MULTIGRAPHS WITH THE MOST EDGE COVERS

By

Bruce Hajek

and

Babu Narayanan

IMA Preprint Series # 1230

April 1994

Multigraphs with the Most Edge Covers

Bruce Hajek   Babu Narayanan

Institute for Mathematics and Its Applications

University of Minnesota, Minneapolis, MN 55455

Among multigraphs with m edges and n vertices, with n even, it is conjectured that the number of edge covers of cardinality r is maximized by the graph with $n/2$ components, each with two vertices and a or $a+1$ parallel edges, where a is the integer part of $2m/n$. In this note, the conjecture is proved for $r = n/2$ (in which case the coverings are perfect matchings) and for $n/2 \leq m \leq n$.

AMS 1980 subject classification: Primary 05C35; Secondary 05C30, 05C70.

Email addresses of authors: b-hajek@uiuc.edu babu@ima.umn.edu

1 INTRODUCTION

Informally, a multigraph is a graph in which there can be multiple edges between pairs of vertices, and both endpoints of an edge can be the same. Formally, a multigraph $G = (V, E)$ consists of a finite set of vertices V and a finite set of edges E. Each edge e has the form (ω, W), where ω is the label of edge e, taking values in some arbitrary set, and $W \subset V$ is the set of endpoints of e. Either $|W| = 1$, in which case edge e is called a self loop, or $|W| = 2$. It is assumed that the labels of distinct edges are distinct. Vertices u and v are called neighbors, if u and v are distinct and at least one edge has endpoints u and v. Let $n(u, v)$ denote the number of edges with endpoints u and v. If two or more edges have the same endpoints, these edges are said to be parallel. In particular, two self loops with the same endpoint are parallel. A subset F of E is called an edge cover of G if every vertex of G is covered by at least one edge in F. Let S_i denote the multigraph with two vertices, no self loops, and i edges, all parallel. Given two graphs G and H, let $G * H$ denote the new graph formed from the disjoint union of G and H, and for $i \geq 1$ let G^{i*} denote the disjoint union of i copies of G. Let $R_k(G)$ denote the number of edge covers of G with cardinality k, and write $R(G)$ to denote the infinite vector $(R_0(G), R_1(G), \ldots)$. Writing $R(G) * R(H)$ for the convolution of the vectors $R(G)$ and $R(H)$, the following basic convolution relation holds: $R(G * H) = R(G) * R(H)$.

Conjecture Let $m \geq n/2$ and $r \geq 0$, where n is even. Among multigraphs with n vertices and m edges, the number of edge covers of cardinality r is maximized by a graph in which every component is isomorphic to $S_{\lfloor 2m/n \rfloor}$ or $S_{\lceil 2m/n \rceil}$.

The conjecture is established for $r = n/2$ in Section 2, and for $n \geq m \geq n/2$ in Section 3. The result in Section 3 also pertains to multigraphs with n odd. It is shown in Section 4 for general n, m, r, that among multigraphs with n vertices and m edges and the maximum number of edge coverings of cardinality r, there is a graph with no four cycles. An interpretation of the conjecture and some related problems are discussed in Section 5.

2 PERFECT MATCHINGS

Let $n \geq 2$ be even and $m \geq n/2$. Let G^{opt} be such that each component is isomorphic to either $S_{\lfloor 2m/n \rfloor}$ or $S_{\lceil 2m/n \rceil}$. This determines G^{opt} up to isomorphism. Note that the set of edge covers of cardinality $n/2$ is simply the set of perfect matchings. The following establishes the conjecture for the case $r = n/2$.

Proposition 2.1 *Let $n \geq 2$ be even and $m \geq n/2$. If among all multigraphs with n vertices and m edges, G has the maximum number of perfect matchings, then G is isomorphic to G^{opt}.*

Proof. Let G be an extremal multigraph, by which we mean G has n vertices, m edges, and the maximum number of perfect matchings among all multigraphs with the same number of vertices and edges. Perfect matchings do not contain self loops, so clearly G has no self loops. Define a *VEE* for G to be given by $(u, \{v, w\})$, where u, v, w are distinct vertices such that v and w are both neighbors of u.

If G has at least one VEE, then we claim that there is an extremal graph $\hat{G}$ with fewer VEE's than G. Indeed, suppose $(u, \{v, w\})$ is a VEE in G. Let $e_1, e_2, \ldots, e_i$ denote the edges with endpoints u and v, and let $f_1, \ldots, f_j$ denote the edges with endpoints u and w. Let n_0 denote the number of perfect matchings disjoint from $\{e_1, \ldots, e_i\} \cup \{f_1, \ldots, f_j\}$, let n_1 denote the number of perfect matchings which contain e_1 and let n_2 denote the number of perfect matchings which contain f_1. Any perfect matching contains at most one edge from $\{e_1, \ldots, e_i\} \cup \{f_1, \ldots, f_j\}$, edges among $\{e_1, \ldots, e_i\}$ are interchangeable, and edges among $\{f_1, \ldots, f_j\}$ are interchangeable. Thus, $R_{n/2}(G) = n_0 + in_1 + jn_2$. The vertices v and w can be exchanged, if necessary, so it is assumed without loss of generality that $n_1 \geq n_2$. Define $\hat{G}$ to be the graph obtained from G by removing $f_1, \ldots, f_j$ from E, and adding j new edges, all with endpoints u and v. Then $R_{n/2}(\hat{G}) = n_0 + (i + j)n_1 \geq R_{n/2}(G)$. Furthermore, $\hat{G}$ has at least one less VEE than G. The claim is proved.

Therefore, there is an extremal graph G with no self loops and no VEEs. Each component of such a graph has two nodes. The number of perfect matchings is the product of the numbers of edges in the $n/2$ components. The product is maximized if and only if the numbers are as close to equal as possible. Thus, G^{opt} is an extremal graph.

