Dynamics of Quasi-Geostrophic Fluid Motions with Rapidly Oscillating Coriolis Force

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Abstract

An averaging principle for quasi-geostrophic fluid motions with rapidly oscillating Coriolis force is proved. This result includes comparison estimate and convergence result between quasi-geostrophic fluid motions and its averaged fluid motions. This averaging principle provides an autonomous system as an approximation for the nonautonomous quasi-geostrophic flows with rapidly oscillating Coriolis force.

Key words: Quasi-geostrophic fluid flows, rapidly oscillating forcing, averaging principle

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1 Introduction

Geophysical flows involve multiple scales in both space and time. The quasi-geostrophic (QG) equation models large scale geophysical flows. It is derived as an approximation of the rotating Navier-Stokes equations by an asymptotic expansion in a small Rossby number. The QG equation is written in terms of the stream function $\psi(x, y, t)$ as in [1]:

$$\triangle\psi_t + J(\psi, \triangle\psi) + (\beta + \alpha(x, y, t))\psi_x = \nu\triangle^2\psi - r\triangle\psi + f(x, y), \tag{1.1}$$

where $\beta > 0$ is the meridional gradient of the Coriolis parameter, $\nu > 0$ is the viscosity, $r > 0$ is the Ekman dissipation constant, $f(x, y)$ is the wind forcing, and $J(h, g) = h_x g_y - h_y g_x$ is the Jacobian operator. Moreover, $\alpha(x, y, t)$ is the fluctuating component of the Coriolis force or the β parameter. This component is usually fast in time. There are a few sources for this fast component [29, 30, 31]: fluctuations in the gravitational force between the Sun and the Earth; fluctuations in the geomagnetic field of the Earth; and fluctuations in the Earth's environment (atmosphere, oceans and land). These fluctuations further affect the Earth's rotation and thus affect the Coriolis force.

The Equation (1.1) can be rewritten in terms of the relative vorticity $\omega = \triangle\psi$ as

$$\omega_t + J(\psi, \omega) + (\beta + \alpha(x, y, t))\psi_x = \nu\triangle\omega - r\omega + f(x, y) \tag{1.2}$$

on an arbitrary bounded planar domain D with sufficiently regular (such as, piecewise smooth) boundary ∂D. This equation is supplemented with homogeneous Dirichlet boundary conditions for both ψ and ω, namely, the no-penetration and free-slip boundary conditions proposed by Pedlosky [2], p. 34 (see also [27]):

$$\psi = \omega = 0, \text{ on } \partial D, \tag{1.3}$$

together with an appropriate initial condition,

$$\omega(x, y, 0) = \omega_0(x, y), \text{ in } D. \tag{1.4}$$

The global well-posedness (i.e., existence and uniqueness of smooth solution) of the dissipative model (1.2)-(1.4) can be obtained similarly as in, for example, [3], [4], [5] or [6]. Most works on this fluid model are for the case of $\alpha = 0$ (constant β parameter). Brannan et al [8] considered the effect of quasi-geostrophic dynamics under random forcing. Duan et al [9] and [10] obtained the existence of time periodic, time almost periodic quasi-geostrophic response, under time periodic and time almost periodic wind forcing, respectively.

We assume that fluctuating Coriolis force term $\alpha(x, y, t)$ of the QG flow model (1.1) is rapidly oscillating, i.e., it has the form $\alpha(x, y, t) = \alpha(x, y, \eta t) \stackrel{\triangle}{=} \alpha(\eta t)$, with parameter $\eta \gg 1$. We also assume that α has a time average in a sense to be specified later. With such Coriolis force, it is desirable to understand the fluid dynamics in some averaged sense, and compare the averaged flows with the original un-averaged flows.

Starting from the fundamental work of Bogolyubov [11] the averaging theory for ordinary differential equations has been developed and generalized by many authors; see [12]–[14] and the references therein. Bogolyubov's main theorems have been further generalized in [15] to the case of differential equations with bounded operator-valued coefficients. Some problems of averaging of differential equations with unbounded operator-valued coefficients have been considered in [16]–[19] in the framework of abstract parabolic equations. More recent works on averaging are by Ilyin [20, 21]. See [28] for a survey on averaging principles of partial differential equations.

The main result of this paper is an averaging principle for quasi-geostrophic motions with rapidly oscillating Coriolis force. This includes comparison estimate, stability estimate, and convergence result (as $\eta \to \infty$) between quasi-geostrophic fluid motions and its averaged fluid motions.

