THE TVD-PROJECTION METHOD FOR SOLVING IMPLICIT NUMERICAL SCHEMES FOR SCALAR CONSERVATION LAWS: A NUMERICAL STUDY OF A SIMPLE CASE

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A. Bourgeat* and B. Cockburn**

Abstract

The stability of Newton's method applied to implicit numerical schemes for scalar conservation laws requires an upper bound on the size of the time steps. We introduce a simple locally defined projection, the TVD-projection, and show how to use it to obtain an always stable extension of Newton's method. A numerical study of this method applied to the Godunov's implicit scheme is presented.

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THE TVD-PROJECTION METHOD FOR SOLVING IMPLICIT NUMERICAL SCHEMES FOR SCALAR CONSERVATION LAWS: A NUMERICAL STUDY

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Introduction

In this paper we consider the problem of actually solving finite difference implicit schemes for the following boundary value problem

(1.1a) $\partial_t u + \partial_x f(u) = 0$ in $(0,T) \times (0,1)$,

(1.1b) $u(x=0) = b_0$ on $(0,T)$,

(1.1c) $u(x=1) = b_1$ on $(0,T)$,

(1.1d) $u(t=0) = u_0$ on $(0,1)$,

where $f \in C^1$, b_0 and $b_1 \in BV(0,T)$, and $u_0 \in BV(0,1)$. The space $BV(\Omega)$ is the space of functions of finite total variation in Ω. It is very well known that the main difficulty concerning the resolution of an implicit scheme is to solve at each time step the nonlinear equation defining the approximate solution. One of the most popular methods to do this is Newton's method. Its quadratic convergence makes it very attractive; however, its lack of global stability makes it useful only if a suitable initial guess can be obtained. A widely used practice is to take the approximate solution at time t^n , u_h^n , as the initial guess for the calculation of u_h^{n+1} :

$$u_h^{(n)} = u_h^n ,$$

$$u_h^{(m+1)} = \Phi_h^n (u_h^{(m)}) \qquad m=1,\dots, M^n .$$

But in this case the sufficient conditions for the convergence of this method given by the now clasical theorem of Kantorovich, see the reference in the book of Rall [7], impose an upper bound on the size of the time-step $\Delta t^n = t^{n+1} - t^n$.

In this way, the main advantage of using an implicit scheme is lost.

In practice, the number of inner iterations M^n is usually taken to be equal to 1. This is justified when the time step is small enough; see, for example, the analysis of this newtonian iterations given by Douglas [3] for the case of the long wave regularization implicit finite difference scheme for (1.1) developed by Douglas, Kendall and Wheeler [4]. This choice is also taken because doing M_h inner iterations with the time step $\Delta t = \Delta t^h$ cost as much as doing M^h time iterations of a single inner iteration with $\Delta t = \Delta t^h/M^h$; this strategy has been adopted, for example, for the implicit-explicit version of the PPM scheme, developed by Fryxell, Woodward, Colella and Winkler [5]. Other authors prefer to do a few inner iterations in order to improve the accuracy of the approximate solution; see, for example, the implicit version of Osher upwind scheme developed by Rai and Chakravarthy [9]. But it should be noted that in any case if the time step is too big the stability of the method is lost.

In this paper we introduce a new and very simple method for overcoming this difficulty. It consists in replacing the newtonian iterations by the following modification

$$\begin{aligned} u_h^{(1)} &= u_h^n \quad , \\ u_h^{(m+1)} &= P(u_h^{(m)})\,(\Phi_h^n)\,u_h^{(m)})) \quad , \quad m=1,\dots, M^n \quad , \end{aligned}$$

where Φ_h^n is the same as before, and the operator $P(v_h)$ is a locally defined projection verifying the following stability properties

$$\begin{aligned} \inf_x \; P(v_h)\, w_h &\geq \inf_x \; v_h \quad , \\ \sup_x \; P(v_h)\, w_h &\leq \sup_x \; v_h \quad , \\ \| P(v_h) w_h \|_{BV} &\leq \| v_h \|_{BV} \quad . \end{aligned}$$

Because of this last property the operator $P(\cdot)$ is called the TVD (total variation diminishing) projection. With this new procedure no oscillations due to the linearization leading to the definition of Φ_h^n appear anymore, and the method is always $L^\infty \cap BV$ - stable regardless of the size of the time step.

In this work we have tested numericaly the proposed method in the case of the Godunov's implicit scheme, the simplest and best known implicit finite difference scheme for (1.1).

An outline of the paper is as follows. In Section 2 Newton's method for solving the implicit Godunov's scheme for (1.1) is written down. In order to have an idea of how the stability properties of these Newtonian iterations depend on f' and f'' we compute explicitly the eigenvalues of grad Φ_h^n. Section 3 is devoted to test the stability of Newton's method in five different test problems. In Section 4 we introduce our TVD-projection, and we show how to modify the newtonian iterations in order to always have $L^\infty \cap BV$ - stability. In Section 5 we test this method in the same cases at which Newton's method was tested, and we compare the corresponding performances. We end in Section 6 with same concluding remarks.

2) Newton's Method for Godunov's implicit scheme

2.1) Introduction

In this Section we write down the Godunov's implicit scheme for (1.1), and its Newtonian iterations.

We want to stress the fact that the stability of Newton's iterations depends essentially on the size of $|f''|$ rather than the one of $|f'|$ as it is commonly believed. Indeed, if $f'' \equiv 0$ the Newtonian iterations are trivially stable regardless of the size of $|f'|$. Also, it is intuitively clear that the presence of a discontinuity represents a threat to the stability of the method. In Subsection 2.4) we make these statements precise by calculating explicitly the eigenvalues of the operator $\text{grad}\ \Phi_h^n$ in the simple, but illuminating, case in which $f' \geqslant 0$. Proof of the results therein are given in Subsection 2.5).