It remains to show that G^{opt} is the only extremal graph. By the above proof, given any extremal graph G, there is a finite sequence of extremal graphs $G = G_1, \ldots, G_N$ such that G_N is isomorphic to G^{opt}, and such that for each $i < N$, G_{i+1} is obtained from G_i by one transformation (of the type described in the second paragraph of the proof.) Suppose for the sake of contradiction that $N \geq 1$. Since G_N is obtained from G_{N-1} by one transformation, and since $R_{n/2}(G_{N-1}) \neq 0$, there is a component H of G_{N-1} with four vertices u, v, w, x and $n(u,v) + n(v,w) + n(w,x)$ edges, where the multiplicities $n(u,v), n(v,w), n(w,x)$ are strictly positive. Furthermore, G_{N-1} is obtained from G_N by changing the component H to a subgraph isomorphic to $\hat{H} = S_{n(u,v)+n(v,w)} * S_{n(w,x)}$. Since $R_2(H) = n(u,v)n(w,x) < R_2(\hat{H}) = [n(u,v) + n(v,w)]n(w,x)$, it follows that $R_{n/2}(G_{N-1}) < R_{n/2}(G_N)$. This is a contradiction, so $N = 0$. Thus, G^{opt} is the only extremal graph. □

3 AVERAGE DEGREE AT MOST TWO

Let $n \geq m \geq n/2$, where $n \geq 1$. Throughout this section, the letter G with or without subscripts or superscripts denotes a multigraph with n vertices and m edges such that each vertex has degree greater than or equal to one. If $m = n/2$, then any G is isomorphic to $(S_1)^{(n/2)*}$. If $m = (n+1)/2$, then up to isomorphisms there are only two possibilities for G (G either has an isolated vertex or a single VEE), and both possibilities have the same number of edge covers of a given cardinality.

Suppose $(n/2) + 1 \leq m \leq n$, and let $\triangle$ denote a complete simple graph with three vertices and three edges. Let G^{opt} be such that each component is isomorphic to either S_1, S_2 or $\triangle$, and such that at most one component of G^{opt} is isomorphic to $\triangle$. This determines G^{opt} up to isomorphism. A version can be written explicitly as follows:

$$G^{opt} = \begin{cases} \triangle * (S_2)^{(m-(n+3)/2)*} * (S_1)^{(n-m)*} & n \text{ odd} \\ (S_2)^{(m-n/2)*} * (S_1)^{(n-m)*} & n \text{ even} \end{cases} \tag{3.1}$$

The following proposition is established in this section. It implies the conjecture for $m \leq n$. By $R(G) \geq R(H)$, we mean that for all $r \geq 0$, $R_r(G) \geq R_r(H)$.

Proposition 3.1 *Let $(n/2) + 1 \leq m \leq n$. Then $R(G^{opt}) \geq R(G)$ for all G. If $R_{\lceil n/2 \rceil}(G^{opt}) = R_{\lceil n/2 \rceil}(G)$ then G is isomorphic to G^{opt}.*

A *leaf* of a multigraph is a vertex with exactly one neighboring vertex. A *two-component* of a multigraph is a component with exactly two vertices. The vertices of a two-component are, of course, leaves. Let L_a denote a multigraph with one edge and a self loops.

We now give nine different conditions on a multigraph G. Under each there is a new graph $\hat{G}$ (not necessarily unique) that is obtained from G by a specified transformation.

T_1 : Suppose G has a self loop e at a vertex u that is not isolated. Let v be a neighbor of u. Obtain $\hat{G}$ by reassigning one endpoint of e to be v.

T_2 : Suppose G has no self loops at vertices that are not isolated, and there is a leaf vertex u not in a two-component of G. Let v denote the neighbor of u. Obtain $\hat{G}$ from G as follows. For any edge in G with one endpoint v and some other endpoint $w \neq u$, let both endpoints of the edge be w in $\hat{G}$. (Note that u and v are in a two-component of $\hat{G}$.)

T_3 : Suppose all leaves of G lie in two-components. Suppose there is a component H of G which is not isomorphic to S_2 and which contains edges e and f parallel to each other. Then there is an isolated edge g in G (see Lemma 3.3 below). Let $\hat{G} = G - e + g'$, where g' is a new edge parallel to g.

T_4 : Suppose all the self loops in G are at isolated vertices, and all leaves of G lie in two-components. Suppose there is a component H of G in which every vertex has degree greater than or equal to three. There is an isolated edge g of G. Let graph $\hat{G}$ be obtained by replacing some edge e of H by a new edge parallel to g, where the edge e is selected according to Lemma 3.4 below.

T_5 : Suppose all the self loops of G are at isolated vertices, and all leaves and parallel edges of G lie in two-components. Suppose there is a component H of G, not isomorphic to $\triangle$, with a vertex u of degree two and two distinct neighbors v and w. (See Fig. 1.) Let e denote the edge with endpoints u and v, and let f denote the edge with endpoints u and w. Define the new graph $\hat{G}$ by reassigning the endpoints of some of the edges in G as follows. Change e and f so that they each have endpoints v and w. For any other edge in G that has an endpoint v or w, change that endpoint to u. (If there is an edge in G with endpoints v and w, that edge becomes a self edge loop in $\hat{G}$).

T_6 : Suppose G contains two components isomorphic to L_1. To obtain $\hat{G}$ replace both by a new component isomorphic to S_2.

T_7 : Suppose G contains a component isomorphic to L_1 and a component isomorphic to S_2. To obtain $\hat{G}$ replace both by a new component isomorphic to $\triangle$.

T_8 : Suppose G contains a component isomorphic to L_1 and a component isomorphic to $\triangle$. To obtain $\hat{G}$ replace both by two new components isomorphic to S_2.

T_9 : Suppose two components of G are isomorphic to the triangle graph $\triangle$. To obtain $\hat{G}$ replace both by three new components isomorphic to S_2.

