathematics teaching throughout the world. Klein's urging of the use of functional thinking resulted in a proposal to teach all mathematics from the functional viewpoint and that all nine-class German schools should have the elements of calculus taught in them. These proposals were

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[18] Russell, op. cit., pp. 312-328; and J. W. A. Young, The Teaching of Mathematics in the Higher Schools of Prussia (New York: Longmans, Green and Company, 1900), pp. 43-78.

adopted by The Society of Natural Scientists at Meran in 1905. One result was that the Prussian ministry of education permitted five nine-class schools to experiment. Other schools shortly followed this example, and now introductory calculus is taught in the secondary schools throughout Germany and Austria.

England.[19] During the nineteenth century the stronger public schools of England were offering special classes in mathematics in the upper forms for boys ranging from seventeen to nineteen years of age with some mathematical ability. These boys were planning either to compete for the university scholarships, to take the army entrance examination, or were preparing to take up engineering.[20] The courses were much like the present elementary calculus found in American colleges and universities. The movement to introduce calculus to a wider selection of students at a lower level followed much the same steps as were encountered in France and Germany, but there was a ten year lag behind the leadership of France.

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[19] M. A. Nordgaard, "An Earlier Place for the Calculus in the Curriculum," The Mathematics Teacher, 20: 325-27, October, 1927; and C. S. Jackson, "The Calculus as a School Subject," Special Reports on Educational Subjects (London: Board of Education, 1912), Vol. XXVI: 365-380.

[20] J. M. Kinney, "Calculus in the High School," The Mathematics Teacher, 16: 326, October, 1923.

In 1914 several schools were giving introductory calculus, primarily using the polynomial function. The teaching was not exhaustive, nor exceedingly rigorous, but correct. There has been no backward turn in this movement in England. The policy is not decreed to the English schools by a central ministry, but must be determined by popular and professional discussion. It is, however, becoming quite general in both England and Scotland. Australia also gives the course in certain schools as does Canada, although the latter is governed much by the attitude in the United States.

The United States[21]

The teaching of elementary calculus in the high school did not become a live issue until about 1920, despite the fact that the issue was put before American mathematics teachers around the turn of the century. The leader in this was Professor E. H. Moore, of the University of Chicago. Almost prophetically Moore discussed the problem of primary, secondary and higher education, offering solutions in 1902 that have become realities today, such as correlation of subjects, unified courses, laboratory methods, junior colleges, teacher training. But his suggestion to teach in the secondary schools the advanced courses of trigonometry, analytic

[21] Nordgaard, "An Earlier Place for the Calculus in the Curriculum," pp. 322-23.

geometry and calculus met with less response--the calculus, no response.

Not for several years was the attempt made to teach elementary calculus and analytic geometry in the high school. But progress was being made. In 1916 the National Committee on Mathematical Requirements was created. It labored from 1916 to 1923 and produced monumental work.

The National Committee recommended that elementary calculus be offered as an elective in the senior high school. It was made clear that calculus is not intended for all schools nor for all teachers or all pupils in any school. The Committee justified the inclusion of this "college course" in the high school curriculum by noting the character of the course. It proposed that elementary calculus be introduced by studying rates of change.

> In nature all things change. How much do they change in a given time? How fast do they change? Do they increase or decrease? When does a quantity become largest or smallest? How can rates of change be compared? These are some of the questions which lead us to study the elementary calculus. Without its essential principles these questions cannot be answered with definiteness.[22]

According to the National Committee, the elementary calculus should include:

> (1) The general notion of a derivative as a limit indispensable for the accurate expression of such fundamen-

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[22]The National Committee of Mathematical Requirements, *op. cit.*, p. 40.

tal quantities as velocity of a moving body or slope of a curve.

(2) Applications of derivatives to easy problems in rates and in maxima and minima.

(3) Simple cases of inverse problems; for example, finding distance from velocity.

(4) Approximate methods of summation leading up to integration as a powerful method of summation.

