ON ENTROPY CONSISTENCY OF LARGE TIME STEP GODUNOV AND GLIMM SCHEMES

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Abstract. In this paper we show that for scalar conservation laws with a convex flux function the large time step approximation of weak solutions gives entropy solutions in the limit if the Courant number is between $\frac{1}{2}$ and 1.

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1. Introduction. Let us consider the initial value problem

$$u_t(x,t) + f\big(u(x,t)\big)_x = 0 \qquad (x,t) \in \mathbb{R} \times (0,\infty) \tag{1.1}$$

$$u(x,0) = u_0(x) \qquad x \in \mathbb{R}, \tag{1.2}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a two times continuously differentiable strictly convex flux function, i.e. $f'' > 0$. A locally integrable function u is called a **weak solution** of the initial value problem if it satisfies **Problem (P)**:

$$\iint\limits_{\mathbb{R}\times(0,\infty)} u\varphi_t + f(u)\varphi_x \; dxdt + \int\limits_{-\infty}^{\infty} u_0(x)\varphi(x)\; dx = 0$$

for all $\varphi \in C_0^\infty(\mathbb{R}^2)$.

□

By $BV(\Omega)$, $\Omega \subset \mathbb{R}, \mathbb{R}^2$, we will denote the space of functions which are locally integrable and whose distributional first order derivatives are signed Radon measures on Ω. We

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will consider initial data $u_0 \in L^\infty(\mathsf{R}) \cap BV(\mathsf{R})$. It is well known that the initial value problem does not have a unique solution. If problem (P) is complemented with an entropy inequality, see Section 2, then the problem does have a unique solution in the class of admissible solutions. Conservation laws for physical problems may generally be seen as macroscopic approximations to microscopic models. The inadmissible solutions will then violate some physical principle in the underlying microscopic model (e.g. the second law of thermodynamics). For a numerical method it is, therefore, important to show that the approximated solution is a actually the sought solution.

Both Godunov's [GO] and Glimms scheme [GLI] are based on the use of Riemann solvers. This means that at each discrete time step a sequence of Riemann problems is solved exactly to generate an admissible approximate solution u_h. If the Courant number is less than $\frac{1}{2}$ the neighboring waves of the Riemann solutions do not interact. At the next time step this solution is approximated by a piecewise constant function and again a sequence of Riemann problems is solved. This process is continued and gives an approximate solution to problem (P) that is an admissible weak solution of the equation (1.1) in the interior of each strip between the discrete time steps. The schemes differ in the manner in which the Riemann solutions are converted to piecewise constant functions at each time step. Glimm's scheme takes values of the Riemann solution at certain points involving a random choice, where as in Godunov's method integral averages are taken.

Since the approximate solution is an admissible entropy solution on each strip between the discrete time steps, it can easily be shown that the limit is an admissible solution. One only has to show that the piecewise constant approximation at each time step does not produce any violation of the entropy inequality in the limit.

The large time step schemes involving Riemann solvers modify the above concept by taking time steps involving larger Courant numbers than $\frac{1}{2}$. In this case neighboring waves may interact. The use of the exact weak solution beyond the time of interaction would be computationally difficult and expensive, except for the Godunov scheme with Courant number less than 1, see LeVeque [LEV1]. In the large time step schemes the interaction of the waves is taken to be simply their linear superposition. Amazingly this gives a consistent and convergent approximation to the initial value problem (P) even for arbitrarily large Courant numbers. This has been shown by LeVeque [LEV1], [LEV2] for the Godunov scheme and by Wang [WAN1] for Glimm's random choice method. The latter large time method had been introduced by Brenier [BRE2], LeVeque [LEV2] and Wang [WAN1]. The question whether such schemes approximate an admissible (entropy) solution has been open until now and was conjectured by LeVeque [LEV1]. For scalar conservation laws the entropy solution is unique, see Smoller [SMO].

For both methods it has been proved for scalar conservation laws that the approximations u_h and the limit solutions obtained by the respective methods are bounded in $L^\infty(\mathsf{R} \times (0, \infty))$ and that the schemes are total variation diminishing (TVD). Together these properties imply that the approximations and the solutions are in $BV(\mathsf{R} \times [0, \infty))$.

