The Instanton -Torus Knot duality

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Based on works, Bulycheva, Nechaev , A.G. 1409..
Milekhin , A.G. 1412...,
Milekhin, Sopenko, A.G. 1506...
To appear

Plan of the talk

- Introduction. Where the question comes from?

- Condensate in 5D SUSY QED and QCD from
the torus knot invariants

- Different representations of torus knot invariants
In 5D gauge theory.

- Torus knot invariants in the dual systems

- Conclusion

Examples of condensates. SUSY

- Holomorphy - «instantons» contribute to the condensate
- Longstanding puzzle concerning the gluino condensate in N=1 SYM(NSVZ). If one compact dimension- fractional instanton-monopoles contribute
- Exact result. Squark and monopole condensates vanishes at Argyres-Douglas point in N=1 SQCD- deconfinement (Yung,Vainshtein,A.G)

Old and new questions

- What is the microscopic picture behind the condensate formation? Examples- squark and gluino condensates in SQCD due to the zero modes in the instanton ensemble(NSVZ 83).Topological sector in the theory
- Chiral condensate in QCD. Quasizero modes in the instanton-antiinstanton ensemble.

New tools

- New invariants of knots. Khovanov homologies and superpolynomials which generalize Jones and HOMFLY polynomials
- Seberg-Witten solution to N=2 SYM. Nekrasov partition sums. Explicit results for the instanton sums in the Omega-background
- Topological phases of matter. Classification via ground state degeneracy+ holonomy of Berry phase or via entanglement entropy

Knot invariants in gauge theories

- J(q,K) =<W(K)> in SU(2) 3d Chern-Simons theory — Jones polynomial of knot K (Witten , 89). Can be generalized to all SU(N) groups-

  HOMFLY polynomial H(a,q,K), $a=q^N$
- Generalization to superpolynomial P(a,q,t,K) (Dunfeld, Gukov,Rasmussen 04). Three gradings (a,q,t) in the Hilbert space
- All knot polynomials are particular indices

  $P(a,q,t.K)= \dim H_{ijk}a^i q^j t^k$ counts the multiplicities of the BPS states (Gukov, Scwartz, Vafa 04) in some theory

Torus knots can be drawn on the torus surface.
T(n,m) corresponds to two windings around cycles

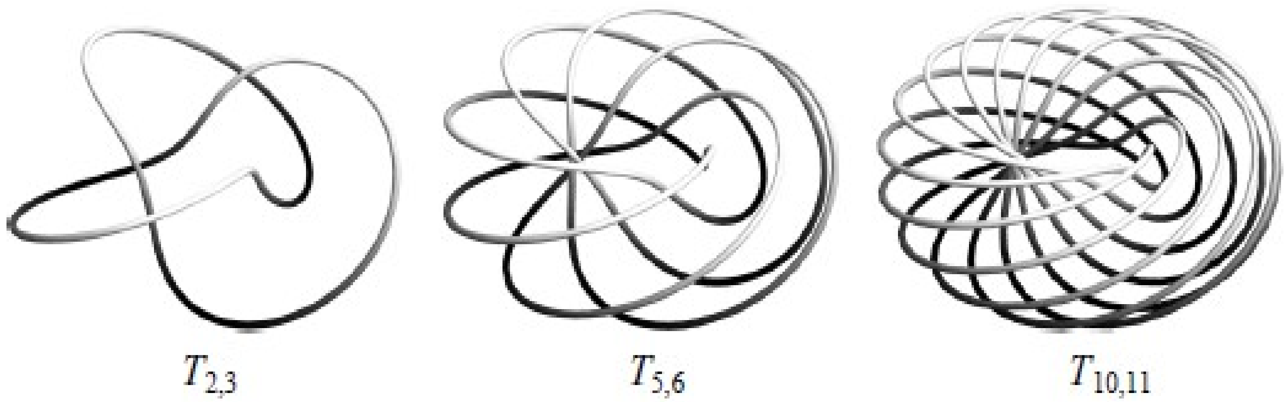

Figure 1: Few samples of torus knots from the series $T_{n,n+1}$: $T_{2,3}$, $T_{5,6}$, $T_{10,11}$.