(5) Applications to simple cases of motion, area, volume, and pressure.[23]

These recommendations startled the general run of mathematicians who had not followed developments in European schools. They did, however, give impetus to movements already in existence: the teaching of calculus to college freshmen at the Massachusetts Institute of Technology and the University of Rochester; and experimentation with secondary school calculus at the laboratory school connected with Teachers College, Columbia University.[24]

II. EXPERIMENTAL PROGRAMS IN THE TEACHING OF CALCULUS

From these beginnings in 1923, calculus in the high school has gone painstakingly, ever-so-slowly forward. In the early schools two plans were used in offering calculus:

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[23] *Ibid.*, p. 38

[24] Nordgaard, "Introductory Calculus as a High School Subject," p. 89.

The first was to employ the methods and notation of calculus in some other course such as algebra. This was basically the plan used in France. The other was to offer a course in calculus as such using what ever block of time was available.

<u>The Early Years</u>

<u>The Horace Mann School for Girls.</u>[25] Pioneer work in this field was done by Miss Vevia Blair in the Horace Mann School for Girls. First offered in 1921 and continued since then, the course has been a senior elective. The groups are college preparatory, meet three periods a week and use no text, although material is drawn from several texts. The syllabus covers the differentiation of the functions $f(x) = x^n$, $f(x) = uv$, $f(x) = u/v$, $f(x) = \sin x$, $f(x) = \cos x$ with problems on rates and maxima and minima. In integral calculus they take up the problems of area, volume, work and fluid pressure. In differentiation the limit of difficulty is found in the development of the MacLaurin series expansion for sin x:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots ;$$

and in integration the limit of difficulty may be found in calculating the volume of a torus made by revolving the circle $x^2 + (y - 7)^2 = 9$ about the x-axis.

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[25] <u>Ibid.</u>, pp. 91-2.

The Lincoln School of Teachers College.[26] The work done in the Lincoln School of Teachers College was led by Dr. Vera Sanford. The experiment was in two parts, the first conducted for three years (1921-24) on eleventh graders, both good and poor students, at the same time algebra and trigonometry were being studied. The work was not elective, but was needed for college entrance.

Rather than borrow time, as it were, from trigonometry and algebra, the calculus course was offered as a senior elective for the second part of the experiment. The actual work in calculus began with the twelfth year and covered roughly the first semester. The key idea was the extension of the function concept to that of the rate of change of a variable quantity. This led directly to the introduction of the derivative and its applications. The indefinite integral as the inverse of the derivative was used for finding areas, volumes, momentums, force and work. The last semester of the senior year was given over to the study of advanced algebra and further work in trigonometry, using methods of calculus whenever possible.

Wadleigh High School, New York.[27] For some years, Mr. John Swenson of the Wadleigh High School of New York has tried the experiment of combining calculus with advanced al-

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[26] Ibid., pp. 92-3. [27] Ibid., pp. 93-4.

gebra, trigonometry and solid geometry. Differentiation and integration of polynomials are introduced in the latter half of the advanced algebra given in the eleventh year. The derivative is used to find maxima and minima, to obtain equations of tangents and normals to conics, and to determine the points of inflection of a curve. Areas are obtained by integration. In the twelfth year the treatment of trigonometric functions is expanded by using trigonometric substitution to integrate certain integrals. While studying solid geometry, integration is used to find volumes and surfaces.

<u>The University High School, Oakland, California.</u>[28] At the University High School of the University of California twelfth year students are given a course in calculus combined with solid geometry. The tenth year's work includes all of plane geometry and five weeks' work in solid geometry, with emphasis on the latter being computational rather than demonstrative. The junior year's work includes algebraic theory, trigonometry, and some work in the natural sciences. In the fall term of the senior year they take up elementary calculus and the more demonstrative parts of solid geometry. About ten weeks is given to the study of calculus, since considerable solid geometry has been studied in the tenth year. Analytic geometry is offered the second semester of the senior year.

[28] *Ibid.*, pp. 93-4.