2.2) The scheme

As usual, let $\{t^n\}_0^{nt}$ be a partition of $[0,T]$, and $\{x_{i+1/2}\}_0^{nx}$ one of $[0,1]$. Let us set $\Delta t^n = t^{n+1} - t^n$, and $\Delta x_i = x_{i+1/2} - x_{i-1/2}$. We associate to $\{t^n\}_0^{nt}$ the space

$$(2.1a) \qquad W_{\Delta t} = \{v_{\Delta t} \in BV(0,T) : v_{\Delta t}(x) = v^n \ , \ t \in (t^n, t^{n+1}) \ , \ n=0,\dots, nt-1\} \ ,$$

and to the partition $\{x_{i+1/2}\}_0^{nx}$, the space

$$(2.1b) \qquad V_h = \{v_h \in BV(0,1) : v_h(x_{1/2}) = v_0 \ , \ v_h(x_{nx+1/2}) = v_{nx+1} \ ,$$
$$v_h(x) = v_i \ , \ x \in (x_{i-1/2}, x_{i+1/2}) \ , \ i=1,\dots,nx\} \ .$$

Define $b_{0\Delta t}$ and $b_{1\Delta t}$ to be the L^2 projection of b_0 and b_1, respectively, on the space $W_{\Delta t}$; and let u_{0h} be the L^2 projection of u_0 on V_h. Then, Godunov's implicit scheme defines an approximate solution $u_h \in W_{\Delta t} \times V_h$ in the following manner:

(2.2a) $u_i^0 = u_{0i}$, $i=1,\dots,nx$;

(2.2b) For $n=1,\dots, nt$:

$$u_0^n = b_0^n \quad ,$$

$$u_{nx+1}^n = b_1^n \quad ,$$

$$(u_i^n - u_i^{n-1}) \,/\, \Delta t^{n-1} + (f_{i+1/2}^{G,n} - f_{i-1/2}^{G,n}) \,/\, \Delta x_i = 0 \ ,$$

$$i=1,\dots, nx \ ;$$

where the Godunov flux $f_{i+1/2}^{G,n} = f^G(u_{i-1}^n , u_i^n)$ is defined by

(2.2c) $f^G(u,v) = f(\xi)$,

where ξ is any point of the interval $I(u,v) = [\min\{u,v\}, \max\{u,v\}]$ such that

(2.2d) $(\mathrm{sgn}(u-v)) \cdot (f(\xi) - f(c)) \leq 0$, $c \in I((u,v)$.

Note that when $f' > 0$ then $f_{i+1/2}^{G,n} = f(u_i^n)$, and when $f' < 0$, $f_{i+1/2}^{G,n} = f(u_{i+1}^n)$. Thus, f^G is a generalization of the classical upwinding flux. Also, among the fluxes of the form $\underset{\sim}{f}_{i+1/2} = \underset{\sim}{f}(u_{i+1}, u_i)$ it is the one that produces the smallest amount of viscosity. See Osher [6], and Brenier and Osher [1] for more details.

2.3) The Newtonian iterations

It is clear that when $f' \geq 0$ or $f' \leq 0$ the Godunov flux is C^1 , provided of course that $f \in C^1$. However, in the general case this is not true. For example, if $f(u) = u(1-u)$, then for $w > 1-w \neq 1/2$

$$\partial_1 f^{G+}(w,1-w) = f'(w) \quad ,$$

$$\partial_2 f^{G-}(w,1-w) = 0 \quad ,$$

and

$$\partial_1 f^{G+}(w,1-w) = 0 \quad ,$$

$$\partial_2 f^{G-}(w,1-w) = f'(1-w) \quad .$$

To define our Newtonian iterations we shall use the following definition for $\partial_1 f^G$, $\partial_2 f^G$:

(2.3a) $$\partial_1 f^G(u,v) = f'(u) \quad \text{if} \quad u = \xi \quad ,$$
$$= 0 \quad \text{otherwise} \quad ;$$

(2.3b) $$\partial_2 f^G(a,v) = f'(v) \quad \text{if} \quad v = \xi \quad ,$$
$$= 0 \quad \text{otherwise} \quad ,$$

where ξ is the same as before.

Thus, we define the operator Φ_h^n as follows:

(2.4a) $$\Phi_h^n : \begin{array}{ccc} V_h & \to & V_h \\ u_h^{(m)} & \to & u_h^{(m+1)} \end{array} \quad ,$$

where

(2.4b) $$u_0^{(m+1)} = b_0^n \quad ,$$

$$u_{nx-1}^{(m+1)} = b_1^n \quad ,$$

$$(u_i^{(m+1)} - u_i^{n-1}) / \Delta t^{n-1} + (f_{i+1/2}^{NG,m+1/2} - f_{i-1/2}^{NG,M+1/2}) / \Delta x_i = 0 \quad ,$$

$$\ell = 1,\dots, nx \quad ,$$

where the flux $f_{i+1/2}^{NG,m+1/2}$ is given by

$$(2.4c)\qquad f_{i+1/2}^{NG,m+1/2} = f^G(u_{i+1}^{(m)}, u_i^{(m)}) + (u_{i+1}^{(m+1)} - u_{i+1}^{(m)})\,\partial_1 f^G(u_{i+1}^{(m)}, u_0^{(m)}) + $$

$$+ (u_i^{(m+1)} - u_i^{(m)})\,\partial_2 f^G(u_{i-1}^{(m)}, u_i^{(m)}) \quad ,$$

$\partial_1 f^G$, $\partial_2 f^G$ being defined by (2.3). It is an easy matter to show that this operator is well defined for every partition $\{t^n\}_0^{nx} \times \{x_{i+1/2}\}_0^{nx}$ of $(0,T) \times (0,1)$; this is guaranteed by the fact that $f^G(u,v)$ is nonincreasing in u , and nondecreasing in v.

Finally, Newton's method can be written as follows:

$$(2.5a)\qquad \begin{aligned} u_0^{(1)} &= b_0^n \quad , \\ u_{nx+1}^{(1)} &= b_i^n \quad , \\ u_i^{(1)} &= u_i^{n-1}, \qquad i=1,\dots, nx \quad ; \end{aligned}$$

(2.5b) For $m=1,\dots,M^n$:

$$u_h^{(m+1)} = \Phi_h^n\,(u_h^{(m)}) \qquad .$$

2.4) The Eigenvalues of Φ_h^n for $f' > 0$.

The convergence of the sequence $\{u_h^{(m)}\}_{m\geq 1}$ generated by (2.5) with $M^n \equiv \infty$ depends on the size of the eigenvalues of Φ_h^n, λ_i. In particular, a sufficient condition for convergence is that the absolute values of λ_i are smaller than one. In what follow we obtain an explicit formula for the λ_i , in the case of $f' > 0$. We introduce the following notation:

$$res_h(w_h)_i = w_i - u_i^{n-1} + \frac{\Delta t^{n-1}}{\Delta x_i} (f(w_i) - f(w_{i-1})) ,$$

$$\gamma_{ij} = [\text{grad } res_h \ (w_h)]_{ij}^{-1} ,$$

$$del_h \ (w_h) = [\text{grad } res_h \ (w_h)]^{-1} \ res_h \ (w_h) .$$

Note that with this notation $\Phi_h^n \ (w_h) = w_h + del_h \ (w_h)$. We have thus the following result.