2 Averaging Principle for Quasi-Geostrophic Flows

In this section, we consider the averaging principle for the QG flows under fluctuating Coriolis force $\alpha(x, y, t) = \alpha(x, y, \eta t) := \alpha(\eta t)$, with parameter $\eta \gg 1$.

First, we provide some preliminaries for later use. Standard abbreviations $L^2 = L^2(D)$, $H_0^k = H_0^k(D), k = 1, 2...$, are used for the common Sobolev spaces, with $< \cdot, \cdot >$ and $\| \cdot \|$ denoting the usual scalar product and norm, respectively, in L^2. We need the following properties and estimates (see [4]) of the Jacobian operator $J : H_0^1 \times H_0^1 \to L^1$:

$$\int_D J(f, g)hdxdy = -\int_D J(f, h)gdxdy, \int_D J(f, g)gdxdy = 0, \tag{2.1}$$

$$|\int_D J(f, g)dxdy| \leq \|\nabla f\| \|\nabla g\|, \tag{2.2}$$

for all $f, g, h \in H_0^1$. We also recall the *Poincaré inequality* [22] for $g \in H_0^1$

$$\|g\|^2 \leq \frac{|D|}{\pi} \int_D |\nabla g|^2 dxdy = \frac{|D|}{\pi} \|\nabla g\|^2, \tag{2.3}$$

with $|D|$ the area of the domain D, and the *Young's inequality* [22] for nonnegative numbers A, B

$$AB \leq \frac{\epsilon}{2} A^2 + \frac{1}{2\epsilon} B^2, \tag{2.4}$$

with ϵ positive.

We further rewrite the QG flow model (1.2). From

$$\triangle \varphi = \omega, \quad (x, y) \in D, \quad \varphi|_{\partial D} = 0, \tag{2.5}$$

we get $\varphi = \triangle^{-1} \omega$. Thus (1.2) can be rewritten as

$$\omega_t + J(\triangle^{-1}\omega, \omega) + \beta \partial_x \triangle^{-1} \omega + \alpha(x, y, t) \partial_x \triangle^{-1} \omega = \nu \triangle \omega - r\omega + f(x, y). \tag{2.6}$$

Let

$$-\mathcal{A} = \nu \triangle - rI - \beta \partial_x \triangle^{-1}.$$

Then by a result in [16], we know that $\mathcal{A}$ is a sectorial operator, and hence $e^{-\mathcal{A}t}$ generates an analytic semigroup in L^2.

We will give a sufficient condition to ensure the smallest eigenvalue of $\mathcal{A}$ to be positive. Consider the eigenvalue equation $\mathcal{A}u = \lambda u$. We have the following energy estimate

$$\begin{aligned}
\lambda \|u||^2 &= \nu \|\nabla u\|^2 + r\|u\|^2 - \int_D \triangle^{-1} u \partial_x u dxdy \\
&\geq \nu \|u\|^2 + r\|u\|^2 - \beta \|\triangle^{-1} u\| \|\nabla u\| \\
&\geq \nu \|\nabla u\|^2 + r\|u\|^2 - \frac{\beta |D|}{\pi} \|u\| \|\nabla u\| \\
&\geq \nu \|\nabla u\|^2 + r\|u\|^2 - \frac{\beta |D|}{\pi} (a_1 \|u\|^2 + a_2 \|\nabla u\|^2) \\
&\geq (\nu - \frac{\beta |D|}{\pi} a_1) \|\nabla u\|^2 + (r - \frac{\beta |D|}{\pi} a_2) \|u\|^2,
\end{aligned} \tag{2.7}$$

where the Poincaré inequality (2.3) is used and where arbitrary positive constants a_1, a_2 satisfy $a_1 a_2 = \frac{1}{4}$. Therefore, when

$$4\nu r > \frac{\beta^2 |D|^2}{\pi^2}, \tag{2.8}$$

we could choose a_1 and a_2 such that the coefficients in (2.7) are positive, then we have

$$\lambda\|u||^2 \geq (\nu - \frac{\beta|D|}{\pi}a_1)\|\nabla u\|^2 + (r - \frac{\beta|D|}{\pi}a_2)\|u\|^2 \geq C\|u\|^2, \qquad (2.9)$$

where $C > 0$ is a constant depending on ν, r, β and $|D|$. So, when ν, r, β and $|D|$ satisfy the condition (2.8), the smallest eigenvalue λ_1 of $\mathcal{A}$ is positive. In this case, the QG flow model

$$\omega_t + \mathcal{A}\omega + J(\triangle^{-1}\omega, \omega) + \alpha(x, y, t)\partial_x\triangle^{-1}\omega = f(x, y)$$

is a dissipative dynamical system [24].