The University High School of the University of Iowa.[29] The manner in which a course in calculus was inaugurated at the University High School, Iowa City offers a good example of what may be expected at other schools which attempt this revision. Here there was no fear about offering the course for the value and feasibility of such a course seemed well established to those in charge. The problem was to fit it into the school's needs and resources. The enrolled class was a select group--selected, not by the school, but by their own interests and abilities and future plans--which wanted an advanced course in mathematics. The course could have no bearing upon college entrance for the students had three semesters of algebra and provision had been made for solid geometry. They wished to have a course that would be of value to them if they did not take up college work, and which should help them in their work in mathematics, in science, or in engineering in case they went to college. The school had formerly given a fourth semester of algebra, but it was decided to substitute a course in elementary calculus and related material.

The teacher had much else to do and had not had any experience in teaching calculus. It was decided to use a basic textbook and to cover in five days a week for a semes-

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[29] Ibid., pp. 94-5.

ter a little less than what college students using the same text cover in four days a week. Supplementary material from other texts was used, but the order of the basic text was followed.

The instructor reports that the students showed keen interest and were impressed by the power of the new mathematics. The students were overwhelmed by the ease with which physics problems could now be worked, and were in agreement that it had been the most useful and helpful mathematics course they had met.

The New Britain High School.[30] In 1922-23, the senior high school in New Britain, Connecticut, began giving elementary calculus as a senior elective, following closely the recommendations of the National Committee.[31] The policy was to give no marks and no assignments. Enough theory and exercises were given to prepare the students for the solution of a set of prepared problems. The main topics considered were variable rates of motion, maxima and minima, and areas and volumes. Mr. Goff, the instructor, mimeographed his own text for the course.

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[30] Robert R. Goff, "A Few Lessons in Calculus for High Schools," The Mathematics Teacher, 15: 307-8, May, 1922.

[31] Cf. pp. 14-5, 27-8.

South Side High School, Newark, New Jersey.[32] At the South Side High School in Newark , Howard F. Fehr, present Head, Department of Teaching of Mathematics, Teachers College, Columbia University, used a syllabus in twelfth year mathematics that incorporated trigonometry, advanced algebra and calculus. The topics studied in calculus follow the same pattern as the National Committee's recommendation of 1923. In an article written after the use of this curriculum for two years, Mr. Fehr reports[33] that:

> (1) A student who has acquired the concept of function can more readily, vividly, and intelligently apply it to the study of physical phenomena.
>
> (2) A student recognizes in the calculus a tool indispensible in modern engineering and science, and is the more appreciative of the modern scientific achievements which he enjoys.
>
> (3) A student familiar with the notation and simple operations of coordinate geometry and calculus who does not continue his education in higher institutions can far more readily pursue the subject without classroom instruction, while the student who does go on to college adapts himself more readily to the new methods of presentation.
>
> (4) A student familiar with the function concept has a general method of attacking geometric problems instead of the "Bag of Tricks" which he used in Euclidean geometry.
>
> (5) As a child matures so should his ideas of algebra, geometry, and trigonometry; and this growth is best obtained for him by showing their applications in the higher branches of mathematics rather than in advanced parts of the same field.

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[32] Fehr, op. cit., pp. 296-302.

[33] Ibid., pp. 301-2.

(6) A student having had such a course can read books and technical papers on recent scientific developments with better understanding.

(7) These simple methods of the coordinate geometry and the calculus unify the mathematical knowledge of the student since they make indiscriminate use of practically all the mathematics the student has previously learned.

III. THE FUTURE EXPERIMENTS

This survey of the historical developments and past experiments in teaching calculus in the secondary school is far from exhaustive, but it should present a cross-section of those experiments which have been tried. Neither does this or previous considerations completely justify the inclusion of calculus in the high school curriculum. One thing is apparent from studying the reports of these experiments: the more able mathematics students are able to grasp the fundamentals of calculus and realize that in calculus they have found the most powerful mathematical tool they have yet encountered. Is it asking too much that these experiments be conducted on a wider scale on groups of students of a wide range of mathematical ability? If they can be taught to recognize the power of this tool and can understand the fundamentals, this is not too great a load to undertake.

CHAPTER VI

A PROPOSAL

Throughout the preceding five chapters, mention has been made of the author's proposal for the teaching of calculus in the high schools, but nowhere has this proposal been given. The purpose of this chapter is to explicitly present these recommendations, and consider the background, the values, the content, and the methods of this suggested course in calculus.