Proposition (2.1) The eigenvalues of the matrix grad $\Phi_h^n \ (w_h)$ are

$$\lambda_i = \frac{\frac{\Delta t^{n-1}}{\Delta x_i} f''(w_i)}{(1 + \frac{\Delta t^{n-1}}{\Delta x_i} f'(w_i))} \cdot del(w_h)_i , \qquad i=1,\ldots,nx . \tag{2.6}$$

This result shows that the role played by the size of the second derivate of f in the stability of (2.5) is essential. It also shows that more stability can be obtained if Δt^{n-1} is smaller. Let us point out that, as $\Phi_h^n(w_h) = w_h + del(w_h)$, the iterations (2.5) converges if and only if $\{del(u_h^{(m)})\}_{m\geq 1}$ converges to zero. As (2.6) shows, when (2.5) converges, the eigenvalues also converge to zero. This fact reflects the quadratic convergence of the method. It is difficult to take into account the dependence of $del(w_h)$ on Δt^{n-1} , f' and f'' , but a simple expression can be obtained in the important case in which w_h is a step function with a single discontinuity. We need first the following lemma.

Lemma (2.2). The matrix $\gamma = [\text{grad } res_h(w_h)]^{-1}$ is given by:

$$\gamma_{ij} = 0 \qquad \text{for } i < j ,$$

$$\gamma_{ii} = 1 - \theta_i$$

$$\gamma_{ij} = \theta_j \ldots \theta_{i-1}(1 - \theta_i) \frac{\Delta x_j}{\Delta x_i} \qquad \text{for } i > j ,$$

where

$$\Theta_i = \frac{(\Delta t^{n-1} / \Delta x_i) f'(w_i)}{1 + \frac{\Delta t^{n-1}}{\Delta x_i} f'(w_i)} \quad .$$

Now we can state the following result, that shows how the presence of a discontinuity influences the eigenvalues of grad Φ_h^n .

Corollary (2.3). Let w_h be the following step function:

$$\begin{aligned} w_i &= b_0^n \qquad , \quad 0 \leq i < i_0 \ , \\ &= b_1^n \qquad \quad i_0 \leq i \leq n_x \ . \end{aligned}$$

Then the eigenvalues of grad Φ_h^n are

$$\lambda_i = \frac{\frac{\Delta t^{n-1}}{\Delta x_i} \quad f''(w_i)}{(1 + \frac{\Delta t^{n-1}}{\Delta x_i} \quad f'(w_i))} \qquad \frac{\Delta t^{n-1}}{\Delta x_{i_0}} \ (\frac{f(b_1^n) - f(b_0^n)}{b_1^n - b_0^n} \qquad \gamma_{ii_0} \ (b_1^n - b_0^n) \ ,$$

where $\gamma_{ii_0} = \gamma_{ii_0} \ (b_0^n, b_i^n)$ is given by Lemma (2.2).

Note that for $\Delta x_i \equiv \Delta x$, $\lambda_{i+1} = \Theta \ \lambda_i$, $i > i_0$, where

$$\Theta = \frac{\frac{\Delta t^{n-1}}{\Delta x} \quad f'(b_i^n)}{(1 + \frac{\Delta t^{n-1}}{\Delta x} \quad f'(b_i^n))} < 1 \qquad ,$$

so, the influence of the discontinuity decreases geometrically with the number of elements to it! Note also that if $f'(b_1^n) = 0$ the only non-zero eigenvalue of grad Φ_h^n is

$$\lambda_{i_0} = \frac{\Delta t^{n-1}}{\Delta x_{i_0}} \ f'(b_1^n) \ \frac{\Delta t^{u-1}}{\Delta x_{i_0}} \ \frac{f(b_1^n) - f(b_0^n)}{b_1^n - b_0^n} \qquad (b_1^n - b_0^n) \qquad .$$

2.5) Proof of Proposition (2.1).

We need the following result.

Lemma (2.4). Let Φ_h^n be the operator defined by (2.4), then,

$$(\text{grad } \Phi_h^n(w_h))_{ij} = - \sum_{\ell,k=1}^{nx} \gamma_{ik} \left(\frac{\partial^2}{\partial w_\ell \partial w_j} \text{res}_h(w_h)_k\right) \text{del } (w_h)_\ell$$

where γ_{ij} is given by Lemma (2.2).

Proof. We use here the standard tensorial notation. By the definition of Φ and γ we have

$$\Phi_i = w_i - \gamma_{im} \text{res}_m \quad .$$

Thus,

$$(\text{grad } \Phi)_{ij} = \delta_{ij} - \gamma_{im,j} \text{res}_m - \gamma_{im} \text{res}_{m,j} \quad .$$

From the definition of γ we obtain easily the following expressions:

$$\gamma_{im} \text{res}_{m,j} = \delta_{ij} \quad ,$$

$$\gamma_{im,j} \text{res}_m = - \gamma_{ik} \text{res}_{k,\ell j} \gamma_{\ell m} \text{res}_m \quad ;$$

As $\text{del}_\ell = \gamma_{\ell m} \text{res}_m$, the result is obtained by simply inserting these experiences in the equality giving (grad Φ_h^n) .

Note that this result is valid regardless of the very form of $\text{res}_h (w_h)$. Now we turn to the proof of Proposition (2.1).

From Lemma (2.2) $\gamma_{ik} = 0$ if $i < k$. Thus, by Lemma (2.3)

$$(2.7a) \qquad (\text{grad } \Phi_h^n)_{ij} = - \sum_{k=1}^{i} \sum_{\ell=1}^{nx} \gamma_{ik} \left(\frac{\partial^2}{\partial w_\ell \partial w_j} \text{res}_k\right) \text{del}_\ell \quad .$$

But, from the definition of res_h :

$$\frac{\partial^2}{\partial w_\ell \partial w_j} res_k = \frac{\Delta t^{n-1}}{\Delta x_k} (f''(w_k)\ \delta_{kj}\delta_{k\ell} - f''(w_{k-1})\ \delta_{jk-1}\ \delta_{\ell k-1}) ,$$

and so, if $k < j$ this quantity is equal to zero.

This implies

$$(2.7b) \qquad (\text{grad } \Phi_h^n)_{ij} = - \sum_{k=j}^{nx} \sum_{\ell=1}^{nx} \gamma_{ik} \frac{\partial}{\partial w_\ell \partial w_j} res_h)\ del_\ell \quad .$$

From (2.7) it is now clear that for $i < j$ $(\text{grad } \Phi_h^n)_{ij} = 0$; i.e., the matrix grad Φ_h^n is lower triangular. This implies that is eigenvalues are

$$\begin{aligned} \lambda_i &= (\text{grad } \Phi_h^n)_{ii} \\ &= \sum_{\ell=1}^{nx} \gamma_{ii} \frac{\partial}{\partial w_\ell \partial w_i} res_i \ \ del_\ell \\ &= \gamma_{ii} \frac{\Delta t^{n-1}}{\Delta x_i} f''(w_i)\ del_i \quad . \end{aligned}$$

Finally, from Lemma (2.2), $\gamma_{ii} = (1 + \frac{\Delta t^{n-1}}{\Delta x_i} f'(w_i))^{-1}$, this completes the proof of Proposition (2.1).

