STOKES' FIRST PROBLEM FOR LINEAR VISCOELASTIC FLUIDS WITH FINITE MEMORY

By

Daniel D. Joseph

and

Michael Renardy

IMA Preprint Series # 557

August 1989

Stokes' first problem for linear viscoelastic fluids with finite memory

Daniel D. Joseph
Department of Aerospace Engineering and Mechanics
University of Minnesota
110 Union St. S.E.
Minneapolis, MN 55455

and

Michael Renardy
Department of Mathematics and ICAM
Virginia Tech
Blacksburg, VA 24061-0123

Abstract

We consider Stokes' first problem for a viscoelastic fluid. The memory of the fluid is truncated to a finite time interval, and discontinuities in the stress relaxation modulus or its derivatives are allowed at the point of truncation. We investigate secondary waves which are generated by the interaction of these singularities in the memory with earlier waves.

1. Introduction

The memory of viscoelastic materials is usually modelled as infinite, e.g. the memory function is assumed to be a linear combination of exponentials. For purposes of approximation, however, one may want to truncate the memory at a finite time T. Depending on how such a truncation is carried out, discontinuities in the stress relaxation modulus or in its higher derivatives are introduced. In wave propagation problems, these discontinuities will generate secondary waves; any discontinuous wavefront will produce an "echo" after a time interval of length T has elapsed. In the present note, we discuss the nature of such echo waves in Stokes' first problem.

We consider a viscoelastic fluid filling the half-space $x > 0$. The fluid is at rest for $t \leq 0$. For $t > 0$, the boundary of the half space moves with constant unit velocity. For simplicity, we set the fluid density equal to 1. With u denoting the velocity and G the stress relaxation modulus, the equations governing the problem are as follows:

$$u_t(x,t) = \int_{-\infty}^{t} G(t-s)u_{xx}(x,s)\,ds;\ x > 0,\ t > 0,$$

$$u(x,t) = 0,\ x > 0,\ t \leq 0;\ u(0,t) = 1,\ t > 0. \tag{1}$$

We are interested in the situation where G has a finite interval of support; we assume that $G(\tau) = 0$ for $\tau > T$, and that $G(\tau)$ is a C^∞-function on the interval $[0, T]$, but we allow for a jump in G or its derivatives at $\tau = T$. We shall see that a jump in G itself leads to pathological behavior; subsequent echo waves have singularities of increasing strength and the solution is physically meaningless. We discuss this situation in Section 2. For the sake of simplicity, we shall limit the discussion to the case where G is a step function.

In Section 3, we discuss the case where only derivatives of G are discontinuous. If the discontinuity is in the first derivative, we find an infinite sequence of jumps in u. If the discontinuity is in higher derivatives of G, the strength of the echo waves is progressively weakening.

2. A step relaxation function

In this section, we shall consider the relaxation function

$$G(\tau) = \begin{cases} 1, & \text{for } 0 \leq \tau \leq T, \\ 0, & \text{for } \tau > T. \end{cases} \tag{2}$$

With this relaxation function, equation (1) yields, after differentiation with respect to time,

$$u_{tt}(x,t) = u_{xx}(x,t) - u_{xx}(x,t-T). \tag{3}$$

As long as the point $(x, t-T)$ lies inside the region where u vanishes, we simply have the wave equation, which, under the boundary conditions given in (1), has the solution

$$u(x,t) = H(t-x). \tag{4}$$

Here H denotes the Heaviside function, which equals 1 for positive argument and 0 for negative argument. It follows that (4) is the solution of (3) for $t < x+T$. However, when t reaches $x+T$, the term $u_{xx}(x,t-T)$ involves the second derivative of the Heaviside function, which is equal to the first derivative of the Dirac delta function. That is, for $t < x+2T$ equation (3) reduces to

