A TRANSPORT MODEL WITH MICRO- AND MACRO-STRUCTURE

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A TRANSPORT MODEL WITH MICRO- AND MACRO-STRUCTURE*

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Summary. In this paper we consider solute transport through porous media where the chemical species undergoes a chemical process through the surface of the porous skeleton. The problem is modeled by a system of differential equations for the macro-concentration $u = u(x,t)$ and the micro-concentration $u' = u'(x, x', t)$. We prove existence and uniqueness, and some properties of the set with positive concentration.

Introduction. In this paper we consider solute transport through porous media, where the chemical species undergoes adsorption, a retention/release reaction with the surface of the porous skeleton. Such processes are of great importance in various fields, e.g., in analytical chemistry (chromatographic separation) and soil science or hydrology (mobility of plant nutrients and pollutants). Often the movement of a solute due to the water movement is so slow that the reaction may be considered to be in equilibrium. Nevertheless sometimes an overall non-equilibrium effect is observed. Usually this is related to the fact that some of the adsorption sites are not directly accessible to the bulk flow of water. The primary solid particles are assumed to be aggregated into porous pellets (aggregates). The intra-aggregate pore sizee is such that bulk flow of water only takes place between the aggregates. Thus solutes move within the aggregates only because of molecular diffusion, whereas they enter or leave the aggregate via another diffusive process (film diffusion).

In principle it is possible to derive a macroscale model from a model in the microscopic scale of single aggregates. Such a macroscale model (which will be made more precise later on) and various versions of it have been studied in the literature. A good reference for models concerning the transport of reacting solutes in porous media is [7]. In the macroscale there appears the porous medium Ω and the inter-aggregate concentration $u, u = u(x,t)$, where x is the spatial variable in Ω (the macro-variable) and t is time; but there is in addition the intra-aggregate concentration $u', u' = u'(x, x', t)$, where x' is the spatial variable in Ω' (the micro-variable) and Ω' is a domain modeling the aggregate. In the differential equations for u', to be set up, x only appears as a parameter.

In this paper we establish, by rigorous analysis, existence and uniqueness of the solution of such a model and some properties of the moving boundary (i.e., the boundary of the set with positive concentration).

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In §1 we set up the model in detail and then proceed, in §2, to define the concept of weak solution. Uniqueness of the weak solution is established in §3. Existence is established first for smooth f_i (in §5) and then for general f_i (§6) by using L^1 estimates derived in §4. Finally, in §7, we establish finite speed of propagation in case the x-space is one dimensional.

§1. Formulation of the problem. Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geq 1$; it is a porous medium throughout which water is flowing. Set

$$Q_T = \Omega \times (0, T] \; .$$

The water content $\Theta : \overline{Q}_T \to \mathbb{R}$ and the water flux $q : \overline{Q}_T \to \mathbb{R}^N$ are assumed to be given functions satisfying

$$\partial_t \Theta = -\nabla \cdot q \; , \tag{1.1}$$

which is the conservation of volume of fluid. The functions Θ, q can be obtained by solving (1.1) supplemented by Darcy's law and a constitutional relationship between pressure Ψ and water content Θ. This leads to a nonlinear parabolic equation for Ψ, possibly degenerate. Our assumption (1.1) means that there is no influence of the dissolved substance on the flow of the water.

Denote by n the outer normal to $\partial\Omega$ and set

$$\alpha = -q \cdot n \; .$$

We assume that $\partial\Omega$ consists of two nonempty closed, disjoint sets S_1 and S_2 such that

$$\begin{aligned} \alpha \geq 0 \quad &\text{on} \quad S_{1T} \equiv S_1 \times (0, T] \; , \\ \alpha \leq 0 \quad &\text{on} \quad S_{2T} \equiv S_2 \times (0, T] \; ; \end{aligned} \tag{1.2}$$

S_1 is, for instance, the inner boundary and S_2 the outer boundary of Ω. Condition (1.2) means that S_1 is the inflow boundary and S_2 is the outflow boundary of the water, for all $t \in (0, T]$.

Let $u : \overline{Q}_T \to \mathbb{R}$ denote the concentration of a substance dissolved in the water. We assume that the solution flux is given by

$$j(u) = qu - D\nabla u \; ; \tag{1.3}$$

qu is the convection term and $-D\nabla u$ is the dispersion and diffusion term. Here

$$D : \overline{Q}_T \to \mathbb{R}^{N,N}$$

is the sum of the molecular diffusion and the mechanical dispersion coefficients. The conservation of mass law is

$$\partial_t(\Theta u) + \nabla \cdot j(u) = Q_1 + Q_2 \tag{1.4}$$

where Q_i are rates of production of chemo-physical processes. The first rate, Q_1, involves the adsorption of u on the outer surface of the aggregates. If the adsorption process is instantaneous, the equilibrium relation between u and the adsorbed concentration v reads

$$v = f_1(u),$$

where f_1 is called the inter-aggregate adsorption isotherm. Then there holds:

$$Q_1 = -\rho\partial_t v = -\rho\partial_t\ f_1(u) \tag{1.5}$$

where $\rho : \Omega \to \mathbb{R}$ is the given bulk density. Typically

$$f_1(s) = As^p\ ,\quad A > 0,\quad 0 < p < 1\ . \tag{1.6}$$

To introduce the rate Q_2 we note that there is diffusion of u into the aggregate through a surrounding quiescent water film; within the aggregate the water is stagnant. We represent the aggregate, appropriately scaled, by a bounded domain Ω' in $\mathbb{R}^M$. At each point x in Ω "sits" one aggregate. The aggregate is modeled as an already averaged/homogenized medium. Let

$$u' : \overline{\Omega} \times \overline{\Omega'} \times [0,T] \to \mathbb{R}$$

denote the concentration of substance in the solution within the aggregate. Then

$$Q_2 = -\gamma \int\limits_{\partial\Omega'} h(x,x')[u(x,t) - u'(x,x',t)]\ dS_{x'} \qquad (\gamma = 1/|\Omega'|)\ ; \tag{1.7}$$

represents a flow across $\partial\Omega'$ due to unequal concentrations inside and outside the aggregate. The function $h : \Omega \times \overline{\Omega'} \to \mathbb{R}$ is given.

The adsorption process within the aggregate at the surfaces of the "micrograins" is also considered to be in equilibrium, i.e., the adsorbed concentration v' is given by

$$v' = f_2(u'),$$

where f_2 is the intra-aggregate adsorption isotherm. Introducing the water content $\Theta'(x,x')$ and the bulk density $\rho'(x,x')$ and the molecular diffusion coefficient $D' : \overline{\Omega} \times \overline{\Omega'} \to R^{M,M}$ in the aggregate, the conservation of mass gives

$$\partial_t(\Theta' u') + \rho'\partial_t f_2(u') - \nabla' \cdot (D'\nabla u') = 0 \quad \text{in} \quad \Omega'\ ; \tag{1.8}$$

typically

$$f_2(s) = A's^{p'} \ , \quad A' > 0, \ p' > 0 \ . \tag{1.9}$$

The transport of substance across $\partial\Omega'$ gives

$$-(D'\nabla' u' \cdot n')(x, x', t) = h(x, x')(u'(x, x', t) - u(x,t)) \quad \text{on} \ \ \partial\Omega' \tag{1.10}$$

where n' is the outward normal to $\partial\Omega'$. This justifies (1.7), as Q_2 is the loss rate due to diffusion into the aggregate, i.e.,

$$\begin{aligned} Q_2 &= -\frac{1}{|\Omega'|} \int_{\Omega'} \partial_t(\Theta' u') + \rho' \partial_t f_2(u') dx' \\ &= -\frac{1}{|\Omega'|} \int_{\partial\Omega'} D'\nabla' u' \cdot n' dS_{x'} = -\gamma \int_{\partial\Omega'} h(u - u') dS_{x'} \ . \end{aligned}$$

