ON THE ABSENCE OF BIFURCATION
FOR ELASTIC BARS IN UNIAXIAL TENSION

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Abstract

We prove that an elasic bar undergoing uniaxial tension will not neck before the axial load on the bar attains a (local) maximum. Further, if the bar is in a hard loading device we show that necking is delayed until after maximum load is achieved. The key ingredient in the latter result is a generalized Korn inequality.

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INTRODUCTION

In this paper we present our initial investigation into the possibility of using finite elastostatics to predict the onset of instability and localization of deformation, such as necking or the formation of Lüders bands, in bars that are undergoing uniaxial tension.

We assume that a bar of uniform cross section is a three-dimensional cylindrical solid composed of a homogeneous isotropic (compressible) elastic material. The bar is put into uniaxial tension by a specification of either the final length of the bar—a hard loading device, or the axial load that is placed upon the bar—a soft loading device. The sides of the bar are assumed to be traction-free.

We begin our analysis with a formulation of the relevant nonlinear boundary-value problem and a proof of the existence of a trivial homogeneous solution to the problem. The crucial assumptions used are growth conditions on the stored energy.

Our main results show that in either loading device a second solution cannot continuously[1] bifurcate off from the trivial solution before the load attains a (local) maximum, if the material that the bar is composed of satisfies certain constitutive hypotheses[2]. Moreover, we prove that in a hard loading device bifurcation cannot occur until after the load[3] attains a (local) maximum. The key ingredient in the latter result is a generalized Korn inequality.

If we make the additional assumption that no solution of the equilibrium equations is admissible unless it is at least metastable[4], according to the energy

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[1] With respect to the C^1 topology.

[2] The relevant constitutive hypotheses are the Baker-Ericksen [7] inequalities, the tension-extension inequalities and a two-dimensional version of the pressure-compression inequality.

[3] In a hard device we use the term load to denote the axial force exerted by the rod upon the loading device.

[4] That is, the solution should at least be a local minimizer of the energy.

criterion, our results lead us to conclude that for the soft loading device finite elastostatics is probably not suitable for predicting the onset of localization as a continuous bifurcation phenomenon.[5] Instead we expect localization to occur discontinuously in a manner similar to "snap buckling."

For the hard loading device no such difficulties are encountered and it therefore may be possible to use finite elastostatics and bifurcation theory to predict the onset of localization.

Most recent work on necking of three-dimensional bars[6] assumes that the bar is composed of an elastic/plastic material and uses Hill's [25, 26] theory of uniqueness and bifurcation.[7] Of particular relevance to our work is a result of Miles [34]. Miles has shown that an elastic/plastic bar will not admit a second solution continuously bifurcating off from the trivial solution <u>before</u> the load attains a local maximum if certain constitutive hypotheses are satisfied.

Our results improve upon Miles' in two respects. First, within the context of finite elasticity we have been able to obtain a clearer statement of the

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[5] Sawyers [38, p. 120] and Rivlin [37, p. 422] have noted that in a soft loading device the trivial solution is unstable after the load achieves a maximum, but bifurcation does not occur immediately. Rivlin further observes that there are no other known solutions immediately after maximum load occurs. Both Sawyers and Rivlin conclude that a strain energy function that allows the load to attain a maximum is probably not a suitable model for an elastic material. Our analysis of the hard loading device indicates that their conclusion may not be correct. At maximum load in a soft device one could abandon continuity rather than finite elasticity. We note that both Sawyers' and Rivlin's observations are based upon their analysis of incompressible rectangular blocks.

[6] Wesolowski [46] has shown that a bar composed of neo-Hookean material never bifurcates in uniaxial tension. Wesolowski [47] and Hill and Hutchinson [27] have shown that a bar composed of Mooney-Rivlin material never bifurcates in plane strain uniaxial tension. Cheng, Ariaratnam and Dubey [10] have concluded that a certain class of elastic materials will not bifurcate in uniaxial tension.

[7] In particular, Cheng, Ariaratnam and Dubey [10], Miles [34] and Hutchinson and Miles [29] have obtained asymptotic and numerical estimates of the loading value at which a circular cylinder will become unstable. Needleman [35] has used the finite element method to calculate the loading value and also the postbuckling behavior of the rod. Hill and Hutchinson [27] have considered the bifurcation problem for a rectangular block. Also see Hutchinson [28] for a survey of other results concerning necking.

required constitutive hypotheses. Second, we have shown that in a hard loading device bifurcation is delayed until after maximum load is achieved[8].

In finite elasticity recent work on localization has concentrated on the implications of failure of ellipticity. In particular Ericksen [14] and James [30] have considered the consequences of a nonconvex strain energy function in a one-dimensional bar theory[9], while Knowles and Sternberg [31, 32, 33] have investigated the loss of ellipticity of the two- and three-dimensional equations.

Alternatively, Antman [2] and Antman and Carbone [4] have examined the possibility of localization using an elastic-rod theory in which ellipticity is never lost. In particular, Antman has shown that the phenomenon of necking is predicted by this theory.

Although none of the results in this paper actually predict the onset of localization, our approach is closer to that of Antman's in that it is committed to explaining localization as a continuous bifurcation phenomenon. It is important to note that in a quasistatic theory it is just as reasonable to expect localization to occur discontinuously (i.e., the bar just jumps to a state of lower energy). In fact, for the soft loading device we have shown that this is the only way in which localization can occur.

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8 Our technique should carry through for elastic/plastic materials. In particular bifurcation, in a hard loading device, should be delayed until after maximum load is achieved.

9 Both Ericksen and James construct stable equilibrium solutions that exhibit the phenomenon of necking.

1. NOTATION

We let

$$\mathrm{Lin} \equiv \text{space of all linear transformations from } \mathbb{R}^3 \text{ into } \mathbb{R}^3$$

with inner product and norm

$$\underset{\sim}{G} \cdot \underset{\sim}{H} \equiv \mathrm{trace}(\underset{\sim}{G}\underset{\sim}{H}^T) , \qquad |\underset{\sim}{G}|^2 \equiv \underset{\sim}{G} \cdot \underset{\sim}{G} ,$$

where $\underset{\sim}{H}^T$ is the transpose of H. We write

$$\mathrm{Lin}^+ \equiv \{\underset{\sim}{H} \in \mathrm{Lin} : \det \underset{\sim}{H} > 0\} ,$$

$$\mathrm{Orth}^+ \equiv \{\underset{\sim}{Q} \in \mathrm{Lin}^+ : \underset{\sim}{Q}^T\underset{\sim}{Q} = \underset{\sim}{I}\},$$

where det is the determinant and $\underset{\sim}{I}$ the identity. Any tensor (linear transformation) $\underset{\sim}{H}$ admits the unique decomposition

$$\underset{\sim}{H} = \underset{\sim}{E} + \underset{\sim}{W}$$

into a symmetric tensor $\underset{\sim}{E}$ and a skew tensor $\underset{\sim}{W}$; in fact

$$\underset{\sim}{E} = \frac{1}{2}(\underset{\sim}{H} + \underset{\sim}{H}^T) , \qquad \underset{\sim}{W} = \frac{1}{2}(\underset{\sim}{H} - \underset{\sim}{H}^T) .$$

We call $\underset{\sim}{E}$ and $\underset{\sim}{W}$, respectively, the <u>symmetric</u> and <u>skew</u> parts of $\underset{\sim}{H}$.

Given two vectors $\underset{\sim}{a}, \underset{\sim}{b} \in \mathbb{R}^3$ we write $\underset{\sim}{a} \otimes \underset{\sim}{b}$ for the tensor product of $\underset{\sim}{a}$ and $\underset{\sim}{b}$; in components

$$(\underset{\sim}{a} \otimes \underset{\sim}{b})_{ij} = a_i b_j .$$

We write ∇ and div for the gradient and divergence operators in $\mathbb{R}^3$: for a vector field $\underset{\sim}{u}$, $\nabla\underset{\sim}{u}$ is the tensor field with components $(\nabla\underset{\sim}{u})_{ij} = \partial u_i/\partial x_j$; for a tensor field S, div S is the vector field with components $\sum_j \partial S_{ij}/\partial x_j$. Given any function $\phi(\underset{\sim}{F})$, we denote the Frechet derivative of ϕ by

$$\frac{d}{d\underset{\sim}{F}} \phi(\underset{\sim}{F}) .$$

Throughout this paper $\mathcal{R}$ will denote a properly regular[10] region in $\mathbb{R}^2$; thus, in particular, the closure of $\mathcal{R}$, $\mathrm{cl}(\mathcal{R})$, is compact, connected, and has piecewise C^1 boundary $\partial\mathcal{R}$. Further we let (x_1, x_2) be cartesian coordinates on $\mathbb{R}^2$ such that the centroid of $\mathcal{R}$ is at the origin; thus

$$\int_{\mathcal{R}} x_1 = \int_{\mathcal{R}} x_2 = 0 . \tag{1.1}$$

Throughout this paper we let $\mathcal{B}$ be the region in $\mathbb{R}^3$ defined by

$$\mathcal{B} \equiv \mathcal{R} \times (0, L) .$$

We shall sometimes refer to $\mathcal{B}$ as a cylindrical solid.

Let

$$\mathrm{Var} = \{ \underset{\sim}{u} \in C^1(\mathcal{B}, \mathbb{R}^3) : \underset{\sim}{u} \neq 0 \text{ and } \underset{\sim}{u} \text{ satisfies (1.2) - (1.4)} \} ,$$

where

$$u_3 = 0 \quad \text{on} \quad \mathrm{cl}(\mathcal{R}) \times \{0, L\} , \tag{1.2}$$

$$\int_{\mathcal{B}} u_1 = \int_{\mathcal{B}} u_2 = 0 , \tag{1.3}$$

$$\int_{\mathcal{B}} (u_{1,2} - u_{2,1}) = 0 . \tag{1.4}$$

10 Cf. Fichera [16, p. 351].

Lemma. There exists a $k > 0$ such that

$$\int_{\mathcal{B}} |\mathbf{E}|^2 \geqslant k \int_{\mathcal{B}} |\nabla \mathbf{u}|^2 , \qquad \text{(Korn Inequality)} \qquad (1.5)$$ [11]

for all $\mathbf{u} \in \text{Var}$, where $\mathbf{E}$ is the symmetric part of $\nabla \mathbf{u}$.

2. THE CONSTITUTIVE RELATION

We consider a homogeneous body that occupies the region $\mathcal{B}$ in a fixed homogeneous reference configuration. For convenience we identify the body with the region $\mathcal{B}$. A *deformation* $\mathbf{f}$ of the body is a member of the space

$$\text{Def} \equiv \left\{ \mathbf{f} \in C^1(\mathcal{B}, \mathbb{R}^3) : \det \nabla \mathbf{f} > 0 \right\} .$$

We assume that the body is hyperelastic with response function $\sigma : \text{Lin}^+ \to \mathbb{R}$. σ gives the *stored energy*

$$\sigma\left(\nabla \mathbf{f}(\mathbf{x})\right)$$

at any point $\mathbf{x} \in \mathcal{B}$ when the body is deformed by $\mathbf{f}$. Writing $\mathbf{F}$ for $\nabla \mathbf{f}(\mathbf{x})$, we assume that[12]

$$\sigma(\mathbf{Q}\mathbf{F}) = \sigma(\mathbf{F})$$

for all $\mathbf{F} \in \text{Lin}^+$ and $\mathbf{Q} \in \text{Orth}^+$.