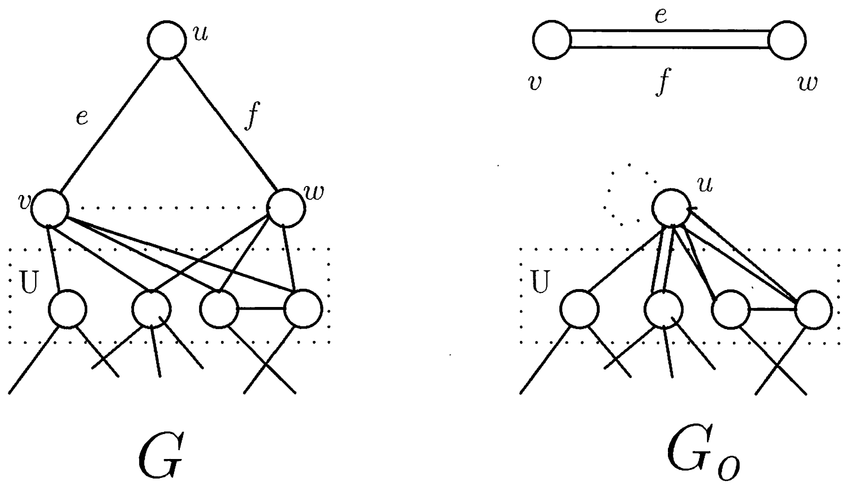

Figure 1: A transformation of type T_4.

Let $\tau(G)$ denote the sum: the number of components of G which are isomorphic to either S_2 or $\triangle$, plus the total number of edges of G contained in two-components and in components isomorphic to $\triangle$.

Lemma 3.1 *Let $r \geq 0$. If $\hat{G}$ is obtained from G by a transformation of one of the nine types above, then $R_r(\hat{G}) \geq R_r(G)$ and $\tau(\hat{G}) \geq \tau(G)$. If the transformation is not of type T_1, then $\tau(\hat{G}) > \tau(G)$.*

Proof. The statments regarding τ are easily checked, so it remains to prove that $R_r(\hat{G}) \geq R_r(G)$ if $\hat{G}$ is obtained from G by a transformation of any one of the nine types. If the transformation is of type T_1 or T_2, then any edge cover for G is also an edge cover for $\hat{G}$. Therefore, $R(\hat{G}) \geq R(G)$ if the transformation is of type T_1 or T_2. The following four inequalities and the basic convolution relation show that if the transformation is of type T_6, T_7, T_8,or T_9, then again $R(\hat{G}) \geq R(G)$:

$$\begin{aligned} R(S_2) = (0,2,1,0,\ldots) &\geq R((L_1)^{2*}) = (0,0,1,0,\ldots) \\ R(\triangle) = (0,0,3,1,0,\ldots) &\geq R(S_2 * L_1) = (0,0,2,1,\ldots) \\ R(S_2^{2*}) = (0,0,4,4,1,0,\ldots) &\geq R(L_1 * \triangle) = (0,0,0,3,1,0,\ldots) \\ R(S_2^{3*}) = (0,0,0,8,12,6,1,0,\ldots) &\geq R(\triangle^{2*}) = (0,0,0,0,9,6,1,0,\ldots) \end{aligned}$$

The remainder of the proof of the lemma is given as Lemmata 3.3-3.5 below. □

Suppose H and an isolated edge f are both components of a graph G. The next lemma is useful in determining whether an edge e can be removed from H and placed parallel to f, without decreasing the number of edge covers. First some notation is introduced. Given a multigraph H and an edge e of H, and given $k \geq 0$, let $R_k(H, e)$ denote the number of edge covers of H of cardinality k that contain e. (Continue to let $R(H)$ with one argument, $R_k(H)$, denote the number of edge covers of H of cardinality k.) An edge e is called *essential* for an edge cover F if $e \in F$ and $F - e$ is not an edge cover. Let $\tilde{R}_k(H, e)$ denote the number of edge covers of H of cardinality k for which e is essential.

Lemma 3.2 *For any k, $R_{k+1}((H-e) * S_2) - R_{k+1}(H * S_1) = R_k(H-e) - \tilde{R}_k(H, e)$.*

Proof. By the basic convolution relation,

$$R_{k+1}((H-e) * S_2) = 2R_{k-1}(H-e) + R_k(H-e). \tag{3.2}$$

On the other hand,

$$\begin{aligned} R_{k+1}(H * S_1) &= R_k(H) \\ &= R_k(H, e) + R_k(H-e) \\ &= \tilde{R}_k(H, e) + R_{k-1}(H-e) + R_k(H-e). \end{aligned} \tag{3.3}$$

Comparing (3.2) and (3.3) proves the lemma. □

Lemma 3.3 *If G satisfies the conditions for a type T_3 transformation, then there is an isolated edge g in G so that $\hat{G}$ can be constructed. If $\hat{G}$ is obtained from G by a transformation of type T_3, then $R(\hat{G}) \geq R(G)$.*

Proof. Suppose G satisfies the conditions for a type T_3 transformation. Then some vertex of G has degree greater than or equal to three (for example, at least one of the endpoints of e has degree at least three). Since all leaves of G lie in two-components, any vertex not in a two-component has degree at least two. Thus, there exists a two-component in which the sum of the vertex degrees is less than or equal to three. Such a two-component must be isomorphic to S_1, so that there is an isolated edge in G as claimed.