Now, we define the fractional power of $\mathcal{A}$ as follows [16] for $0 < \gamma < 1$:

$$\mathcal{A}^\gamma = (\mathcal{A}^{-\gamma})^{-1}, \text{ where } \mathcal{A}^{-\gamma} = \frac{1}{\Gamma(\gamma)}\int_0^\infty t^{\gamma-1}e^{-\mathcal{A}t}dt.$$

The corresponding domain $D(\mathcal{A}^\gamma)$ is a Banach space with the norm defined by

$$\|x\|_\gamma := \|x\|_{D(\mathcal{A}^\gamma)} = \|\mathcal{A}^\gamma x\|.$$

Now we recall some definitions and useful results for later use.

Proposition 2.1 *[16] Suppose $\mathcal{A}$ is sectorial and its spectrum $\sigma(\mathcal{A})$ has positive real parts: Re $\sigma(\mathcal{A}) > a > 0$. Then the following estimates are valid for $b > 0$*

$$\|e^{-\mathcal{A}t}\|_{L^2\to L^2} \leq Ke^{-at}, \qquad t \geq 0, \qquad (2.10)$$

$$\|\mathcal{A}^b e^{-\mathcal{A}t}\|_{L^2\to L^2} \leq \frac{K_b}{t^b}e^{-at}, \quad t > 0, \qquad (2.11)$$

where K, K_b are positive constants.

Proposition 2.2 *[16] Given two sectorial operators A and B in L^2,with $D(A) = D(B)$, Re $\sigma(A) > 0$ and Re$\sigma(B) > 0$. Let the operator $(A - B)A^{-a}$ be bounded in L^2 for some $a \in [0, 1)$. Then for every $\gamma \in [0, 1)$, we have $D(A^\gamma) = D(B^\gamma)$, and the corresponding norms in $D(A^\gamma)$ and $D(B^\gamma)$ are equivalent.*

We now turn to the averaging principle for the QG flow model.

Let $\eta \gg 1$ be a large dimensionless parameter. Setting

$$\tau = \eta t, \;\; \epsilon = \eta^{-1},$$

we obtain the equation in the so-called standard form

$$\omega_\tau + \epsilon \mathcal{A}\omega + \epsilon J(\triangle^{-1}\omega, \omega) = -\epsilon\alpha(x, y, t)\partial_x \triangle^{-1}\omega + \epsilon f(x, y). \tag{2.12}$$

We assume that α has a time average in $H = L^2(D)$. More precisely, let $\alpha(\tau), \alpha_0 \in H$ and suppose that

$$\frac{1}{T}\int_t^{t+T} \|\alpha(\tau)d\tau - \alpha_0\| \leq \min(M_0, \sigma_0(T)), \tag{2.13}$$

where M_0 is positive constant, $\sigma_0(T) \to 0, \;$ as $T \to \infty$.

Remark: We note that in the situation here we can not use the method of [20, 21] directly.

Now, we consider the averaged equation

$$\bar{\omega}_\tau + \epsilon \mathcal{A}\bar{\omega} + \epsilon J(\triangle^{-1}\bar{\omega}, \bar{\omega}) = -\epsilon\alpha_0(x, y)\partial_x \triangle^{-1}\bar{\omega} + \epsilon f(x, y). \tag{2.14}$$

Next, we give the existence of absorbing set (see the following lemma) for (2.12) with initial value ω_0 in the space $H = L^2, V = D(\mathcal{A}^{\frac{1}{2}}) = H_0^1$. The existence of absorbing set for (2.14) with initial value $\bar{\omega}_0$ is similar. The definition of an absorbing set is in, for example, [23, 24, 25]. These sets are certain balls $B(R_0)$ with radius R_0 large enough. This means that for every bounded set B

$$S_t B \subset B(R_0), \; \text{for } t > t_0(B, R_0).$$

In addition, the semigroup is uniformly bounded in these spaces, that is, given any ball, in particular, the ball $B(R_0)$, there exists a ball $B(R)$ such that

$$S_t B(R_0) \subset B(R), \; \text{for } t > 0.$$

By increasing R we may assume that

$$S_t B(R_0) \subset B(R - \rho), \; \text{for } t > 0, \rho > 0,$$

where ρ is a positive constant.