A different large time step method involving multivalued solutions was introduced and studied by Brenier [BRE1]. LeVeque [LEV1] has also used approximate Riemann solvers as introduced by Harten/Lax [HL] with a large time step Godunov scheme. Further, Brenier [BRE2] has studied the large time step Glimm scheme using Roe's Riemann solver. The first author has obtained similar results on consistency and convergence for approximate Riemann solvers and a random choice method [WAN2]. In a forthcoming paper [WW] we will show that schemes using approximate Riemann solvers also give entropy solutions. The restriction to Courant numbers less than 1 seems to be only of a technical nature. The interaction estimates given in Section 4 will have to be extended to include more types of wave interaction in order to give the results for larger Courant numbers.

Our proofs require the use of what seems at first to be a more general entropy inequality. The dissipation measure of the solution is not required to be negative but only to be bounded above in the sense of distributions. For solutions to problem (P) this can be shown to imply that the dissipation measure is actually non-positive. For the approximate solutions generated by the large time step scheme this is generally not the case. The interaction of a shock with a rarefaction wave wave produces a positive absolutely continuous part in the dissipation measure. In an appendix we show how a compensation argument could be used to avoid the extended version of the entropy inequality. But the argument's usefulness is rather limited.

A general outline of our proof is the following. In a few technical lemmas we show that the singular parts of the dissipation measure, given by the discontinuities in the approximate solutions, are negative when applied to constant functions. The absolutely continuous parts are positive unless they vanish, but they are bounded.

For the large time step Godunov scheme we use the boundary integral version of the entropy inequality to show that the dissipation measure is bounded above in the sense of distribution.

For the large time step Glimm scheme we use the weak version of the entropy inequality. The test function is expanded by Taylor's theorem on each rectangle of the mesh in order to apply the measure to a constant plus a higher order term. The contribution from the higher order term and the contribution from the random choice approximation vanish in the limit. The limit is a solution and therefore boundedness in the sense of distributions implies that the measure is negative on its' support.

2. Entropy conditions. We consider a weak solution u to problem (P) to be **admissible** if it additionally satisfies an **entropy inequality**

$$U(u)_t + F(u)_x \leq 0$$

in the sense of distributions. These solutions are also called **entropy solutions**. Here $U(\cdot)$ may be any convex function. It will be called an **entropy**. The function $F(u) :=$

$\int_0^u U'(\xi)f'(\xi)d\xi$ is the associated **entropy flux**, cp. Lax [LAX3], Smoller [SMO]. Since the flux function $f(\cdot)$ in our conservation law is assumed to be convex we may take $U(u) = f(u)$ and $F(u) = \int_0^u f'(\xi)^2 d\xi$. Therefore, from now on we will use the entropy inequality

$$f(u)_t + F(u)_x \leq 0 \tag{2.1}$$

in the sense of distributions, i.e.

$$-\iint_{\mathbb{R}\times(0,\infty)} f(u)\varphi_t + F(u)\varphi_x \, dxdt \leq 0 \tag{2.2}$$

for all $\varphi \in C_0^\infty(\mathbb{R} \times (0,\infty)), \varphi \geq 0$.

Since the solutions to scalar conservation laws obtained by the large time step schemes are in $L^\infty(\mathbb{R} \times (0,\infty)) \cap BV(\mathbb{R} \times [0,\infty))$ the inequality (2.2) is a distributional inequality for the signed Radon measure $\eta(u) = f(u)_t + F(u)_x$. This measure is called the **dissipation measure**, see DiPerna [DPR1]. For BV-solutions to problem (P) this measure is supported on the set of approximate jump continuities $\mathfrak{J}$ of u, cp. DiPerna [DPR2], [DPR3]. The set $\mathfrak{J}$ consists of a countable union of Lipschitz curves $\mathfrak{J}_m, m \in \mathbb{N}$, and a set I with one dimensional Hausdorff measure $H_1(I) = 0$. The set I consists of the points where shock waves collide or are formed, see DiPerna [DPR3].