$$u_{tt} = u_{xx} - \delta'(t-T-x), \tag{5}$$

which can be shown to have the solution

$$u(x,t) = H(t-x) - \frac{t-T+x}{4}\delta(t-T-x). \tag{6}$$

Note that (6) satisfies the boundary condition $u(0,t) = 1$, even at $t = T$. Proceeding in the same fashion, we find the equation

$$u_{tt} = u_{xx} - \delta'(t-T-x) - \frac{1}{2}\delta'(t-2T-x) + \frac{t-2T+x}{4}\delta''(t-2T-x) \tag{7}$$

for $t < x+3T$, which yields the solution

$$u(x,t) = H(t-x) - \frac{t-T+x}{4}\delta(t-T-x) - \frac{t-2T+x}{8}\delta(t-2T-x) + \frac{(t-2T+x)^2}{32}\delta'(t-2T-x). \tag{8}$$

We can continue the calculation and get an explicit expression for u on any time interval. The singularities get progressively worse; we have a jump at $t = x$, a delta function at $t = x+T$, a derivative of a delta function at $t = x+2T$, etc. Clearly, this solution is not physically reasonable. Indeed, a sufficient condition for stability of the rest state of viscoelastic fluids is that G is positive, monotone decreasing and convex [2]. A function which jumps from a positive value to zero at a finite time is not convex. The preceding calculation should be viewed as a dramatic example of an instability. In fact, it is an example of ill-posedness; since arbitrarily high derivatives of the delta function occur, the solution does not remain within any Sobolev class.

3. Relaxation functions with discontinuities in higher derivatives

We shall deduce the singularities of u from the asymptotic behavior of its Laplace transform (cf. e.g. [1]). Let $\hat{u}(x,\lambda)$ denote the Laplace transform of u:

$$\hat{u}(x,\lambda) = \int_0^\infty e^{-\lambda t} u(x,t)\, dt. \tag{9}$$

Equation (1) becomes

$$\lambda \hat{u}(x,\lambda) = \hat{G}(\lambda)\hat{u}_{xx}(x,\lambda),\ \hat{u}(0,\lambda) = \frac{1}{\lambda}, \tag{10}$$

which has the solution

$$\hat{u}(x,\lambda) = \frac{1}{\lambda}\exp\Big(-x\sqrt{\frac{\lambda}{\hat{G}(\lambda)}}\Big). \tag{11}$$

The solution u is given by the inverse transform

$$u(x,t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t}\hat{u}(x,\lambda)\, d\lambda, \tag{12}$$

where γ is any positive real number.

We assume that G is given by a smooth (C^∞) function on the interval $[0,T]$, that $G(T)=0$, and $G(t)=0$ for $t>T$. For large λ, we use repeated integration by parts to expand $\hat{G}(\lambda)$ as follows:

$$\hat{G}(\lambda) = \int_0^T e^{-\lambda t}G(t)\, dt = \frac{1}{\lambda}G(0) + \frac{1}{\lambda}\int_0^T e^{-\lambda t}G'(t)\, dt$$

$$= \frac{1}{\lambda}G(0) + \frac{1}{\lambda^2}(G'(0) - G'(T)e^{-\lambda T}) + \ldots + \frac{1}{\lambda^{n+1}}(G^{(n)}(0) - G^{(n)}(T)e^{-\lambda T}) + O(|\lambda|^{-(n+2)}). \tag{13}$$

By inserting this into (11), we obtain

$$\hat{u}(x,\lambda) = \frac{1}{\lambda}\exp\Big(\frac{-\lambda x}{\sqrt{G(0)}}\Big)\exp\Big(\frac{G'(0)x}{2G(0)^{3/2}}\Big)\exp\Big(-\frac{G'(T)e^{-\lambda T}x}{2G(0)^{3/2}}\Big) + O(|\lambda|^{-2}). \tag{14}$$