Lemma 2.3 *Let $\omega_0 \in H^1(D)$, $f \in L^\infty(0, \infty, L^2(D))$, and let the dissipativity condition (2.8) be satisfied and $\frac{\lambda_1}{2M^2} > \|\alpha\|^2$, where λ_1 is the smallest eigenvalue of $\mathcal{A}$ and M is the*

Sobolev embedding constant in $\|\partial_x \triangle^{-1}\omega\|_{L^\infty} \leq M\|\mathcal{A}^{\frac{1}{2}}\omega\|$. *Then we have the following estimate for the solution of (2.12) with initial value* ω_0*:*

$$\|\mathcal{A}^{\frac{1}{2}}\omega\| \leq R_0, \ \textit{for} \ \ t \geq T_1(\omega_0),$$

where $T_1(\omega_0)$ *is some positive function of* ω_0*.*

PROOF. Multiplying (2.12) by ω, integrating over D and integrating by parts, we get

$$\frac{1}{2}\frac{d}{dt}\|\omega\|^2 + \|\mathcal{A}^{\frac{1}{2}}\omega\|^2 + \int_D \alpha\partial_x\triangle^{-1}\omega\omega = \int_D f\omega.$$

By Cauchy inequality, Soblove inequality and **Proposition 2.2**, we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\omega\|^2 + \|\mathcal{A}^{\frac{1}{2}}\omega\|^2 &\leq \|f\|\|\omega\| + \|\partial_x\triangle^{-1}\omega\|_{L^\infty}\|\alpha\|\|\omega\| \\ &\leq \|f\|\|\omega\| + M\|\mathcal{A}^{\frac{1}{2}}\omega\|\|\alpha\|\|\omega\|. \end{aligned} \tag{2.15}$$

So,

$$\frac{d}{dt}\|\omega\|^2 + \|\mathcal{A}^{\frac{1}{2}}\omega\|^2 + \lambda_1\|\omega\|^2 \leq \frac{\lambda_1}{2}\|\omega\|^2 + \frac{2}{\lambda_1}\|f\|^2 + \|\mathcal{A}^{\frac{1}{2}}\omega\|^2 + M^2\|\alpha\|^2\|\omega\|^2,$$

that is

$$\frac{d}{dt}\|\omega\|^2 + (\frac{\lambda_1}{2} - M^2\|\alpha\|^2)\|\omega\|^2 \leq \frac{2}{\lambda_1}\|f\|^2,$$

where λ_1 is the smallest eigenvalue of $\mathcal{A}$ (we know $\lambda_1 > 0$ under (2.8)). Under the condition of **Lemma 2.3**, there exists $\gamma = \frac{\lambda_1}{2} - M^2\|\alpha\|^2 > 0$. Using Gronwall inequality, we get

$$\|\omega\|^2 \leq e^{-\gamma t}\|\omega_0\|^2 + \frac{2}{\lambda_1\gamma}\|f\|^2(1 - e^{-\gamma t}).$$

There exists $T(\omega_0) > 0$ such that for every $t \geq T$, we have

$$\|\omega\|^2 \leq M_1.$$

M_1 is a constant depends only on $\lambda_1, M, \|f\|$ and $\|\alpha\|$. We go back to (2.4), we have

$$\int_t^{t+1} \|\mathcal{A}^{\frac{1}{2}}\omega\|^2 \leq M_2, \ t \geq 0,$$

where M_2 is a constant depends on $M_1, \|f\|, \lambda_1$ and mM..

Multiplying (2.12) by $\mathcal{A}\omega$, integrating over D and integrating by parts, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\mathcal{A}^{\frac{1}{2}}\omega\|^2 + \|\mathcal{A}\omega\|^2 + \int_D \alpha\partial_x\triangle^{-1}\omega\mathcal{A}\omega + \int_D J(\triangle^{-1}\omega,\omega)\mathcal{A}\omega = \int_D f\mathcal{A}\omega.$$