§2. **Classical and weak formulation.** We make the following assumptions: Ω and Ω' have $C^{1,\beta}$ boundary, for some $\beta \in (0,1)$, D and D' are measurable symmetric matrices,

$$\begin{aligned} &\nu|\xi|^2 \le \xi^* D(x,t)\xi \le \frac{1}{\nu} \ |\xi|^2 \qquad \text{for} \ \ (x,t) \in \overline{Q_T} \ , \\ &\nu|\xi|^2 \le \xi^* D'(x,x')\xi \le \frac{1}{\nu} \ |\xi|^2 \quad \text{for} \ \ (x,x') \in \overline{\Omega} \times \overline{\Omega'} \ , \\ &\partial_t D \in L^\infty(Q_T), \end{aligned}$$

where ν is a positive constant;

$$\begin{aligned} &\Theta \in C([0,T], L^\infty(\Omega)), \quad \Theta' \in L^\infty(\Omega \times \Omega'), \ q \in L^\infty(Q_T) \ , \\ &\Theta(x,t), \Theta'(x,x') \ge \Theta_0 > 0 \quad \text{for} \quad (x,t) \in \overline{Q_T}, (x,x') \in \overline{\Omega} \times \overline{\Omega'} \ , \\ &\partial_t \Theta = -\nabla \cdot q \in L^\infty(Q_T) \ , \\ &\alpha \equiv -q \cdot n \quad \text{exists on} \quad S_T = \partial\Omega \times (0,T] \quad \text{in the trace class}, \\ &\alpha \in L^\infty(S_T), \text{ and } \ \alpha \ge 0 \quad \text{on} \quad S_{1T} \equiv S_1 \times (0,T] \ , \quad \alpha \le 0 \ \text{ on } S_{2T} \equiv S_2 \times (0,T] \\ &\quad \text{and} \ \ \alpha_t \in L^\infty(S_{2T}), \end{aligned}$$

where S_1, S_2 are nonempty, closed disjoint subsets of $\partial\Omega$ with $S_1 \cup S_2 = \partial\Omega$ and n is the outward normal to $\partial\Omega$;

$$\begin{aligned} &\rho \in L^\infty(\Omega), \ \rho' \in L^\infty(\Omega \times \Omega'), \quad \rho \ge 0, \ \rho' \ge 0 \ , \\ &F \in L^2(S_{1T}) \ ; \end{aligned}$$

$$
\begin{aligned}
&u_0 \in L^2(\Omega), \\
&u_0' \in L^2(\Omega \times \Omega'), \\
&h \in L^\infty(\overline{\Omega} \times \partial\Omega'),\ h \geq 0\ , \\
&f_i \in C(-\infty,\infty) \cap C^1(0,\infty),\ f_i(s) = 0 \quad \text{if} \quad s \leq 0\ , \\
&f_i \quad \text{is monotone nondecreasing.}
\end{aligned}
$$

Set

$$
Q_T' = \Omega' \times (0,T], \quad S_T' = \partial\Omega' \times (0,T]\ .
$$

We seek functions $u : \overline{Q_T} \to \mathbb{R}$ and $u' : \overline{\Omega} \times \overline{Q_T'} \to \mathbb{R}$ satisfying the following parabolic equations, and boundary and initial conditions:

$$
\begin{aligned}
&(\partial_t(\Theta u + \rho f_1(u)) - \nabla \cdot (D\nabla u - qu))(x,t) \\
&\qquad + \int_{\partial\Omega'} \gamma h(x,x')(u(x,t) - u'(x,x',t))\, dS_{x'} = 0\ , \qquad (x,t) \in Q_T\ ,
\end{aligned}
\tag{2.1}
$$

$$
\begin{aligned}
(D\nabla u - qu) \cdot n = F \quad &\text{on} \quad S_{1T}\ , \\
D\nabla u \cdot n = 0 \quad &\text{on} \quad S_{2T}\ ,
\end{aligned}
\tag{2.2}
$$

$$
(\partial_t(\Theta' u' + \rho' f_2(u')) - \nabla' \cdot D'\nabla' u')(x,x',t) = 0\ , \qquad x \in \Omega, (x',t) \in Q_T'\ ,
\tag{2.3}
$$

$$
(D'\nabla' u' \cdot n' + h(u' - u))(x,x',t) = 0\ , \qquad x \in \Omega,\ (x',t) \in S_T'
\tag{2.4}
$$

where n' is the outward normal to $\partial\Omega'$,

$$
u(x,0) = u_0(x), \qquad x \in \Omega\ ,
\tag{2.5}
$$

$$
u'(x,x',0) = u_0'(x,x'), \qquad x \in \Omega, \quad x' \in \Omega'\ .
\tag{2.6}
$$

In (2.1) γ is a given positive constant.

The system (2.1)–(2.6) is similar to the reaction-diffusion system studied by Friedman and Tzavaras [3]. However due to the fact that non-smooth functions f_i of the form (1.6), (1.9) should and will be included in the present model, the methods of the present paper are different in several respects from the methods in [3]. They have more in common with the methods used in [5] to study a pde/ode system of similar nature. Note also that in [8], among other things, a simplified linear version of (2.1)–(2.6) has been studied by means of Hilbert space methods.

We shall later on use the notation:

$$\tilde{f}_1(x,t,s) = \Theta(x,t)s + \rho(x)f_1(s) \ , \tag{2.7}$$

$$\tilde{f}_2(x,x',s) = \Theta'(x,x')s + \rho(x,x')f_2(s) \ . \tag{2.8}$$

By a *classical solution* (u,u') of (2.1)–(2.6) we mean a pair (u,u') such that u, u' and all their derivatives appearing in (2.1)–(2.4) are continuous for $x \in \overline{\Omega}$, $x' \in \overline{\Omega}'$, $0 \le t \le T$, and satisfy (2.1)–(2.6).

For the purpose of establishing existence and uniqueness we shall work with the concept of weak solutions.

DEFINITION 2.1. A pair (u,u') is a *weak lower (upper) solution* of (2.1)–(2.6) if

$$\begin{aligned} &u \in C([0,T], L^2(\Omega)), u' \in C([0,T],\ L^2(\Omega \times \Omega')) \ , \\ &\partial_{x_i} u \in L^2(Q_T), \quad \partial_{x'_i} u' \in L^2(\Omega \times Q'_T) \ , \\ &f_1(u(\cdot,0)) \in L^2(\Omega),\ f_1(u) \in L^2(Q_T), \\ &f_2(u'(\cdot,0)) \in L^2(\Omega \times \Omega'),\ f_2(u') \in L^2(\Omega \times Q'_T), \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} &-\int_{\Omega} \tilde{f}_1(\cdot,0,u(\cdot,0))\varphi(\cdot,0) - \int_{Q_T} \tilde{f}_1(\cdot,u)\partial_t\varphi + \int_{Q_T} (D\nabla u - qu)\cdot\nabla\varphi \\ &\underset{(\ge)}{\le} \int_{S_{1T}} F\varphi + \int_{S_{2T}} \alpha u\varphi - \int_{Q_T}\int_{\partial\Omega'} \gamma h(u-u')\varphi \\ &\text{for any } \ \varphi \in W_2^{1,1}(Q_T),\ \varphi(\cdot,T) = 0,\ \varphi \ge 0 \quad \text{in} \quad Q_T \ , \end{aligned} \tag{2.10}$$

$$u(\cdot,0) \underset{(\ge)}{\le} u_0 \ , \tag{2.11}$$

$$\begin{aligned} &-\int_{\Omega}\int_{\Omega'} \tilde{f}_2(\cdot,u'(\cdot,0))\psi(\cdot,0) - \int_{\Omega}\int_{Q'_T} \tilde{f}_2(\cdot,u')\partial_t\psi + \int_{\Omega}\int_{Q'_T} D'\nabla'u'\cdot\nabla'\psi \\ &\underset{(\ge)}{\le} \int_{\Omega}\int_{S'_T} h(u-u')\psi \\ &\text{for any } \psi \in L^2(\Omega \times Q'_T) \text{ such that } \partial_t\psi,\ \partial_{x'}\ \psi \in L^2(\Omega \times Q'_T) \ , \\ &\psi(\cdot,T) = 0 \ , \quad \psi \ge 0 \quad \text{in} \quad \Omega \times Q'_T \ , \end{aligned} \tag{2.12}$$

$$u'(\cdot,0) \underset{(\ge)}{\le} u'_0 \ . \tag{2.13}$$

If (u,u') is a weak lower and upper solution then we say that it is a *weak solution*. In that case (2.10), (2.12) hold with equality, without any sign restriction on φ and ψ, and (2.11), (2.13) hold with equality.