Let us consider the distributional inequality

$$-\iint_{\mathbb{R}\times(0,\infty)} f(u)\varphi_t + F(u)\varphi_x \, dxdt \leq K \iint_{\mathbb{R}\times(0,\infty)} \varphi \, dxdt, \tag{2.3}$$

for some $K > 0$ and all $\varphi \in C_0^\infty(\mathbb{R} \times (0,\infty)), \varphi \geq 0$. The inequality (2.3) implies that the positive part of the measure $\eta(u)$ is absolutely continuous with respect to Lebesgue measure. If u is a BV-solution to problem (P) this implies (2.2).

In order to illustrate this point we give a simple proof for a special piecewise smooth case.

LEMMA 2.1. *Suppose that $\Omega \subset \mathbb{R} \times (0,\infty)$ is an open bounded set that is divided into the disjoint open parts Ω^-, Ω^+ by a C^1-curve Γ. We assume that the curve is parametrized as $\Gamma = \{(x(t),t) | t \in (a,b) \subset (0,\infty)\}$ for appropriate $a, b \in [0,\infty), a < b$. The notation Ω^-, Ω^+ is understood to mean that $(x,t) \in \Omega^-$ implies that $x < x(t)$. Further, we suppose we are given a function u on Ω with $u|_{\Omega^-} \in C^1(\overline{\Omega^-})$ and $u|_{\Omega^+} \in C^1(\overline{\Omega^+})$. The function u may have a jump across Γ. By u^-, u^+ we denote the limit values of u on Γ when approaching Γ from within Ω^- resp. Ω^+. If for some $K > 0$ the function u satisfies the inequality*

$$-\int_\Omega \varphi_x u \, dxdt \leq K \int_\Omega \varphi dxdt \tag{2.4}$$

for all $\varphi \in C_0^\infty(\Omega), \varphi \geq 0,$ *then* $u_- > u_+$ *on* Γ.

Proof. By integration by parts on Ω^- and Ω^+ separately one has

$$-\int_\Omega \varphi_x u \, dxdt = \int_{\Omega^- \cup \Omega^+} \varphi u_x \, dxdt - \int_\Gamma \varphi[u^- - u^+] \, ds. \tag{2.5}$$

Now suppose at some point on $p \in \Gamma$ we have $u_- < u_+$. By continuity there is a whole neighborhood where this holds, i.e. we may find an open ball $B(p)$ in Ω, centered a p, such that $u_- < u_+$ on $B(p) \cap \Gamma$. Now we choose a $\phi \in C_0^\infty(B(p))$, $\varphi(p) > 0$ and $\varphi \geq 0$. We set $L = \int_\Omega \varphi \, dxdt$ and $\varphi_\epsilon(x,t) := \frac{1}{\epsilon} \, \varphi\big(x(t) + \frac{x - x(t)}{\epsilon} \, , t\big)$ for $0 < \epsilon < 1$. Then $\int_\Omega \varphi_\epsilon \, dxdt = L$. Further there is a constant $M > 0$ such that $u_x > -M$ on Ω. Now (2.4) and (2.5) give

$$\begin{aligned} KL &\geq \int_{\Omega^- \cup \Omega^+} \varphi_\epsilon u_x \, dxdt - \int_\Gamma \varphi_\epsilon [u^- - u^+] \, dS \\ &\geq -ML - \int_\Gamma \frac{1}{\epsilon} \, \varphi\big(x(t) + \frac{x - x(t)}{\epsilon} \, , t\big)[u^- - u^+] \, dS \end{aligned}$$

or

$$(K + M) \geq -\int_\Gamma \frac{1}{\epsilon} \, \varphi\big(x(t), t\big)[u^- - u^+] \, dS.$$

If, as assumed, $u^- < u^+$ then the right hand side is not bounded above for $\epsilon \to 0$, giving a contradiction.