By Cauchy inequality, Soblove inequality and **Proposition 2.2**, we have

$$\frac{d}{dt}\|\mathcal{A}^{\frac{1}{2}}\omega\|^2 \le C\|\mathcal{A}^{\frac{1}{2}}\omega\|^2 + M_3,$$

where M_3 depends on $M_1, \|f\|, \lambda_1$ and M. By uniform Gronwall inequality, we have

$$\|\mathcal{A}^{\frac{1}{2}}\omega\|^2 \le M_4, \text{ for } t \ge T_1 \ge T.$$

Take $R_0 = M_4$, the proof of **Lemma 2.3** is completed. ■

Now, we consider the averaging principle in the space $V = H_0^1$. Given a point ω_0 in $B_V(R_0)$, we compare the trajectories (solutions) $\omega(\tau)$, $\bar{\omega}(\tau)$ of system (2.12) and (2.14) starting from this initial point. Consider their difference on the interval $\tau \in [0, \frac{T}{\epsilon}]$, T being arbitrary but fixed. We suppose for the moment that $\omega(\tau) \in B_V(R)$. Then the difference $z(\tau) = \omega(\tau) - \bar{\omega}(\tau)$ satisfies the equation

$$\partial_t z + \epsilon\mathcal{A} + \epsilon[J(\triangle^{-1}\omega,\omega) - J(\triangle^{-1}\bar{\omega},\bar{\omega})] = -\epsilon(\alpha(x,y,t)\partial_x\triangle^{-1}\omega - \alpha_0(x,y)\partial_x\triangle^{-1}\bar{\omega}). \tag{2.16}$$

We first have some estimates on the nonlinear terms.

Lemma 2.4 *The nonlinear operator $J(u,v)$ is a bounded Lipschitz map in the following sense:*

$$\|J(u_1,u_2) - J(v_1,v_2)\| \le$$
$$C_{\frac{1}{2}}(\|u_1\|_{\frac{1}{2}} + \|u_2\|_{\frac{1}{2}} + \|v_1\|_{\frac{1}{2}} + \|v_2\|_{\frac{1}{2}})(\|u_1 - v_1\|_{\frac{1}{2}} + \|u_2 - v_2\|_{\frac{1}{2}}), \tag{2.17}$$
$$\|J(u_1,u_2) - J(v_1,v_2)\|_{\frac{1}{2}} \le$$
$$C_0(\|u_1\|_{D(\mathcal{A})} + \|u_2\|_{D(\mathcal{A})} + \|v_1\|_{D(\mathcal{A})} + \|v_2\|_{D(\mathcal{A})})(\|u_1 - v_1\|_{D(\mathcal{A})} + \|u_2 - v_2\|_{D(\mathcal{A})}), \tag{2.18}$$

where $C_{\frac{1}{2}}$ and C_0 are some positive constants.

PROOF. Since

$$\begin{aligned} J(u_1,u_2) - J(v_1,v_2) &= \\ (u_{1x} - v_{1x})u_{2y} + (u_{1y} - v_{1y})v_{1x} &- (u_{2x} - v_{2x})u_{1y} - (u_{2y} - v_{2y})v_{2x}, \end{aligned} \tag{2.19}$$

(2.17) and (2.18) are obtained by direct estimates. Here the equivalence of norms $\|\cdot\|_{H^2}$ and $\|\cdot\|_{D(\mathcal{A})}$ is used. ∎

Now we get back to equation (2.16). Inverting the linear operator we come to an equivalent integral equation

$$\begin{aligned} z(\tau) &= \epsilon\int_0^\tau e^{-\epsilon\mathcal{A}(\tau-s)}[J(\triangle^{-1}\omega,\omega)-J(\triangle^{-1}\bar\omega,\bar\omega)]ds \\ &- \epsilon\int_0^\tau e^{-\epsilon\mathcal{A}(\tau-s)}(\alpha(x,y,t)\partial_x\triangle^{-1}\omega-\alpha_0(x,y)\partial_x\triangle^{-1}\bar\omega)ds. \end{aligned} \tag{2.20}$$

Using (2.11) and (2.17), the $D(\mathcal{A}^{\frac{1}{2}})$-norm of the first term in the right hand side satisfies the inequality

$$\begin{aligned} &\|\epsilon\int_0^\tau \mathcal{A}^{\frac{1}{2}}e^{-\epsilon\mathcal{A}(\tau-s)}[J(\triangle^{-1}\omega,\omega)-J(\triangle^{-1}\bar\omega,\bar\omega)]ds\| \\ &\le \epsilon\int_0^\tau K_{\frac{1}{2}}\epsilon^{-\frac{1}{2}}(\tau-s)^{-\frac{1}{2}}e^{-\epsilon a(\tau-s)}2R\|z(s)\|_{\frac{1}{2}}ds \\ &= 2RK_{\frac{1}{2}}\epsilon^{\frac{1}{2}}\int_0^\tau(\tau-s)^{-\frac{1}{2}}e^{-\epsilon a(\tau-s)}\|z(s)\|_{\frac{1}{2}}ds. \end{aligned} \tag{2.21}$$

Here $\|z\|_{\frac{1}{2}} = \|\mathcal{A}^{\frac{1}{2}}z\|$.