3. Uniqueness.

THEOREM 3.1. *There exists at most one weak solution.*

Proof. Consider two solutions (u_i, u_i') $(i = 1, 2)$ and set $u = u_1 - u_2, u' = u_1' - u_2'$. Fix $\tau \in (0, T]$. We subtract (2.10) for u_2 from (2.10) for u_1 and use the test function

$$\varphi(x,t) = \begin{cases} \int_t^\tau u(x,s)\, ds & \text{from} \quad t \le \tau \\ 0 & \text{for} \quad t > \tau\ . \end{cases} \tag{3.1}$$

With (2.11) we proceed analogously, taking the test function

$$\psi(x,x',t) = \begin{cases} \int_t^\tau u'(x,x',s)\, ds & \text{for} \quad t \le \tau \\ 0 \quad \text{for} \quad t > \tau\ . \end{cases} \tag{3.2}$$

Adding γ times the second equation to the first equation leads to

$$\begin{aligned} &\int_{Q_\tau} (\tilde f_1(\cdot, u_1) - \tilde f_1(\cdot, u_2))u + \gamma \int_\Omega \int_{Q'_\tau} (\tilde f_2(\cdot, u_1') - \tilde f_2(\cdot, u_2'))u' \\ &+ \int_{Q_\tau} D\nabla u \cdot \nabla\varphi + \gamma \int_\Omega \int_{Q'_\tau} D'\nabla u' \cdot \nabla'\psi - \int_{Q_\tau} uq \cdot \nabla\varphi \\ &- \int_{S_{2\tau}} \alpha u\varphi + \int_{Q_\tau} \int_{\partial\Omega'} \gamma h(u - u')(\varphi - \psi) = 0\ . \end{aligned} \tag{3.3}$$

We proceed to evaluate some of the terms. The first one is

$$\begin{aligned} \int_{Q_\tau} (\tilde f_1(\cdot, u_1) - \tilde f_1(\cdot, u_2))u = \int_{Q_\tau} \Theta u^2 \\ + \int_{Q_\tau} \rho(f_1(u_1) - f_1(u_2))(u_1 - u_2) \ge \int_{Q_\tau} \Theta u^2 \end{aligned} \tag{3.4}$$

and analogously, for the second term,

$$\int_\Omega \int_{Q'_\tau} (\tilde f_2(\cdot, u_1') - \tilde f_2(\cdot, u_2'))u' \ge \int_\Omega \int_{Q'_\tau} \Theta' {u'}^2\ . \tag{3.5}$$

Next, since $\partial_t \varphi = -u$,

$$(3.6)\qquad \begin{aligned} \int_{Q_\tau} D\nabla u \cdot \nabla\varphi &= -\int_{Q_\tau} \frac{1}{2}\,[\partial_t(D\nabla\varphi.\nabla\varphi) - (\partial_t D)\nabla\varphi\cdot\nabla\varphi] \\ &\geq \frac{\nu}{2}\int_\Omega |\nabla\varphi(\cdot,0)|^2 - \frac{1}{2}\,\|\partial_t D\|_\infty \int_{Q_\tau} |\nabla\varphi|^2 \end{aligned}$$

and, analogously,

$$(3.7)\qquad \int_\Omega\int_{Q'_\tau} D'\nabla' u'\cdot\nabla'\psi \geq \frac{\nu}{2}\int_\Omega\int_{\Omega'} |\nabla'\psi(\cdot,0)|^2\,.$$

The boundary term for $S_{2\tau}$ can be estimated by

$$(3.8)\qquad \begin{aligned} -\int_{S_{2\tau}} \alpha u\varphi &= \int_{S_{2\tau}} [\partial_t\left(\frac{1}{2}\alpha\varphi^2\right) - \frac{1}{2}\,(\partial_t\alpha)\varphi^2] \\ &= -\int_{S_2}\frac{1}{2}\,\alpha\varphi^2(\cdot,0) - \int_{S_{2\tau}}\frac{1}{2}\,(\partial_t\alpha)\varphi^2 \\ &\geq -C\left[\int_{Q_\tau}|\nabla\varphi|^2 + \tau\int_{Q_\tau} u^2\right] \end{aligned}$$

by the trace estimate

$$\int_{S_2} w^2 \leq C\int_\Omega (w^2 + |\nabla w|^2)$$

and the definition of φ; recall that $\alpha \leq 0$ on S_{2T}.

Next

$$-\int_{Q_\tau} uq\cdot\nabla\varphi \geq -\varepsilon\int_{Q_\tau} u^2 - \frac{C}{\varepsilon}\int_{Q_\tau}|\nabla\varphi|^2 \qquad \text{for} \qquad 0<\varepsilon<1$$

and, finally, for the coupling term,

$$\begin{aligned} \int_{Q_\tau}\int_{\partial\Omega'} \gamma h(u-u')(\varphi-\psi) &= -\int_{Q_\tau}\int_{\partial\Omega'} \gamma h\partial_t(\varphi-\psi)(\varphi-\psi) \\ = \int_\Omega\int_{\partial\Omega'} \frac{\gamma}{2}\,h(\varphi-\psi)^2(\cdot,0) &\geq 0 \quad \text{since} \quad h\geq 0\,. \end{aligned}$$

Substituting these estimates into (3.3) and choosing ε small enough, we find that

$$\int_{Q_\tau} u^2 + \int_\Omega \int_{Q'_\tau} u'^2 + \int_\Omega |\nabla\varphi(\cdot,0)|^2 + \int_\Omega \int_{\Omega'} |\nabla'\psi(\cdot,0)|^2 \le C\left(\int_{Q_\tau} |\nabla\varphi|^2 + \tau \int_{Q_\tau} u^2\right) \tag{3.9}$$

for some constant $C > 0$.

Set

$$g(\tau) = \int_\Omega \left| \int_0^\tau \nabla u(x,s)ds \right|^2 dx + \int_\Omega \int_{\Omega'} \left| \int_0^\tau \nabla' u'(x,x',s)ds \right|^2 dx'dx \ .$$

Then g is well defined and belongs to $C[0,T]$, and (3.9) implies that

$$g(\tau) + \int_{Q_\tau} u^2 + \int_\Omega \int_{Q'_\tau} u'^2 \le C \left[\int_0^\tau (g(\tau) + g(t))dt + \tau \int_{Q_\tau} u^2 \right] \ . \tag{3.10}$$

There is a small $\overline{\tau} > 0$ such that, for $\tau \in (0, \overline{\tau}]$, (3.10) implies

$$g(\tau) + \int_{Q_\tau} u^2 + \int_\Omega \int_{Q'_\tau} u'^2 \le C \int_0^\tau g(t)dt$$

and then, by Gronwall's inequality,

$$u \equiv 0 \quad \text{in} \quad Q_{\overline{\tau}} \ , \quad u' \equiv 0 \quad \text{in} \quad \Omega \times Q'_{\overline{\tau}} \ .$$

The choice of $\overline{\tau}$ is independent of the initial time. Therefore we can repeat the whole argument on $[\overline{\tau}, 2\overline{\tau}]$, etc, and conclude that $u \equiv 0$ in Q_T and $u' = 0$ in $\Omega \times Q'_T$.

§4. $\mathbf{L^1}$ estimates. In this section we establish L^1 stability estimates under additional assumptions which will be removed in subsequent sections.

THEOREM 4.1. *Let (u_1, u'_1) be a weak lower solution corresponding to the inflow F_1 and let (u_2, u'_2) be a weak upper solution corresponding to F_2. Assume that*

$$\begin{aligned} &\partial_t u_i, \ \partial_t f_1(u_i) \in L^2(Q_T), \\ &\partial_t u'_i, \ \partial_t f_2(u'_i) \in L^2(\Omega \times Q'_T) \ . \end{aligned} \tag{4.1}$$

Then, for $t \in [0,T]$,

$$
\left\{ \int\limits_{\Omega} [\Theta(u_1 - u_2)^+(\cdot, s) + \rho(f_1(u_1) - f_1(u_2))^+(\cdot, s)] \right.
$$

$$
(4.2) \qquad \left. + \gamma \int\limits_{\Omega} \int\limits_{\Omega'} [\Theta'(u_1' - u_2')^+(\cdot,\cdot, s) + \rho'(f_2(u_1') - f_2(u_2'))^+(\cdot,\cdot, s)] \right\} \Bigg|_0^t
$$

$$
\leq \int\limits_{S_{1t}} (F_1 - F_2)^+(\cdot, s) \ .
$$

Proof. By virtue of (4.1), we can integrate by parts to get from the first two terms in (2.10)

$$
\int\limits_{\Omega} \partial_t \tilde{f}_1(\cdot, u)\varphi
$$

and thus, by density argument, allow test functions φ satisfying:

$$
\varphi \in W_2^{1,0}(Q_T) \ , \quad \varphi \geq 0 \ .
$$

We can similarly integrate by parts in (2.12) and allow test functions

$$
\psi \in L^2(\Omega \times Q_T'), \partial_{x_j}\psi \in L^2(\Omega \times Q_T') \ , \quad \psi \geq 0 \ .
$$

Set $u = u_1 - u_2$, $u' = u_1' - u_2'$. We want to test with $\operatorname{sgn}(u^+)$ and $\operatorname{sgn}(u'^+)$. But we can do so only with smooth approximations

$$
\varphi(x,t) = \begin{cases} g^n(u^+(x,t)) & \text{for} \quad t \leq \tau \\ 0 \quad \text{for} \quad t > \tau \ , \end{cases}
$$

$$
\psi(x,x',t) = \begin{cases} g^n(u'^+(x,x',t)) & \text{for} \quad t \leq \tau \\ 0 \quad \text{for} \quad t > \tau \end{cases}
$$

where $\tau \in (0,T]$, and

$$
g^n(r) = \begin{cases} 1 & \text{if} \quad r > \dfrac{1}{n} \\ nr & \text{if} \quad |r| < \dfrac{1}{n} \\ -1 & \text{if} \quad r < -\dfrac{1}{n} \ . \end{cases}
$$