□

Another form of the entropy inequality is to require for $\eta = \eta(u)$ that

$$\iint_Q d\eta \leq K \cdot \text{vol}(Q) \tag{2.6}$$

holds for all rectangles $Q \subset \mathbb{R} \times (0, \infty)$ with sides parallel to the axes, cp. Harten/Lax/vanLeer [HLV]. Suppose $Q = \{x | x_1 < x < x_2, t_1 < t < t_2\}$ then Green's formula for BV functions, see Vol'pert [VOL], gives

$$\begin{aligned} \iint_Q d\eta = &\int_{x_1}^{x_2} f\big(u(x, t_2 - 0)\big) - f\big(u(x, t_1 + 0)\big) \, dx \\ &+ \int_{t_1}^{t_2} F\big(u(x_2 - 0, t)\big) - F\big(u(x_1 + 0, t)\big) \, dt \end{aligned}$$

where ± 0 is used to denote the inward traces of u on ∂Q. Together with (2.6) this gives the entropy inequality

(2.7)

$$\int_{x_1}^{x_2} f\big(u(x,t_2-0)\big)-f\big(u(x,t_1+0)\big)\,dx+\int_{t_1}^{t_2} F\big(u(x_2-0,t)\big)-F\big(u(x_1+0,t)\big)dt \le K(x_2-x_1)(t_2-t_1).$$

This version will be more convenient for the large time step Godunov scheme.

At H_1 almost every point of its' support $\mathfrak{J}$ the entropy inequality implies, see DiPerna [DPR1],

$$s[f(u_0)-f(u_1)]-[F(u_0)-F(u_1)]\le 0 \tag{2.8}$$

where u_0, u_1 are the respective left and right states, w.r.t. the x-axis, of u. Here s is the appropriately defined shock speed. The values u_0, u_1 are well defined since the points on the shock curves are regular points [VOL]. In the same manner the equation (1.1) implies the Rankine-Hugoniot jump condition

$$s[u_0-u_1]-[f(u_0)-f(u_1)]=0, \tag{2.9}$$

which holds H_1 almost everywhere on $\mathfrak{J}$. It follows that

$$s=\frac{f(u_0)-f(u_1)}{u_0-u_1}\,. \tag{2.10}$$

It is well known that the left hand side of (2.8) is of third order in the shock strength $|u_0-u_1|$, i.e. the "dissipation of entropy" is of third order, see Lax [LAX2]. This is reflected in the following estimates.

LEMMA 2.2. *Let u_0, u_1 , $u_0 > u_1$ be the respective left and right states at a shock satisfying the entropy inequality*

$$f(u)_t+F(u)_x\le 0$$

in the sense of distributions. Let $c_1 \le f''(\xi) \le c_2$ for $\xi \in [u_1, u_0]$. Then we have

$$\frac{-c_2^2}{12}\,(u_0-u_1)^3 \le [f(u_0)-f(u_1)]\frac{f(u_0)-f(u_1)}{u_0-u_1}\;-[F(u_0)-F(u_1)] \le \frac{-c_1^2}{12}\,(u_0-u_1)^3. \tag{2.11}$$

Proof. Set $P(u)=\big(f(u)-f(u_1)\big)^2-(u-u_1)\big(F(u)-F(u_1)\big)$. Then one has $P(u_1)=P'(u_1)=0$ and $-c_2^2(u-u_1)^2 \le P''(u) \le -c_1^2(u-u_1)^2$. Application of the mean value theorem to $\frac{P(u_0)}{(u_0-u_1)^4}$ gives the desired result.

∎

By the Lebesgue decomposition theorem, see Federer [FED] 2.9.2., the dissipation measure may be decomposed with respect to Lebesgue measure as $\eta = \Delta_{\mathrm{abs}} + \Delta_{\mathrm{sing}}$. The measure Δ_{abs} is absolutely continuous with respect to Lebesgue measure and is supported on a Borel set $A \subset \mathbb{R} \times [0, \infty)$. The support of Δ_{sing} is a Lebesgue null set and must be the set $\mathfrak{J}$.