I. THE BACKGROUND

Study of past trends, recommendations, experiments, and conditions reveals no less than four criteria upon which the teaching of calculus in the high school can be based, at least partially.

1. Calculus has been taught for many years in the secondary schools of Europe to students of ability equal to that of American students without any adverse effects.

2. Small scale experimentation in the teaching of calculus to American youth has shown that the pupils have been able to master the elementary concepts and operations, and have recognized the importance of this mathematical tool.

3. Other mathematical subject matter that, like

calculus, was once thought to be university-level, has, with success, been introduced to the secondary school mathematics curriculum.

4. The inclusion of calculus in the secondary school curriculum has been advocated for many years by the world's leading mathematical educators.

II. THE VALUES

Certain values, many of which may be inestimable, can be realized from the study of calculus in the high school.

1. Calculus is a valuable tool in many fields of endeavor; the inclusion of calculus early in the curriculum will facilitate the learning of these applied fields and will provide a more meaningful learning experience.

2. Young students of mathematics will be provided a greater opportunity to understand and appreciate the importance of mathematics in modern-day living, and will be given a deeper insight into the nature of mathematics.

3. The study of calculus in the secondary school will accelerate the learning of those students preparing for a scientific vocation and will thus provide the opportunity for them to be released into society with a greater knowledge. There is a definite possibility, also, that these people may be delivered to the community at an earlier age, so that they can provide humanity with greater service during their years of greatest productivity.

4. Calculus can serve as the unifying course in high school mathematics, for it makes use of all mathematics previously learned in school. Furthermore, it serves as the connecting link between pure and applied mathematics because of which this first introduction to theoretical mathematics can be made meaningful by the use of applications.

III. THE CONTENT

While the content of this proposed course is outlined at length in Chapter VII, a brief preview of this content is in order.

1. The differentiation of the algebraic function.
2. The integration of the algebraic polynomial.
3. Elementary applications of calculus. Among these applications are: maxima and minima, elementary curve tracing, rectilinear motion, areas and volumes.
4. While the subject is not included in the proposed syllabus, differentiation and integration of trigonometric functions, with appropriate applications, could be added by the teacher whose students are sufficiently prepared.

IV. THE METHOD

The methods of teaching a high school subject are, rightfully, the teacher's own trade secrets, so to speak.

As such, very little can be said concerning the specific methods one should use in teaching calculus. However, certain principles should be followed by the teacher who is offering a course in high school calculus.

Since most present high school mathematics students are not sufficiently mathematically mature to understand a rigorous, theoretical approach to calculus, a psychological rather than logical presentation must be utilized. Wherever possible, theory must be related either to experience or to graphical presentation. In this way, learning experiences will become more meaningful, and knowledges can be acquired much more rapidly.

The proof of essential theorems, however, will not be overlooked by assuming a more informal approach. Rather than neglect the proof of an important theorem, the teacher will be asked to argue from a geometric standpoint, if at all possible. Otherwise, it will be required that the teacher explain to his students that the proof is too much for the present level, and must be assumed to be valid. This assumption, however, does not mean that content will be presented from the standpoint of rote learning.

The content included in the proposed course for high school calculus is outlined in Chapter VII. While the content is so arranged that it could be taught in a separate semester course, this is not the only reason for the given arrangement. The author realizes the difficulties that can arise from the proposal to include calculus in the high school curriculum. With these limitations in mind, he envisions the possibility of including topics from the syllabus in other mathematics courses, rather than offering a separate elective course. If this case did arise, the topics in the syllabus could be taught in these other courses as the teacher so determines. As an example, the author suggests that the subject of graphing the linear function by using the difference quotient could be introduced to a class in first year algebra.

As a final aid to the teacher, the author strongly suggests that the teacher who plans to offer a course in high school calculus frequently refer to his college analytic geometry and calculus texts to refresh his memory and further his understanding of the elementary topics outlined in this paper.