3) Testing the stability of Newton's method

3.1) Introduction

In this Section we shall test numerically the stability of the Newtonian iterations (2.5) for (2.2) in five different problems of increasing difficulty. All of them are Riemann problems that have a self-similar solution easy to calculate. The first three of them have a non-decreasing nonlinearity; i.e. $f' \geqslant 0$. The last two have a non convex f, and their solution presents a sonic point; i.e., a point u^* for which $f'(u^*) = 0$. After displaying the test problems, we define a function that measures the stability of (2.2), (2.5), and we evaluate it numerically. We end by a brief discussion of the numerical results that will serve as a reference for evaluating the performance of our proposed method.

3.2) The test problems

The test problems are of the form (1.1) where

(3.1) $$f_i(u) = z_i u^3 \, (1-r_i(1-u)^3 \, (z_i u^3 + (1-u)^3)^{-1} \quad , \quad i=1,\dots,5 \; ,$$

(3.2a)	$u_{0i}(x) = 0$	$x \in (0,1)$	$i=1,2,3$	,
	$b_{0i}(t) = 1$	$t \in (0,T=0.3)$	$i=1,2,3$	,
	$b_{1i}(t) = 0$	$t \in (0,T=0.3)$	$i=1,2,3$	,
(3.2b)	$u_{0i}(x) = 0$	$x \in (0,0.6)$,	$i=4,5$	,
	$= 1$	$x \in (06,1)$	$i=4,5$	,
	$b_{0i}(t) = 0$	$t \in (0,T=0.05)$	$i=4,5$	,
	$b_{1i}(t) = 1$	$t \in (0,T=0.05)$	$i=4,5$	.

See figures (1),(2) and tables (1) and (2).

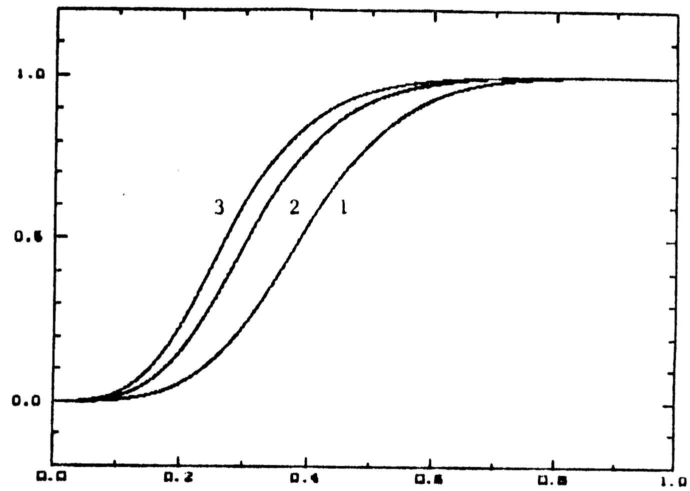


Figure (1.a) The functions $\mu \mapsto f_i(\mu)$, i=1,2,3.

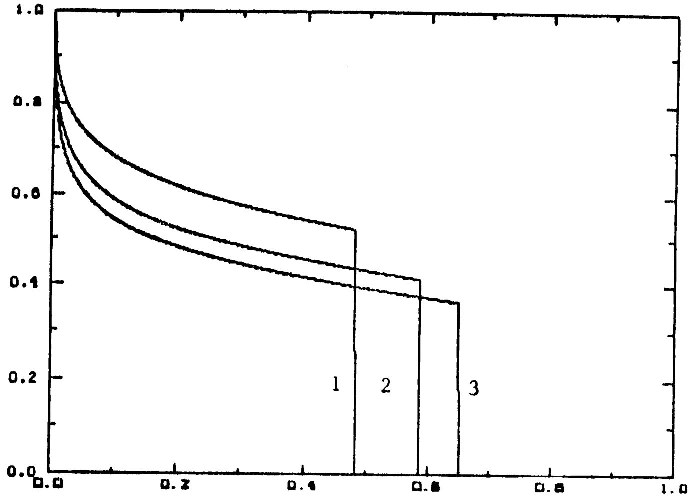


Figure (1.b). The solutions $x \mapsto u_i(T=0.3,x)$, i=1,2,3.

i	z_i	r_i	$\|f'_i\|_{L^\infty(0,1)}$	$\|f''_i\|_{L^\infty(0,1)}$	s_i
1	4	0	3.18	16.67	1.61
2	12	0	3.61	23.18	1.95
3	20	0	3.91	28.19	2.17

Table (1). The index i indicates the number of the test problem, and s_i is the speed of the discontinuity of the solution of problem i .

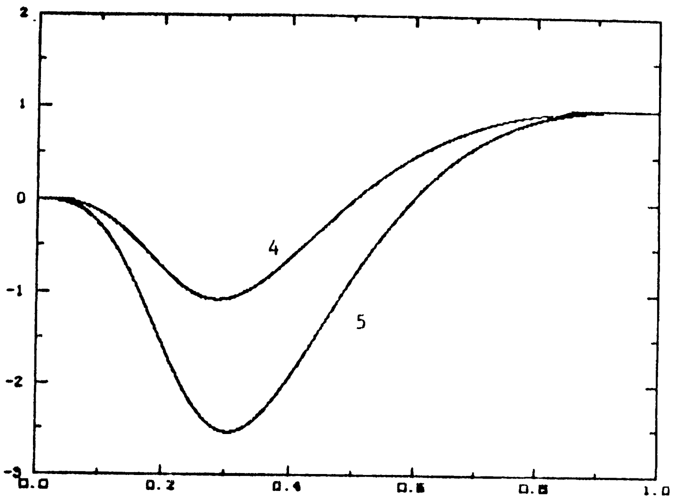


Figure (2.a) - the functions $u \mapsto f_i(u)$ $i=4,5$.

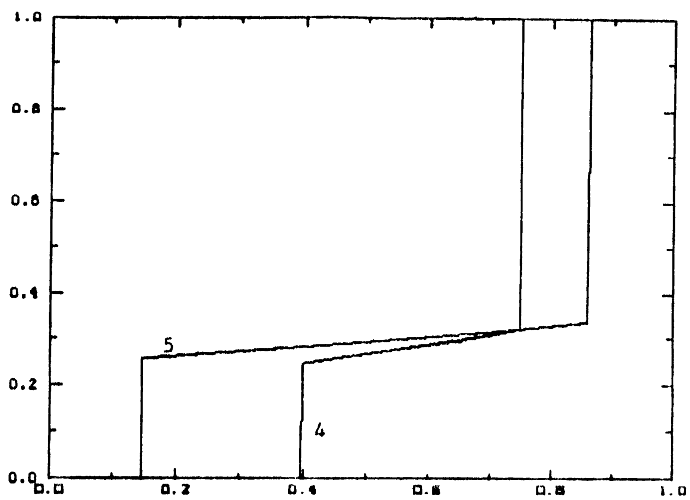


Figure (2.2b) - the solutions $x \mapsto u_i(T=0.05,x)$, $i=4,5$.

i	x_i	r_i	$\|f_i'\|_{L^\infty(0,1)}$	$\|f_i''\|_{L^\infty(0,1)}$	s_i^+	s_i^-
4	20	8	7.17	99.19	2.98	4.05
5	20	15	16.10	194.99	5.19	9.08

Table (2). As before i indicates the number of the problem; s_i^+ and s_i^- are the speeds of the discontinuities (of the solution of problem i) traveling to the right and to the left, respectively.