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[11] Cf., e.g., Fichera [16, p. 384] whose proof, with minor modifications, applies in the present circumstance.

[12] This restriction is a consequence of frame indifference.

We assume that the body is isotropic so that

$$\sigma(\mathbf{F}\mathbf{Q}) = \sigma(\mathbf{F})$$

for all $\mathbf{F} \in \mathrm{Lin}^+$ and $\mathbf{Q} \in \mathrm{Orth}^+$. A well-known[13] consequence of this assumption is that there is a function $\psi : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ such that

$$\sigma(\mathbf{F}) = \psi\left(\frac{1}{2}\mathbf{F}\cdot\mathbf{F}, \frac{1}{4}\mathbf{F}\mathbf{F}^T \cdot \mathbf{F}\mathbf{F}^T, \det \mathbf{F}\right) .$$

We shall assume that ψ, and hence σ, are C^2.

The (Piola-Kirchhoff) stress $\mathbf{S} : \mathrm{Lin}^+ \to \mathrm{Lin}$ is given by

$$\mathbf{S}(\mathbf{F}) \equiv \frac{d}{d\mathbf{F}} \sigma(\mathbf{F}) ,$$

while the linear transformation $\mathbf{A}(\mathbf{F}) : \mathrm{Lin} \to \mathrm{Lin}$ defined by

$$\mathbf{A}(\mathbf{F}) \equiv \frac{d}{d\mathbf{F}} \mathbf{S}(\mathbf{F})$$

is called the elasticity tensor.

We say that the elasticity tensor is strongly-elliptic at a deformation $\mathbf{f}$ if, for each $\mathbf{x} \in \mathcal{B}$

$$\mathbf{H} \cdot \mathbf{A}\big(\nabla \mathbf{f}(\mathbf{x})\big)\mathbf{H} > 0$$

whenever $\mathbf{H} = \mathbf{a} \otimes \mathbf{b}$ with $\mathbf{a} \neq \mathbf{0}$, $\mathbf{b} \neq \mathbf{0}$.

Definition. The reference configuration is

(a) natural if $\mathbf{S}(\mathbf{I}) = \mathbf{0}$;

(b) positive if $\mathbf{E} \cdot \mathbf{A}(\mathbf{I})\mathbf{E} > 0$

for all symmetric $\mathbf{E} \neq \mathbf{0}$.

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13 Cf., e.g., Gurtin [21] or Truesdell and Noll [44]. The function $\hat{\psi}$ is not defined on the indicated set, but only on the subset that defines possible simultaneous values of the principal invariants. We are assuming that the function $\hat{\psi}$ has a C^1 extension to the entire first octant.

Proposition 2.1. For any $\mathbf{F} \in \mathrm{Lin}^+$ and $\mathbf{H} \in \mathrm{Lin}$

$$\mathbf{S}(\mathbf{F}) = \psi_{,1}\mathbf{F} + \psi_{,2}\mathbf{F}\mathbf{F}^T\mathbf{F} + \psi_{,3}(\det \mathbf{F})\mathbf{F}^{-T} \tag{2.1}$$

$$\begin{aligned}\mathbf{A}(\mathbf{F})\mathbf{H} = {} & \psi_{,1}\mathbf{H} + \psi_{,2}\left[\mathbf{H}\mathbf{F}^T\mathbf{F} + \mathbf{F}\mathbf{H}^T\mathbf{F} + \mathbf{F}\mathbf{F}^T\mathbf{H}\right] \\ & + \psi_{,3}(\det \mathbf{F})\left[(\mathbf{F}^{-T}\cdot \mathbf{H})\mathbf{F}^{-T} - \mathbf{F}^{-T}\mathbf{H}^T\mathbf{F}^{-T}\right] + \sum_{i,j=1}^{3}(\mathbf{G}^i\cdot \mathbf{H})\mathbf{G}^j\psi_{,ij}\,,\end{aligned} \tag{2.2}$$

where

$$\mathbf{G}^1 = \mathbf{F}, \quad \mathbf{G}^2 = \mathbf{F}\mathbf{F}^T\mathbf{F}, \quad \mathbf{G}^3 = (\det \mathbf{F})\mathbf{F}^{-T}$$

$$\psi_{,i} = \psi_{,i}\left(\tfrac{1}{2}\mathbf{F}\cdot\mathbf{F},\ \tfrac{1}{4}\mathbf{F}\mathbf{F}^T\cdot\mathbf{F}\mathbf{F}^T,\ \det \mathbf{F}\right).$$

Proof. Equation (2.1) follows upon differentiating ψ and noting that

$$\frac{d}{d\mathbf{F}}(\mathbf{F}\mathbf{F}^T\cdot\mathbf{F}\mathbf{F}^T) = 4\mathbf{F}\mathbf{F}^T\mathbf{F}, \quad \frac{d}{d\mathbf{F}}(\det \mathbf{F}) = (\det \mathbf{F})\mathbf{F}^{-T}.$$

Equation (2.2) follows upon differentiating (2.1) and noting that

$$\frac{d}{d\mathbf{F}}(\mathbf{F}^{-T})[\mathbf{H}] = -\mathbf{F}^{-T}\mathbf{H}^T\mathbf{F}^{-T}.$$

3. CONSTITUTIVE INEQUALITIES

Let $\mathbf{f} \in \mathrm{Def}$, $\mathbf{x} \in \mathcal{B}$, and $\mathbf{F} = \nabla \mathbf{f}(\mathbf{x})$. Then $\mathbf{F}$ has polar decomposition $\mathbf{F} = \mathbf{V}\mathbf{R}$ where $\mathbf{R} \in \mathrm{Orth}^+$ and $\mathbf{V}$ is symmetric and positive definite. By the spectral theorem there is an orthonormal basis $\{\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3\}$ such that

$$\mathbf{V} = \sum_{i=1}^{3}\lambda_i\,\mathbf{e}^i\otimes\mathbf{e}^i. \tag{3.1}$$

The scalars λ_i are called the local *principal stretches* (of the deformation $\mathbf{f}$ at the point $\mathbf{x}$) while the vectors $\mathbf{e}^i$ are called the *principal axes of strain*.

The Cauchy Stress $\underset{\sim}{T}$ is given by the relation

$$\underset{\sim}{T}(\underset{\sim}{F}) \equiv \underset{\sim}{S}(\underset{\sim}{F})\underset{\sim}{F}^T/(\det \underset{\sim}{F}) . \tag{3.2}$$

We note that as a consequence of (2.1) $\underset{\sim}{T}$ is symmetric. The eigenvalues of $\underset{\sim}{T}$ are called the local principal stresses. If we combine (2.1), (3.1), and (3.2) we find that

$$\underset{\sim}{T}(\underset{\sim}{F}) = \sum_{i=1}^{3} t_i \underset{\sim}{e}^i \otimes \underset{\sim}{e}^i ,$$

where the principal stresses t_i are given by

$$t_i(\lambda_1, \lambda_2, \lambda_3) = (\lambda_i^2 \psi_{,1} + \lambda_i^4 \psi_{,2})/\lambda_1\lambda_2\lambda_3 + \psi_{,3} \tag{3.3}$$

The Baker-Ericksen [7] inequality is the requirement that the principal stresses have the same order as the principal stretches:

$$(t_i - t_j)(\lambda_i - \lambda_j) > 0 , \quad \lambda_i \neq \lambda_j .$$

In view of (3.3) a slightly stronger requirement is that

$$BE_i \equiv \psi_{,1} + \psi_{,2}(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - \lambda_i^2) > 0 \tag{3.4}$$

(even if $\lambda_i = \lambda_j$).

The tension-extension inequality is the requirement that each principal stress is a strictly increasing function of the corresponding principal stretch. Slightly stronger than this is the requirement

$$TE_i \equiv (\lambda_1\lambda_2\lambda_3\lambda_i^{-1}) \frac{\partial t_i}{\partial \lambda_i} > 0 . \tag{3.5}$$

Fix $\underset{\sim}{x}_0 \in \mathcal{B}$ and let us consider deformations whose principal stretches (at $\underset{\sim}{x}_0$) are equal. It follows from (3.3) that the Cauchy Stress at $\underset{\sim}{x}_0$ is a pressure. Define

$$\lambda \equiv \lambda_1 = \lambda_2 = \lambda_3 \; , \qquad t \equiv t_1 = t_2 = t_3 \; .$$

The classical pressure-compression inequality is the requirement that this pressure be a strictly increasing function of volume (of the deformed state). Slightly stronger than this is the requirement that

$$PC_c \equiv \lambda^2 \frac{dt}{d\lambda} > 0 \; . \tag{3.6}$$

Remark. The constitutive inequalities mentioned thus far have been used and studied extensively. For a more complete discussion of them see Truesdell and Noll [44] or Wang and Truesdell [45].

We now observe that the tension-extension inequality might possibly be viewed as a one-dimensional version of the pressure-compression inequality, i.e., that the one-dimensional pressure (the tension t_i) is a strictly increasing function of the one-dimensional volume (the length or stretch λ_i). This viewpoint has motivated us to formulate a two-dimensional version of the pressure-compression inequality.

Fix $\underset{\sim}{x}_0 \in \mathcal{B}$ and let us consider deformations in which two of the principal stretches (at $\underset{\sim}{x}_0$) are equal. It follows from (3.3) that the corresponding stresses are equal. Define

$$\alpha \equiv \lambda_1 = \lambda_2 \; , \qquad t(\alpha,\lambda) \equiv t_1(\alpha,\alpha,\lambda) = t_2(\alpha,\alpha,\lambda) \; .$$

The two-dimensional pressure-compression inequality is the requirement that the (two-dimensional) pressure t is a strictly increasing function of the area α^2. Slightly stronger than this is the requirement that

$$PC_{2d} \equiv \lambda\alpha \frac{\partial t}{\partial \alpha} > 0 \; . \tag{3.7}$$

We note that the above inequalities are not all independent and for future reference we derive the relationships that connect them. If we first differentiate t_i with respect to λ_j we find, with the aid of (3.3), for example, that

$$\frac{\partial t_1}{\partial \lambda_1} = (\psi_{,1} + 3\lambda_1^2 \psi_{,2})\lambda_2^{-1}\lambda_3^{-1} + B_1$$

$$\frac{\partial t_1}{\partial \lambda_2} = -(\lambda_1 \psi_{,1} + \lambda_1^3 \psi_{,2})\lambda_2^{-2}\lambda_3^{-1} + B_2 \ , \tag{3.8}$$

$$\frac{\partial t_1}{\partial \lambda_3} = -(\lambda_1 \psi_{,1} + \lambda_1^3 \psi_{,2})\lambda_2^{-1}\lambda_3^{-2} + B_3$$

where

$$\lambda_2\lambda_3 B_1 = \lambda_1^2 \psi_{,11} + 2\lambda_1^4 \psi_{,12} + 2\lambda_1\lambda_2\lambda_3 \psi_{,13} + \lambda_1^6 \psi_{,22} + 2\lambda_1^3\lambda_2\lambda_3 \psi_{,23} + \lambda_2^2\lambda_3^2 \psi_{,33} \ ,$$

$$\lambda_2\lambda_3 B_2 = \lambda_1\lambda_2 \psi_{,11} + (\lambda_1^2 + \lambda_2^2)\lambda_1\lambda_2 \psi_{,12} + (\lambda_1^2 + \lambda_2^2)\lambda_3 \psi_{,13} + \lambda_1^3\lambda_2^3 \psi_{,22} + (\lambda_1^4 + \lambda_2^4)\lambda_3 \psi_{,23} + \lambda_1\lambda_2\lambda_3^2 \psi_{,33} \ ,$$

$$\lambda_2\lambda_3 B_3 = \lambda_1\lambda_3 \psi_{,11} + (\lambda_1^2 + \lambda_3^2)\lambda_1\lambda_3 \psi_{,12} + (\lambda_1^2 + \lambda_3^2)\lambda_2 \psi_{,13} + \lambda_1^3\lambda_3^3 \psi_{,22} + (\lambda_1^4 + \lambda_3^4)\lambda_2 \psi_{,23} + \lambda_1\lambda_2^2\lambda_3 \psi_{,33} \ .$$