Knot invariants. New approaches

- The old evaluation- it is vev of electric Wilson loop along the knot in 3d CS. Is there the S-dual «magnetic» version of the knot invariants? (Witten 10, Witten, Gaiotto 11). Wittten,s idea — knot invariants somehow count instantons in 4d and 5d N=4 SUSY gauge theory. Only partial success.
- The superpolynomials of the torus knots are expressed as very specific integrals over moduli space of points in C^2 (E.Gorsky-Negut , 13). Way to zero-size instantons in 5d theory.

5D SQED and SQCD

- Consider the U(1) 5d SUSY gauge theory with N_f=2 or N_f=3. One dimension is compact S^1. Add 5d CS term k AFF. Introduce Omega-deformation= two intependent rotations (angular velocities) in R^4.
- There is the explicit answer for the instanton partition function in this theory due to Nekrasov localization
- Surprise. The condensate of the massless flavor « is sum over the invariants of the T(n,m) torus knots in the momentum space»

Some facts on 5d SQED

- The BPS particles in the theory are W-bosons, instantons. Due to the CS term the instanton charge induces the electric charge
- Complicated dyonic instantons(both charges). Even more complicated states with 3 charges(+flavor). Not fully classified. Monopoles are loops(monopole particles lifted tp 5d)
- One -loop effect of all BPS particles in 5d D with compact dimension reproduces all instanton partition sum in D=4 SYM theory(Nekrasov-Lawrence)

$$\Omega^m = \Omega^{mn} x_n, \qquad \Omega^{mn} = \frac{1}{2\sqrt{2}} \begin{pmatrix} 0 & i\epsilon_1 & 0 & 0 \\ -i\epsilon_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i\epsilon_2 \\ 0 & 0 & i\epsilon_2 & 0 \end{pmatrix}.$$

Omega-deformation by external field

$$\mathcal{L}_m = -\frac{1}{4g^2} F_{mn} F^{mn} + \frac{1}{g^2} (\partial_m \phi + F_{mn} \Omega^n)(\partial^m \phi - F^{mn} \Omega^n) +$$
$$\frac{1}{2} |D_m Q|^2 + \frac{1}{2} |D_m \tilde{Q}|^2 + \frac{2}{g^2} (i\partial_m (\Omega^m \bar{\phi} + \Omega^m \phi) + g^2 (\bar{Q} Q - \bar{Q} \tilde{Q}))^2 +$$
$$\frac{1}{2} |(\phi - m - i\Omega^m D_m) q|^2 + \frac{1}{2} |(\phi - \tilde{m} - i\Omega^m D_m) \tilde{q}|^2 + 2g^2 |\tilde{q} q|^2$$

Spectrum of BPS particles

$$Z = \frac{1}{g^2} n_I + n_e a + \sum_i n_{f_i} m_{f_i}$$

String web (Aharony,Hanany, Kol)

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Instanton-torus knot duality,

$$\frac{e^{\beta M}}{(1+a)\beta^2}\frac{d^2 Z_{nek}(q,t,m,M,m_a,Q)}{dM\,dm}\bigg|_{m\to 0, M\to\infty} = \sum_n Q^n (tq)^{n/2} P_{n,nk+1}(q,t,a) \qquad (2)$$

where m_a, m, M are masses of three hypemultiplets and Q is the counting parameter for the instantons. The mapping between the parameters at the lhs and rhs goes as follows

$$t = \exp(-\beta\epsilon_1) \qquad (3)$$

$$q = \exp(-\beta\epsilon_2) \qquad (4)$$

$$a = -\exp(\beta m) \qquad (5)$$

$$Q = \exp(-\beta/g^2) \qquad (6)$$

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Instanton-torus knot duality