Let us estimate the second term in the right hand side of (2.20). Since

$$\alpha(x,y,t)\partial_x\triangle^{-1}\omega-\alpha_0(x,y)\partial_x\triangle^{-1}\bar\omega =$$

$$(\alpha(x,y,t)-\alpha_0(x,y))\partial_x\triangle^{-1}\omega+\alpha_0(x,y,t)\partial_x\triangle^{-1}z,$$

then

$$\|-\epsilon\int_0^\tau e^{-\epsilon\mathcal{A}(\tau-s)}(\alpha(x,y,t)\partial_x\triangle^{-1}\omega-\alpha_0(x,y)\partial_x\triangle^{-1}\bar\omega)ds\|_{\frac{1}{2}}$$

$$\begin{aligned} &= \|\epsilon\int_0^\tau e^{-\epsilon\mathcal{A}(\tau-s)}(-(\alpha(x,y,t)-\alpha_0(x,y))\partial_x\triangle^{-1}\omega-\alpha_0(x,y,t)\partial_x\triangle^{-1}z)ds\|_{\frac{1}{2}} \\ &\le \epsilon\|\int_0^\tau e^{-\epsilon\mathcal{A}(\tau-s)}\alpha_0(x,y)\partial_x\triangle^{-1}zds\|_{\frac{1}{2}} \\ &+\epsilon\|\int_0^\tau e^{-\epsilon\mathcal{A}(\tau-s)}(\alpha(x,y,t)-\alpha_0(x,y))\partial_x\triangle^{-1}\omega\|_{\frac{1}{2}}. \end{aligned} \tag{2.22}$$

Since

$$\epsilon\|\int_0^\tau e^{-\epsilon\mathcal{A}(\tau-s)}\alpha_0(x,y)\partial_x\triangle^{-1}zds\|_{\frac{1}{2}}$$

$$\leq \epsilon^{\frac{1}{2}} K_{\frac{1}{2}} \int_0^\tau (\tau - s)^{-\frac{1}{2}} e^{-\epsilon a(\tau - s)} \|\alpha_0(x,y)\partial_x \triangle^{-1} z\| ds$$
$$\leq \epsilon^{\frac{1}{2}} K_{\frac{1}{2}} \|\alpha_0\| \int_0^\tau (\tau - s)^{-\frac{1}{2}} e^{-\epsilon a(\tau - s)} \|\partial_x \triangle^{-1} z\|_{L^\infty} ds$$
$$\leq \epsilon^{\frac{1}{2}} K_{\frac{1}{2}} \|\alpha_0\| M \int_0^\tau (\tau - s)^{-\frac{1}{2}} e^{-\epsilon a(\tau - s)} \|z\|_{\frac{1}{2}} ds,$$

here

$$\|\partial_x \triangle^{-1} z\|_{L^\infty} \leq M\|z\|_{\frac{1}{2}}$$

is used. Integrating by parts and using (2.11) we have

$$\epsilon\| \int_0^\tau e^{-\epsilon \mathcal{A}(\tau - s)} (\alpha(x,y,t) - \alpha_0(x,y))\partial_x \triangle^{-1} \omega\|_{\frac{1}{2}}$$
$$= \epsilon\|\epsilon \int_0^\tau \mathcal{A} e^{-\epsilon \mathcal{A}(\tau - s)} \int_s^\tau (\alpha(x,y,t) - \alpha_0(x,y))\partial_x \triangle^{-1} \omega dt ds$$
$$+ e^{-\epsilon \mathcal{A}(\tau - s)} \int_s^\tau (\alpha(x,y,t) - \alpha_0(x,y))\partial_x \triangle^{-1} \omega dt|_0^\tau\|_{\frac{1}{2}}$$
$$\leq \|\epsilon \mathcal{A}^{\frac{1}{2}} e^{-\epsilon \mathcal{A}\tau} \int_0^\tau (\alpha(x,y,t) - \alpha_0(x,y))\partial_x \triangle^{-1} \omega dt\|$$
$$+ \|\epsilon^2 \int_0^\tau \mathcal{A}^{\frac{3}{2}} e^{-\epsilon \mathcal{A}(\tau - s)} \int_s^\tau (\alpha(x,y,t) - \alpha_0(x,y))\partial_x \triangle^{-1} \omega dt ds\|.$$
$$\leq (\epsilon\tau)^{\frac{1}{2}} e^{-\epsilon a\tau} K_{\frac{1}{2}} \frac{1}{\tau} \| \int_0^\tau (\alpha(x,y,t) - \alpha_0(x,y))\partial_x \triangle^{-1} \omega dt\|$$
$$+ \epsilon^{\frac{1}{2}} \int_0^\tau (t - s)^{-\frac{3}{2}} K_{\frac{3}{2}} e^{-\epsilon a(\tau - s)} \| \int_0^\tau (\alpha(x,y,t) - \alpha_0(x,y))\partial_x \triangle^{-1} \omega dt\| ds. \qquad (2.23)$$