We note that $B_j = B_1$ whenever $\lambda_j = \lambda_1$. Thus if we add equations (3.8) we conclude that

$$\frac{\partial t_1}{\partial \lambda_1} + \frac{\partial t_2}{\partial \lambda_2} = 2\,\frac{\partial t_1}{\partial \lambda_1} - (\psi_{,1} + 3\alpha^2 \psi_{,2})\alpha^{-1}\lambda_3^{-1} - (\alpha \psi_{,1} + \alpha^3 \psi_{,2})\alpha^{-2}\lambda_3^{-1} \ , \tag{3.9}$$

whenever $\lambda_1 = \lambda_2 \equiv \alpha$ and

$$\frac{\partial t_1}{\partial \lambda_1} + \frac{\partial t_2}{\partial \lambda_2} + \frac{\partial t_2}{\partial \lambda_3} = 3\,\frac{\partial t_1}{\partial \lambda_1} - 2(\psi_{,1} + 3\lambda^2 \psi_{,2})\lambda^{-2} - 2(\lambda \psi_{,1} + \lambda^3 \psi_{,2})\lambda^{-3} \ , \tag{3.10}$$

whenever $\lambda_1 = \lambda_2 = \lambda_3 \equiv \lambda$.

It now follows from (3.4), (3.5), (3.7), and (3.9) that

$$PC_{2d} = 2(TE_1 - BE_3) , \tag{3.11}$$

while by (3.4), (3.5), (3.6), and (3.10)

$$PC_c = 3TE_1 - 4BE_1 . \tag{3.12}$$

In the above calculation we have used the fact that $TE_i = TE_j$ and $BE_i = BE_j$ whenever $\lambda_i = \lambda_j$.

4. UNIAXIAL TENSION AND COMPRESSION. THE HARD DEVICE.

We consider deformations of the body which are states of uniaxial tension or compression. More precisely, suppose that $\lambda \in (0, \infty)$ and consider the boundary-value problem

$$\begin{aligned} &\operatorname{div} \underset{\sim}{S}(\nabla \underset{\sim}{f}) = \underset{\sim}{0} \quad \text{in } \mathcal{B} , \\ &\underset{\sim}{S}(\nabla \underset{\sim}{f})\underset{\sim}{n} = \underset{\sim}{0} \quad \text{on } \partial\mathcal{R} \times [0, L] , \\ &S_{13}(\nabla \underset{\sim}{f}) = S_{23}(\nabla \underset{\sim}{f}) = 0 \quad \text{on } \mathcal{R} \times \{0, L\} , \\ &f_3 = 0 \quad \text{on } \mathcal{R} \times \{0\} , \\ &f_3 = \lambda L \quad \text{on } \mathcal{R} \times \{L\} , \end{aligned} \tag{4.1}$$

where $\underset{\sim}{n}$ is the outward unit normal to the lateral surface and λ is the ratio of the final to the initial height of the cylindrical solid.

We note that if a deformation $\underset{\sim}{f}$ satisfies (4.1) then so does $\underset{\sim}{g} \circ \underset{\sim}{f}$ if $\underset{\sim}{g}$ is either a translation perpendicular to the axis of the cylinder or any rotation about this axis, i.e.,

$$\underset{\sim}{g} = \underset{\sim}{Q}\underset{\sim}{x} + (\beta, \gamma, 0)^T ,$$

where $\beta, \gamma \in \mathbb{R}$,

$$Q = \begin{pmatrix} & R & & \vline & 0 \\ & & & \vline & 0 \\ \hline 0 & & 0 & \vline & 1 \end{pmatrix},$$

and $RR^T = I$.

In order to eliminate this trivial nonuniqueness of solution we impose additional constraints upon the deformation f. We require

$$\int_{\mathcal{B}} (f - i)_1 = \int_{\mathcal{B}} (f - i)_2 = 0 ,$$

$$\int_{\mathcal{B}} (f_{1,2} - f_{2,1}) = 0 , \tag{4.2}$$

where $i(x) = x$ is the identity deformation.

Theorem 4.1 [39]. Assume that[14]

(i) $\sigma(F) \to \infty$ as $\det F \to 0^+$;

(ii) $\sigma(F) \to \infty$ as $\|F\| \to \infty$.

Then for any $\lambda \in (0, \infty)$ there is an $\alpha = \alpha(\lambda)$ such that

$$f_\lambda(x) = \begin{pmatrix} \alpha(\lambda) & & \\ & \alpha(\lambda) & \\ & & \lambda \end{pmatrix} x \tag{4.3}$$

is a solution to (4.1) and (4.2).

Proof. It is clear that f_λ satisfies $(4.1)_1$, $(4.1)_{3-5}$, and (4.2) for any α independent of (i) and (ii). By $(4.1)_2$ and (2.1) we want to determine α so that

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14 For a discussion of assumptions of this type see Ball [6]. The idea for this kind of assumption was developed by Antman in a series of papers concerning nonlinear rod and shell theories, cf., e.g., [3].

$$\alpha\psi_{,1} + \alpha^3\psi_{,2} + \alpha\lambda\psi_{,3} = 0, \tag{4.4}$$

where

$$\psi_{,i} = \psi_{,i}(\alpha^2 + \lambda^2/2,\ \alpha^4/2 + \lambda^4/4,\ \alpha^2\lambda) .$$

For fixed λ we find that

$$\frac{1}{2}\frac{\partial}{\partial\alpha}\sigma(\nabla \underset{\sim}{f}_\lambda) = \alpha\psi_{,1} + \alpha^3\psi_{,2} + \alpha\lambda\psi_{,3} . \tag{4.5}$$

Finally, by (i) and (ii) we find that $\hat{\sigma}\uparrow\infty$ as both $\alpha \to 0^+$ and $\alpha \to \infty$. The mean value theorem and (4.5) yield (4.4) for some α. This is the desired result. ∎

We note that the function $\alpha : (0,\infty) \to \mathbb{R}$, constructed in the previous theorem, may not be uniquely defined since for each λ there may be many values of α that satisfy (4.4). We will fix our attention on one such function and make the additional assumption that its restriction to some interval is smooth. More precisely we assume that

$$\alpha \in C^1\left((\lambda_m, \lambda_M),\ \mathbb{R}^+\right) \tag{4.6}$$

is such that (4.3) is a solution to the boundary-value problem (4.1) and (4.2).

5. HADAMARD STABILITY.

We now consider the linearization stability of solutions to (4.1) and (4.2).

Definition. A deformation $f \in \mathrm{Def}$ is (*strictly*) *Hadamard-stable*[15] with respect to perturbations in Var if

$$HS(\underset{\sim}{f}, \underset{\sim}{u}) \equiv \int_{\mathcal{B}} \nabla\underset{\sim}{u}\cdot\underset{\sim}{A}(\nabla\underset{\sim}{f})\nabla\underset{\sim}{u} \geqslant 0$$

[15] Cf. Gurtin and Spector [22] and Spector [43]. Hadamard [23, p. 252] used the phrase "stabilité de l'equilibre interne." See also Pearson [36], Hill [24], Green and Adkins [19], Truesdell and Noll [44].

(HS > 0) for all $\underset{\sim}{u} \in \mathrm{Var}$. A set $\Omega \subset \mathrm{Def}$ is *uniformly Hadamard-stable* with respect to perturbations in Var if for some $k > 0$

$$\int_{\mathcal{B}} \nabla \underset{\sim}{u} \cdot \underset{\sim}{A}(\nabla \underset{\sim}{f}) \nabla \underset{\sim}{u} \geq k \int_{\mathcal{B}} |\nabla \underset{\sim}{u}|^2$$

for all $\underset{\sim}{f} \in \Omega$ and $\underset{\sim}{u} \in \mathrm{Var}$.

If we now endow Def with the topology generated by the semi-norm

$$\sup_{\mathcal{B}} |\nabla \underset{\sim}{f}| ,$$

then in the context of our problem, (4.1) and (4.2), the following theorems can be proven using techniques of Gurtin and Spector [22] and Spector [43].

Theorem 5.1 Every uniformly Hadamard-stable deformation has a neighborhood which is uniformly Hadamard-stable.

Theorem 5.2. Given λ_0 let $\underset{\sim}{f} \in \mathrm{Def}$ satisfy (4.1) and (4.2), with $\lambda = \lambda_0$, and suppose that $\underset{\sim}{f}$ is uniformly Hadamard-stable. Then there is a neighborhood of $\underset{\sim}{f}$ in which there are no other solutions to (4.1) and (4.2) for any $\lambda \in (0, \infty)$.

Remark. If we view (4.3) as a one-parameter family of trivial solutions to our problem then Theorem 5.2 implies that a second solution branch cannot continuously bifurcate from the trivial solution branch as long as the trivial solution is uniformly Hadamard-stable. We note that our choice of topology is crucial to this interpretation and that the choice of topology is not appropriate if one wants to consider deformations that are not smooth. In particular Ball [6A] has shown that cavitation can be viewed as a continuous bifurcation phenomenon, independent of the Hadamard stability of the critical state.

In preparation to examining the stability of solutions (4.1) and (4.2), which have the form (4.3), we compute the tensor

$$\underset{\sim}{D} \equiv \underset{\sim}{A}(\nabla \underset{\sim}{f}_\lambda) \underset{\sim}{U} \qquad (5.1)$$

for $U \in \mathrm{Lin}$. By (2.2), (3.4), (3.5), (3.7), (3.11), (4.3), and (4.4), we find that

$$D_{ii} = 2BE_3 U_{ii} + (PC_{2d} - TE_1)(U_{11} + U_{22}) + XU_{33} ,$$

$$D_{33} = TE_3 U_{33} + X(U_{11} + U_{22}) ,$$

$$D_{12} = D_{21} = BE_3(U_{12} + U_{21}) ,$$

$$D_{i3} = BE_1(U_{i3} + \frac{\alpha}{\lambda} U_{3i}) ,$$

$$D_{3i} = BE_1(U_{3i} + \frac{\alpha}{\lambda} U_{i3}) ,$$

for $i = 1, 2$ and where

$$\alpha^{-1}X = \lambda\psi_{,11} + (\alpha^2\lambda + \lambda^3)\psi_{,12} + (\lambda^2 + \alpha^2)\psi_{,13} + (\lambda^4 + \alpha^4)\psi_{,23} + \alpha^2\lambda^3 \psi_{,22} + \alpha^2\lambda\psi_{,33} + \psi_{,3} . \tag{5.2}$$

In deriving the above equations we have made use of the fact that $BE_1 = BE_2$ and $TE_1 = TE_2$ at deformations of the form (4.3).