Corresponding to every edge cover F of H (consisting of edges from H) for which e is essential, there is the distinct edge cover $F - e + f$ of H which does not contain e. Therefore $\tilde{R}_k(H, e) \leq R_k(H - e)$ for $k \geq 0$. By lemma 3.2, it follows that $R((H - e) * S_2) \geq R(H * S_1)$. Let K denote the graph formed from the union of all components of G other than H and the component consisting of g alone. Then $R(\hat{G})$ (resp. $R(G)$) is the convolution of $R(K)$ with $R((H - e) * S_2)$ (resp. with $R(H * S_1)$). Since convolution with nonnegative vectors is order preserving, the desired inequality $R(\hat{G}) \geq R(G)$ ensues. □

Lemma 3.4 *Let $r \geq 0$. If G and H satisfy the conditions for a transformation of type T_4, then there is an edge e of H such that the transformed graph $\hat{G}$ satisfies $R_r(\hat{G}) \geq R_r(G)$.*

Proof. Note that an isolated edge g must exist since some vertex has degree three, hence some vertex has degree one, and the edge incident to such vertex must be isolated. Given an edge e of the component H, let G_e denote the graph obtained from G by removing e and adding a new edge parallel to g. It must be shown that there is a choice of e so that $R_r(G_e) \geq R_r(G)$.

Let K denote the graph formed from the union of all components of G other than H and the component consisting of g alone. Observe from the convolution relation and Lemma 3.2 , that

$$\sum_e R_r(G_e) - R_r(G) = \sum_e \sum_{k=0}^{r+1} R_{r-k-1}(K)\,[R_{k+1}((H-e)*S_2) - R_{k+1}(H*S_1)] \qquad (3.4)$$

$$= \sum_e \sum_{k=0}^{r+1} R_{r-k-1}(K)\left[R_k(H-e) - \tilde{R}_k(H,e)\right] \qquad (3.5)$$

where e ranges over all edges of H.

Fix $k \geq 0$ and suppose F is an edge cover of H of cardinality k. We claim that the number of edges in F that are essential (for covering H) is less than or equal to the number of edges in the component H that are not in the edge cover F. Indeed, the number of essential edges in F is less than the number of vertices of H that are covered by F exactly once. Each vertex that is covered exactly once by F, having degree at least three, is the endpoint of at least two edges of H that are not in F. Since an edge of H but not in F can be selected at most twice in this way, once for each endpoint, the claim is proved. Using the claim and summing over all edge covers of H containing k edges implies that

$$\sum_e R_k(H-e) - \tilde{R}_k(H,e) \geq 0 \qquad (3.6)$$

Multiplying each side of (3.6) by $R_{r-k-1}(H)$, summing over k, interchanging the order of summation, and comparing the result to (3.5) yields that the lefthand side of (3.4) is nonnegative. There thus exists an edge e such that $R_r(G_e) \geq R_r(G)$, as required. □

Lemma 3.5 *If $\hat{G}$ is obtained from G by a transformation of type T_5, then $R(\hat{G}) \geq R(G)$.*

Proof. As indicated in Figure 1, let U denote the set of vertices in $\hat{G}$ that are neighbors of u. Equivalently, U is the set of vertices (other than u, v or w) which are neighbors of either v or w in G. Let E_o denote the set of edges that are not incident to u, v, or w (in either graph). Fix $r \geq 1$. Partition the set $\mathcal{C}$ of edge covers of G of cardinality r into $r+1$ sets, $\mathcal{C}(0), \ldots, \mathcal{C}(r)$, as follows. Let $F \in \mathcal{C}(0)$ if and only if $F \cap E_o$ does not cover every vertex in U. For $k \geq 1$, let $F \in \mathcal{C}(k)$ if and only if $F \cap E_o$ covers every vertex in U and $|F \cap E_o| = r - k$. Partition the set $\mathcal{C}_o$ of edge covers of $\hat{G}$ of cardinality r into $r+1$ sets, $\mathcal{C}_o(0), \ldots, \mathcal{C}_o(r)$, in the same fashion. To complete the proof, it is shown that $|\mathcal{C}(k)| \leq |\mathcal{C}_o(k)|$ for each k.

To begin with, we claim that $\mathcal{C}(0) \subset \mathcal{C}_o(0)$. (Recall that G and $\hat{G}$ have the same set of edges—the construction of $\hat{G}$ only involves reassigning the endpoints of some of the edges.) Indeed, if $F \in \mathcal{C}(0)$, then either $e \in F$ or $f \in F$ since F covers u in G, and therefore F covers v and w in $\hat{G}$. Also, F contains at least one edge in $E - E_o$ which covers a vertex in U, so that F also covers u in $\hat{G}$. Finally, F covers all other vertices in $\hat{G}$ since it covers them in G. The claim is true so the case $k = 0$ is verified.

Observe that $\mathcal{C}(1) = \emptyset$, so the case $k = 1$ is also verified.

Suppose $k \geq 2$. Let $x(r-k)$ denote the number of distinct subsets of E_o of cardinality $r - k$ which cover $V - \{u, v, w\}$, and let $y(k)$ (resp. $y_o(k)$) denote the number of distinct subsets of $E - E_o$ of cardinality k which cover $\{u, v, w\}$ in G (resp. in $\hat{G}$). Then $|\mathcal{C}(k)| = x(r-k)y(k)$ and $|\mathcal{C}_o(k)| = x(r-k)y_o(k)$, so that it remains to show that $y_o(k) \geq y(k)$ for $k \geq 2$. Let $l = 1$ if there is an edge in G with endpoints v and w, and let $l = 0$ otherwise. Let j_v (resp. j_w) denote the number of edges in G incident to v (resp. w) and some vertex of U. Then

$$y(k) = \binom{j_v + j_w + l}{k-2} + 2\binom{j_v + j_w + l}{k-1} - \binom{j_v}{k-1} - \binom{j_w}{k-1} \tag{3.7}$$

whereas

$$y_o(k) = I_{\{k \geq 3\}}\binom{j_v + j_w + l}{k-2} + 2\binom{j_v + j_w + l}{k-1} \tag{3.8}$$

Comparison of (3.7) and (3.8) immediately yields that $y_o(k) \geq y(k)$ for $k \geq 3$. Finally, to check the inequality for $k = 2$, notice that $y_o(2) - y(2) = j_v + j_w - 1$, which is nonnegative by the assumption that H is not isomorphic to the triangle graph $\triangle$. □

Proof of Proposition 3.1. Fix $r \geq 0$ and let G be arbitrary. Construct a sequence of graphs $G = G_0, G_1, \ldots, G_N$ as follows. Given G_i, if the conditions for at least one of the nine types of transformations holds, apply such a transformation to produce G_{i+1}. Let N be such that the graph G_N obtained in this way does not satisfy the conditions for any of the nine types of transformations.