Since

$$\begin{aligned} \| \textstyle\int_0^\tau (\alpha(x,y,t) - \alpha_0(x,y))\partial_x \triangle^{-1} \omega dt\| &\leq \sup_{t \in [0,\tau]} \|\partial_x \triangle^{-1} \omega\|_{L^\infty} \textstyle\int_0^\tau \|\alpha(x,y,t) - \alpha_0(x,y)\| dt \\ &\leq M \sup_{t \in [0,\tau]} \|\omega\|_{\frac{1}{2}} \textstyle\int_0^\tau \|\alpha(x,y,t) - \alpha_0(x,y)\| dt. \end{aligned}$$

Similarly,

$$\| \int_s^\tau (\alpha(x,y,t) - \alpha_0(x,y))\partial_x \triangle^{-1} \omega dt\| \leq M \sup_{t \in [0,\tau]} \|\omega\|_{\frac{1}{2}} \int_s^\tau \|\alpha(x,y,t) - \alpha_0(x,y)\| dt.$$

By (2.13) and **Lemma 2.3**, we have

$$\epsilon\| \int_0^\tau e^{-\epsilon \mathcal{A}(\tau - s)} (\alpha(x,y,t) - \alpha_0(x,y))\partial_x \triangle^{-1} \omega\|_{\frac{1}{2}} \leq (\epsilon\tau)^{\frac{1}{2}} K_{\frac{1}{2}} M R_0 \min\{M_0, \sigma(\tau)\}$$
$$+ \epsilon^{\frac{1}{2}} K_{\frac{3}{2}} M R_0 \min\{M_0, \sigma(\tau)\} \int_0^\tau (\tau - s)^{-\frac{1}{2}} ds.$$

$$\leq (\epsilon\tau)^{\frac{1}{2}}(K_{\frac{1}{2}} + 2K_{\frac{3}{2}})MR_0 \min\{M_0, \sigma(\tau)\} =: L(\tau). \tag{2.24}$$

For any $\delta > 0$, let τ_δ be so large that for $\tau \geq \tau_\delta, \sigma_\gamma \leq \delta$. Let ϵ_0 be so small that for $\epsilon < \epsilon_0$ then inequality $\frac{T}{\epsilon} > \tau_\delta$ is valid. Then

$$L(\tau) \leq G(T,\epsilon) := e^{-\epsilon a\tau} \begin{cases} T^{\frac{1}{2}}(K_{\frac{1}{2}} + 2K_{\frac{3}{2}})\delta, & \text{if } \tau \geq \tau_\delta, \\ (\epsilon\tau)^{\frac{1}{2}}(K_{\frac{1}{2}} + 2K_{\frac{3}{2}})M_0, & \text{if } \tau < \tau_\delta. \end{cases}$$

Since τ_δ does not depend on ϵ, we let $\delta \to 0$ and then $\epsilon \to 0$. We obtain

$$\| - \epsilon \int_0^\tau e^{-\epsilon\mathcal{A}(\tau - s)}(\alpha(x,y,t)\partial_x\triangle^{-1}\omega - \alpha_0(x,y)\partial_x\triangle^{-1}\bar{\omega})ds\|_{\frac{1}{2}}$$
$$\leq G(T,\epsilon) \to 0 \text{ when } \epsilon \to 0. \tag{2.25}$$

Thus, by (2.20)–(2.25) we obtain the following inequality:

$$\|z(\tau)\|_{\frac{1}{2}} \leq K\epsilon^{\frac{1}{2}} \int_0^\tau (\tau - s)^{-\frac{1}{2}} \|z(s)\|_{\frac{1}{2}} ds + G(T,\epsilon), \tag{2.26}$$

where $K = 2RK_{\frac{1}{2}}$.

We need the following fact.