If we now take the inner product of (5.1) with U we conclude that

$$\underset{\sim}{U} \cdot \underset{\sim}{A}(\nabla \underset{\sim}{f}_{\lambda})\underset{\sim}{U} = \underset{\sim}{d} \cdot \underset{\sim}{M}\underset{\sim}{d} + BE_3(U_{12} + U_{21})^2 + BE_1(\underset{\sim}{r} \cdot \underset{\sim}{N}\underset{\sim}{r}) + BE_1(\underset{\sim}{s} \cdot \underset{\sim}{N}\underset{\sim}{s}) . \tag{5.3}$$

Here

$$\underset{\sim}{M} \equiv \begin{bmatrix} TE_1 & PC_{2d} - TE_1 & X \\ PC_{2d} - TE_1 & TE_1 & X \\ X & X & TE_3 \end{bmatrix} , \tag{5.4}$$

$$\underset{\sim}{N} \equiv \begin{bmatrix} 1 & \alpha/\lambda \\ \alpha/\lambda & 1 \end{bmatrix} , \tag{5.5}$$

$$\underset{\sim}{d} \equiv \begin{pmatrix} U_{11} \\ U_{22} \\ U_{33} \end{pmatrix} , \qquad \underset{\sim}{r} \equiv \begin{pmatrix} U_{13} \\ U_{31} \end{pmatrix} , \qquad \underset{\sim}{s} \equiv \begin{pmatrix} U_{23} \\ U_{32} \end{pmatrix} . \tag{5.6}$$

If the reference configuration is natural then $\alpha = \lambda = 1$ in (4.3) will be a solution of (4.1) and (4.2). Thus, for the remainder of this section we assume that the reference configuration is natural and that the loading path contains this reference, i.e., $1 \in (\lambda_m, \lambda_M)$ and $\alpha(1) = 1$.

First, we note that by (3.3), (3.4), and (3.5) all of the BE_i are equal and all of the TE_i are equal. Let us denote by BE and TE, respectively, the common values of the BE_i and TE_i. In addition, it follows from (3.4), (3.5), (3.6), (3.8), (3.12), (4.4), and (5.2) that at the reference configuration

$$X = \frac{1}{3}(PC_c - 2BE) . \tag{5.7}$$

If we combine the above observations with (5.3) - (5.6) we conclude that

$$\underset{\sim}{U} \cdot \underset{\sim}{A}(\underset{\sim}{I})\underset{\sim}{U} = 2BE|\underset{\sim}{E}|^2 + \frac{1}{3}(PC_c - 2BE)(\underset{\sim}{E} \cdot \underset{\sim}{I})^2 \tag{5.8}$$

where $\underset{\sim}{E}$ is the symmetric part of $\underset{\sim}{U}$. The following proposition is then standard[16].

Proposition 5.3. *Assume that the reference configuration is natural. Then necessary and sufficient conditions for the reference configuration to be positive are that*

$$BE > 0 , \qquad PC_c > 0 , \tag{5.9}$$

in the reference configuration.

[16] Cf., e.g., Gurtin [20, p. 85].

Finally, we will make use of the following theorem.

Theorem 5.4 [22, 43]. Assume that the reference configuration is natural and positive. Then the reference configuration has a neighborhood that is uniformly Hadamard-stable.

6. PRELIMINARY RESULTS.

We denote by $BE_i(\lambda)$, $TE_i(\lambda)$, $PC_{2d}(\lambda)$, and $S_{33}(\lambda)$, respectively, the values of BE_i, TE_i, PC_{2d}, and $S_{33}(\nabla f_\lambda)$ when $\lambda_3 = \lambda$ and $\lambda_1 = \lambda_2 = \alpha = \alpha(\lambda)$.

Theorem 6.1. Suppose that for some $\lambda \in (\lambda_m, \lambda_M)$

(i) $BE_i(\lambda) > 0, \quad TE_i(\lambda) > 0, \quad PC_{2d}(\lambda) > 0$; (6.1)

(ii) $\frac{d}{d\lambda} S_{33}(\lambda) > 0$; (6.2)

(iii) $\alpha(\lambda) < \lambda$. (6.3)

Then f_λ is uniformly Hadamard-stable.

Remark. The loading path considered in Theorem 6.1 need not include a natural state. We therefore think that it is premature to assert that (6.2) and (6.3) (or for that matter (6.1)) are physically expected conditions. In fact, at this point we cannot even assert that elongation ($\lambda \in (1, \infty)$) produces uniaxial tension. As we shall see later, these difficulties will not occur when the loading path includes a positive natural state.

Proof of Theorem 6.1. We will prove that the matrices M and N given in (5.4) and (5.5) are positive definite. It will then follow, with the aid of (6.1), that the quadratic form (5.3) satisfies

$$U \cdot A(\nabla f_\lambda) U \geqslant k |E|^2$$

for all $U \in \mathrm{Lin}$, where E is the symmetric part of U and $k > 0$ is a fixed constant.

Thus, if we take $\underset{\sim}{U} = \nabla\underset{\sim}{u}(\underset{\sim}{x})$, integrate over $\mathcal{B}$, and use the Korn inequality $(1.5)_1$, we infer the existence of a $k_1 > 0$ such that

$$\int_{\mathcal{B}} \nabla\underset{\sim}{u} \cdot \underset{\sim}{A}(\nabla\underset{\sim}{f}_{\lambda})\nabla\underset{\sim}{u} \geqslant k_1 \int_{\mathcal{B}} |\nabla\underset{\sim}{u}|^2$$

for all $\underset{\sim}{u} \in \mathrm{Var}$. This is the desired result.

We first note that equation (6.3) immediately yields $\underset{\sim}{N}$ positive definite. To prove that $\underset{\sim}{M}$ is positive definite it suffices to show that

$$TE_1 > 0 , \tag{6.4}$$

$$\det \begin{bmatrix} TE_1 & PC_{2d} - TE_1 \\ PC_{2d} - TE_1 & TE_1 \end{bmatrix} > 0 , \tag{6.5}$$

$$\det \underset{\sim}{M} > 0 . \tag{6.6}$$

Clearly (6.4) and (6.5) follow from (3.11) and (6.1). A simple computation, using (5.4) and (3.11), yields

$$\det \underset{\sim}{M} = 2BE_3 \left[TE_3 PC_{2d} - 2X^2\right] .$$

Let us now compute the derivative of $S_{33}(\lambda)$. By (2.1) and (4.3)

$$S_{33}(\lambda) = \lambda\psi_{,1} + \lambda^3\psi_{,2} + \alpha^2\psi_{,3} , \tag{6.7}$$

where

$$\psi_{,i} = \psi_{,i}(\alpha^2 + \lambda^2/2,\ \alpha^4/2 + \lambda^4/4,\ \alpha^2\lambda) .$$

If we let $\alpha = \alpha(\lambda)$ and take the <u>total</u> derivative of (6.7) with respect to λ, we find, with the aid of (3.5) and (5.2), that

$$\frac{d}{d\lambda} S_{33} = TE_3 + 2X \frac{d\alpha}{d\lambda} \quad .$$

Next, we differentiate (4.4) with respect to λ, to conclude, with the aid of (3.7) and (5.2), that

$$\frac{d\alpha}{d\lambda} = - \frac{X}{PC_{2d}} \tag{6.8}$$

Thus

$$\frac{d}{d\lambda} S_{33} = TE_3 - \frac{2X^2}{PC_{2d}} \quad , \tag{6.9}$$

and hence

$$\det \underset{\sim}{M} = 2BE_3 PC_{2d} \frac{dS_{33}}{d\lambda} \quad .$$

Equation (6.6) now follows from (6.1) and (6.2). This concludes the proof. ■

The previous theorem gives sufficient conditions for Hadamard-stability, but does not address the question of their necessity. It is fairly easy to see that (6.3) is not necessary and as we shall prove later, neither is (6.2). The next result does show, however, that the Baker-Ericksen, tension-extension, and two-dimensional pressure-compression inequalities are necessary for Hadamard-stability.

Theorem 6.2.[17] *Suppose that* $\underset{\sim}{f}_\lambda$ *is Hadamard-stable. Then*

$$BE_i(\lambda) \geqslant 0, \qquad TE_i(\lambda) \geqslant 0, \qquad PC_{2d}(\lambda) \geqslant 0.$$

Moreover, if $\underset{\sim}{f}_\lambda$ *is strictly Hadamard-stable then all of the inequalities are strict.*

[17] The Baker-Ericksen and tension-extension inequalities are well-known consequences of strong ellipticity; cf., e.g., Truesdell and Noll [44, p.168], Knowles and Sternberg [32], or Simpson and Spector [40]. Strong-ellipticity is a well-known consequence of uniform Hadamard-stability. See, for example, Coral [11], Graves [18] or Truesdell and Noll [44, p.252].

Proof. Consider the functions

$$\tilde{v}^1(\tilde{x}) = \begin{pmatrix} x_1 \\ 0 \\ 0 \end{pmatrix}, \qquad \tilde{v}^2(\tilde{x}) = \begin{pmatrix} 0 \\ 0 \\ \sin(x_3\pi/L) \end{pmatrix}$$

$$\tilde{v}^3(\tilde{x}) = \begin{pmatrix} x_3 \\ 0 \\ 0 \end{pmatrix}, \quad \tilde{v}^4(\tilde{x}) = \begin{pmatrix} x_2 \\ x_1 \\ 0 \end{pmatrix}, \quad \tilde{v}^5(\tilde{x}) = \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix}.$$

It is clear from (1.1)-(1.3) that each $\tilde{v}^i \in \mathrm{Var}$. Thus, if we let $\tilde{U} = \nabla\tilde{v}^i(\tilde{x})$ in (5.3) and integrate over $\mathcal{B}$, we conclude that the integrals

$$\phi(\tilde{v}^i) = \left[\mathrm{Vol}(\mathcal{B})\right]^{-1} \int_{\mathcal{B}} \nabla\tilde{v}^i \cdot \tilde{A}(\nabla\tilde{f}_\lambda)\nabla\tilde{v}^i$$

are given by

$$\phi(\tilde{v}^1) = TE_1, \qquad \phi(\tilde{v}^2) = \frac{\pi^2}{L^2} TE_3 \left(\int_{\mathcal{B}} \cos^2 \frac{x_3\pi}{L}\right) \mathrm{Vol}(\mathcal{B})$$

$$\phi(\tilde{v}^3) = BE_1, \qquad \phi(\tilde{v}^4) = 4BE_3, \qquad \phi(\tilde{v}^5) = 2PC_{2d}.$$

The desired result is now immediate. ∎

We now return to our study of sufficient conditions for Hadamard-stability. If the reference configuration is natural, then $\alpha = \lambda = 1$ in (4.3) will be a solution of (4.1) and (4.2). Thus, for the remainder of this section we assume that the reference configuration is natural and that the loading path contains this reference, that is,

$$\alpha \in C^1\left((\lambda_m, \lambda_M), \mathbb{R}^+\right)$$

with

$$\lambda_m < 1 < \lambda_M , \qquad \alpha(1) = 1.$$

We are now able to show that elongation produces uniaxial tension and that hypothesis (6.3) of Theorem 6.1 is a consequence of hypothesis (6.2) for loading path that contains a positive natural state.