3.3) Measuring the stability (and convergence of the method)

To measure the stability properties of (2.2), (2.5) we proceed as follows. First, we fix Δx (all the grids will be taken uniform) small enough in order to ensure that the error already entered in the region of its asymptotic behavior. Then, we define the quantity λ as follows:

$$\lambda = \Delta x^{-1} \sup \{\Delta t_0 : \quad rc(\Delta t, \Delta x) > 1/2 \quad \forall \, \Delta t < \Delta t_0\} \tag{3.2a}$$

where

$$rc(\Delta t, \Delta x) = \frac{\log \left[\dfrac{\| u(T) - u_h(\Delta t, \Delta x; T) \|_{L^1(0,1)}}{\| u(T) - u_h(\frac{\Delta t}{2}, \frac{\Delta x}{2}; T) \|_{L^1(0,1)}} \right]}{\log 2} , \tag{3.2b}$$

is the estimated rate of convergence. We denoted by $u_h(\Delta t, \Delta x; T)$ the aproximate solution, obtained by (2.2), (2.5) with $\Delta t^n \equiv \Delta t$ and $\Delta x_{i+1/2} \equiv \Delta x$, at time T ; $u(T)$ is the exact solution at that time.

We recall the fact that $1/2$ is the more rate of convergence for the scheme (2.2). (This is also true for 3-point implicit monotone schemes; see Sanders [8]). Thus, we shall consider λ as a measure of the stability of the scheme. The bigger λ is, the more stable is the scheme. We shall take $\Delta x = 0.001$ in all the examples.

3.4) The numerical results.

For each of our five test problems we have calculated $\lambda(M^n \equiv 1, \equiv 2, \equiv 3)$ (see Table 3), the corresponding total number of time iterations nt(i.e., M^n times the numbers of time steps; see Table 4), the error $\| u(T) - u_h(T) \|_{L^1(0,1)}$ (see Table 5), and finally, the efficiency of the algorithm (Table 6) that is defined as follows:

(3.4a) $$\text{eff} = \{\|u(T) - u_h(T)\|_{L^1(0,1)} \cdot nt\}^{-1} .$$

test problem	$M^n \equiv 1$	$M^n \equiv 2$	$M^n \equiv 3$
1	0.617	1.327	1.240
2	0.507	0.811	0.777
3	0.457	0.694	0.682
4	0.250	0.595	0.342
5	0.110	0.221	0.160

Table 3. Values of λ defined by (3.2) for each of the test problems "i" , i=1 → and $M^n \equiv 1,2,3$ in M^n is the number of total inner iterations in the Newtonian iterations (2.5).

test problem	$M^n \equiv 1$	$M^n \equiv 2$	$M^n \equiv 3$
1	486	452	726
2	592	740	1158
3	656	864	1320
4	200	168	438
5	452	936	936

Table 4. Values of the total number of time iterations $nt = M^n \times$ number of time steps. Note that $nt = M^n \times 1000 \times T / \lambda$.

test problem	$M^n \equiv 1$	$M^n \equiv 2$	$M^n \equiv 3$
1	2.18	62.34	30.50
2	2.24	16.05	12.38
3	2.23	137.82	6.30
4	2.51	31.61	8.46
5	1.41	8.66	4.42

Table 5. Values of $10^3 \times \|u(T) - u_h(T)\|_{L^1(0,1)}$.

test problem	$M^n \equiv 1$	$M^n \equiv 2$	$M^n \equiv 3$
1	9.44	0.35	0.45
2	7.54	0.84	0.70
3	6.84	0.84	1.20
4	19.92	1.88	2.70
5	15.62	2.55	2.42

Table 6. Values of $10^4 \times$ eff, defined by (3.3).

From table (3) we see that the stability of (2.5) increases when passing from 1 inner iteration to 2, then it decreases when passing to 3 inner iterations. (For more than 3 inner iterations λ can possibly oscilate slightly around $\lambda(M^n \equiv 3)$ and then decrease slowly to an asymptotic value; we have not shown these results). It is also interesting to note that for a fixed value of M^n, the value of λ decreases monotonically with i, the number of our test problems. This corresponds to the increasing values of $\|f_i'\|_{L_\infty}$ and $\|f_i''\|_{L_\infty}$; see tables 1 and 2. Though it is not easy to find a simple relation between the values of λ and those of f', f'', table 6 seems to indicate that the following relationship is verified:

$$\lambda_i(M^n \equiv 1) \cdot \mathrm{Max}\{s_i^+, s_i^-\} = 1 \quad .$$

test problem	$\lambda_i(M^n \equiv 1) \cdot \mathrm{Max}\{s_i^+, s_i^-)$
1	0.993
2	0.988
3	0.991
4	1.010
5	0.999

Table 6. The cfℓ-number $\lambda_i(M^n \equiv 1) \cdot \mathrm{Max}\{s_i^+, s_i^-\}$.
For i=1,2,3 $\max\{s_i^+, s_i)$ is simply s_i.

This is rather surprising result for, on the one hand, it does not involve the values of f'', and on the other hand, it is not true when $f'' \equiv 0$. It indicates that the stability of the Newtonian iterations (2.5) with $M^n \equiv 1$ depends only on the cfℓ-number, exactly as if it were an explicit scheme!

Table 4 shows that the amount of work for $M^n \equiv 1,2,3$ is of the same order of magnitude. Tables 5 and 6 show that with $M^n \equiv 1$ not only the iterations (5) are more efficient but that they give raise to a better approximate solution.

4) The TVD-projection method

4.1) Introduction

In this section we define and study the stability properties of the so-called TVD (total variation diminishing) projection $P(w_h)$. Then, we write down our TVD - projection method and study its stability. We end by some remarks about the relationship the convergence and the conservativity of the method.

4.2) Definition of $P(w_h)$.