V. A WORD OF CAUTION

The point does not need to be pressed that all students will not be able to achieve a high level of understanding in elementary calculus. Few students who achieved the enviable record of "straight A" in high school mathematics have been able to achieve equal success in college calculus. A student who does not earn such a record in high school calculus should not be punished severely for his lack of ability. Far more advantageous to the student and humanity would be the policy of guiding this student out of mathematics and into a field more commensurate to his abilities, perhaps a field in which mathematics plays a limited role.

Neither should the teacher expect the work of all high school students of calculus to be as thorough as that of the college student. Thoroughness, conciseness, and clarity of expression come only with experience; this experience can not normally be achieved by the senior year of high school. The mathematics teacher should not, however, accept major omissions under this criterion. Work must be complete to the extent of the presentation to the students. Thus, for example, the teacher must consider that

$$\int 3x^2 dx = x^3$$

is incorrect, for the concept of an additive constant in an integral is fundamentally simple. Following the same criteria, the statement that

$$\lim_{x \to 0} 1/x = \infty$$

is a perfectly legitimate statement, despite the fact that nothing "is equal to" infinity, provided that the teacher has taken the appropriate precautions in explaining the concept of infinity, and has, with equal discretion, defined the symbol ∞.

The high school calculus teacher must be prepared. Under no circumstances should he attempt to teach that material which he himself does not fully understand. Disregard for this rule can lead only to gross error and misunderstanding on the part of the student, and can do nothing but corroborate a feeling among many college mathematics teachers that high school mathematics teachers are incompetent.

CHAPTER VII

THE PROPOSED CONTENT

It is the author's intention to give the reasons why calculus should be taught to secondary school students and propose the topics of a high school course, and not to write a textbook on calculus. Thus, these topics as presented in this chapter are in skeletal outline form. The more important theorems have been stated at length, and it has been assumed that the teacher well-enough informed to teach high school calculus can refer to his college calculus text or any of the many excellent texts listed in the bibliography to this paper in order to fill in the details.

The content as outlined here is more than minimal for most high school students, but it is also not maximal. Omitted from discussion are the more advanced topics of differentiation such as rates of change and appoximation by differentials and of integration as centroids, fluid pressure and multiple integration. Furthermore, the differentiation and integration of transcendental functions have, with full intention, been omitted, for the author believes that even the superior high school student is not mathematically mature for an adequate discussion of these topics.

Before the study of calculus can properly begin, there are certain prerequisite elementary knowledges that must either be recalled or learned. While it is possible to introduce many of these fundamentals at other places in the development of calculus, they are grouped in the beginning for convenience. Those fundamentals needed later can then be recalled.

I. Fundamental concepts
  A. Variable
    1. Definition of variable
      a) Discrete
      b) Bounded
      c) Infinite
      d) Finite
  B. Function
    1. Definition of function
      a) Independent variable
      b) Dependent variable
    2. Definition of range
      a) Single-valued function
      b) Multiple-valued function
      c) Bounded
      d) Infinite
      e) Finite
      f) Discrete
  C. Functional notation
    1. Even function

2. Odd function

D. Functions of more than one variable

E. Algebraic translation

1. Definition of locus

2. Definition of algebraic translation

a) Formation of a function

b) Change of variable in a function

The study of calculus is dependent upon the notion of the increment of a variable. This increment is symbolically represented by use of the Greek letter "Delta," Δ. In this section, many uses are made of the operator Δ.

II. Introduction to the increment

A. Linear function

1. Definition of linear function

2. Naming of linear function from graph

B. The operator Δ

1. Definition of the operator

2. The difference quotient

3. The increment of a variable

C. The use of the difference quotient in graphing

1. Slope of a line

2. Angle of inclination of a line

3. Equality of slope, difference quotient, and tangent of angle of inclination

4. Slope-intercept form of linear function

D. The distance formula

E. Increasing and decreasing linear functions

1. Definition of increasing function

2. Definition of decreasing function

F. The quadratic function

1. Definition of quadratic function

2. Graph of quadratic function

3. Tangent line

4. Difference between slope of secant line and tangent line from the same point

a) Slope of secant line approaches slope of tangent line

5. Definition of slope of a curve

G. The cubic function

1. Definition of cubic function

2. Graph of cubic function

3. Slope of cubic function

The concept of a limit (whether it be the limit of a sequence, a variable, or a function), and the associated concept of continuity are considerably too sophisticated for an analytic study in the high school calculus course. For this reason the notion of the limit will be discussed only in the most intuitive manner.