Proposition 6.3. Assume that the reference configuration is natural and positive. Further, suppose that $\lambda_m < 1 < \Lambda < \lambda_M$, $\alpha(1) = 1$, and

$$\frac{d}{d\lambda} S_{33}(\lambda) \geqslant 0 \qquad \text{for all} \quad \lambda \in [1, \Lambda] \quad . \tag{6.10}$$

Then

$$\begin{aligned} &S_{33}(\lambda) > 0 \qquad \text{for all} \qquad \lambda \in (1, \Lambda] , \\ &\alpha(\lambda) < \lambda \qquad \text{for all} \qquad \lambda \in (1, \Lambda] \quad . \end{aligned} \tag{6.11}$$

Proof. We first consider $(6.11)_1$. Note that at $\lambda = 1$, $TE_1 = TE_2 = TE_3 \equiv TE$ and $BE_1 = BE_2 = BE_3 \equiv BE$. Thus, by (6.9),

$$\frac{dS_{33}}{d\lambda} = (PC_{2d})^{-1} \left[(TE)(PC_{2d}) - 2X^2\right] \quad \text{at } \lambda = 1 .$$

If we now use (3.11), (3.12), and (5.7) to simplify the last expression, we find

$$\frac{dS_{33}}{d\lambda} = 3(BE)(PC_c)(PC_c + BE)^{-1} \quad \text{at } \lambda = 1 .$$

Thus, by Proposition 5.3 and the smoothness of α, we conclude that $dS_{33}/d\lambda$ is strictly positive in a neighborhood of $\lambda = 1$. Equation $(6.11)_1$ now follows from (6.10) and the fact that $S_{33}(1) = 0$.

To prove $(6.11)_2$ we first differentiate $\alpha\lambda^{-1}$ to conclude that

$$\frac{d}{d\lambda}\left(\frac{\alpha}{\lambda}\right) = \frac{d\alpha}{d\lambda} - 1 \qquad \text{at} \quad \lambda = 1 .$$

If we combine the last expression with (3.11), (3.12), (5.7), and (6.8) we discover that

$$\frac{d}{d\lambda}\left(\frac{\alpha}{\lambda}\right) = -\frac{3}{2}(PC_c)(PC_c + BE)^{-1} \qquad \text{at} \quad \lambda = 1 \ .$$

Thus, by Proposition 5.3 and the smoothness of α, we conclude that $\frac{d}{d\lambda}(\alpha\lambda^{-1})$ is negative in a neighborhood of $\lambda = 1$ and hence that there is a $\lambda_1 > 1$ such that

$$\alpha(\lambda) < \lambda \qquad \text{for all} \quad \lambda \in (1, \lambda_1) \ . \tag{6.12}$$

Suppose now, for the sake of contradiction, that $(6.11)_2$ does not hold. It is then clear from (6.12) and the continuity of α that there must exist $\lambda \in [\lambda_1, \Lambda]$ such that

$$\alpha(\lambda) = \lambda \ .$$

At this λ, $\underset{\sim}{f}_\lambda(\underset{\sim}{x}) = \lambda\underset{\sim}{x}$, $\nabla\underset{\sim}{f}_\lambda = \lambda\underset{\sim}{I}$, and hence, by (2.1)

$$\underset{\sim}{S}(\lambda) = k\underset{\sim}{I}$$

for some k. We note that $\underset{\sim}{f}_\lambda$ is a solution of (4.1) so that by $(4.1)_5$ we must have $k = 0$ and hence $S_{33}(\lambda) = 0$. This contradicts $(6.11)_1$. Thus $(6.11)_2$ must be satisfied for all $\lambda \in (1, \Lambda]$. ∎

Finally, by Theorem 5.4, Theorem 6.1, and Proposition 6.3 we arrive at the following theorem.

Theorem 6.4. *Assume that the reference configuration is natural and positive. Further, suppose that* $\lambda_m < 1 < \Lambda < \lambda_M$; $\alpha(1) = 1$;

$$BE_i(\lambda) > 1 \ , \quad TE_i(\lambda) > 0 \ , \quad PC_{2d}(\lambda) > 0 \quad \text{for all} \quad \lambda \in (1, \Lambda);$$

$$\frac{d}{d\lambda} S_{33}(\lambda) > 0 \qquad \text{for all} \quad \lambda \in (1, \Lambda) \ .$$

Then there exists an $\epsilon > 0$ *such that* $\underset{\sim}{f}_\lambda$ *is uniformly Hadamard-stable for all* $\lambda \in (1 - \epsilon, \Lambda)$.

7. A GENERALIZED KORN INEQUALITY

Let Ω denote an arbitrary bounded open subset of $\mathbb{R}^3$ and, as is usual, let us denote by $L^2(\Omega)$ the space (of equivalence classes) of square integrable functions on Ω. Define[18]

$$H^1(\Omega) \equiv \text{completion of } \left\{ \underset{\sim}{u} \in C^1(\Omega, \mathbb{R}^3) : \|\underset{\sim}{u}\|_{1,\Omega} < \infty \right\} ,$$

$$H^1_{Var}(\mathcal{B}) \equiv \text{completion of } \left\{ \underset{\sim}{u} \in \text{Var} : \|\underset{\sim}{u}\|_{1,\mathcal{B}} < \infty \right\} ,$$

with respect to the norm

$$\|u\|^2_{1,\Omega} \equiv \int_\Omega |u|^2 + \int_\Omega |\nabla u|^2 .$$

For the remainder of this section $\mathcal{B}$ will denote an arbitrary properly regular region in $\mathbb{R}^3$. We shall write

$$\Omega \subset\subset \mathcal{B}$$

provided that the closure of Ω is contained in $\mathcal{B}$. Define

$$J(\underset{\sim}{x}) \equiv \begin{cases} k \exp\left[-1/(1 - |\underset{\sim}{x}|^2)\right] & \text{if } |\underset{\sim}{x}| < 1 \\ 0 & \text{if } |\underset{\sim}{x}| \geqslant 1 \end{cases} ,$$

where $k > 0$ is chosen so that

$$\int_{\mathbb{R}^3} J(\underset{\sim}{x}) = 1 . \qquad (7.1)$$

For $\epsilon > 0$ the function

$$J_\epsilon(\underset{\sim}{x}) \equiv \epsilon^{-3} J(\epsilon^{-1} \underset{\sim}{x})$$

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[18] Cf., e.g., Adams [1].

is called a mollifier. J_ϵ is a C^∞ function, it satisfies (7.1), and it is identically zero outside a ball of radius ϵ. If $\underset{\sim}{u} \in L^2(\mathcal{B})$ we denote the ϵ-mollification of $\underset{\sim}{u}$ by

$$\underset{\sim}{u}^\epsilon(\underset{\sim}{x}) \equiv (J_\epsilon * u)(\underset{\sim}{x}) \equiv \int_{\mathcal{B}} J_\epsilon(\underset{\sim}{x}-\underset{\sim}{y})\underset{\sim}{u}(\underset{\sim}{y})d\underset{\sim}{y} \quad .$$

The following properties of the ϵ-mollification are well known

Proposition 7.1.[19] Let $\underset{\sim}{u} \in H^1(\mathcal{B})$ and $\Omega \subset\subset \mathcal{B}$. Then $\underset{\sim}{u}^\epsilon \in H^1(\Omega)$, $\underset{\sim}{u}^\epsilon \in C^\infty(\Omega, \mathbb{R}^3)$,

$$\nabla(J_\epsilon * \underset{\sim}{u}) = J_\epsilon * (\nabla \underset{\sim}{u}) \quad \text{in} \quad H^1(\Omega) \ ,$$

$$\lim_{\epsilon \to 0^+} \|\underset{\sim}{u}^\epsilon - \underset{\sim}{u}\|_{1,\Omega} = 0 \ .$$

Proposition 7.2[20] (Rellich Compactness Theorem). If $\underset{\sim}{u}^m \in H^1(\mathcal{B})$ is a bounded sequence then there exists a subsequence $\underset{\sim}{u}^{m_j}$ that is a Cauchy sequence in $L^2(\mathcal{B})$.

Finally, we will need the following generalization of Korn's inequality[21].

Theorem 7.3. Let $\underset{\sim}{L}^j \in \mathrm{Lin}$ be given for $j = 1, 2, \ldots, J$. Further, suppose that there are constants α_{ijkpq}, $i, k, p, q = 1, 2, 3$ and $j = 1, 2, \ldots, J$, such that

$$\frac{\partial u_i}{\partial x_p \partial x_q} = \sum_{j=1}^{J} \sum_{k=1}^{3} \alpha_{ijkpq} \frac{\partial}{\partial x_k} (\underset{\sim}{L}^j \cdot \nabla \underset{\sim}{u}) \tag{7.2}$$

for any $u \in C^2(\mathcal{B}, \mathbb{R}^3)$. Then there exists a constant $\kappa > 0$ such that

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[19] Cf., e.g., Adams [1, pp. 30 and 52].

[20] Cf., e.g., Fichera [16, p. 352].

[21] Aronszajn [5] developed inequalities of this type for scalar equations on domains with smooth boundaries. Smith [42] extended Aronszajn's result to more general domains by developing an alternative method of proof involving singular integrals. Gobert [17] used Smith's method to prove the standard Korn inequality. Another extension of Aronszajn's result is due to De Figueiredo [12] who considered a large class of systems on domains with smooth boundary. See also Campanato [9].

$$\sum_{j=1}^{J} \int_{\mathcal{B}} (\underset{\sim}{L}^j \cdot \nabla \underset{\sim}{u})^2 + \int_{\mathcal{B}} |\underset{\sim}{u}|^2 \geqslant \kappa \|\underset{\sim}{u}\|_{1,\mathcal{B}}^2 \qquad (7.3)$$

for all $\underset{\sim}{u} \in H^1(\mathcal{B})$.

This generalization of Korn's inequality is a simple consequence of either Duvaut and Lions' [13] proof or Gobert's [17] proof of the Korn inequality[22]. If one examines either proof one observes that a crucial step in the proof is the identity[23]

$$\frac{\partial^2 u_i}{\partial x_p \partial x_q} = \frac{1}{2} \frac{\partial}{\partial x_p}(u_{i,q} + u_{q,i}) + \frac{1}{2} \frac{\partial}{\partial x_q}(u_{i,p} + u_{p,i}) - \frac{1}{2} \frac{\partial}{\partial x_i}(u_{p,q} + u_{q,p}) .$$

If this identity is replaced with (7.2), it is clear that either proof will yield (7.3) instead of the standard Korn inequality.