Set

$$
G^n(r) = \int\limits_0^r g^n(s)ds \ .
$$

Subtracting the corresponding inequalities for (u_1, u_1') and (u_2, u_2') with φ, ψ as above and adding the second inequality multiplied by γ to the first inequality, leads to

$$
\begin{aligned}
&\int_{Q_\tau} (1-\varepsilon)\partial_t(\Theta u) g^n(u^+) + \int_{Q_\tau} \partial_t[\varepsilon \Theta u + \rho(f_1(u_1) - f_1(u_2))] g^n(u^+) \\
&+ \gamma \int_\Omega \int_{Q'_\tau} (1-\varepsilon)\partial_t(\Theta' u') g^n(u'^+) + \gamma \int_\Omega \int_{Q'_\tau} \partial_t[\varepsilon \Theta' u' + \rho'(f_2(u_1') - f_2(u_2'))] g^n(u'^+) \\
&+ \int_{Q_\tau} D\nabla u \cdot \nabla g^n(u^+) + \gamma \int_\Omega \int_{Q'_\tau} D'\nabla' u' \cdot \nabla' g^n(u'^+) \\
&- \int_{Q_\tau} qu \cdot \nabla g^n(u^+) - \int_{S_{2\tau}} \alpha u g^n(u^+) + \int_{Q_\tau} \int_{\partial\Omega'} \gamma h(u - u')(g^n(u^+) - g^n(u'^+)) \\
&\le \int_{S_{1\tau}} (F_1 - F_2) g^n(u^+)
\end{aligned}
\tag{4.3}
$$

for any $0 < \varepsilon < 1$. We denote the terms on the left-hand side of (4.3) by $A_1, \ldots, A_9$. Then

$$
\begin{aligned}
A_1 &= (1-\varepsilon) \int_{Q_\tau} \partial_t \Theta \cdot u g^n(u^+) + (1-\varepsilon) \int_{Q_\tau} \partial_t (\Theta G^n(u^+)) \\
&\quad - (1-\varepsilon) \int_{Q_\tau} \partial_t \Theta \cdot G^n(u^+) \equiv (1-\varepsilon)(B_1 + B_2 + B_3), \\
A_3 &= \gamma(1-\varepsilon) \int_\Omega \int_{Q'_\tau} \partial_t (\Theta' G^n(u^+)), \\
A_5 &= \int_{Q_\tau} D\nabla u^+ \cdot \nabla u^+ \frac{d}{dr}\, g^n(u^+) \ge 0
\end{aligned}
$$

and similarly

$$
A_6 \ge 0 \ .
$$

Next

$$
A_7 = \int_{Q_\tau} (\nabla \cdot (qu)) g^n(u^+) + \int_{S_\tau} \alpha u g^n(u^+) \equiv B_4 + B_5 \ ,
$$

and

$$B_5 + A_8 = \int\limits_{S_{1\tau}} \alpha u g^n(u^+) = \int\limits_{S_{1\tau}} \alpha u^+ g^n(u^+) \ ,$$

$$\begin{aligned} B_4 &= \int\limits_{Q_\tau} (\nabla \cdot q) u g^n(u^+) + \int\limits_{Q_\tau} q \cdot \nabla G^n(u^+) \\ &= -B_1 - \int\limits_{Q_\tau} (\nabla \cdot q) G^n(u^+) - \int\limits_{S_\tau} \alpha G^n(u^+) \\ &= -B_1 - B_3 - \int\limits_{S_\tau} \alpha G^n(u^+) \quad \text{since} \quad \partial_t \Theta = -\nabla \cdot q \ , \end{aligned}$$

so that

$$\begin{aligned} A_7 + A_8 = -B_1 - B_3 &+ \int\limits_{S_{1\tau}} \alpha(u^+ g^n(u^+) - G^n(u^+)) - \int\limits_{S_{2\tau}} \alpha G^n(u^+) \\ &\geq -B_1 - B_3 \end{aligned}$$

because $\alpha \geq 0$ on $S_{1\tau}$ and $\alpha \leq 0$ on $S_{2\tau}$ and $G^n(r) \leq r g^n(r)$. Finally, $A_9 \geq 0$ since $g^n(r)$ is monotone in r. We substitute these estimates into (4.3) and get

$$\begin{aligned} &(1-\varepsilon) \int\limits_\Omega \Theta G^n(u^+(\cdot,t))\Big|_0^\tau + \int\limits_{Q_\tau} \partial_t[\varepsilon \Theta u + \rho(f_1(u_1) - f_1(u_2))] g^n(u^+) \\ (4.4) \quad &+ (1-\varepsilon)\gamma \int\limits_\Omega \int\limits_{\Omega'} \Theta' G^n(u'^+(\cdot,t))\Big|_0^\tau + \gamma \int\limits_\Omega \int\limits_{Q'_\tau} \partial_t[\varepsilon \Theta' u' + \rho'(f_2(u'_1) - f_2(u'_2))] g^n(u'^+) \\ &- \varepsilon \int\limits_{Q_\tau} (\partial_t \Theta) u g^n(u^+) + \varepsilon \int\limits_{Q_\tau} (\partial_t \Theta) G^n(u^+) \leq \int\limits_{S_{1\tau}} (F_1 - F_2)^+(\cdot, t). \end{aligned}$$

As $n \to \infty$

$$\begin{aligned} &g^n(u^+) \to \operatorname{sgn}(u^+) \quad \text{in} \quad L^p(\Omega) \quad \text{for any} \quad p > 1 \ , \\ &G^n(u^+) \to u^+ \quad \text{in} \quad L^\infty(\Omega) \ , \end{aligned}$$

and analogously

$$g^n(u'^+) \to \operatorname{sgn}(u'^+), \quad G^n(u'^+) \to u'^+ \ .$$

Therefore the second term on the left-hand side of (4.4) converges to

$$\begin{aligned} &\int\limits_{Q_\tau} \partial_t[\varepsilon \Theta u + \rho(f_1(u_1) - f_1(u_2))] \operatorname{sgn}(u^+) \\ &\quad = \int\limits_{Q_\tau} \partial_t[(\varepsilon \Theta u + \rho(f_1(u_1) - f_1(u_2)))^+] \end{aligned}$$

as the function $r \to \varepsilon\Theta(x,t)r + \rho(x)f_1(r)$ is strictly increasing for $\varepsilon > 0$. The same applies to the fourth term in (4.4). In the limiting inequality we take $\varepsilon \to 0$ and arrive at the assertion (4.2).

Theorem 4.1 has two immediate consequences.

COROLLARY 4.2. *Under the asumptions of Theorem 4.1 we have: if* $F_1 \le F_2$ *then*

$$u_1 \le u_2 \quad \text{in} \quad Q_T \quad \text{and} \quad u_1' \le u_2' \quad \text{in} \quad \Omega \times Q_T' \; .$$

COROLLARY 4.3. *Let* (u_i, u_i') *be a weak solution for data* F_i *satisfying (4.1). Then the following* L^1 *stability estimate holds:*

$$\begin{aligned} &\Big\{ \|\Theta(u_1 - u_2)(\cdot,s)\|_1 + \|\rho(f_1(u_1) - f_1(u_2))(\cdot,s)\|_1 \\ &+ \gamma[\|\Theta'(u_1' - u_2')(\cdot,s)\|_1 + \|\rho'(f_2(u_1') - f_2(u_2'))(\cdot,s)\|_1]\Big\}\Big|_0^t \\ &\le \|F_1 - F_2\|_{1,S_{1t}} \quad , \quad \forall\, t \in (0,T] \; . \end{aligned} \tag{4.5}$$

The term in the braces represents the difference in total mass. Inequality (4.5) asserts that the total mass can increase only if there is inflow, and that there may be loss due to outflow.

The comparison principle of Corollary 4.3 leads to appropriate bounds provided we can construct comparison functions. These are furnished by:

LEMMA 4.4. *Let* u *be a weak lower (upper) solution of*

$$\partial_t \tilde{f}_1(\cdot,u) - \nabla\cdot(D\nabla u - qu) = 0 \qquad \text{in} \qquad Q_T \; ,$$

$u(\cdot,0) = u_0$, and boundary condition (2.2), defined analogously to (2.9), (2.10), (2.11) with $h \equiv 0$.