III. The Limit concept

A. Sequences

1. Definition of sequence

2. Limit of a sequence

3. The concept of infinity

B. Theorems on sequences

1. The limit of the product of a constant and a sequence is the product of the constant and the limit of the sequence.

2. The limit of the sum of two sequences is the sum of the limits of the sequences.

3. The limit of the difference of two sequences is the difference of the limits of the sequences.

4. The limit of the product of two sequences equals the product of the limits of the sequences.

5. The limit of the quotient of two sequences equals the quotient of the limits of the sequences, provided the limit of the divisor is not zero.

C. Functions

1. Limit of a variable

2. Limit of a function

3. Definition of a continuous function

D. Limit theorems on functions

1. The limit of the product of a constant and a function equals the product of the constant and the limit of the function.

2. The limit of the sum of two functions equals the sum of the limits of the functions.

3. The limit of the difference of two functions is the difference of the limits of the functions.

4. The limit of the product of two functions equals the product of the limits of the functions.

5. The limit of the quotient of two functions equals the quotient of the limits of the functions, provided the limit of the divisor is not zero.

E. Areas by summation

F. Volumes by summation

While many mathematicians (and the author) believe that the integral is the fundamental backbone of calculus, the discussion of the derivative and its applications will provide a more meaningful learning experience for the high school student, and should be studied prior to the discussion of the integral.

IV. The operations of calculus

A. The derivative

1. Definition of derivative

2. Geometric significance

3. Physical significance

4. Notations for derivatives

B. The Δ-process of differentiating

C. The derivative of x^n

1. The constant function

a) Theorem: The derivative of a constant function is zero.

b) Theorem: If a function has a derivative which is zero at each point of an interval, the function is constant on that interval.

D. The derivative of the algebraic polynomial

1. Linear combination of two functions

2. Definition of the differentiable function

3. Theorem:

$$\frac{d}{dx}cu = c\frac{du}{dx}$$

a) Corollary:

$$\frac{d}{dx}(-u) = -\frac{du}{dx}$$

4. Theorem:

$$\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}$$

a) Corollary:

$$\frac{d}{dx}(u_1 + \cdots + u_n) = \frac{du_1}{dx} + \cdots + \frac{du_n}{dx}$$

b) Corollary:

$$\frac{d}{dx}(u - v) = \frac{du}{dx} - \frac{dv}{dx}$$

E. Increasing and decreasing functions

1. Definition of increasing function

2. Definition of decreasing function

3. Theorem: If the derivative is positive throughout a given interval, the function is increasing there; if the derivative is negative, the function is decreasing.

4. Stationary point of a function

F. Elementary curve tracing

1. Sketch of the graph of a function by finding

a) Zero points of the function, if any

b) Stationary points of the function, if any

c) Intervals where the function increases

d) Intervals where the function decreases

G. Finding a function with a known derivative

1. Theorem: The most general function $F(x)$ which has $f(x)$ as its derivative is

$$F(x) = g(x) + c,$$

where $g(x)$ is any particular function such that $g'(x) = f(x)$ and where c is any constant.

2. Definition of antiderivative

3. Theorem: If $y' = ax^n$, where a and n are constants, and $n \neq -1$, then y has the form

$$y = \frac{ax^{n+1}}{n+1} + c,$$

where c is an arbitrary constant.

H. Rectilinear motion

1. Velocity as derivative of distance

2. Acceleration as derivative of velocity

3. Freely falling bodies

I. Areas by antidifferentiation

J. Differentiation of products and quotients

1. Theorem:

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

a) Corollary

$$\frac{d}{dx}(uvw) = uv\frac{dw}{dx} + uw\frac{dv}{dx} + vw\frac{du}{dx}$$

2. Theorem:

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

K. Derivatives of higher order

1. Definition of higher order derivatives

2. Notation for higher order derivatives

The more elementary applications of the derivative have been discussed in the previous section of the outline.