We note that equation (7.2) can be characterized algebraically. Let us define nine 3-tensors $\underset{\sim}{B}^{ipq}$, $i, p, q = 1, 2, 3$, by

$$B^{ipq}_{\ell mk} \equiv \sum_{j=1}^{J} \alpha_{ijkpq} L^j_{\ell m} \quad . \qquad (7.4)$$

Then (7.2) is equivalent to

$$\sum_{k,\ell,m=1}^{3} B^{ipq}_{\ell mk} E_{\ell mk} = E_{ipq} \qquad (7.5)$$

for every 3-tensor $\underset{\sim}{E}$ that is symmetric in its last two arguments, i.e., $E_{ipq} = E_{iqp}$.

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22 Gobert's proof is the appropriate one for our purposes since the region we consider does not have C^1 boundary.

23 Cf. Gobert [17, p. 138] and Duvaut and Lions [13, eq. 3.16].

It is clear that for the $\underset{\sim}{B}^{ipq}$ to satisfy equation (7.5) they must be of the form

$$B^{ipq}_{\ell mk} = \delta^{\ell}_{i}\delta^{m}_{p}\delta^{k}_{q} + W^{ipq}_{\ell mk} \quad , \tag{7.6}$$

where $\underset{\sim}{W}^{ipq}$ are any 3-tensors that are skew in their last two arguments, i.e., $W^{ipq}_{\ell mk} = -W^{ipq}_{\ell km}$. It follows from (7.6) that

$$B^{ipq}_{\ell mk} + B^{ipq}_{\ell km} = \delta^{\ell}_{i}\left[\delta^{m}_{p}\delta^{k}_{q} + \delta^{k}_{p}\delta^{m}_{q}\right] . \tag{7.7}$$

Define three $J \times 3$ matrices $\underset{\sim}{\Lambda}^{\ell}$ and nine $3 \times J$ matrices $\underset{\sim}{A}^{ipq}$ by

$$\Lambda^{\ell}_{jm} \equiv L^{j}_{\ell m} \quad , \qquad A^{ipq}_{kj} \equiv \alpha_{ijkpq} \; .$$

Then (7.7) can be rewritten, with the aid of (7.4), as

$$\underset{\sim}{H}_{\ell}(\underset{\sim}{A}^{ipq}) = \delta^{\ell}_{i}\left[\underset{\sim}{e}^{p} \otimes \underset{\sim}{e}^{q} + \underset{\sim}{e}^{q} \otimes \underset{\sim}{e}^{p}\right] , \tag{7.8}$$

where $\underset{\sim}{H}_{\ell}$ is the linear mapping from $J \times 3$ matrices to symmetric 3×3 matrices defined by

$$\underset{\sim}{H}_{\ell}(\underset{\sim}{A}) = \underset{\sim}{A}\underset{\sim}{\Lambda}^{\ell} + (\underset{\sim}{A}\underset{\sim}{\Lambda}^{\ell})^{T} \; .$$

We note the equation (7.8) is just the requirement that the mapping $\underset{\sim}{H} \equiv (H_1, H_2, H_3)$,

$$\underset{\sim}{H} : M^{J \times 3} \longrightarrow \mathrm{Sym} \times \mathrm{Sym} \times \mathrm{Sym} \quad ,$$

be onto. Since the dimension of Sym is 6 this requires that[24] $J \geqslant 6$.

One possible sufficient condition for $\underset{\sim}{H}$ to be onto is that the dimension of the null space be equal to $3J - 6$. To date we have been unable to find a simple method for calculating the dimension of the null space and have therefore found it easier to find the α_{ijkpq} that satisfy (7.2).

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[24] In n dimensions this generalizes to $J \geqslant n(n+1)/2$. A Korn type inequality that only requires $2n - 1$ operators has been obtained by De Figueiredo [12, p. 66], but all of his operators L^{j} are rank one.

8. FINAL RESULTS

By using the generalized Korn inequality we can prove a stronger version of Theorem 6.4.

Theorem 8.1. *Assume that the reference configuration is natural and positive. Further, suppose that* $\lambda_m < 1 < \Lambda < \lambda_M$, $\alpha(1) = 1$;

(i) $BE_i(\lambda) > 0, \quad TE_i(\lambda) > 0 \ , \quad PC_{2d}(\lambda) > 0$ *for all* $\lambda \in (1, \Lambda]$;

(ii) $\frac{d}{d\lambda} S_{33}(\lambda) \geqslant 0$ *for all* $\lambda \in [1, \Lambda]$.

Then there exists an $\epsilon > 0$ *such that* $\underset{\sim}{f}_\lambda$ *is uniformly Hadamard-stable for all* $\lambda \in (1 - \epsilon, \Lambda + \epsilon)$.

Remark. If one assumes that the Baker-Ericksen, tension-extension, and two-dimensional pressure-compression inequalities do not fail in uniaxial tension, then Theorem 8.1 says that bifurcation cannot occur until *after* the load attains its maximum value.

We note that Theorem 8.1 is an immediate consequence of Theorem 5.1, Theorem 5.4, Proposition 6.3, and the following theorem.

Theorem 8.2. *Suppose that for some* $\lambda \in (\lambda_m, \lambda_M)$

(i) $BE_i(\lambda) > 0 \ , \quad TE_i(\lambda) > 0 \ , \quad PC_{2d}(\lambda) > 0$;

(ii) $\frac{d}{d\lambda} S_{33}(\lambda) \geqslant 0$;

(iii) $\alpha(\lambda) < \lambda$.

Then $\underset{\sim}{f}_\lambda$ *is uniformly Hadamard-stable.*

Proof. If the inequality in (ii) is strict then the result follows from Theorem 6.1. Thus we assume that

$$\frac{d}{d\lambda} S_{33}(\lambda) = 0 \ . \tag{8.1}$$

We first obtain a lower bound for the quadratic form (5.3). By direct calculation we find that the matrix $\underset{\sim}{M}$, given by (5.4), satisfies

$$\underset{\sim}{d} \cdot \underset{\sim}{M}\underset{\sim}{d} = (TE_3)^{-1}(Xd_1 + Xd_2 + TE_3 d_3)^2 + BE_3(d_1 - d_2)^2 \quad . \tag{8.2}$$

Define

$$k_0 = \min\left\{(TE_3)^{-1},\ BE_3,\ BE_1(1 - \frac{\alpha}{\lambda})\right\}$$

and note that $k_0 > 0$ by (i) and (iii). It follows from (5.3) - (5.6) and (8.2) that

$$\underset{\sim}{U} \cdot \underset{\sim}{A}(\nabla \underset{\sim}{f}_\lambda)\underset{\sim}{U} \geqslant k_0 \sum_{i=1}^{7} |\underset{\sim}{L}^i \cdot \underset{\sim}{U}|^2 \tag{8.3}$$

for all $\underset{\sim}{U} \in \mathrm{Lin}$, where $\underset{\sim}{L}^i \in \mathrm{Lin}$ are defined by the relations

$$\underset{\sim}{L}^1 \cdot \underset{\sim}{U} \equiv XU_{11} + XU_{22} + TE_3 U_{33} \ ,$$

$$\underset{\sim}{L}^2 \cdot \underset{\sim}{U} \equiv U_{11} - U_{22} \ , \qquad \underset{\sim}{L}^3 \cdot \underset{\sim}{U} \equiv U_{12} + U_{21} \ , \tag{8.4}$$

$$\underset{\sim}{L}^4 \cdot \underset{\sim}{U} \equiv U_{13} \ , \quad \underset{\sim}{L}^5 \cdot \underset{\sim}{U} \equiv U_{31} \ , \quad \underset{\sim}{L}^6 \cdot \underset{\sim}{U} \equiv U_{23} \ , \quad \underset{\sim}{L}^7 \cdot \underset{\sim}{U} \equiv U_{32} \ .$$

If we now take $\underset{\sim}{U} = \nabla \underset{\sim}{u}(\underset{\sim}{x})$ in (8.3) and integrate over $\mathcal{B}$ we conclude that

$$\int_{\mathcal{B}} \nabla \underset{\sim}{u} \cdot \underset{\sim}{A}(\nabla \underset{\sim}{f}_\lambda)\nabla \underset{\sim}{u} \geqslant k_0 \sum_{i=1}^{7} \int_{\mathcal{B}} |\underset{\sim}{L}^i \cdot \nabla \underset{\sim}{u}|^2 \tag{8.5}$$

for all $\underset{\sim}{u} \in C^1(\mathcal{B}, \mathbb{R}^3)$.

Since $X \neq 0$ (by (i), (6.9), and (8.1)) the desired result now follows from (i), (8.5), and the following proposition. ∎

Proposition 8.3. *Let $\underset{\sim}{L}^i \in \mathrm{Lin}$ be defined by (8.4). Further suppose that $TE_3, X \neq 0$. Then there exists a constant $k_1 > 0$ such that*

$$\sum_{i=1}^{7} \int_{\mathcal{B}} |\underset{\sim}{L}^i \cdot \nabla \underset{\sim}{u}|^2 \geqslant k_1 \|\underset{\sim}{u}\|^2_{1,\mathcal{B}} \tag{8.6}$$

for all $\underset{\sim}{u} \in H^1_{Var}(\mathcal{B})$.

To prove this result we will need the following

Lemma. *Let $\underset{\sim}{L}^i \in \text{Lin}$ be defined by (8.4) and suppose that $TE_3, X \neq 0$. If $\underset{\sim}{u} \in C^1(\Omega, \mathbb{R}^3)$ satisfies*

$$\underset{\sim}{L}^i \cdot \nabla \underset{\sim}{u} = 0 \quad \text{in } \Omega \tag{8.7}$$

then there exist constants $\alpha, \beta \in \mathbb{R}$, $\underset{\sim}{b} \in \mathbb{R}^3$ such that

$$\underset{\sim}{u}(\underset{\sim}{x}) = \underset{\sim}{W}\underset{\sim}{x} + \underset{\sim}{b} , \tag{8.8}$$

$$\underset{\sim}{W} \equiv \begin{pmatrix} \alpha TE_3 & \beta & 0 \\ -\beta & \alpha TE_3 & 0 \\ 0 & 0 & -2\alpha X \end{pmatrix} . \tag{8.9}$$

Proof.[25] If we let $i = 4, 5, 6, 7$ in (8.7) and use $(8.4)_3$ we find that

$$u_1 = u_1(x_1, x_2) , \quad u_2 = u_2(x_1, x_2) , \quad u_3 = u_3(x_3) . \tag{8.10}$$

Also, if we let $i = 1, 2$ in (8.7) and use $(8.4)_{1,2}$ we find that

$$\begin{aligned} 2Xu_{1,1} + TE_3 u_{3,3} &= 0 , \\ u_{1,1} - u_{2,2} &= 0 . \end{aligned} \tag{8.11}$$

---

[25] Alternatively, we could show that the operators L^j satisfy (7.2) and hence that $\underset{\sim}{u}$ must be affine in order to satisfy (8.7). Equation (8.9) then follows from the side conditions.