If

$$\partial_t \tilde{f}_2(\cdot,u) \in L^1(\Omega \times Q_T') \quad \text{and} \quad \partial_t \tilde{f}_2(\cdot,u) \le 0 \quad (\ge 0)$$

and

$$u(x,0) \underset{(\ge)}{\le} u_0'(x,x') \quad \text{for} \quad x \in \Omega,\; x' \in \Omega' \; ,$$

then (u,u) is a weak lower (upper) solution of (2.1)–(2.6).

The proof follows immediately if we recast (2.10), (2.12) by using the integration by parts described in the first paragraph of the proof of Theorem 4.1.

From now on we assume that the data are bounded and non-negative, i.e.

$$\begin{aligned} &u_0 \in L^\infty(\Omega), u_0 \geq 0, \quad u_0' \in L^\infty(\Omega \times \Omega'), \quad u_0' \geq 0, \\ &F \in L^\infty(S_{1T}), \quad F \geq 0, \quad \frac{F}{\alpha} \in L^\infty(S_{1T}) \end{aligned} \tag{4.6}$$

(the last condition means that $F = 0$ on the subset of S_{1T} where $\alpha = 0$ and F/α is in L^∞ in the compliment of this subset).

From Lemma 4.4 it follows that (M, M) is an upper solution if

$$M = \max\{\|u_0\|_\infty \ , \|u_0'\|_\infty, \ \|F/\alpha\|_{\infty, S_{2T}}\} \tag{4.7}$$

and $(0,0)$ is a lower solution. Therefore:

COROLLARY 4.5. *Let (u, u') be a weak solution of (2.1)–(2.6) satisfying (4.1). Then*

$$0 \leq u \leq M \quad in \quad Q_T, \quad 0 \leq u' \leq M \quad in \quad \Omega \times Q_T' \ .$$

§5. Existence of solution for smooth f_i. In this section we establish existence for smooth f_i; general f_i will be considered in the next section. We shall use the Schauder fixed point theorem. For this we need to consider the following auxiliary problems:

Given $g \in L^\infty(Q_T)$, find a weak solution (defined analogously to (2.9), (2.10), (2.11)) of

$$\begin{aligned} &\partial_t \tilde{f}_1(\cdot, u) - \nabla \cdot (D\nabla u - qu) + \gamma \left(\int_{\partial\Omega'} h(\cdot, x') dS_{x'} \right) u = g \quad \text{in} \quad Q_T \ , \\ &u(\cdot, 0) = u_0 \quad \text{and boundary condition (2.2)} \ . \end{aligned} \tag{5.1}$$

Given $g' \in L^\infty(\Omega \times S_T')$, find a weak solution (defined analogously to (2.9), (2.12), (2.13)) of

$$\begin{aligned} &\partial_t \tilde{f}_2(\cdot, u') - \nabla' \cdot (D'\nabla u') = 0 \quad \text{in} \quad \Omega \times Q_T' \ , \\ &D'\nabla' u' \cdot n' + hu' = g' \quad \text{in} \quad \Omega \times S_T' \ , \\ &u'(\cdot, 0) = u_0' \quad \text{in} \quad \Omega \times \Omega' \ . \end{aligned} \tag{5.2}$$

For these two scalar degenerate parabolic problems the L^1-stability estimates are basically well known and follow from simplification of the proof of Theorem 4.1.

From now on we assume additionally that

$$u_0 \in C^\delta(\overline{\Omega})\ ,\ u_0' \in C^\delta(\overline{\Omega} \times \overline{\Omega'}) \quad \text{for some} \quad \delta \in (0,1),$$

(5.3)
$$\partial_{x_i} u_0 \in L^2(\Omega), \quad \partial_{x_i'} u_0' \in L^2(\Omega \times \Omega'),$$
$$\partial_t F \in L^2(S_{1T}).$$

In this section only we also require that f_i is smooth, i.e.,

(5.4)
$$f_i \in C^1[0,\infty)\ .$$

Then, the differential equation in (5.1) can be written as

(5.5)
$$(\Theta + \rho \frac{d}{du}\ f_1(u))\partial_t\, u - \nabla \cdot D\nabla u + q \cdot \nabla u + \gamma \left(\int\limits_{\partial\Omega'} h(\cdot, x') dS_{x'} \right) u = g$$

using $\partial_t \Theta = -\nabla \cdot q$. As the coefficient of $\partial_t u$ is bounded from below by $\Theta_0 > 0$, we are dealing with an initial-boundary value problem for a non-degenerate quasilinear parabolic equation. The same applies to (5.2). From [6] we then conclude:

There exists a unique weak solution u of (5.1) and $u \in C^\sigma(\overline{Q}_T)$; further, if $\|g\|_\infty \leq M_0 < \infty$ then

(5.6)
$$\|u\|_{C^\sigma(\overline{Q}_T)} \leq C$$

where C depends on M_0 but not on g. Here $\sigma \in (0,1)$ is independent of g, M_0.

If u_n and u are solutions of (5.1) corresponding to data g_n and g then

(5.7)
$$\|g_n - g\|_{\infty, Q_T} \to 0 \quad \text{implies} \quad \|u_n - u\|_{\infty, Q_T} \to 0\ .$$

Indeed this follows by using (5.6) applied to u_n.

Similarly, there exists a unique weak solution u' of (5.2) and, if $\|g'\|_\infty \leq M_0 < \infty$, $u(x,\cdot) \in C^\sigma(\overline{Q_T'})$ uniformly in $x \in \overline{\Omega}$ and in g', for some $\sigma \in (0,1)$. We have

(5.8)
$$\|u'(x,\cdot)\|_{C^\sigma(Q_T')} \leq C\ ,$$

where C depends on M_0, but not on g' and $x \in \overline{\Omega}$. If u_n' and u' are the solutions of (5.2) corresponding to data g_n' and g', respectively, and if u_n', u' are equicontinuous in $\overline{\Omega} \times \overline{\Omega'} \times [0,T]$, then

(5.9)
$$\|g_n' - g'\|_{\infty, \Omega \times S_T'} \to 0 \quad \text{implies} \quad \|u_n' - u\|_{\infty, \Omega \times Q_T'} \to 0\ .$$

We shall use these facts to prove:

THEOREM 5.1. *Let (5.4) hold. Then there exists a weak solution of (2.1)–(2.6) such that (4.1) holds and*

$$u \in C^\sigma(\overline{Q_T}), \quad u' \in C^\sigma(\overline{\Omega} \times \overline{Q'_T}) \tag{5.10}$$

for some $\sigma \in (0,1)$.

Proof. Let $(\underline{u}, \underline{u}')$ and $(\overline{u}, \overline{u}')$ be lower and upper solutions of (2.1)–(2.6); for example (see §4)

$$(\underline{u}, \underline{u}') = (0,0), \quad (\overline{u}, \overline{u}') = (M, M) \ . \tag{5.11}$$

Introduce the set

$$A = \{u' \in C(\overline{\Omega} \times \overline{Q'_T}), \underline{u}' \le u' \le \overline{u}' \quad \text{on} \quad \overline{\Omega} \times \overline{Q'_T}\} \ ,$$

and consider the following operator W defined on A:

For $u' \in A$ let u be the weak solution of (5.1) for

$$g \equiv \int_{\partial\Omega'} \gamma h(\cdot, x') u'(\cdot, x', \cdot) dS_{x'} \ , \tag{5.11}$$

and let $\tilde{u}'$ be the weak solution of (5.2) for

$$g' = hu \ , \tag{5.12}$$

Set $Wu' = \tilde{u}'$.

A fixed point of W is obviously a weak solution of (2.1)–(2.6) satisfying (4.1) (by [6]). We want to apply Schauder's fixed point theorem. For this it is sufficient to show:

(a) $W(A) \subset A$,

(b) W is compact, and

(c) W is continuous;

here A is provided with the L^∞-norm.

The assertion (b) follows from

$$\begin{aligned} |\tilde{u}'(x,x',t) - \tilde{u}'(y,y',s)| \ &\le C(|x-y|^\sigma + |x'-y'|^\sigma + |t-s|^\sigma) \\ \text{for } \ x,y \in \overline{\Omega}, \quad x',y' &\in \overline{\Omega'} \ , \quad t,s \in [0,T] \ , \end{aligned} \tag{5.13}$$

provided C and σ can be chosen uniformly for $u' \in A$. But this can be shown by an argument from Friedman and Tzavaras [3; pp. 185–6] based on (5.6) and (5.8).

The assertion (c) is a consequence of (5.7) and (5.9). Thus it remains to prove (a).