Terms of order $|\lambda|^{-2}$ lead to an absolutely convergent integral in (12) and hence to a continuous contribution to u. Any jump discontinuities in u are hence determined by the terms included in (14). We analyze (14) further by expanding the last exponential:

$$\exp\Big(-\frac{G'(T)e^{-\lambda T}x}{2G(0)^{3/2}}\Big) = \sum_{n=0}^\infty \frac{1}{n!}\Big(-\frac{G'(T)x}{2G(0)^{3/2}}\Big)^n e^{-n\lambda T}. \tag{15}$$

Hence we find

$$\hat{u}(x,\lambda) = \sum_{n=0}^{\infty} \frac{1}{\lambda}\frac{1}{n!}\Big(-\frac{G'(T)x}{2G(0)^{3/2}}\Big)^n \exp\Big(\frac{-\lambda x}{\sqrt{G(0)}} - n\lambda T\Big)\exp\Big(\frac{G'(0)x}{2G(0)^{3/2}}\Big) + O(|\lambda|^{-2}). \quad (16)$$

Carrying out the inverse Laplace transform, we find

$$u(x,t) = \sum_{n=0}^{\infty} \frac{1}{n!}\Big(-\frac{G'(T)x}{2G(0)^{3/2}}\Big)^n \exp\Big(\frac{G'(0)x}{2G(0)^{3/2}}\Big) H(t - nT - \frac{x}{\sqrt{G(0)}}) + \ ..., \quad (17)$$

where the dots indicate a remainder term given by a continuous function. Hence u has a jump across $t = nT + x/\sqrt{G(0)}$ with amplitude

$$\frac{1}{n!}\Big(-\frac{G'(T)x}{2G(0)^{3/2}}\Big)^n \exp\Big(\frac{G'(0)x}{2G(0)^{3/2}}\Big). \quad (18)$$

If $G'(T) = 0$, then the only jump discontinuity of u is at $t = x/\sqrt{G(0)}$. Only higher derivatives of u have jumps at later times. The jumps in these higher derivatives can be computed by carrying out the expansion of $\hat{u}(x,\lambda)$ to higher orders. If $G(T) = G'(T) = 0$, but $G''(T) \neq 0$, we obtain

$$\hat{u}(x,\lambda) = \exp(-\frac{\lambda x}{\sqrt{G(0)}})\exp(\frac{G'(0)x}{2G(0)^{3/2}})\Big(\frac{1}{\lambda} - \frac{3G'(0)^2 x}{8G(0)^{5/2}\lambda^2} + \frac{G''(0)x}{2G(0)^{3/2}\lambda^2}$$
$$-\frac{G''(T)e^{-\lambda T}x}{G(0)^{3/2}\lambda^2} + O(|\lambda|^{-3})\Big). \quad (19)$$

We carry out the Laplace transform inversion, and obtain

$$u(x,t) = \exp(\frac{G'(0)x}{2G(0)^{3/2}})\Big[H(t - \frac{x}{\sqrt{G(0)}}) - \frac{3G'(0)^2 x}{8G(0)^{5/2}}(t - \frac{x}{\sqrt{G(0)}})H(t - \frac{x}{\sqrt{G(0)}})$$
$$+\frac{G''(0)x}{2G(0)^{3/2}}(t - \frac{x}{\sqrt{G(0)}})H(t - \frac{x}{\sqrt{G(0)}}) - \frac{G''(T)x}{G(0)^{3/2}}(t - T - \frac{x}{\sqrt{G(0)}})H(t - T - \frac{x}{\sqrt{G(0)}})\Big] + \ ..., \quad (20)$$

where the dots indicate a remainder term which is continuously differentiable. We can see that the derivative of u rather than u itself jumps across $t = T + x/\sqrt{G(0)}$. By carrying the expansion further, it can be seen that across $t = nT + x/\sqrt{G(0)}$ there is a jump in the nth derivative of u.

If $G(T) = G'(T) = ... = G^{(m)}(T) = 0$, but $G^{(m+1)}(T) \neq 0$, a similar analysis shows that the nmth derivative of u has a jump across $t = nT + x/\sqrt{G(0)}$.