One can easily show from (8.10) and (8.11) that

$$u_1(x_1, x_2) = \alpha TE_3 x_1 + f(x_2) ,$$
$$u_2(x_1, x_2) = \alpha TE_3 x_2 + g(x_1) , \qquad (8.12)$$
$$u_3(x_3) = 2Xx_3 + b_3 .$$

If we now differentiate $(8.12)_1$ with respect to x_2 and $(8.12)_2$ with respect to x_1 we find, with the aid of $(8.4)_2$ and (8.7) (with $i = 3$), that $f'(x_2) + g'(x_1) = 0$ and hence that

$$f(x_2) = \beta x_2 + b_1 , \qquad g(x_1) = -\beta x_1 + b_2 . \qquad (8.13)$$

Equations (8.8) and (8.9) now follow from (8.12) and (8.13). ∎

Proof of Proposition 8.3. A simple computation[26] shows that the $\underset{\sim}{L}^i$ satisfy (7.2) provided that $X, TE_3 \neq 0$. Thus we can apply Theorem 7.3 to conclude that

$$\sum_{i=1}^{7} \int_{\mathcal{B}} |\underset{\sim}{L}^i \cdot \nabla \underset{\sim}{u}|^2 + \int_{\mathcal{B}} |\underset{\sim}{u}|^2 \geqslant k \|\underset{\sim}{u}\|^2_{1,\mathcal{B}} \qquad (8.14)$$

for all $\underset{\sim}{u} \in H^1(\mathcal{B})$.

Let us assume, for the sake of contradiction, that there is a sequence $\underset{\sim}{u}^m \in H^1_{Var}(\mathcal{B})$ such that

$$\|\underset{\sim}{u}^m\|_{1,\mathcal{B}} = 1 , \quad \sum_{i=1}^{7} \int_{\mathcal{B}} |\underset{\sim}{L}^i \cdot \nabla \underset{\sim}{u}^m|^2 \longrightarrow 0 . \qquad (8.15)$$

Since $\underset{\sim}{u}^m$ is bounded in $H^1(\mathcal{B})$ we can apply the Rellich Compactness Theorem to deduce the existence of a subsequence $\underset{\sim}{u}^{m_j}$ that is Cauchy in $L^2(\mathcal{B})$. It then follows from (8.14) and the second equation in (8.15) that $\underset{\sim}{u}^{m_j}$ is Cauchy in $H^1(\mathcal{B})$. We conclude that there exists a $\underset{\sim}{u} \in H^1_{Var}(\mathcal{B})$ such that

[26] Cf. Appendix.

$$\underset{\sim}{L}^i \cdot \nabla \underset{\sim}{u} = 0 \quad \text{a.e. in } \mathcal{B} \,, \qquad \|\underset{\sim}{u}\|_{1,\mathcal{B}} = 1 \,. \tag{8.16}$$

Now[27], let $\underset{\sim}{u}^\epsilon$ be the ϵ-mollification of $\underset{\sim}{u}$. Then by Proposition 7.1 and equation (8.16) we find that for any $\Omega \subset\subset \mathcal{B}$, $\underset{\sim}{u}^\epsilon \in C^\infty(\Omega, \mathbb{R}^3)$ and

$$\underset{\sim}{L}^i \cdot \nabla \underset{\sim}{u}^\epsilon = 0 \quad \text{in} \quad \Omega \,.$$

Thus, by the Lemma, we find that

$$\underset{\sim}{u}^\epsilon(\underset{\sim}{x}) = \underset{\sim}{W}^\epsilon \underset{\sim}{x} + \underset{\sim}{b}^\epsilon \quad \text{in} \quad \Omega \,,$$

where $\underset{\sim}{W}^\epsilon$ has the form (8.9).

If we once again use Proposition 7.1 we conclude that there is a vector $\underset{\sim}{b}^\Omega \in \mathbb{R}^3$ and a matrix $W^\Omega \in \text{Lin}$ (of the form (8.9)) such that

$$\underset{\sim}{u}(\underset{\sim}{x}) = \underset{\sim}{W}^\Omega \underset{\sim}{x} + \underset{\sim}{b}^\Omega \quad \text{in} \quad \Omega \,. \tag{8.17}$$

Define

$$\Omega_i = \left\{ \underset{\sim}{x} \in \mathcal{B} : \text{dist}(\underset{\sim}{x}, \partial\mathcal{B}) > 1/i \right\}$$

so that

$$\Omega_i \subset \Omega_{i+1} \,, \qquad \bigcup_{i=1}^{\infty} \Omega_i = \mathcal{B} \,.$$

It is clear from this construction and (8.17) that

$$\underset{\sim}{u}(\underset{\sim}{x}) = \underset{\sim}{W} \underset{\sim}{x} + \underset{\sim}{b} \quad \text{in} \quad \mathcal{B} \,,$$

where $\underset{\sim}{W}$ has the form (8.9).

[27] Alternatively, following Fichera [16], we could show that $\underset{\sim}{u}$ is a (weak) solution of a homogeneous linear elliptic system with constant coefficients and hence that it is C^∞ by elliptic regularity theory. The use of mollifiers to obtain a more elementary proof was suggested to us by Professor D. Kinderlehrer.

We now observe that by (1.1) and (1.3) $b_1 = b_2 = 0$, while by (1.4) $\beta = 0$ (cf. (8.9)). In addition we note that u_3 is a linear function that is zero at $x_3 = 0, L$ (by (1.2)). Thus $\underset{\sim}{u}$ is identically zero on $\mathcal{B}$. This contradicts the second equation in (8.16) and hence no such sequence $\underset{\sim}{u}^m$ can exist. Equation (8.6) is now immediate. ∎

9. SOME EXAMPLES

We consider two special constitutive relations that were proposed by Blatz and Ko [8] to fit experimental data[28]. For solid rubbers they derived the relation

$$\sigma(\underset{\sim}{F}) = \frac{1}{2} \underset{\sim}{F} \cdot \underset{\sim}{F} + \frac{1}{m} (\det \underset{\sim}{F})^{-m} , \tag{9.1}$$

with $m = 13.3$, while for foam rubbers they concluded

$$\sigma(\underset{\sim}{F}) = \frac{1}{4} \left[(\underset{\sim}{F} \cdot \underset{\sim}{F})^2 - \underset{\sim}{F}\underset{\sim}{F}^T \cdot \underset{\sim}{F}\underset{\sim}{F}^T \right] (\det \underset{\sim}{F})^{-2} + \det \underset{\sim}{F} . \tag{9.2}$$

We first consider equation (9.2). A simple computation shows that[29]

$$t_i = 1 - (\lambda_1 \lambda_2 \lambda_3)^{-1} \lambda_i^{-2} ,$$

and hence

$$BE_i = (\lambda_1 \lambda_2 \lambda_3)^{-2} \lambda_i^{+2} > 0 ,$$

$$TE_i = 3\lambda_i^{-4} > 0 ,$$

$$PC_{2d} = 4\alpha^{-5} > 0 .$$

Also

$$\alpha = \lambda^{-1/4} , \qquad S_{33}(\lambda) = \lambda^{-1/2} - \lambda^{-3}$$

---

[28] Cf. Knowles and Sternberg [31, pp. 349-350] for a discussion of the validity of Blatz and Ko's [8] data fitting.

[29] Cf. Knowles and Sternberg [31].

so that (see figure 1)

$$\frac{d}{d\lambda} S_{33}(\lambda) > 0 \quad \text{for} \quad \lambda \in (0, 6^{2/5} \approx 2.05) .$$

We conclude, by Theorem 8.1, that the material given by (9.2) is uniformly Hadamard-stable in uniaxial tension for $\lambda \in (1-\epsilon, 6^{2/5}+\epsilon)$. We note that Knowles and Sternberg [31] have shown that the underlying equations are strongly-elliptic if and only if $\lambda \in (r^{-1}, r)$, where $r \equiv (2-3^{1/2})^{-4/5} \approx 2.87$. It follows that the material is unstable in uniaxial tension for $\lambda \in (r, \infty)$. No conclusion can be drawn at this time in the interval $(6^{2/5}+\epsilon, r)$.

Let us now consider equation (9.1). A simple computation shows that[30]

$$t_i = \lambda_i^2(\lambda_1\lambda_2\lambda_3)^{-1} - (\lambda_1\lambda_2\lambda_3)^{-m-1}$$

and hence

$$BE_i = 1 > 0 ,$$

$$TE_i = 1 + (m+1)(\lambda_1\lambda_2\lambda_3)^{-m}\lambda_i^{-2} > 0 ,$$

$$PC_{2d} = (m+1)\lambda(\alpha\lambda)^{-m-1} > 0 .$$

Also

$$\alpha = \lambda^{-m/(m+2)} , \qquad S_{33}(\lambda) = \lambda - \lambda^{-(2m+1)/(m+1)} ,$$

so that (see figure 1)

$$\frac{d}{d\lambda} S_{33}(\lambda) > 0 \quad \text{for} \quad \lambda \in (0, \infty) .$$

We conclude, by Theorem 8.1, that the material given by (9.1) is <u>always</u> uniformly Hadamard-stable in uniaxial tension.

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[30] Cf. Simpson and Spector [41].

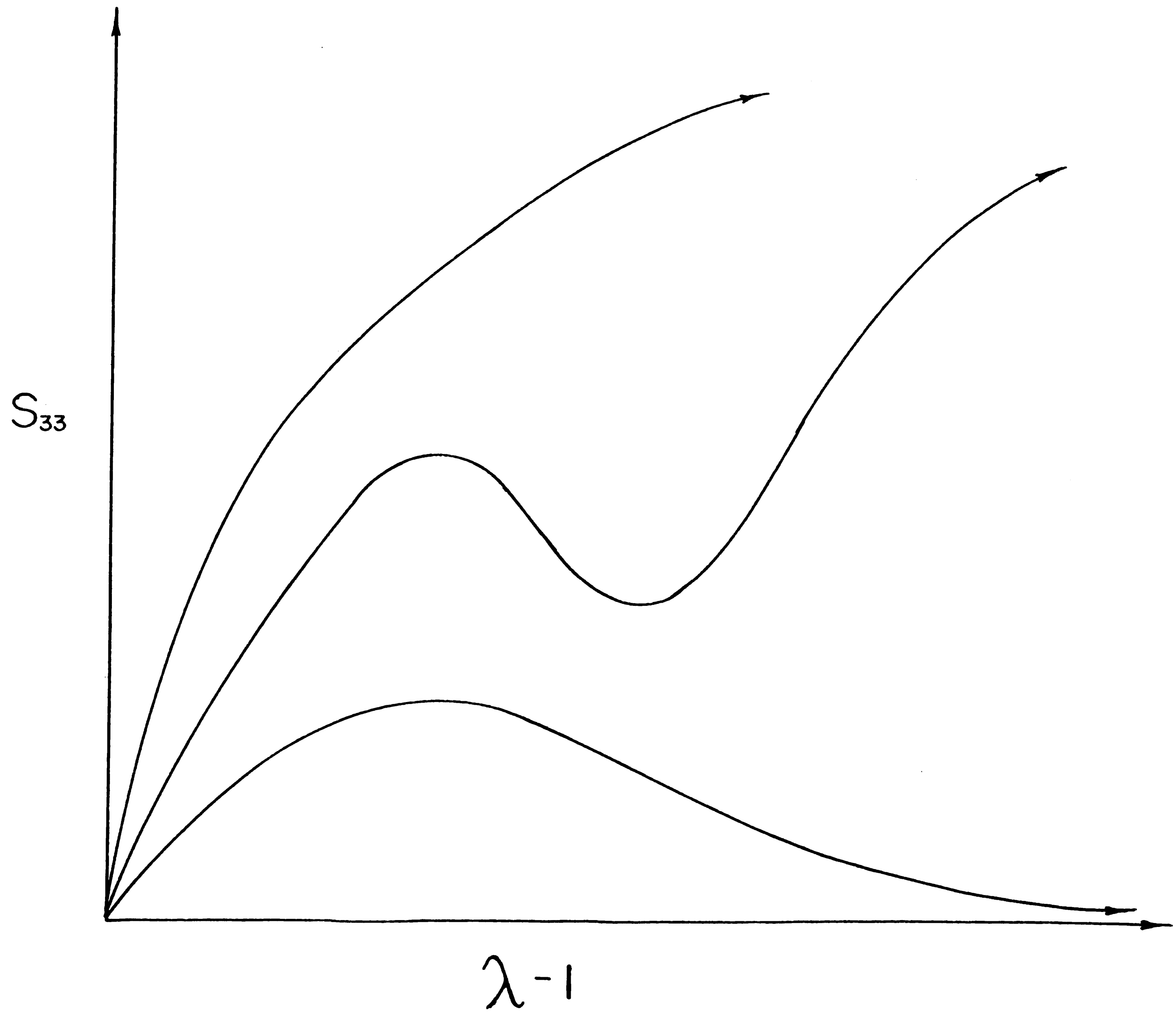

Fig. 1. The uppermost curve gives S_{33} vs. λ for the constitutive relation (9.1) while the lowermost curve corresponds to equation (9.2). We conjecture that there are stored energy functions that yield the middle curve yet never lose Hadamard stability in uniaxial tension.