The following are more advanced applications that can be studied by the whole class for a better background, or by only the better students if time is limited.

V. Further applications of calculus

A. Maximum and minimum values

1. Definition of absolute maximum and minimum

2. Definition of relative maximum and minimum

3. Theorem: If the function f(x) has a relative maximum or minimum at a point and if the derivative exists at that point, it must be zero.

4. Theorem: A function has a relative maximum at a point if the derivative is positive to the left of the point and negative to the right of the point, and a relative minimum at that point if the derivative is negative to the left of the point and positive to the right.

B. Significance of the second derivative in finding maxima and minima

1. Definition of concave upward and concave downward

2. Theorem: The graph of a function is concave upward if the second derivative is positive and concave downward if the second derivative is negative.

3. Definition of a point of inflection

4. Theorem: If a point is a point of inflection of a curve, the second derivative is zero there.

5. Theorem: The function $y = f(x)$ will have a relative maximum at $x = x_0$ provided that

$$y' = 0, \quad y'' < 0, \text{ at } x = x_0$$

and a relative minimum at $x = x_0$ provided that

$$y' = 0, \quad y'' > 0, \text{ at } x = x_0.$$

C. Applications of maxima and minima

1. Theorem: If f(x) is defined and continuous in a closed interval, then there is at least one point in the interval at which the value f(x) is an absolute maximum and at least one point at which it is an absolute minimum.

An informal discussion of the integral has been covered in the discussion of areas and volumes by summation and antidifferentiation. These topics of discussion are the springboard to the formal discussion of the integral and will be related here.

VI. The integral

A. Properties of the integral

1. Definition of the integral as the limit of a sum

2. Definition of the integral as the general antiderivative

3. Notation for the integral

a) Use of the differential dx

B. Integral formulas

1. $\int k \, du = ku + c$

2. $\int u^n \, du = \frac{u^{n+1}}{n+1} + c$

C. The definite integral

1. Definition of the definite integral

a) Definition of the limits of integration

b) Definition of integrand

D. Properties of definite integrals

1. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

2. $\int_a^a f(x)\,dx = 0$

3. $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

4. $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$

5. $\int_a^b \left[f(x) + g(x)\right]\,dx = \int_a^b f(x)\,dx + \int_a^b f(x)\,dx$

E. Calculating definite integrals

1. Theorem: If $F'(x) = f(x)$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

F. Areas by integration

G. Volumes by integration

It is hardly necessary to say that frequent illustrative problems should be worked on the chalk board. Equally important is a good cross-sectional set of problems for the student to solve. These are the author's last two suggestions. From this point on, the content and method of the high school calculus course is left in the hands of the teacher. Only he knows what his students can manipulate and assimilate.

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APPENDIX

APPENDIX

The author unfortunately has had no opportunity to validate his proposed plan for teaching calculus in the high school. While it has been assumed that much of the content of this plan can be handled by high school students because of its similarity to successful experiments (cited in Chapter V), the first-hand opinions of high school teachers have been obtained for analysis in order to obtain validity by authority.

This was done in the following way: The outline of content contained in Chapter VII was distributed to forty-five junior and senior high school mathematics teachers attending a National Science Foundation Institute at the University of Minnesota, Duluth Branch during the summer of 1958. The purpose of this plan was explained to them, with emphasis placed on the fact that the author has conceived this calculus outline to be used either as the basis for a semester course in high school calculus, or as a source for appropriate topics in calculus that could be studied in any high school mathematics course where they might be found suitable.

These teachers were asked to read the outline and, so that the author could get a cross-section of opinion and obtain some sort of validity for his project, answer the following questions contained in a questionnaire:

1. At what level do you teach?
2. What subjects or grades do you teach?
3. How many years have you been teaching?
4. How many quarter-credits of mathematics have you taken in college?
5. What is the enrollment in your school?
6. Is there any special class in mathematics taught (or planned to be taught) in your school beyond the usual curriculum?
7. If so, of what does this course consist?
8. Do you think high school students could understand the material in the outline? Explain your answer.
9. Do you think that a semester course in calculus, using the topics in the outline, would be practical for the better high school mathematics student? Why?