If $G(T) \neq 0$, we find

$$\hat{G}(\lambda) = \frac{1}{\lambda}(G(0) - e^{-\lambda T}G(T)) + \frac{1}{\lambda^2}(G'(0) - G'(T)e^{-\lambda T}) + O(|\lambda|^{-3}), \quad (21)$$

and hence

$$\exp\Big(-x\sqrt{\frac{\lambda}{\hat{G}(\lambda)}}\Big) = \exp\Big(-\frac{\lambda x}{\sqrt{G(0)-G(T)e^{-\lambda T}}}\Big)\exp\Big(\frac{1}{2}x\frac{G'(0)-G'(T)e^{-\lambda T}}{(G(0)-G(T)e^{-\lambda T})^{3/2}}\Big)$$

$$\Big(1+O(|\lambda|^{-1})\Big). \tag{22}$$

The singularities of the solution can now be worked out by expanding (22) in powers of $e^{-\lambda T}$ and inverting the transform term by term. In this fashion, it is possible to recover the results which we found by a more explicit method in Section 2. We omit the details of the calculation.

Acknowledgement

This research was completed while M. Renardy was visiting the Institute for Mathematics and Applications at the University of Minnesota. Financial support from the IMA and from the National Science Foundation under Grant DMS-8796241 is gratefully acknowledged.

References

[1] M. Renardy, W.J. Hrusa and J.A. Nohel, *Mathematical Problems in Viscoelasticity*, Longman 1987

[2] M. Slemrod, *Arch. Rat. Mech. Anal.* **62** (1976), 303

Recent IMA Preprints

#	Author/s	Title

495 **Lucas Hsu, Niky Kamran and Peter J. Olver**, Equivalence of Higher Order Lagrangians II. The Cartan Form for Particle Lagrangians

496 **D.J. Kaup and Peter J. Olver**, Quantization of BiHamiltonian Systems

497 **Metin Arik, Fahrünisa Neyzi, Yavuz Nutku, Peter J. Olver and John M. Verosky** Multi-Hamiltonian Structure of the Born-Infeld Equation

498 **David H. Wagner**, Detonation Waves and Deflagration Waves in the One Dimensional ZND Model for High Mach Number Combustion

499 **Jerrold R. Griggs and Daniel J. Kleitman**, Minimum Cutsets for an Element of a Boolean Lattice

500 **Dieter Jungnickel**, On Affine Difference Sets

501 **Pierre Leroux**, Reduced Matrices and q-log Concavity Properties of q-Stirling Numbers

502 **A. Narain and Y. Kizilyalli**, The Flow of Pure Vapor Undergoing Film Condensation Between Parallel Plates

503 **Donald A. French**, On the Convergence of Finite Element Approximations of a Relaxed Variational Problem

504 **Yisong Yang**, Computation, Dimensionality, and Zero Dissipation Limit of the Ginzburg-Landau Wave Equation

505 **Jürgen Sprekels**, One-Dimensional Thermomechanical Phase Transitions with Non-Convex Potentials of Ginzburg-Landau Type

506 **Yisong Yang**, A Note On Nonabelian Vortices

507 **Yisong Yang**, On the Abelian Higgs Models with Sources

508 **Chjan. C. Lim**, Existence of Kam Tori in the Phase Space of Vortex Systems

509 **John Weiss**, Bäcklund Transformations and the Painlevé Property

510 **Pu Fu-cho and D.H. Sattinger**, The Yang-Baxter Equation for Integrable Systems

511 **E. Bruce Pitman and David G. Schaeffer**, Instability and Ill-Posedness in Granular Flow

512 **Brian A. Coomes**, Polynomial Flows on $\mathbb{C}^{n*}$

513 **Bernardo Cockburn, Suchung Hou and Chi-Wang Shu**, The Runge-Kutta Local Projection Discontinuous Galerkin Finite Element Method for Conservation Laws IV: The Multidimensional Case

514 **Peter J. Olver**, Invariant Theory, Equivalence Problems, and the Calculus of Variations