The superpolynomial is the complicated product in terms of the Young tableou

$$P(A,q,t)_{nk+1,n} =$$

$$\sum_{\lambda:|\lambda|=n} \frac{t^{(k+1)\sum l} q^{(k+1)\sum a}(1-t)(1-q)\prod^{0,0}(1-Aq^{-a'}t^{-l'})\prod^{0,0}(1-q^{a'}t^{l'})(\sum q^{a'}t^{l'})}{\prod(q^{a}-t^{l+1})\prod(t^{l}-q^{a+1})}$$

In this formula there is only one independent index — n (instanton number)

New findings

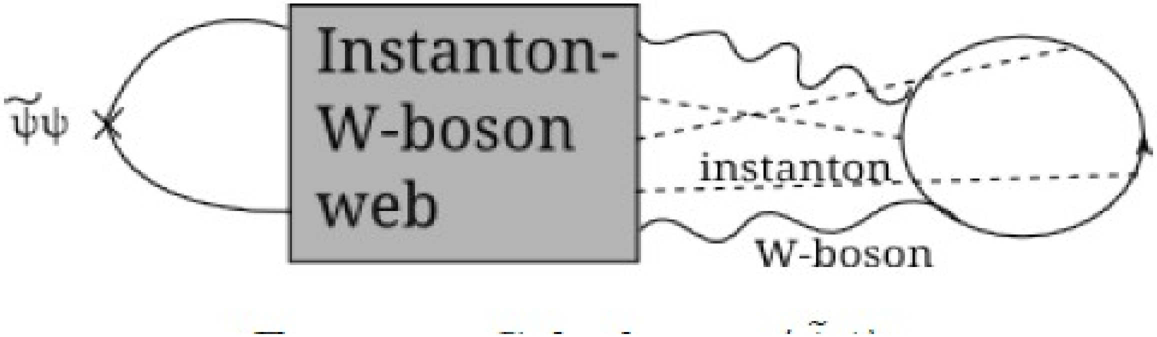
- The information about the knots is encoded in the condensate. Torus knots T(m,n) are important. The physical identifications of the numbers: n- instanton charge, m -electric charge
- The physical variable is expressed in terms of the sum over the knots. The first example of such situation!
- The rank of the gauge group in the CS picture is the mass of the antifundamental(!)

Interpretation

- The knot invariants describe the multiplicity of states at fixed 4d quantum numbers (n,m)
- In some sence they count the 2d instantons on the nonabelian strings at fixed 4d instanton number. «Knotting the fermionic zero modes?»
- Instantons are membranes in the internal space and draw the knots on the «flavor branes». Knots live in the internal Calabi-Yau («momentum») space

Knot invariant as entropic factor

- The place of the knots in the diagrams



HOMFLY for generic (n,m) knots

- Consider the N_f=2 theory with Lagrangian brane. Count the contribution of states with (n,m) quantum numbers into condensate
- Consider the SU(2) theory with N_f=4. Two masses fixed, one mass vanishes, one is arbitrary. Expand the condensate in series in two quantum numbers
- Consider N_f=2 U(1) with fractional 5d CS number k=m/n. Extract n- instanton contribution

Sum over the (n,m)

- Double series for the condensate

$$\langle\tilde{\psi}\psi\rangle_{LB} = \frac{\partial Z_{inst}}{\partial m_f}\Big|_{m_f=0} = \sum_{n,m} Q_c^n z^m P_{n,nk+m}(A,q,t)$$

$$P(A,q,t)_{n,nk+m} =$$

$$\sum_{\lambda:|\lambda|=n} \frac{t^{(k+1)\sum l} q^{(k+1)\sum a}(1-t)\prod^{0,0}(1+Aq^{-a'}t^{-l'})\prod^{0,0}(1-q^{a'}t^{l'})}{\prod(q^a - t^{l+1})\prod(t^l - q^{a+1})} \times$$

$$Coef_{z^m} M(z)$$

where $M(z)$ is the contribution from the Lagrangian brane with zero framing:

$$M(z) = \prod_{j=1}^{l(\lambda)} \frac{1 - zt^{j-1}q^{\lambda_j}}{1 - zt^{j-1}}$$