We begin by proving that for any $u' \in A$ the solution u of (5.1) with g defined by (5.11) satisfies:

$$\underline{u} \le u \le \overline{u} \quad \text{in} \quad \overline{Q_T} \, . \tag{5.14}$$

Indeed, we can proceed as in the proof of Theorem 4.1 and test the integral inequality for $\underline{u} - u$ (from (2.10)) by $g^n((\underline{u} - u)^+)$. The only difference occurs in the coupling term on the left-hand side, which now reads

$$\int_{Q_T} \int_{\partial\Omega'} \gamma h((u' - \underline{u}') + (\underline{u} - u)) g^n((\underline{u} - u)^+) \ge 0 \, .$$

Therefore, for $t \in (0, T]$,

$$\int_{\Omega} (\Theta(\underline{u} - u)^+ + \rho(f_1(\underline{u}) - f_1(u))^+(\cdot, s)\Big|_0^t \le 0$$

and thus, because of the monotonicity of f_1, $\underline{u} \le u$ in $\overline{Q_T}$. The proof of the other inequality in (5.14) is similar.

We can proceed in this way with $\tilde{u}'$, the solution of (5.2) with g' given by (5.12), and test the integral inequality for $\underline{u}' - \tilde{u}'$ (from (2.12)) with $g^n((\underline{u}' - \tilde{u}')^+)$. The coupling term on the left hand side is

$$\int_{\Omega} \int_{S'_T} h((u - \underline{u}) + (\underline{u}' - \tilde{u}')) g^n((\underline{u}' - \tilde{u}')^+) \ge 0 \, ,$$

using (5.14). This shows that $\underline{u}' \le \tilde{u}'$, and similarly $\tilde{u}' \le \overline{u}'$. Thus $Wu' \in A$. The property (4.1) follows along the lines of [6; pp. 173–178], leading to an estimate of $\partial_t u$ and $\partial_t u'$ in L^2-norms in terms of u_0, u_0' and F in appropriate norms.

§6. Existence for general $\mathbf{f_i}$. In this section we assume that

$$\begin{aligned} &f_1(u) = u^p f_{10}(u), \quad f_2(u) = u^q f_{20}(u) \, , \quad f_i(u) \le C(1 + u) \quad \text{for some} \quad C > 0 \, , \\ &f_{i0} \in C^1[0, \infty), \quad \frac{d}{du} f_{i0}(u) > 0 \quad \text{if} \quad u \ge 0 \, , \\ &\text{and} \quad 0 < p < 1, \quad 0 < q \le 1 \, . \end{aligned} \tag{6.1}$$

THEOREM 6.1. *If (6.1) holds then there exists a weak solution of (2.1)–(2.6) satisfying (4.1) and, for any compact subsets $K_1 \subset Q_T$ and $K_2 \subset \Omega \times Q'_T$,*

$$u \in C^\sigma(K_1), \quad u' \in C^\sigma(K_2) \tag{6.2}$$

for some $\sigma \in (0,1)$ which may depend on K_1, K_2.

Proof. For any $\varepsilon \in (0,1)$ introduce $C^1[0,\infty)$ functions

$$f_i^\varepsilon(r) = \begin{cases} f_i(r) & \text{for} \quad r > \varepsilon \\ \dfrac{f_i(r)}{\varepsilon/2}\, r & \text{for} \quad 0 \le r < \varepsilon/2 \ , \end{cases}$$

$$\frac{d}{dr}\, f_i^\varepsilon(r) \ge 0 \ ,$$

and define

$$u_0^\varepsilon = u_0 + \varepsilon \ , \quad F^\varepsilon = F + \alpha\varepsilon, \quad u_0'^\varepsilon = u_0' + \varepsilon \ .$$

Theorem 5.1 applies to f_i^ε, so that there exist weak solutions $(u^\varepsilon, u'^\varepsilon)$ of (2.1)–(2.6) for f_i^ε and the data $u_0^\varepsilon, F^\varepsilon, u_0'^\varepsilon$, satisfying (4.1). By Lemma 4.4, weak lower and upper solutions of $(u^\varepsilon, u'^\varepsilon)$ are given by

$$(\varepsilon, \varepsilon) \qquad \text{and} \quad (M+1,\ M+1)$$

where M is defined by (4.7). Analogously to Corollary 4.5 we then have

$$\varepsilon \le u^\varepsilon \le M+1 \quad \text{in} \quad Q_T \ , \quad \varepsilon \le u'^\varepsilon \le M+1 \quad \text{in} \quad \Omega \times Q_T' \ . \tag{6.3}$$

From the lower bounds it follows that $(u_\varepsilon, u_\varepsilon')$ is a solution of (2.1)–(2.6) also for f_i and the data $u_0^\varepsilon, F^\varepsilon, u_0'^\varepsilon$.

We now apply the local Hölder regularity results of DiBenedetto and Friedman [1] (our assumptions are somewhat different than in [1] but the proof is essentially the same) for degenerate parabolic equation to conclude that

$$u^\varepsilon \in C^\sigma(K_1) \ , \ u'^\varepsilon(x,\cdot) \in C^\sigma(\widehat{K}_2) \quad \text{uniformly in} \ \ x \in K_0 \tag{6.4}$$

where σ and the Hölder coefficients are independent of ε; here K_0, K_1 and $\widehat{K}_2$ are any compact subsets of Ω, Q_T and Q_T', respectively. We conclude, again by [3], that also

$$u'^\varepsilon \in C^\sigma(K_2) \tag{6.5}$$

for any compact subset K_2 of $\Omega \times Q_T'$, with Hölder constant and exponent independent of ε.

From (6.3)–(6.5) we see that there exist functions $u : \overline{Q_T} \to \mathbb{R}$, $u' : \overline{\Omega} \times \overline{Q_T'} \to \mathbb{R}$ such that, for a subsequence $\varepsilon \to 0$,

$$u^\varepsilon \to u \ , \ u'^\varepsilon \to u' \tag{6.6}$$

uniformly in compact subsets of Q_T and $\Omega \times Q'_T$, respectively. By testing (2.10) with u^ε and (2.12) with u'^ε and using (6.3) we can show that

$$\int\limits_{Q_T} |\nabla u^\varepsilon|^2 \le C_0 \ , \qquad \int\limits_{\Omega}\int\limits_{Q'_T} |\nabla' u'^\varepsilon|^2 \le C_0 \tag{6.7}$$

where C_0 is a constant independent of ε. Therefore, for a subsequence,

$$\begin{aligned} \nabla u^\varepsilon &\to \nabla u \quad \text{weakly in} \quad L^2(Q_T) \ , \\ \nabla' u'^\varepsilon &\to \nabla' u' \quad \text{weakly in} \quad L^2(\Omega \times Q'_T) \ . \end{aligned} \tag{6.8}$$

Using (6.6), (6.8) and the trace theorem we can pass to the limit in (2.10), (2.12) and conclude that (u, u') is a weak solution to (2.1)–(2.6), satisfying (6.2). The continuity in time with values in L^2 carries over from $(u^\varepsilon, u'^\varepsilon)$ to (u, u') by means of (4.5) and (6.3), showing that $(u^\varepsilon, u'^\varepsilon)$ is a Cauchy sequence in the corresponding spaces.

COROLLARY 6.2. *Theorem 4.1 and Corollaries 4.2, 4.3, 4.5 hold true if (u_1, u'_1) or (u_2, u'_2) is a weak solution (not necessarily satisfying (4.1)).*

REMARK 6.1. The regularity of the data can be relaxed by an approximation procedure.

Proof. As weak solutions are unique by Theorem 2.1, we can approximate them by $(u_i^\varepsilon, u_i'^\varepsilon)$ satisfying (4.1) as done in the proof of Theorem 6.1 and Remark 6.1. The corresponding inequalities thus hold for $(u_i^\varepsilon, u_i'^\varepsilon)$ in place of (u_i, u'_i). Passing to the limit and using (6.6), (6.7), gives the assertion.

REMARK 6.2. In the subsequent analysis we shall use comparison principles for (2.1)–(2.6) and for the scalar auxiliary problems (5.1) and (5.2) with other boundary conditions than considered up to now. By inspecting the arguments we see that everything carries over as long as we do not impose Neumann conditions on the inflow boundary S_{1T}. In particular, the whole theory carries over to the following situations:

(i) Dirichlet boundary conditions on S_T or S'_T for (5.2); here no assumptions on inflow and outflow boundaries are necessary;

(ii) Dirichlet on S_{1T} and Neumann on S_{2T};

(iii) Flux on S_{1T} and Dirichlet on S_{2T}.

We require in all these cases that the solution be continuous in a neighborhood of the Dirichlet boundary, so that the Dirichlet conditions are satisfied in the usual sense.

§7. **Finite speed of propagation.** In this section we assume in addition to (5.3) and (6.1) that

$$h(x, x') \ge h_0 > 0 \ . \tag{7.1}$$

We only consider solutions such that u is continuous in Q_T and u' is continuous in $\Omega \times Q'_T$, i.e., as established in §6.