515 **Daniel D. Joseph and Thomas S. Lundgren** with an appendix by **R. Jackson and D.A. Saville**, Ensemble Averaged and Mixture Theory Equations

516 **P. Singh, Ph. Caussignac, A. Fortes, D.D. Joseph and T. Lundgren**, Stability of Periodic Arrays of Cylinders Across the Stream by Direct Simulation

517 **Daniel D. Joseph**, Generalization of the Foscolo-Gibilaro Analysis of Dynamic Waves

518 **A. Narain and D.D. Joseph**, Note on the Balance of Energy at a Phase Change Interface

519 **Daniel D. Joseph**, Remarks on inertial radii, persistent normal stresses, secondary motions, and non-elastic extensional viscosities

520 **D. D. Joseph**, Mathematical Problems Associated with the Elasticity of Liquids

521 **Henry C. Simpson and Scott J. Spector**, Some Necessary Conditions at an Internal Boundary for Minimizers in Finite Elasticity

522 **Peter Gritzmann and Victor Klee**, On the 0-1 Maximization of Positive Definite Quadratic Forms

523 **Fu-Cho Pu and D.H. Sattinger**, The Yang-Baxter Equations and Differential Identities

524 **Avner Friedman and Fernando Reitich**, A Hyperbolic Inverse Problem Arising in the Evolution of Combustion Aerosol

525 **E.G. Kalnins, Raphael D. Levine and Willard Miller, Jr.**, Conformal Symmetries and Generalized Recurrences for Heat and Schrödinger Equations in One Spatial Dimension

526 **Wang Jinghua and Gerald Warnecke**, On Entropy Consistency of Large Time Step Godunov and Glimm Schemes

527 **C. Guillopé and J.C. Saut**, Existence Results for the Flow of Viscoelastic Fluids with a Differential Constitutive Law

528 **H.L. Bodlaender, P. Gritzmann, V. Klee and J. Van Leeuwen** Computational Complexity of Norm-Maximization

529 **Li Ta-tsien (Li Da-qian) and Yu Xin**, Life-Span of Classical Solutions to Fully Nonlinear Wave Equations

530 **Jong-Shenq Guo**, A Variational Inequality Associated with a Lubrication Problem

531 **Jong-Shenq Guo**, On the Semilinear Elliptic Equation $\Delta w - \frac{1}{2}y \cdot \nabla w + \lambda w - w^{-\beta} = 0$ in R^n

532 **Andrew E. Yagle**, Inversion of the Bloch transform in magnetic resonance imaging using asymmetric two-component inverse scattering

Recent IMA Preprints (Continued)