Lemma 2.5 *[16] Let $\gamma \in (0,1]$ and for $t \in [0,T]$*

$$u(t) \leq a + b\int_0^t (t-s)^{\gamma-1}u(s)ds.$$

Then

$$u(t) \leq aE_\gamma((b\Gamma(\gamma))^{\frac{1}{\gamma}}t),$$

where the function $E_\gamma(z)$ is increasing and $E_\gamma(z) \sim \gamma^{-1}e^z$ as $z \to \infty$.

Applying this lemma to the inequality (2.26) on $\tau \in [0, \frac{T}{\epsilon}]$, we obtain

$$\|z(t)\|_{\frac{1}{2}} \leq G(T,\epsilon)E_{\frac{1}{2}}(\epsilon\tau\pi K^2) \leq G_\gamma(T,\epsilon)E_{\frac{1}{2}}(T\pi K^2) := \eta_T(\epsilon). \tag{2.27}$$

We thus have proved the proximity of solutions of (2.12) and (2.14) in $V = H_0^1$, assuming that the trajectory $\omega(t)$ with initial condition $\omega_0 = \omega(0) \in B_V(R_0)$ stays in the ball $B(R)$ on the interval $[0, \frac{T}{\epsilon}]$.

Let ϵ be so small that the right-hand side of (2.27) are less than $\frac{\rho}{2}$, where ρ is defined earlier in this section when we discuss absorbing sets. Suppose that the trajectory $\omega(t)$

leaves the ball $B(R)$ during the interval $[0, \frac{T}{\epsilon}]$ and let τ^* be the first momemt where $\|\omega(\tau^*)\|_{\frac{1}{2}} = R$. However, on the interval $\tau \in [0, \tau^*]$ both trajectories stay in the ball $B(R)$ and what we have proved so far shows that the inequality $\|\omega(\tau) - \bar{\omega}(\tau)\|_{\frac{1}{2}} \leq \frac{\rho}{2}$ is valid. In particular, it is valid for $\tau = \tau^*$. This together with the inequality $\|\bar{\omega}(\tau^*)\|_{\frac{1}{2}} \leq R - \rho$, which holds by the hypothesis of the following theorem and the property of the semigroup $S(t)$, gives the contradiction

$$\|\omega(\tau^*)\|_{\frac{1}{2}} \leq \|\omega(\tau^*) - \bar{\omega}(\tau^*)\|_{\frac{1}{2}} + \|\bar{\omega}(\tau^*)\|_{\frac{1}{2}} \leq R - \frac{\rho}{2}.$$

Therefore we have the following main result in this paper.

Theorem 2.6 *(Averaging Principle for Quasi-Geostrophic Fluid Motions) Let* $\omega_0 \in H^1(D)$, $f \in L^\infty(0, \infty, L^2(D))$, *and let the dissipativity condition (2.8) be satisfied and* $\frac{\lambda_1}{2M^2} > \|\alpha\|^2$, *where* λ_1 *is the smallest eigenvalue of* $\mathcal{A}$ *and* M *is the Sobolev embedding constant in* $\|\partial_x \triangle^{-1} \omega\|_{L^\infty} \leq M \|\mathcal{A}^{\frac{1}{2}} \omega\|$. *Assume that the right-hand side of equation (2.12) has an average in the sense of (2.13). Let* $T > 0$ *be arbitrary and fixed.*

If $\omega(0) = \bar{\omega}(0) \in B_V(R_0)$, *that is, the initial values coincide and belong to the absorbing ball, then for* $\tau \in [0, \frac{T}{\epsilon}]$,

$$\|\omega(\tau) - \bar{\omega}(\tau)\|_{\frac{1}{2}} \leq \eta_T(\epsilon) \to 0 \text{ as } \epsilon \to 0,$$

where $\eta_T(\epsilon)$ *is defined in (2.27).*

This theorem gives comparison estimate and convergence result (as $\eta \to \infty$) between the QG flows and averaged QG flows, on finite but large time intevals.

3 Summary

We have obtained an averaging principle, Theorem 2.6, for quasi-geostrophic fluid motions with rapidly oscillating Coriolis force (characterizied by a large dimensionless parameter η). We have derived comparison estimate and proved convergence result (as $\eta \to \infty$) between quasi-geostrophic fluid motions and its averaged fluid motions.

The averaged fluid model is an autonomous system while the original quasi-geostrophic fluid system with rapidly oscillating Coriolis force is a nonautonomous system. Thus the averaging principle obtained in this paper provides an autonomous fluid system as an approximation for the original nonautonomous fluid system.

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