Let w_h and v_h be two arbitrary elements of V_h . To define $P(w_h)v_h$ we proceed as follows. First, to each point of the partition $\{x_{i+1/2}\}_0^{nx}$ we associate a value $\eta_{i+1/2}$ defined in the following way:

(4.1a) $$\eta_{i+1/2} = w_i$$
$$\text{if } \operatorname{sgn}(v_{i+1} - w_{i+1}) = \operatorname{sgn}(v_i - w_i) \neq \operatorname{sgn}(w_{i+1} - w_i) \quad ,$$

(4.1b) $$\eta_{i+1/2} = w_{i+1}$$
$$\text{if } \operatorname{sgn}(v_{i+1} - w_{i+1}) = \operatorname{sgn}(v_i - w_i) = \operatorname{sgn}(w_{i+1} - w_i) \quad ,$$

(4.1c) $$\eta_{i+1/2} = \theta_{i+1/2}\, w_i + (1-\theta_{i+1/2}) w_{i+1} \qquad \text{otherwise,}$$

where

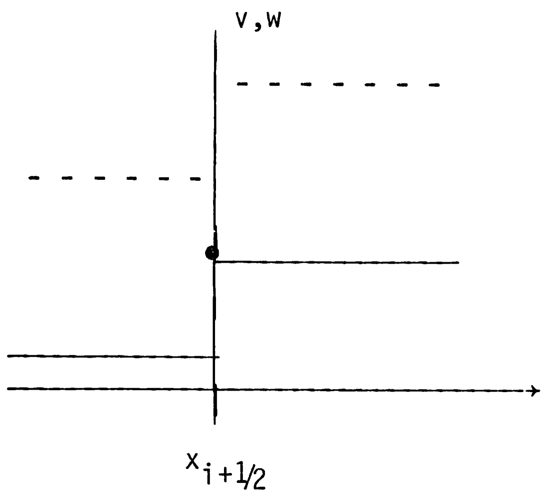
(4.1d) $$\theta_{i+1/2} = \frac{v_{i+1} - w_{i+1}}{(v_{i+1} - w_{i+1}) - v_i - w_i)} \quad .$$

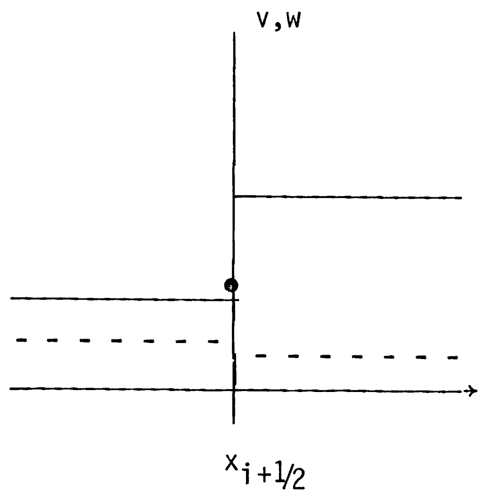
We also define $\eta_{-1/2}$ and $\eta_{nx+3/2}$ as follows:

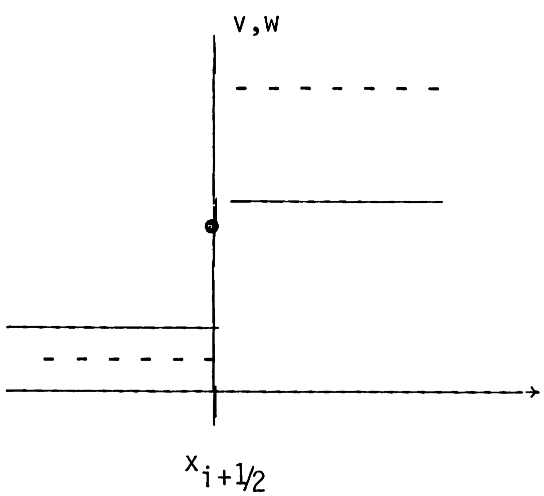
(4.1e) $$\eta_{-1/2} = w_0 \quad ,$$

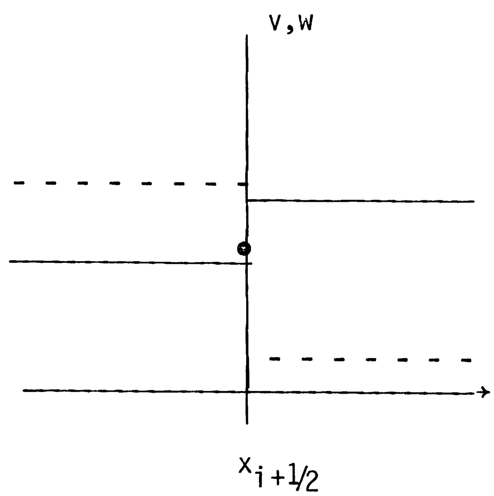
(4.1f) $$\eta_{nx+3/2} = w_{nx+1} \quad .$$

The definition of $\eta_{i+1/2}$ is illustrated in the figures that follow. In there, the continuous lines represent the values of the function w_h, and the dotted ones the values of the function v_h to be projected. The bid dot represents the value of $\eta_{i+1/2}$









Note that $\eta_{i+1/2}$ always lies in the interval $I(w_i, w_{i+1})$ regardless of the values v_i, v_{i+1}. We shall prove this later.

Next, we define the intervals I_i as follows:

$$I_i = I(w_i, \eta_{i+1/2}) \cup I(w_i, \eta_{i-1/2})$$

$$\text{if} \quad \text{sign}\,(w_i - \eta_{i+1/2})(w_i - \eta_{i-1/2}) < 0 \;,$$

$$= I(w_i, \eta_{i+1/2}) \cap I(w_i, \eta_{i-1/2})$$

$$\text{otherwise;} \qquad i = 0, \ldots, nx+1 \;.$$

Now, we introduce the L^∞-projection on the interval $[a,b]$:

$$(4.2) \qquad P^\infty_{[a,b]}(c) = \begin{cases} b & \text{if } c > b \;, \\ c & \text{if } c \in [a,b] \;, \\ a & \text{if } c < a \;. \end{cases}$$

Finally, we define the projection $P(w_h)\, v_h$ as follows:

$$(4.3a) \qquad P(w_h) : \begin{array}{l} V_h \to V_h \;, \\ v_h \to u_h \;, \end{array}$$

where

$$(4.3b) \qquad u_i = P^\infty_{I_i}(v_i) \qquad i=0, \ldots, nx+1 \;,$$

where the $\eta_{i+1/2}$ are defined by (4.1). Note that the operator $P(w_h)$ is a projection for it is easy to check that

$$P(w_h)\; P(w_h)\, v_h = P(w_h)\, v_h \;.$$

4.3) Stability properties of $P(w_h)$.

Let us begin with the following result.