We argue that N is finite. Since $\tau(G_{i+1}) \geq \tau(G_i)$ for all $i < N$, and since $\tau(G_{i+1}) > \tau(G_i)$ if the transformation applied to G_i is of any type except type T_1, there is a finite N_o so that only transformations of type T_1 are used for $i \geq N_o$. Only finitely many transformations of type T_1 can be applied consecutively, since the number of self loops decreases by one for each application. Thus N is finite.

Next, it is argued that G_N is isomorphic to G^{opt}. Since G_N does not satisfy the conditions for transformations of type T_1 or T_2, it has no self loops at vertices that are not isolated, and all leaves of G_N lie in two-components. Since transformations of type T_3 can't be applied, all two-components of G are isomorphic to S_1 or S_2, and all isolated vertices have exactly one self loop attached. Let H denote a component of G with three or more nodes. Since H has no leaf vertices, all vertices of H have at least two neighbors. Not all vertices of H have degree greater than two, since a transformation of type T_4 cannot be applied. Thus some vertex of H has degree two and has two neighbors. Since transformations of type T_5 can't be applied, H must be isomorphic to $\triangle$. In summary, every component of G_N is isomorphic to L_1, S_1, S_2, or $\triangle$.

Suppose for the sake of contradiction that some component of G is isomorphic to L_1. Since transformations of type T_6, T_7 or T_8 can't be applied, there can be no other component isomorphic to L_1, and no component isomorphic to S_2 or to $\triangle$. This implies that $m = (n+1)/2$, which is excluded by the conditions of the proposition. Thus, all components of G_N are isomorphic to S_1, S_2 or $\triangle$. Finally, since transformations of type T_9 can't be applied, at most one component can be isomorphic to $\triangle$. Thus, G_N is isomorphic to G^{opt} as claimed. Since $R(G_N) \geq R(G_1) = R(G)$, it follows that $R(G^{opt}) \geq R(G)$ as required.

To complete the proof of the proposition, it remains to show that if G is not isomorphic to G^{opt}, then $R_{\lceil n/2 \rceil}(G^{opt}) < R_{\lceil n/2 \rceil}(G)$. (For n even this follows from the last part of Proposition 2.1, though this fact isn't used here.) Let G be given such that G is not isomorphic to G^{opt}. By the proof above, there is a sequence of graphs $G = G_0, G_1, \ldots, G_N$ such that G_N is isomorphic to G_{opt}, $R_{\lceil n/2 \rceil}(G_{j+1}) \geq R_{\lceil n/2 \rceil}(G_j)$ for each j, and G_{j+1} is obtained from G_j by a transformation of one of the nine types.

It is argued next that

$$R_{\lceil n/2 \rceil}(G_N) > R_{\lceil n/2 \rceil}(G_{N-1}) \tag{3.9}$$

which, since G^{opt} is isomorphic to G_N and $R_{\lceil n/2 \rceil}(G_{N-1}) \geq R_{\lceil n/2 \rceil}(G)$, implies the desired result. The argument is broken into nine cases, according to the type of the transformation used to obtain G_N from G_{N-1}. In what follows, we use repeatedly the following fact, which follows from the basic convolution relation and the fact that $\lceil n/2 \rceil$ is the minimum number of edges needed to cover n vertices: if $G = H * K$ then $R_{\lceil n/2 \rceil}(H * K) = R_i(H) R_{\lceil n/2 \rceil - i}(K)$, where $i = \lceil(\text{number of vertices in } H)/2\rceil$.

(1) Observe that if S' is the multigraph with two vertices, two edges and exactly one self loop, then $R_1(S_2) = 2 > 1 = R_1(S')$. Thus if T_1 is the last transformation used, then (3.9) holds. (2) Transformation T_2 can't be the last transformation used, since G_N does not have any self loops. (3) Suppose T_3 is the last transformation used. Then $H - e$, being a component of G_N, is isomorphic to either S_1 or S_2. Therefore H is isomorphic to S_2 or S_3, but it is assumed that H is not isomorphic to S_2. Thus H is isomorphic to S_3. Since $R_2(S_2 * S_2) > R_2(S_3 * S_1)$, it follows that (3.9) holds. (4) Suppose T_4 is the last transformation used. Then $H - e$ is isomorphic to S_1 or S_2, which since all vertices of H have degree at least three is possible only if H is isomorphic to S_3. Since $R_2(S_2 * S_2) > R_2(S_3 * S_1)$, (3.9) holds. (5) Suppose T_5 is the last transformation used. Note that there is no edge with endpoints v and w in G_{N-1}, since G_N has no self loops. The component of G_N containing u is either isomorphic to S_1 or to S_2. Thus the component H of S_{N-1} contains four vertices and either three or four edges. Since H contains no leaves, it must be isomorphic to the simple graph C_4 consisting of a single four-edge cycle. Since $R_2(S_2 * S_2) = 4 > 2 = R_2(C_4)$, (3.9) holds. (6-9) Similarly, if T_6, T_7, T_8 or T_9 is the last transformation used, then (3.9) holds since

$$\begin{aligned}
R_1(S_2) = 2 &> R_1((L_1)^{2*}) = 0 \\
R_2(\triangle) = 3 &> R_2(S_2 * L_1) = 2 \\
R_2(S_2^{2*}) = 4 &> R_2(L_1 * \triangle) = 0 \\
R_3(S_2^{3*}) = 8 &> R_3(\triangle^{2*}) = 0
\end{aligned}$$

Thus, in any case, (3.9) holds as claimed, and the proposition is proved. □.