10. A MODIFIED PROBLEM. THE SOFT DEVICE.

We now consider the boundary-value problem (4.1) - (4.2) with the displacement condition $(4.1)_5$ replaced by the traction condition

$$S_{33}(\nabla \underset{\sim}{f}) = \hat{p} \qquad \text{on} \quad \mathcal{R} \times \{L\} , \tag{10.1}$$

where $\hat{p} \in \mathbb{R}$ is a given (dead) load that is applied to the top surface of the cylinder.

Let us define $p : (\lambda_m, \lambda_M) \longrightarrow \mathbb{R}$ by

$$p(\lambda) = S_{33}(\nabla \underset{\sim}{f}_\lambda) \quad , \tag{10.2}$$

where $\underset{\sim}{f}_\lambda$ is the solution to (4.1) and (4.2) that is given by (4.3) and (4.6). Then $\underset{\sim}{f}_\lambda$ is also a solution to $(4.1)_{1-4}$, (10.1), and (10.2) if $\hat{p} = p(\lambda)$. Thus, we will view (10.2) as a parametrization of the loading and consider the solution $\underset{\sim}{f}_\lambda$ to our modified problem.

Define

$$\text{Var}^* \equiv \left\{ \underset{\sim}{u} \in C^1(\mathcal{B}, \mathbb{R}^3) : \underset{\sim}{u} \neq 0 \text{ and } \underset{\sim}{u} \text{ satisfies (1.3), (1.4), and (10.3)} \right\} ,$$

where

$$u_3 = 0 \quad \text{on} \quad \text{cl}(\mathcal{R}) \times \{0\} , \tag{10.3}$$

and note that the Korn and Poincaré inequalities (in the form (1.5)) are still valid if Var is replaced by Var^*. Also, let us replace Var by Var^* in the definition of Hadamard stability. Then using the techniques of Section 6 we obtain necessary and sufficient conditions for Hadamard-stability.

Theorem 10.1. *Suppose that $\underset{\sim}{f}_\lambda$ is Hadamard-stable. Then*

$$BE_i(\lambda) \geqslant 0 , \quad TE_i(\lambda) \geqslant 0 , \quad PC_{2d}(\lambda) \geqslant 0 ,$$

and

$$\frac{dp}{d\lambda} = \frac{d}{d\lambda} S_{33}(\underset{\sim}{f}_\lambda) \geqslant 0 . \tag{10.4}$$

Moreover, if $\underset{\sim}{f}_\lambda$ is strictly Hadamard-stable then all of the inequalities are strict.

Proof. Equation $(10.4)_1$ follows from Theorem 6.2. To prove $(10.4)_2$ consider the function

$$\underset{\sim}{v}(\underset{\sim}{x}) = \begin{pmatrix} a_1 x_1 \\ a_2 x_2 \\ a_3 x_3 \end{pmatrix} .$$

It is clear from (1.3), (1.4), and (10.3) that $\underset{\sim}{v} \in \mathrm{Var}^*$. Thus, if we let $\underset{\sim}{U} = \nabla \underset{\sim}{v}(\underset{\sim}{x})$ in (5.3) and integrate over $\mathcal{B}$, we conclude that

$$\int_{\mathcal{B}} \nabla \underset{\sim}{v} \cdot \underset{\sim}{A}(\nabla \underset{\sim}{f}_\lambda) \nabla \underset{\sim}{v} = (\underset{\sim}{a} \cdot \underset{\sim}{M} \underset{\sim}{a}) \mathrm{Vol}(\mathcal{B}) .$$

Thus $\underset{\sim}{M}$ must be positive definite for $\underset{\sim}{f}_\lambda$ to be Hadamard-stable. The desired result now follows from $(6.9)_2$. ∎

Theorem 10.2. Assume that the reference configuration is natural and positive. Further, suppose that $\lambda_m < 1 < \Lambda < \lambda_M$; $\alpha(1) = 1$;

$$BE_i(\lambda) > 0 , \quad TE_i(\lambda) > 0 , \quad PC_{2d}(\lambda) > 0 \qquad \underline{\text{for all}} \ \lambda \in (1, \Lambda) ;$$

$$\frac{dp}{d\lambda} = \frac{d}{d\lambda} S_{33}(\lambda) > 0 \qquad \underline{\text{for all}} \ \lambda \in (1, \Lambda) .$$

Then there exists an $\epsilon > 0$ such that $\underset{\sim}{f}_\lambda$ is uniformly Hadamard-stable for all $\lambda \in (1 - \epsilon, \Lambda)$.

Remark. If one assumes that the Baker-Ericksen, tension-extension and two-dimensional pressure compression inequalities do not fail in uniaxial tension, then Theorems 10.1 and 10.2 imply that bifurcation cannot occur before the load attains its maximum value, but that it might occur as soon as this maximum value is attained.

11. DISCUSSION

We first note that it would probably be of interest to investigate the consequences of the failure of either the Baker-Ericksen, tension-extension or two-dimensional pressure-compression inequalities. In particular, the failure of the tension-extension inequality seems to give rise to multiple homogeneous solutions (of the form (4.3)), while the failure of the two-dimensional pressure-compression inequality appears to lead to symmetry breaking.

We will henceforth assume that the above-mentioned inequalities do not fail in uniaxial tension and discuss the possibility of using the maximum-load criteria and finite elastostatics to predict localization.

The total energy is given by the sum of the total stored energy and the energy potential of the loads. For the soft loading device the total energy is given by

$$E(\underset{\sim}{f}) = \int_{\mathcal{B}} \sigma(\nabla \underset{\sim}{f}) - \int_{\mathcal{R} \times \{L\}} p f_3$$

while for the hard loading device

$$E(\underset{\sim}{f}) = \int_{\mathcal{B}} \sigma(\nabla \underset{\sim}{f}) \; .$$

Since we are attempting to describe localization as a continuous quasistatic phenomenon it seems reasonable that we should require each solution $\underset{\sim}{f}_{\lambda}$ be (at least) a local[31] minimizer of the appropriate energy. If we differentiate the energy twice and apply the second derivative test[32], we find that a necessary condition for a solution to be a local minimizer is that it is Hadamard-stable. Thus,

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[31] We do not require the solutions to be global minimum since we are also interested in metastable states.

[32] Cf., e.g., Truesdell and Noll [44, p. 327].

we will not consider solutions of the equilibrium equations[33] that are not Hadamard-stable.

When we consider the soft loading device from the viewpoint of the preceding paragraph, we find that either bifurcation must occur when the load reaches a local maximum or localization does not occur by continuous bifurcation. Since it is unlikely that a nonhomogeneous solution bifurcates off at maximum load[34] we conclude that <u>for the soft loading device finite elastostatics is probably not suitable for predicting localization as a continuous bifurcation phenomenon.</u>

No such difficulties are encountered in the hard loading device since Hadamard-stability is not lost at maximum load. Thus, it may be possible to use finite elastostatics and bifurcation theory to predict the onset of localization.

APPENDIX

We show that the operators $\underset{\sim}{L}^j$ defined by (8.4) satisfy equation (7.2).

$$u_{3,k1} = u_{3,1k} = \frac{\partial}{\partial x_k}(u_{3,1}) , \qquad \text{(5 components)}$$

$$u_{3,k2} = u_{3,2k} = \frac{\partial}{\partial x_k}(u_{3,2}) , \qquad \text{(3 components)}$$

$$u_{1,k3} = u_{1,3k} = \frac{\partial}{\partial x_k}(u_{1,3}) , \qquad \text{(5 components)}$$

$$u_{2,k3} = u_{2,3k} = \frac{\partial}{\partial x_k}(u_{2,3}) , \qquad \text{(5 components)}$$

$$2Xu_{1,11} = \frac{\partial}{\partial x_1}(Xu_{1,1} + Xu_{2,2} + TE_3u_{3,3}) + X\frac{\partial}{\partial x_1}(u_{1,1} - u_{2,2}) - TE_3u_{3,13} ,$$

---

[33] Equations (4.1) and (4.2) or Equations $(4.1)_{1-4}$, (4.2), and (10.1).

[34] One can show that at maximum load a second <u>homogeneous</u> solution bifurcates off and that the null space of the linearized problem is one-dimensional. Thus, if standard bifurcation theory applies no other solutions bifurcate off at maximum load.

$$2Xu_{2,22} = \frac{\partial}{\partial x_2}(Xu_{1,1}+Xu_{2,2}+TE_3u_{3,3}) - X\frac{\partial}{\partial x_2}(u_{1,1}-u_{2,2}) - TE_3u_{3,23} ,$$

$$TE_3u_{3,33} = \frac{\partial}{\partial x_3}(Xu_{1,1}+Xu_{2,2}+TE_3u_{3,3}) - Xu_{1,31} - Xu_{2,32} ,$$

$$u_{1,22} = \frac{\partial}{\partial x_2}(u_{1,2}+u_{2,1}) + \frac{\partial}{\partial x_1}(u_{1,1}-u_{2,2}) - u_{1,11} ,$$

$$u_{2,11} = \frac{\partial}{\partial x_1}(u_{1,2}+u_{2,1}) - \frac{\partial}{\partial x_2}(u_{1,1}-u_{2,2}) - u_{2,22} ,$$

$$u_{1,12} = u_{1,21} = \frac{\partial}{\partial x_1}(u_{1,2}+u_{2,1}) - u_{2,11} , \qquad \text{(2 components)}$$

$$u_{2,12} = u_{2,21} = \frac{\partial}{\partial x_2}(u_{1,2}+u_{2,1}) - u_{1,22} . \qquad \text{(2 components)}$$

Thus all 27 components, $u_{i,pq}$, have been written in the form (7.2).

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