#	Author/s Title
533	**Bei Hu**, A Fiber Tapering Problem
534	**Peter J. Olver**, Canonical Variables for BiHamiltonian Systems
535	**Michael Renardy**, A Well-Posed Boundary Value Problem for Supercritical Flow of Viscoelastic Fluids of Maxwell Type
536	**Michael Renardy**, Ill-Posedness Resulting from Slip As a Possible Explanation of Melt Fracture
537	**Michael Renardy**, Compatibility Conditions at Corners Between Walls and Inflow Boundaries for Fluids of Maxwell Type
538	**Rolf Rees**, The Spectrum of Restricted Resolvable Designs with $r = 2$
539	**D. Lewis and J.C. Simo**, Nonlinear stability of rotating pseudo-rigid bodies
540	**Robert Hardt and David Kinderlehrer**, Variational Principles with Linear Growth
541	**San Yih Lin and Yisong Yang**, Computation of Superconductivity in Thin Films
542	**A. Narain**, Pressure Driven Flow of Pure Vapor Undergoing Laminar Film Condensation Between Parallel Plates
543	**P.J. Vassiliou**, On Local Equivalence for Vector Field Systems
544	**Brian A. Coomes**, On Conditions Sufficient for Injectivity of Maps
545	**Yanchun Zhao**, A Class of Global Smooth Solutions of the One Dimensional Gas Dynamics System
546	**H. Holden, L. Holden and N.H. Risebro**, Some Qualitative Properties of 2×2 Systems of Conservation Laws of Mixed Type
547	**M. Slemrod**, Dynamics of Measured Valued Solutions to a Backward-Forward Heat Equation
548	**Avner Friedman and Jürgen Sprekels**, Steady States of Austenitic–Martensitic–Domains in the Ginzburg–Landau Theory of Shape Memory Alloys
549	**Avner Friedman and Bei Hu**, Degenerate Hamilton–Jacobi–Bellman Equations in a Bounded Domain
550	**E.G. Kalnins, Willard Miller, Jr., and M.V. Tratnik**, Families of Orthogonal and Biorthogonal Polynomials on the N-Sphere
551	**Heinrich Freistühler**, On Compact Linear Degeneracy
552	**Matthew Witten**, Quantifying the Concepts of Rate and Acceleration/Deceleration of Aging
553	**J.P. Albert and J.L. Bona**, Total Positivity and the Stability of Internal Waves in Stratified Fluids of Finite Depth
554	**Brian Coomes and Victor Zurkowski**, Linearization of Polynomial Flows and Spectra of Derivations
555	**Yuriko Renardy**, A Couette-Poiseuille Flow of Two Fluids in a Channel
556	**Michael Renardy**, Short wave instabilities resulting from memory slip
557	**Daniel D. Joseph and Michael Renardy**, Stokes' first problem for linear viscoelastic fluids with finite memory
558	**Xiaxi Ding**, Superlinear Conservation Law with Viscosity
559	**J.L. Ericksen**, Liquid Crystals with Variable Degree of Orientation
560	**F. Robert Ore, Jr. and Xinfu Chen**, Electro-Optic Modulation in an Arbitrary Cross-Section Waveguide
561	**M.V. Tratnik**, Multivariable biorthogonal continuous-discrete Wilson and Racah polynomials
562	**Yisong Yang**, Existence of Solutions for a Generalized Yang-Mills Theory
563	**Peter Gritzmann, Laurent Habsieger and Victor Klee**, Good and Bad Radii of Convex Polygons
564	**Martin Golubitsky, Martin Krupa and Chjan. C. Lim**, Time-Reversibility and Particle Sedimentation
565	**G. Yin**, Recent Progress in Parallel Stochastic Approximations
566	**G. Yin**, On H-Valued SA: Finite Dimensional Approximations
567	**Chien-Cheng Chang**, Accurate Evaluation of the Effect of Diffusion and Conductivity in Certain Equations
568	**Chien-Cheng Chang and Ruey-Ling Chern**, The Effect of Viscous Diffusion in Discrete Vortex Dynamics for Slightly Viscous Flows
569	**Li Ta-Tsien (Li Da-qian) and Zhao Yan-Chun**, Global Existence of Classical Solutions to the Typical Free Boundary Problem for General Quasilinear Hyperbolic Systems and its Applications
570	**Thierry Cazenave and Fred B. Weissler**, The Structure of Solutions to the Pseudo-Conformally Invariant Nonlinear Schrödinger Equation
571	**Marshall Slemrod and Athanasios E. Tzavaras**, A Limiting Viscosity Approach for the Riemann Problem in Isentropic Gas Dynamics
572	**Richard D. James and Scott J. Spector**, The Formation of Filamentary Voids in Solids
573	**P.J. Vassiliou**, On the Geometry of Semi-Linear Hyperbolic Partial Differential Equations in the Plane Integrable by the Method of Darboux
574	**Jerome V. Moloney and Alan C. Newell**, Nonlinear Optics
575	**Keti Tenenblat**, A Note on Solutions for the Intrinsic Generalized Wave and Sine-Gordon Equations
576	**P. Szmolyan**, Heteroclinic Orbits in Singularly Perturbed Differential Equations
577	**Wenxiong Liu**, A Parabolic System Arising In Film Development