4 RULING OUT FOUR CYCLES

The result of this section is stated as a lemma, since it is our hope that it will prove useful in establishing the conjecture in general. Let a four cycle in a multigraph be given by a sequence of four distinct vertices (u, v, w, x) such that the multiplicities $n(u, v), n(v, w), n(w, x)$, and $n(x, u)$ are all strictly positive.

Lemma 4.1 *Let $m \geq n/2$ and $r \geq 0$. Among multigraphs with n vertices and m edges which have the maximum number of edge covers of cardinality r, there is a graph with no four cycles.*

Proof. Let the letter G with or without subscripts or superscripts denote a multigraph with n vertices and m edges. Fix $r \geq 0$ and consider a graph G with a four-cycle (u, v, w, x). It suffices to produce a graph $\hat{G}$ with strictly fewer four-cycles than G and with $R_r(\hat{G}) \geq R_r(G)$. Construct a random sequence of graphs $G = G_0, G_1, \ldots, G_N$ as follows. If G_i has been constructed and (u, v, w, x) is not a four-cycle of G_i, then let $N = i$. Otherwise, (u, v, w, x) is a four-cycle of G_i, and the graph G_{i+1} is obtained by moving two edges of G_i. Let e, f, g, and h denote four edges in G_i with endpoints $\{u, v\}$, $\{v, w\}$, $\{w, x\}$, and $\{x, u\}$, respectively. Move e and g to become parallel to f and h, respectively, with probability one half. Otherwise move f and h to become parallel to e and g, respectively. We claim that $E[R_r(G_{i+1})|G_i] \geq R_r(G_i)$. Indeed, let $S \subset \{u, v, w, x\}$ and $0 \leq j \leq 4$. Consider the number of ways to select j edges from among $\{e, f, g, h\}$ in order to cover S. The mean of this number for G_{i+1} is greater than or equal to the number for G_i, as can be easily checked separately for each possible choice of S and j. On the other hand, the number of ways to choose $r - j$ edges from $E - \{e, f, g, h\}$ to cover all vertices except those in S is the same for G_i and G_{i+1}. Multiplying the numbers and summing over S and j establishes the claim. Thus, the sequence $\{R_r(G_i)\}_{i \geq 0}$ is a martingale.

Let $n_i(u, v)$ denote the number of edges with endpoints u and v in G_i. Then given $i < N$, $n_{i+1}(v, v) = n_i(u, v) + 1$ or $n_{i+1}(v, v) = n_i(u, v) - 1$, each with probability one half. That is, $n_i(u, v)$ corresponds to a fair coin toss process. Since it is bounded, if follows that N has a finite mean. Thus, the optional sampling theorem of martingale theory implies that $E[R_r(G_N)] \geq R_r(G)$. There are only two possible outcomes for the random graph G_N. Let $\hat{G}$ denote the possible outcome with the greater number of cardinality r edge covers, taking $\hat{G}$ to be either possibility in case of a tie. Then $\hat{G}$ has fewer cycles than G, and $R_r(\hat{G}) \geq R_r(G)$ as desired. □

5 PURITY OF MISSION

The conjecture in this paper is a strengthening of our earlier conjecture reported by D. West [3]. The conjecture was originally motivated by our desire to obtain an upper bound on the error probability in a code-division multiple-access optical system with correlation constraint $\lambda = 2$ (see [1] for definitions). However, as reported in [3], there is a scenario in which the question might arise more directly. Suppose there are n targets to be bombed by m bombers, with each bomber assigned a pair of targets. A "successful" bomber reaches both assigned targets; an "unsuccessful" one reaches neither target. Suppose it is known that r of the bombers will be successful, all $\binom{m}{r}$ possibilities being equally likely. If r itself is random and binomially distributed with parameters m and p, then each bomber is successful with probability p, independently of the outcome of other bombers.

The question is, how should bombers be assigned to targets in order to maximize the probability that all targets are bombed? The conjecture states that if n is even, then the targets should be partitioned into pairs, and then bombers should be assigned to the pairs as evenly as possible. In particular, the missions of any two bombers are either identical or disjoint. Proof of the conjecture would thus establish an instance of optimality of the "purity of mission" principle of military doctrine, which states that a mission should be divided into disjoint pieces, with separate pieces targeted by disjoint sets of resources.

This interpretation of the conjecture suggests other conjectures. For example, we may allow each bomber to bomb Q targets, where Q may be larger than two. Or we may have the number of targets successfully reached by a bomber take the values 0,1, or 2 with probabilities $p_0, 2p_1$, or p_2, respectively. If exactly one target is reached, the two choices are assumed to have equal probability. We conjecture that if $p_2 p_0 \geq p_1^2$ (so the events of a bomber reaching the respective targets are positively correlated) then the purity of mission assignment again maximizes the probability that all targets are bombed.

The conjecture has some relation to the well-known theorem of Brègman, which states that the number of perfect matchings in a bipartite simple graph with two sets of $n/2$ vertices and specified degrees $d_1, \ldots, d_{n/2}$ of the vertices in one set is at most $\prod_i (d_i!)^{1/d_i}$ [2]. In fact, in the spirit of Brègman's theorem, and our main conjecture above, we formulate another conjecture as follows. Let n be even, and let $d_1, \ldots, d_{n/2}$ denote positive integers as before. Set $d_{i+(n/2)} = d_i$ for $1 \leq i \leq n/2$. Then the conjecture is the following: Among all multigraphs with vertex set $\{1, \ldots, n\}$ such that vertex i has degree d_i for all i, the number of edge coverings of a specified cardinality r is maximized by the graph $S_{d_1} * \ldots * S_{d_{n/2}}$. This conjecture is true under the two additional assumptions: (1) the multigraph is bipartite with vertices $\{1, \ldots, n/2\}$ forming one set and (2) $r = n/2$ (so that perfect matchings are considered). In fact, under these two conditions, the degree constraint need only be enforced for vertices $\{1, \ldots, n/2\}$—as a proof based on transforming VEEs as in Section 2 shows. The conjecture under (1) and (2) also follows readily from an extension of Brègman's result attributed by Schrijver [2] to Brouwer.