Lemma (4.1). Set $u_h = P(w_h)\, v_h$. Then

(4.4a) $$u_i \quad \varepsilon \quad I_i \qquad\qquad i=0,\ldots, nx+1 \quad ,$$

(4.4b) $$\eta_{i+1/2} \quad \varepsilon \quad I(w_i \;, w_{i+1}) \qquad\qquad i=0,\ \ldots,\ nx \quad .$$

Proof. Property (4.4a) is a direct consequence of (4.2) and (4.3). The second is a consequence of the definition of $\eta_{i+1/2}$, (4.1).

We can state now our key result.

Proposition (4.2). Set $u_h = P(w_h)\, v_h$. Then,

(4.5a) $$\inf_x u_h \geq \inf_x w_h \quad ,$$

(4.5b) $$\operatorname{sub}_x u_h \leq \sup_x w_h \quad ,$$

and

(4.6) $$\|u_h\|_{BV} \leq \|w_h\|_{BV} \qquad .$$

Proof. Inequalities (4.5) are a corollary of (4.4). To prove (4.6) we proceed as follows:

$$\|u_h\|_{BV} = \sum_{i=0}^{nx} |u_{i+1} - u_i|$$

$$\leq \sum_{i=0}^{nx} (|u_{i+1} - \eta_{i+1/2}| + |\eta_{i+1/2} - u_i|)$$

$$= |\eta_{1/2} - u_0| + |\eta_{nx+1/2} - u_{nx+1}| +$$

$$\sum_{i=1}^{nx} (|\eta_{i+1/2} - u_i| + |u_i - \eta_{i-1/2}|) \quad .$$

But, from (4.1) $u_0 = \eta_{-1/2}$, $u_{nx+1} = \eta_{nx+3/2}$; and from (4.4a)

$$|\eta_{i+1/2} - u_i| + |u_i - \eta_{i-1/2}| \leq |\eta_{i+1/2} - w_i| + |w_i - \eta_{i-1/2}|$$

Thus,

$$\|u_h\|_{BV} = \leq \sum_{i=0}^{nx+1} (|\eta_{i+1/2} - w_i| + |w_i - \eta_{i-1/2}|)$$

$$= |w_0 - \eta_{-1/2}| + |\eta_{nx+3/2} - w_{nx+1}|$$

$$+ \sum_{i=1}^{nx+1} (|w_i - \eta_{i-1/2}| + |\eta_{i-1/2} - w_{i-1}|) \quad .$$

But again, from (4.1) $w_0 = \eta_{-1/2}$, $w_{hx+1} = \eta_{nx+3/2}$; and from (4.4b)

$$|w_i - \eta_{i-1/2}| + |\eta_{i-1/2} - w_{i-1}| = |w_i - w_{i-1}| \quad .$$

Thus,

$$\|u_h\|_{BV} \leq \sum_{i=1}^{nx+1} |w_i - w_{i-1}|$$

$$= \|w_h\|_{BV} \quad ,$$

This completes the proof.

4.4) The TVD-projection method

With the notation introduced previously, we can write our algorithm for the inner iterations of (2.2) in the following manner:

(4.7a) $$u_0^{(1)} = b_0^n \quad ,$$

$$u_{hx+1)}^{(1)} = b_1^n \quad ,$$

$$u_i^{(1)} = u_i^{n-1} \quad , \qquad \ell=1,\ldots,nx \quad ;$$

(4.7b) For $m = 1, \ldots, M^n$:

$$u_h^{(m+1)} = P(u_h^{(m)})(\Phi_h^n(u_h^{(m)})$$

Compare this with the Newtonian iterations (2.5). This method has the advantage of being stable regardless the characteristics of the partitions $\{t^h\}_0^{nt}$ and $\{x_{i+1/2}\}_0^{nx}$. Moreover, it also guarantees the compactness of the sequence $\{u_h\}_{h>0}$ determined by (2.2), (4.7) in the $L^\infty(0,T;L^1(0,1))$ - norm. This is a direct consequence of the following result.

Theorem (4.3). Let u_h be the approximate solution obtained by (2.2) and (4.7). Then,

$$\inf u_h \geq \inf_{t,x} [b_0(t), b_1(t), u_0(x)] ,$$

$$\sup u_h \leq \sup_{t,x} [b_0(t), b_1(t), u_0(x)] ,$$

$$\|u_h\|_{L^\infty(0,t;BV(0,1))} \leq \|b_0\|_{BV(0,T)} + \|b_1\|_{BV(0,T)} + \|u_h\|_{BV(0,1)} .$$

The proof is straightforward: it is a direct consequence of Proposition (4.2) and the definition of $b_{0\Delta t}, b_{1\Delta t}, u_{0h}$. Thus, the sequence $\{u_h\}_{h>0}$ has a subsequence converging strongly in $L^\infty(0,T;L^1(0,1))$ to a limit. The problem of proving that this limit is indeed the entropy solution of (1.1) is very delicate and will not be considered here.

4.5) The price of stability

From the definition of the Newtonian iterations (2.4), (2.5) we obtain easily the following expression for the conservations of mass:

$$\sum_{i=1}^{nx} u_i^{(m+1)} \Delta x_i = \sum_{i=1}^{nx} u_i^{n-1} \Delta x_i -$$

$$- \Delta t^{n-1} (f_{nx+1/2}^{NG,m+1/2} - f_{1/2}^{NG,m+1/2}) ,$$

where $u_h^{(m+1)} = \Phi_h^n (u_h^{(m)})$. On the other hand, by (2.2) we have for the approximate solution of the Godunov scheme

$$\sum_{i=1}^{nx} u_i^n \Delta x_i = \sum_{i=1}^{nx} u_i^{n-1} \Delta x_i -$$

$$- \Delta t^{n-1} (f_{nx+1/2}^{G,n} - f_{i/2}^{G,n}) ,$$

Thus, the error in the total amount of mass at the time step (t^{n-1},t^n) introduced by Newton's method is simply

$$\text{(4.8a)} \qquad - \Delta t^{n-1} ((f_{nx+1/2}^{G,n} - f_{nx+1/2}^{NG,m+1/2}) - (f_{1/2}^{G,n} - f_{1/2}^{NG,m+1/2})) ,$$

with $m = M^h$.