LEMMA 7.1. *(a) Let $x \in \Omega,\ x' \in \Omega',\ 0 \le t_1 < t_2$. Then*

$$u'(x,x',t_1) > 0 \quad \textit{implies} \quad u'(x,x',t_2) > 0\ ;$$

(b) Let $x \in \Omega, \quad 0 \le t_1 < t_2$. Then

$$u(x,t_1) > 0 \quad \textit{implies} \quad u(x,t_2) > 0\ ;$$

there is a subset B of Ω with $\text{meas}(\Omega \backslash B) = 0$ *such that for $x \in B, \quad 0 < t < T$:*
(c) $u(x,t) > 0$ implies $\int_{\Omega'} u'(x,x',t)dx' > 0$;
(d) $\int_{\Omega'} u'(x,x',t)dx' > 0$ implies

$$u(x,t) > 0\ , \quad \textit{if in addition}\ \ u'_0 \equiv 0\ .$$

Proof. To prove (a) note that by continuity there is a σ-neighborhood $B_\sigma(x')$ such that for some $\delta > 0$

$$u'^{\varepsilon}(x,\cdot,t_1) \ge u'(x,\cdot,t_1) \ge \delta \quad \text{in} \quad B_\sigma(x')\ ; \tag{7.2}$$

here $(u^\varepsilon, u'^\varepsilon)$ is the approximating sequence of solutions constructed in the proof of Theorem 6.1.

For $\varepsilon \ge 0$, let w^ε denote the solution of

$$\begin{aligned} \Theta' \partial_t w^\varepsilon - \nabla' \cdot (D' \nabla' w^\varepsilon) = 0 \quad &\text{in} \quad A \equiv B_\sigma(x') \times (t_1, t_2)\ , \\ w^\varepsilon = \varepsilon \quad &\text{on} \quad \partial B_\sigma(x') \times (t_1,t_2)\ , \\ w(\cdot, t_1) = \delta \quad &\text{in} \quad B_\sigma(x')\ . \end{aligned}$$

If $\varepsilon < \delta$ then by comparison $\varepsilon \le w^\varepsilon \le \delta$ and therefore, for any small $\mu > 0, w^\varepsilon(x,t+\mu) - w^\varepsilon(x,t) \le 0$ on $t = t_1$. Applying the maximum principle to this function we find that it is ≤ 0 throughout $A \cap \{t < t_2 - \mu\}$. Hence $\partial_t w^\varepsilon \le 0$ and, consequently,

$$\partial_t f_2(w^\varepsilon) = \left(\frac{d}{du}\ f_2(w^\varepsilon) \right) \partial_t w^\varepsilon \le 0\ .$$

It follows that w^ε is a lower solution of

$$\partial_t(\Theta' \tilde{u}' + \rho f_2(\tilde{u}')) - \nabla' \cdot (D' \nabla' \tilde{u}') = 0 \tag{7.3}$$

with Dirichlet conditions $\tilde{u}' = u'^\varepsilon$. Clearly also

$$w^\varepsilon \ge w^0 > 0 \quad \text{for} \quad (x,t) \in A\ .$$

By the comparison principle for (7.3) with Dirichlet conditions we conclude,

$$u'^{\varepsilon} \geq w^{\varepsilon} \geq w_0 > 0 \quad \text{in} \quad A$$

and (a) follows.

The proof of (b) is similar.

To prove (c) we assume that $\int\limits_{\Omega'} u'(x,x',t)dx' = 0$ and show that $u(x,t) = 0$. From the assumed equality it follows that $u'(x,x',t) = 0$ for all $x' \in \Omega$ and, by (a),

$$u'(x,x',\tau) = 0 \quad \text{if} \quad 0 < \tau \leq t \; .$$

For almost every $x \in \Omega$ (2.12) is valid pointwise in x. But then,

$$\int\limits_{S'_t} h(x,x')u(x,\tau)\psi(x,x')dS_{x'}d\tau = 0$$

for any test function ψ, and thus $u(x,\tau) = 0$ for $0 < \tau \leq t$.

To prove (d) we assume that $u(x,t) = 0$. By (b) this implies that $u(x,\tau) = 0$ if $0 < \tau \leq t$. But then for almost every $x \in \Omega$ (2.12) reduces to a scalar degenerate parabolic problem with homogeneous data. Uniqueness to this problem implies

$$u'(x,x',\tau) = 0 \quad \text{for} \quad x' \in \Omega' \;,\; \tau \in (0,t)$$

and thus the assumption in (d) on u' is not satisfied, a contradiction.

We concentrate, from now on, on the one-dimensional situation for the macroscale, i.e., $\Omega = (0,l)$ for some $l > 0$; Ω' is still a domain in $\mathbb{R}^M$, $M \geq 1$.

We take $x = 0$ as the inflow boundary, i.e., $q(0,t) > 0$ and $x = l$ as the outflow boundary, i.e., $q(l,t) \geq 0$. We would like to consider the behavior of the supports of u and $\int\limits_{\Omega'} u'(\cdot,x',\cdot)dx'$.

Definition 7.1. Let (u,u') be the solution of (2.1)–(2.6) and $t \in [0,T]$. Set

$$\begin{aligned} s_1(t) &= \sup\{x \in [0,l], \quad u(x,t) > 0\} \;, \\ s_2(t) &= \sup\{x \in [0,l], \quad \int\limits_{\Omega'} u'(x,x',t)dx' > 0\} \;, \\ s(t) &= \sup\{x \in [0,l], \quad u(x,t) + \int\limits_{\Omega'} u'(x,x',t)dx' > 0\} \;. \end{aligned}$$

If the corresponding sets are empty, the definition is modified to $s_1(t) = 0$, etc.

In the same way we define $r_i(t)$, $r(t)$ as the infima of the sets above with the modification $r_1(t) = l$ etc. for empty sets.

Obviously

$$s_1(t) \leq s(t), \quad s_2(t) \leq s(t) \tag{7.4}$$

and s_i, s (r_i, r) are lower (upper) semicontinuous.

THEOREM 7.2. *(a) For $0 < t \le T$, $s_2(t) = s(t)$; if $u_0' = 0$ then $s_1(t) = s(t)$.*
(b) s_1, s_2 and s are monotone nondecreasing for $t \in [0,T]$.

Proof. Let

$$v(x,t) = \int_{\Omega'} u'(x,x',t)dx' .$$

To prove the first part of (a) it suffices to consider the case $s(t) > 0$. For any small ε there is $x \in (s(t)-\varepsilon, s(t))$ such that $u(x,t)+v(x,t) > 0$. By Lemma 7.1 (c) $v(x,t) > 0$ and thus $s_2(t) \ge x > s(t) - \varepsilon$. Taking $\varepsilon \to 0$ the assertion follows.

The second assertion in (a) follows by the same argument, using Lemma 7.1 (d).

The monotonicity of s_1 and s_2 follows from Lemma 7.1, and since $s(t) = s_2(t)$ for $0 < t < T$, the proof of (b) is complete.

REMARK 7.1. Similarly one can show that $r_2(t) = r(t)$, $r_1(t) \ge r(t)$ and $r_1(t) = r(t)$ if $u_0' = 0$; the functions r_1, r_2, r are monotone nonincreasing.

We want to prove that solutions of (2.1)–(2.6) have finite speed of propagation. We therefore need comparison functions with the same properties. To this end we consider a travelling wave solution of

$$\begin{aligned} &\overline{\Theta}\partial_t u + \overline{\rho}\partial_t f_1(u) - \overline{D}\partial_{xx}u + \overline{q}\partial_x u = 0 , \quad x \in \mathbb{R},\ t > 0 , \\ &u(+\infty,t) = 0 \quad \text{for} \quad t > 0 \end{aligned} \tag{7.5}$$

with $\overline{\Theta}, \overline{\rho}, \overline{D}$ positive constants and $\overline{q}$ a real constant. A travelling wave solution of (7.5) is given by a function

$$u : E \to \mathbb{R}, \qquad E = (\eta_{-\infty}, \infty) \qquad (\eta_{-\infty} \ge -\infty)$$

and a wave speed c ($c > 0$) such that $u = u(\eta)$ satisfies (7.5) for $\eta = x - ct$, i.e.,

$$\begin{aligned} -c\overline{\Theta}u' - c\overline{\rho}(f_1(u))' - \overline{D}u'' + \overline{q}u' = 0 \quad \text{in} \quad E , \\ u(+\infty) = 0 , \end{aligned} \tag{7.6}$$

where the derivatives are with respect to η. By integration, (7.6) reduces to

$$(\overline{q} - c\overline{\Theta})u - c\overline{\rho}f_1(u) - \overline{D}u' = 0 . \tag{7.7}$$