ACKNOWLEDGEMENT The authors are grateful to the Institute for Mathematics and Its Applications for bringing them together and providing the opportunity to pursue this work. They also thank their officemate Craig Zirbel for useful discussions.

References

[1] R.K. Chung, J.W. Salehi, and V.K. Wei. Optical orthogonal codes: Design, analysis, and applications. *IEEE Transactions on Information Theory*, 35:595–604, May 1989.

[2] A. Schrijver. A short proof of Minc's conjecture. *Journal of Combinatorial Theory, Series A*, 25:80–83, 1978.

[3] D. West. Open problem #8. *Newsletter, SIAM Activity Group on Discrete Mathematics*, 3(1):7–11, Fall 1992.

#	Author/s	Title
1143	**M.H.A. Davis & M. Zervos**, A problem of singular stochastic control with discretionary stopping	
1144	**Bernardo Cockburn & Pierre-Alain Gremaud**, An error estimate for finite element methods for scalar conservation laws	
1145	**David C. Dobson & Fadil Santosa**, An image enhancement technique for electrical impedance tomography	
1146	**Jin Ma, Philip Protter, & Jiongmin Yong**, Solving forward-backward stochastic differential equations explicitly — a four step scheme	
1147	**Yong Liu**, The equilibrium plasma subject to skin effect	
1148	**Ulrich Hornung**, Models for flow and transport through porous media derived by homogenization	
1149	**Avner Friedman, Chaocheng Huang, & Jiongmin Yong**, Effective permeability of the boundary of a domain	
1150	**Gang Bao**, A uniqueness theorem for an inverse problem in periodic diffractive optics	
1151	**Angelo Favini, Mary Ann Horn,& Irena Lasiecka**, Global existence and uniqueness of regular solutions to the dynamic von Kármán system with nonlinear boundary dissipation	
1152	**E.G. Kalnins & Willard Miller, Jr.**, Models of q-algebra representations: q-integral transforms and "addition theorems"	
1153	**E.G. Kalnins, V.B. Kuznetsov & Willard Miller, Jr.**, Quadrics on complex Riemannian spaces of constant curvature, separation of variables and the Gaudin magnet	
1154	**A. Kersch, W. Morokoff & Chr. Werner**, Selfconsistent simulation of sputtering with the DSMC method	
1155	**Bing-Yu Zhang**, A remark on the Cauchy problem for the Korteweg-de Vries equation on a periodic domain	
1156	**Gang Bao**, Finite element approximation of time harmonic waves in periodic structures	
1157	**Tao Lin & Hong Wang**, Recovering the gradients of the solutions of second-order hyperbolic equations by interpolating the finite element solutions	
1158	**Zhangxin Chen**, L^p-posteriori error analysis of mixed methods for linear and quasilinear elliptic problems	
1159	**Todd Arbogast & Zhangxin Chen**, Homogenization of compositional flow in fractured porous media	
1160	**L. Qiu, B. Bernhardsson, A. Rantzer, E.J. Davison, P.M. Young & J.C. Doyle**, A formula for computation of the real stability radius	
1161	**Maria Inés Troparevsky**, Adaptive control of linear discrete time systems with external disturbances under inaccurate modelling: A case study	
1162	**Petr Klouček & Franz S. Rys**, Stability of the fractional step Θ-scheme for the nonstationary Navier-Stokes equations	
1163	**Eduardo Casas, Luis A. Fernández & Jiongmin Yong**, Optimal control of quasilinear parabolic equations	
1164	**Darrell Duffie, Jin Ma & Jiongmin Yong**, Black's consol rate conjecture	
1165	**D.G. Aronson & J.L. Vazquez**, Anomalous exponents in nonlinear diffusion	
1166	**Ruben D. Spies**, Local existence and regularity of solutions for a mathematical model of thermomechanical phase transitions in shape memory materials with Landau-Ginzburg free energy	
1167	**Pu Sun**, On circular pipe Poiseuille flow instabilities	
1168	**Angelo Favini, Mary Ann Horn, Irena Lasiecka & Daniel Tataru**, Global existence, uniqueness and regularity of solutions to a Von Kármán system with nonlinear boundary dissipation	
1169	**A. Dontchev, Tz. Donchev & I. Slavov**, On the upper semicontinuity of the set of solutions of differential inclusions with a small parameter in the derivative	
1170	**Jin Ma & Jiongmin Yong**, Regular-singular stochastic controls for higher dimensional diffusions — dynamic programming approach	
1171	**Alex Solomonoff**, Bayes finite difference schemes	
1172	**Todd Arbogast & Zhangxin Chen**, On the implementation of mixed methods as nonconforming methods for second order elliptic problems	
1173	**Zhangxin Chen & Bernardo Cockburn**, Convergence of a finite element method for the drift-diffusion semiconductor device equations: The multidimensional case	
1174	**Boris Mordukhovich**, Optimization and finite difference approximations of nonconvex differential inclusions with free time	
1175	**Avner Friedman, David S. Ross, and Jianhua Zhang**, A Stefan problem for reaction-diffusion system	
1176	**Alex Solomonoff**, Fast algorithms for micromagnetic computations	
1177	**Nikan B. Firoozye**, Homogenization on lattices: Small parameter limits, H-measures, and discrete Wigner measures	
1178	**G. Yin**, Adaptive filtering with averaging	
1179	**Wlodzimierz Byrc and Amir Dembo**, Large deviations for quadratic functionals of Gaussian processes	
1180	**Ilja Schmelzer**, 3D anisotropic grid generation with intersection-based geometry interface	
1181	**Alex Solomonoff**, Application of multipole methods to two matrix eigenproblems	
1182	**A.M. Latypov**, Numerical solution of steady euler equations in streamline-aligned orthogonal coordinates	
1183	**Bei Hu & Hong-Ming Yin**, Semilinear parabolic equations with prescribed energy	
1184	**Bei Hu & Jianhua Zhang**, Global existence for a class of Non-Fickian polymer-penetrant systems	