In our method the error in the total amount of mass has a rather different form, for it cannot be written in conservation form like the Newtonian iterations (2.4), (2.5). Set $w_h = \Phi_h^n (u_h^{(m)})$, and $u_h^{(m+1)} = P(u_h^{(m)}) w_h$, then

$$\sum_{i=1}^{nx} u_i^{(m+1)} \Delta x_i = \sum_{i=1}^{nx} w_i \Delta x_i + \sum_{i=1}^{nx} (u_i^{(m+1)} - w_i) \Delta x_i$$

$$= \sum_{i=1}^{nx} u_i^{n-1} \Delta x +$$

$$- \Delta t^{n-1} (f_{nx+1/2}^{NG,m+1/2} - f_{1/2}^{NG,m+1/2}) +$$

$$+ \sum_{i=1}^{nx} (u_i^{(m+1)} - w_i) \Delta x_i ,$$

where, of course $f^{NG} = f^{NG}(u_h^{(m)}, w_h)$. This implies that the error in the total amount of mass produced by our method is the sum of a term similar to (4.8a) and the following one

$$(4.8b) \qquad \sum_{i=1}^{nx} (u_i^{(m+1)} - w_i)\ \Delta x_i \ = \ \int_0^1 (\ P(u_h^{(m)}) - Id)\ w_h(x)dx \quad .$$

The term (4.8a) is due to the linearization procedure, and the term (4.8b) is only due to the projection $P(u_h^{(m)})$. In other words, the price that we have to pay for stability is an increase of the loss of mass!

Nevertheless, we have to point out that if $w_h = u_h^{(m)}$ then $P(u_h^{(m)})\ u_h^{(m)} = u_h^{(m)}$ and the term (4.8b) is equal to zero. Thus, if del $(u_h^{(m)}) = \Phi_h^n\ (u_h^{(m)}) - u_h^{(m)} = w_h - u_h^{(m)}$ is small enough then $P(u_h^{(m)})\ w_h = w_h$ and (4.8b) vanishes. If this is the case, the inner iteration of our method reduces simply to a Newtonian one. In fact, this happens when Δt^{n-1} is small enough. Thus, our method can be considered as an extension of Newton's method. When Δt^{n-1} is big, the term (4.8b) is not anymore equal to zero, and indeed it can be very big. If our method converges we expect the term (4.8b) to go to zero with the number of iterations m . Our numerical experience indicates that this is always the case, and that once (4.8b) becomes zero it stays equal to zero, so that the method coincides with Newton's method after a certain number of iterations. Once this happens, convergence is achieved very quickly, but it is extremely inefficient to ask the term (4.8b) to be equal to zero in order to stop the iterations. It suffices to stop the inner interations when (4.8b) is "small enough". We consider this problem in next Section.

5) Testing the stability of the TVD-projection method

5.1) Introduction

In this Section we test numerically the stability properties of our method on the same six test problems on which the method (2.2), (2.5) was already tested. Our stopping criterion is based on a control of the amount of mass lost by the use of our projection. After defining it we calculate the quantities λ defined by (3.2), and we compare them with the λ obtained for Newton's method. The same Δx has been taken.

5.2) The stopping criterion

To define a stopping criterion for (4.7) is to define M^n. We define such a quantity as follows:

$$M^n = \min \{m : LM(u_h^{(m)}) > \varepsilon \} \quad , \tag{5.1a}$$

where the loss of mass LM is defined by

$$LM(u_h^{(m)}) = \sum_{i=1}^{nx} |(P(u_h^{(m)}) - I_d) \, \Phi_h^n(m_h^{(m)}))_i | \Delta x_i \quad . \tag{5.1b}$$

In our numerical experiences we shall take ε as follows:

$$\varepsilon = \Delta t \cdot (\Delta x)^{j/2} \qquad j = 1,2,3,4 \, . \tag{5.2}$$

Roughly speaking at each time step we are allowing ourselves to lose an amount of mass equal to ε ; thus, in nt iterations we allowed ourselves to lose a total amount of mass equal to $T \cdot (\Delta x)^{j/2}$. Taking into account that the error behaves like $C_0(\Delta t)^\alpha$ for $\alpha \in (0.5,1)$, and that $\Delta t >> \Delta x$, we think that this is a reasonable criterion.

5.3) The numerical results

Our numerical results are shown in the tables 8 to 11.

test problem	j = 1	j = 2	j = 3	j = 4
1	75	75	150	150
2	50	75	150	150
3	50	50	75	75
4	25	25	25	25
5	25	25	25	25

Table 8. Values of λ for the TVD-projection method. The value j is associated to the stopping criterion (5.1) by (5.2).

test problem	j = 1	j = 2	j = 3	j = 4
1	583	618	727	728
2	648	769	926	927
3	761	780	889	889
4	196	394	645	645
5	365	369	1465	1465

Table 9. The total number of iterations nt:
$$nt = \sum_{\text{time steps}} (\text{number of inner iterations}).$$

test problem	j = 1	j = 2	j = 3	j = 4
1	64.29	71.75	115.69	115.69
2	50.43	73.71	122.14	122.14
3	50.16	48.33	80.06	80.06
4	38.54	55.26	49.69	49.69
5	96.95	96.42	93.00	93.00

Table 10. The values of $10^3 \, \|u(T) - u_h(T)\|_{L^1(0,1)}$

test problem	j = 1	j = 2	j = 3	j = 4
1	0.27	0.23	0.12	0.12
2	0.31	0.18	0.09	0.09
3	0.26	0.27	0.14	0.14
4	1.32	0.46	0.31	0.31
5	0.28	0.28	0.07	0.07

Table 11. The values of 10^4 eff .

Note that the the minimum value for the number of time steps is 2, see the definition of λ (3.2), and so we have:

$$\lambda_{max} = \frac{T}{2} \cdot 1000 = 500\ T \quad .$$

As $T = 0.3$ for problems 1,2,3 and $T = 0.05$ for problems 4,5 the maximum value for λ is 150 for problems 1,2,3 and 25 for problems 4,5.

From these results we can conclude that the stopping criterion (5.1) works well, and works better for $j = 1$; see the errors and efficiency in Tables 10 and 11, respectively. Note that the results obtained with $j = 3$ and 4 are essentially the same.

By comparing the values of λ^N for the Newtonian iterations (2.5), table 3, and those for our method, table 8, we see that our λ are at least two orders of magnitude bigger than those of the Newtonian iteration. Thus, we can say that the use of the TVD-projection enhances enormously the stability of Newton's iteration.

6) Conclusion

In this paper we have introduced a very simple and locally defined projection, the TVD (total variation diminishing) - projection, and we have shown how to use it to define an always-stable extension of Newton's method, applied to solve implicit schemes for the scalar conservation law (1.1). After proving that the TVD-projection makes the inner iterations $L^\infty \cap BV$ - stable, we tested our method on Godunov's implicit scheme (1.1). Our numerical results show that the time step sizes allowed by our method are at least two orders of magnitude bigger than those allowed by the standard newtonian iterations.

As first-order schemes are not very efficient, we are planning to apply this technique for solving more accurate implicit schemes (...with variable space-discretization). In particular, we are interested in solving the implicit finite element quasi-monotones schemes introduced recently by one of the authors; see [2]. These schemes are TVDM (total variation diminishing in the means), and seem to be a suitable application of our TVD-projection method.

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