The return from this sampling was only twenty, but this small group does seem to provide a definite, but hardly conclusive, trend of opinion. The facts collected from this survey can best be analyzed question by question.

<u>At what level do you teach</u>? Of the twenty teachers replying, four taught only in junior high schools (grades 7 to 9), and sixteen taught only in a senior high school (grades 10 to 12) or in a combination junior-senior high school.

What subjects or grades do you teach? The subjects taught by these teachers represented the full mathematics curriculum of the secondary school. A breakdown of the courses taught together with the number of teachers teaching them is included: seventh and eighth grade arithmetic or general mathematics, 5; elementary algebra, 10; plane geometry, 12; advanced algebra, 11; solid geometry, 7; trigonometry, 7. The seven teachers that taught solid geometry also taught trigonometry. Thus only thirty-five percent of the teachers responding have experience working with the calibre of student this proposed course is designed to aid.

How many years have you been teaching? The average number of years taught by these teachers was 12.2, with the experience ranging from four to twenty-five years. Those teachers that had taught the greatest number of years generally taught the more advanced courses.

How many quarter-credits of mathematics have you taken at college? The number of quarter-credits of college mathematics preparation averaged 39.9. The variance was from eighteen to sixty-four. This average represents about six quarter-credits over the minimum required of majors in many teacher-training institutions. Of the eight teachers with a preparation greater than the median, only three were teaching solid geometry and trigonometry. Experience, not preparation was the basis for selection of these teachers.

What is the enrollment of your school? The average enrollment in the schools represented by these teachers was 695 with a low of 125 and a high of 2000. All of the schools that were represented by affirmative answers to the next question had enrollments greater than the average.

Is there any special class in mathematics taught (or planned to be taught) in your school beyond the usual curriculum? If so, of what does this course consist? Of the twenty schools represented, ten offered no special mathematics courses of any type, four schools were planning programs to be offered in the near future, and six schools now offered some kind of special mathematics program for the better student. Of these six, one school provided for ability grouping in plane geometry only, assuming that grouping was a natural process for advanced algebra and trigonometry. Four schools offered enriched programs in all courses from plane geometry up to allow the better student to get "something extra" from the course. This plan was adopted in these schools because the mathematics staffs were too small to allow for any other plan. The sixth school taught plane and solid geometry in one year, advanced algebra and trigonometry were accelerated to less than the normal three semesters, the rest of the senior year spent studying modern algebra and statistics. It should be noted that this is the type of special program recommended by the author if one is to "find time" to teach some of the elements of calculus in the twelfth year.

Do you think high school students could understand the material in the outline? Explain. Three of the responding teachers felt that the outlined material was sufficiently easy for high school seniors, while four thought that no high school student could understand any of the elements of calculus. The remaining thirteen teachers estimated that only a portion of twelfth grade mathematics students could comprehend calculus, the estimates varying from five percent to thirty percent.

Do you think that a semester course in calculus, using topics in the outline, would be practical for the better high school mathematics student? Why? Thirteen teachers felt that calculus should be offered as an elective course in the twelfth grade, for the reason that it would better prepare a student for college mathematics. Four more teachers thought that calculus should not be offered as a separate course, citing the recommendations of the Commission on Mathematics of the College Entrance Examination Board as authority. The last three felt that the outline contained too much material for consumption in a semester course.

Thus the consensus of the group seems to be this: Calculus should be offered in the high school as an elective twelfth grade subject, with a content somewhat more restrictive than the outline in Chapter VII, but would be understood by only a fraction of the students in present twelfth grade mathematics courses. These seem to be much similar to the opinions reached by the author in the text of this paper.

It is the hope of the author that a number of enterprising secondary schools will show a desire to experiment in the teaching of calculus. Only with continued efforts along these lines can data be gathered to determine whether or not programs of advanced mathematics can adequately be taught on the pre-college level.