1185 **Rongze Zhao & Thomas A. Posbergh**, Robust stabilization of a uniformly rotating rigid body
1186 **Mary Ann Horn & Irena Lasiecka**, Uniform decay of weak solutions to a von Kármán plate with nonlinear boundary dissipation
1187 **Mary Ann Horn, Irena Lasiecka & Daniel Tataru**, Well-posedness and uniform decay rates for weak solutions to a von Kármán system with nonlinear dissipative boundary conditions
1188 **Mary Ann Horn**, Nonlinear boundary stabilization of a von Kármán plate via bending moments only
1189 **Frank H. Shaw & Charles J. Geyer**, Constrained covariance component models
1190 **Tomasz Luczaka**, A greedy algorithm estimating the height of random trees
1191 **Timo Seppäläinen**, Maximum entropy principles for disordered spins
1192 **Yuandan Lin, Eduardo D. Sontag & Yuan Wang**, Recent results on Lyapunov-theoretic techniques for nonlinear stability
1193 **Svante Janson**, Random regular graphs: Asymptotic distributions and contiguity
1194 **Rachid Ababou**, Random porous media flow on large 3-D grids: Numerics, performance, & application to homogenization
1195 **Moshe Fridman**, Hidden Markov model regression
1196 **Petr Klouček, Bo Li & Mitchell Luskin**, Analysis of a class of nonconforming finite elements for Crystalline microstructures
1197 **Steven P. Lalley**, Random series in inverse Pisot powers
1198 **Rudy Yaksick**, Expected optimal exercise time of a perpetual American option: A closed-form solution
1199 **Rudy Yaksick**, Valuation of an American put catastrophe insurance futures option: A Martingale approach
1200 **János Pach, Farhad Shahrokhi & Mario Szegedy**, Application of the crossing number
1201 **Avner Friedman & Chaocheng Huang**, Averaged motion of charged particles under their self-induced electric field
1202 **Joel Spencer**, The Erdös-Hanani conjecture via Talagrand's inequality
1203 **Zhangxin Chen**, Superconvergence results for Galerkin methods for wave propagation in various porous media
1204 **Russell Lyons, Robin Pemantle & Yuval Peres**, When does a branching process grow like its mean? Conceptua proofs of $L \log L$ criteria
1205 **Robin Pemantle**, Maximum variation of total risk
1206 **Robin Pemantle & Yuval Peres**, Galton-Watson trees with the same mean have the same polar sets
1207 **Robin Pemantle**, A shuffle that mixes sets of any fixed size much faster than it mixes the whole deck
1208 **Itai Benjamini, Robin Pemantle & Yuval Peres**, Martin capacity for Markov chains and random walks in varying dimensions
1209 **Wlodzimierz Bryc & Amir Dembo**, On large deviations of empirical measures for stationary Gaussian processes
1210 **Martin Hildebrand**, Some random processes related to affine random walks
1211 **Alexander E. Mazel & Yurii M. Suhov**, Ground states of a Boson quantum lattice model
1212 **Roger L. Fosdick & Darren E. Mason**, Single phase energy minimizers for materials with nonlocal spatial dependence
1213 **Bruce Hajek**, Load balancing in infinite networks
1214 **Petr Klouček**, The transonic flow problems stability analysis and numerical results
1215 **Petr Klouček**, On the existence of the entropic solutions for the transonic flow problem
1216 **David A. Schmidt & Chjan C. Lim**, Full sign-invertibility and symplectic matrices
1217 **Piermarco Cannarsa & Maria Elisabetta Tessitore**, Infinite dimensional Hamilton-Jacobi equations and Dirichlet boundary control problems of parabolic type
1218 **Zhangxin Chen**, Multigrid algorithms for mixed methods for second order elliptic problems
1219 **Zhangxin Chen**, Expanded mixed finite element methods for linear second order elliptic problems I
1220 **Gang Bao**, A note on the uniqueness for an inverse diffraction problem
1221 **Moshe Fridman**, A two state capital asset pricing model
1222 **Paolo Baldi**, Exact asymptotics for the probability of exit from a domain and applications to simulation
1223 **Carl Dou & Martin Hildebrand**, Enumeration and random random walks on finite groups
1224 **Jaksa Cvitanic & Ioannis Karatzas**, On portfolio optimization under "drawdown" constraints
1225 **Avner Friedman & Yong Liu**, A free boundary problem arising in magnetohydrodynamic system
1226 **Dominic Welsh**, Randomised approximation schemes for Tutte-Gröthendieck invariants
1227 **Zhangxin Chen, Bernardo Cockburn, Carl L. Gardner, & Joseph W. Jerome**, Quantum hydrodynamic simulation of hysteresis in the resonant tunneling diode
1228 **E.G. Kalnins, G.C. Williams, & William Miller, Jr.**, Intrinsic characterisation of the separation constant for spin one and gravitational perturbations in Kerr geometry
1229 **Zhangxin Chen**, Large-scale averaging analysis of multiphase flow in fractured reservoirs
1230 **Bruce Hajek & Babu Narayanan**, Multigraphs with the most edge covers
1231 **K.B. Athreya**, Entropy maximization
1232 **F.I. Karpelevich & Yu.M. Suhov**, Functional equations in the problem of boundedness of stochastic branching dynamics
1233 **E. Dibenedetto & V. Vespri**, On the singular equation $\beta(u)_t = \Delta u$