Restricting c by $c > \overline{q}/\overline{\Theta}$ if $\overline{q} \ge 0$, (7.7) can be solved by

$$\eta = \int_{u(\eta)}^{a} \frac{\overline{D}ds}{(c\overline{\Theta} - \overline{q})s + c\overline{\rho}f_1(s)} \equiv G(u(\eta)) \tag{7.8}$$

where we have fixed a translation of the independent variable of u so that $u(a) = 0$. Notice that the denominator in (7.8) is always positive if $u(\eta) \geq 0$. The function G is strictly decreasing in $u(\eta)$, and a solution of (7.8) is given by

$$u(\eta) = G^{-1}(\eta) \quad \text{for} \quad \eta_{-\infty} < \eta < a$$

where

$$\eta_{-\infty} = G(+\infty) \quad \text{and} \quad a = G^{-1}(0) \; .$$

As a consequence of (6.1),

$$\eta_{-\infty} = -\infty \quad \text{and} \quad a < \infty \; .$$

Setting $u(\eta) = 0$ if $\eta > a$ we obtain a finite travelling wave. Further, by (7.7),

$$u'(\eta) < 0 \quad \text{for} \quad \eta < a \; , \quad u'(a-) = 0 \; . \tag{7.9}$$

It follows that $u \in C^1(E)$ and, by (7.7),

$$\overline{D}u' + c\overline{\rho} f_1(u) \in C^1(E) \; .$$

From (7.9) we deduce that

$$(f_i(u))' \in L^1_{\text{loc}}(E) \; , \tag{7.10}$$

and from (7.6)

$$u'' \in L^1_{\text{loc}}(E) \; . \tag{7.11}$$

Thus $u(x,t) \equiv u(x-ct)$ is a strong solution to (7.5) in half plane

$$\mathbb{R}^+_2 = \{(x,t); \; x \in \mathbb{R}^1, \; t > 0\} \; ,$$

satisfying

$$\begin{aligned} &u, \partial_x u, \partial_t u \in C(\overline{\mathbb{R}^+_2}), \\ &\partial_t f_i(u), \; \partial_{xx} u \in L^1_{\text{loc}}(\mathbb{R}^+_2) \; . \end{aligned} \tag{7.12}$$

From (7.6), (7.9) we conclude that

$$(f_i(u))' \leq 0 \quad (i = 1,2), \quad u'' \geq 0 \quad \text{in} \quad E \; . \tag{7.13}$$

Set

$$\tilde{u}(z) = u(z + a)$$

Then

$$\eta = \int_0^{\tilde{u}(\eta)} \frac{\overline{D} ds}{(\overline{q} - c\overline{\Theta})s - c\overline{\rho} f_1(s)} \ , \quad \tilde{u}(\eta) > 0 \ ; \tag{7.14}$$

the denominator is negative.

Observe that for any large $M > 0$ and small $\delta > 0$ there is a wave speed c sufficiently large such that

$$\tilde{u}(-\delta) = M \ , \quad \tilde{u}(\eta) > M \quad \text{for} \quad \eta < -\delta \ . \tag{7.15}$$

From now on we assume additionally that

$$\begin{aligned} &\rho(x) \geq \rho_0 > 0 \quad \text{for} \quad x \in \overline{\Omega} \ , \\ &\partial_x D \in L^\infty(Q_T) \ . \end{aligned} \tag{7.16}$$

THEOREM 7.3. *Let (u, u') be a weak solution of (2.1)–(2.6). If $t_1 \in [0, T)$ and $s(t_1) < l$ then for any small δ and large c satisfying (7.15) with M defined by (4.7) there holds:*

$$s(t) \leq s(t_1) + c(t - t_1) + \delta \quad \textit{if} \quad t_1 \leq t \leq t_1 + \varepsilon \ , \tag{7.17}$$

provided $s(t_1) + c\varepsilon + \delta \leq l$.

Proof. Let $b = s(t_1), \overline{\Theta} = \Theta_0, \ \overline{\rho} = \rho_0 \ , \ \overline{D} = \dfrac{1}{\nu}$ (from the general assumptions in §2), and

$$\overline{q} \geq \sup_{Q_T}(q - \partial_x D) \ .$$

Consider the travelling wave $(\tilde{u}, c)$ with $c > \overline{q}/\overline{\Theta}$ such that (7.15) is satisfied for some $0 < \delta < l - b$. Let

$$\begin{aligned} &A = (b, l) \times (t_1, t_1 + \varepsilon] \ , \ \varepsilon > 0 \ , \\ &\hat{u}(x, t) = \tilde{u}(x - c(t - t_1) - b - \delta) \ . \end{aligned}$$

Using (7.9), (7.13), we find that

$$\begin{aligned} &\Theta \partial_t \hat{u} + \rho \partial_t f_1(\hat{u}) - D \partial_{xx} \hat{u} + (q - \partial_x D) \partial_x \hat{u} \\ &\geq \overline{\Theta} \partial_t \hat{u} + \overline{\rho} \partial_t f_1(\hat{u}) - \overline{D} \partial_{xx} \hat{u} + \overline{q} \partial_x \hat{u} = 0 \end{aligned}$$

and

$$\partial_t f_2(\hat{u}) \geq 0$$

in A. Further, for $x \in (b, l)$,

$$\hat{u}(x, t_1) \geq 0 = u(x, t_1) \ ,$$
$$\hat{u}(x, t_1) \geq 0 = u'(x, x', t_1) \ , \ x' \in \Omega'$$

and at $x = l$

$$\partial_x \hat{u}(l, t) = \partial_x u(l, t) = 0$$

if $\varepsilon < (l - b - \delta)/c$. Finally, by (7.15),

$$\hat{u}(b, t) \geq \hat{u}(b, t_1) = M \geq u(b, t) \ .$$

Using the comparison results of §4 (see Remark 6.2) for $(\hat{u}, \hat{u}), (u, u')$ we deduce that

$$\hat{u}(x, t) \geq u(x, t) \ , \ \hat{u}(x, t) \geq u'(x, x', t)$$

for $(x, t) \in A$, $x' \in \Omega'$. This implies (7.17).

COROLLARY 7.4. *The functions s, s_2 are continuous for $t \in (0, T]$; if $u_0' \equiv 0$ then s_1 is also continuous in $(0, T]$.*

Since s is lower semicontinuous and, by a similar argument as in the proof of Theorem 7.3 and the monotonicity of s, also upper semicontinuous, it follows that s is continuous. The rest follows by Theorem 7.2.

COROLLARY 7.5. *Assume that $q - \partial_x D \leq 0$. There exists a constant K such that*

$$s(t) \leq s(t_1) + 2K\sqrt{t - t_1} \quad \text{if} \quad t > t_1, \quad s(0) + K\sqrt{t - t_1} < l \ . \tag{7.18}$$

This follows by the proof of (7.17). Indeed, we have proved that

$$s(t) \leq s(t_1) + \delta + c(\delta)(t - t_1)$$

where $c(\delta)$ is determined by $\tilde{u}(-\delta) = M$, i.e. (by (7.14) with $\overline{q} = 0$), $\delta = A/c$, A constant independent of t_1, δ. Hence

$$s(t) \leq s(t_1) + \frac{A}{c} + c(t - t_1) \quad \text{provided} \quad c \geq \frac{A}{l - s(t_1)} \ .$$

If for fixed t we minimize on c, then (7.18) follows.

REMARK 7.2. If $r(t_1) > 0$ then analogously to (7.17) we have

$$r(t) \geq r(t_1) - c(t - t_1) - \delta \ , \quad t_1 < t < t_1 + \varepsilon \ .$$

This follows from Theorem 7.3 after making the transformation $x \to l - x$. An estimate from below analogous to (7.18) also holds provided $q + \partial_x D \geq 0$.

REMARK 7.3. One could expect that for $q - \partial_x D \leq 0$, i.e., against the direction of convection, the free boundary does not move at all. This seems to be wrong, as it is wrong for the scalar problem (5.1) (with $g \equiv 0$) according to [4]. Nevertheless (7.18) seems to be by no means sharp, as it does not take into account the strength of the convection.

REMARK 7.4. For $q - \partial_x D \geq 0$, i.e., in the direction of the convection, if the mass inflow is positive ($F \geq F_0$ for some constant $F_0 > 0$) one expects that the free boundary increases at least linearly. This can be made rigorous for $f_2 \equiv 0$. Then $(u, \int_{\Omega'} u(\cdot, x', \cdot)dx')$ satisfies a pde/ode system, for which travelling wave solutions have been studied in [2]. There are finite travelling waves, which can be used to compare with from below.

REMARK 7.5. If $s_1(0) < s(0)$ then, by an argument as for Lemma 7.1 (d), $s(t) = s(0)$ as long as $s_1(t) < s(0)$. If $u_0' \not\equiv 0$ then in general $s_1(t) < s(t)$ and s_1 is not continuous in $(0, T]$.

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