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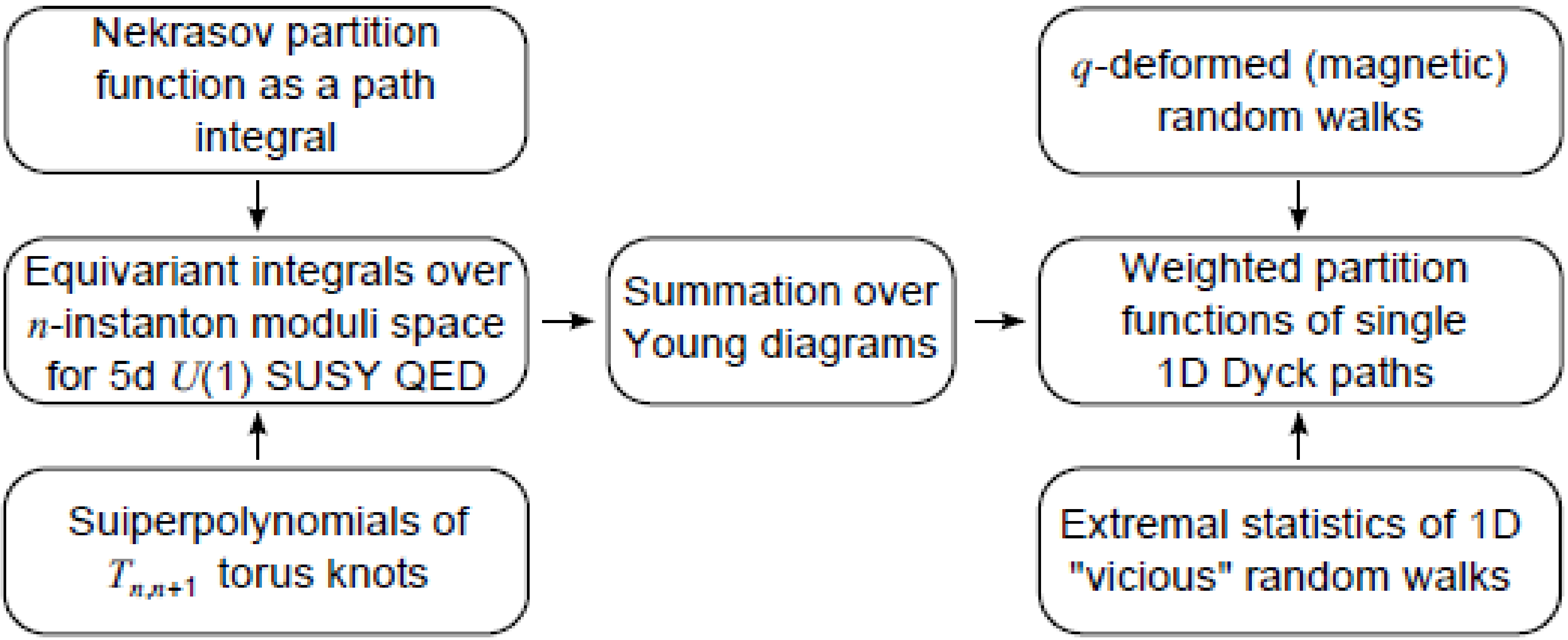
Complimentary views on the instanton sums

Bulycheva, Nechaev, A.G. 1409



Knot invariants in the dual systems

- Nekrasov partition function in 5d is related to the q-Liouville conformal block. The derivative of the conformal block- generating function for knot invariants
- The Nekrasov partition function — wave function of the holomorphic Hamiltonian system. Perturbative and nonperturbative effects in SYM= similar effects in QM(example Toda, Basar-Dunne).Linking of pert and nonpert corrections via torus knot invariants (in progress)

Conclusion

- Just touch tip of the iseberg. Many surprises
- The «knotting» between electric degrees of freedom and instantons is important for the condensate formation
- Unexpected appearance of knot invariants in Liouville conformal blocks. General linking and knotting of perturbative and nonperturbative contributions.
- Some interplay with the solid state order parameters

A lot of open questions...........