Let $\Omega \subset \mathbb{R} \times [0, \infty)$ be any set and $f \in C^0(\Omega)$ then we use the notations

$$\langle \eta, f \rangle_\Omega = \int_\Omega f d\eta \tag{2.12}$$

to denote the restriction of η to Ω. We will denote the density associated to Δ_{abs} by the Radon-Nikodyin theorem also by Δ_{abs}, i.e. write

$$\langle \Delta_{\mathrm{abs}}, f \rangle_\Omega = \int_\Omega f d\Delta_{\mathrm{abs}} = \int_\Omega f \Delta_{\mathrm{abs}}\, dx. \tag{2.13}$$

3. Large time step Godunov and Glimm schemes. Let us introduce some convenient notations to describe the methods more accurately. We will suppose that x-axis is partitioned into intervals of length h by the set of points $x_i = hi, h > 0,\ i \in \mathbb{Z}$. Likewise the positive time axis is partitioned into intervals by the points $t_j = \tau j, \tau > 0, j \in \mathbb{N}_0$. The points (x_i, t_j) define a rectangular mesh on $\mathbb{R} \times [0, \infty)$. We will always assume that the time step $\tau = \lambda h$ for some fixed mesh ratio $\lambda > 0$. Thereby, the approximations will only depend on the parameter h. It will be convenient to introduce the open rectangles

$$Y_{h,i,j} = \{(x,t) | ih < x < (i+1)h, j\tau < t(j+1)\tau\}. \tag{3.1}$$

A basic ingredient in the method is the solution of **Riemann problems**. These are initial value problems for $t \geq t_j$ for some $j \in \mathbb{N}_0$ involving only two constant initial states $u_{i-1}^j, u_i^j \in \mathbb{R}$ with a jump at the point x_i for some $i \in \mathbb{Z}$, i.e. initial data

$$U_i^j(x) = \begin{cases} u_{i-1}^j & x < x_i \\ u_i^j & x \geq x_i \end{cases}. \tag{3.2}$$

Since equation (1.1) is invariant under the transformations $(x,t) \to (\alpha x, \alpha t), \alpha \in \mathbb{R}$, and translations these problems may be solved exactly using similarity solutions involving only shocks and rarefaction waves, cp. Lax [LAX1], [LAX3]. Smoller [SMO].

Suppose for some $j \in \mathbb{N}_0$ we are given a piecewise constant function $u_h^j(x) = u_i^j$ for $x \in [x_i, x_{i+1})$. For each $i \in \mathbb{Z}$ we may solve the Riemann problem at x_i involving the states $u_{i-1}^j,\quad u_i^j$ to give the solution $u_{h,i}^j(x,t)$ for $t > t_j$. We assume that the set $\{u_i^j\}_{i \in \mathbb{Z}}$ is bounded, then we may set

$$A_j = \sup_{i \in \mathbb{Z}} \sup_{x \in \mathbb{R}} |f'(u_{h,i}^j(x, t_j))| = \sup_{i \in \mathbb{Z}} |f'(u_i^j)|$$

to be the upper bound to the characteristic speeds involved in the solutions to the Riemann problems at time t_j. Setting $A = \sup_{j\in\mathbb{N}_0} A_j$ we may define the Courant number of the scheme to be $c = \lambda A$. As long as $c < \frac{1}{2}$ the neighboring Riemann solutions will be separated by the intermediate constant state. Therefore, setting

$$u_h^j(x,t) = u_h^j(x) + \sum_{i\in\mathbb{Z}} \left(u_{h,i}^j(x,t) - u_{h,i}^j(x,t_j)\right) \tag{3.3}$$

gives an exact weak solution on the strip $t_j \leq t < t_{j+1}$ to the initial value problem at time t_j with initial data $u_h^j(x)$. Taking $c > \frac{1}{2}$ there will be interactions between neighboring waves and (3.3) will fail to be a weak solution to equation (1.1) on the whole strip $t_j \leq t < t_{j+1}$. In the large time step methods the solution (3.3) is taken despite $c > \frac{1}{2}$.

The difference between the schemes occurs in the calculation of the piecewise constant initial data $u_h^{j+1}(x)$ from the function $u_h^j(\cdot, t_{j+1})$. The initial data $u_h^{j+1}(x)$ are then used to calculate $u_h^{j+1}(x,t)$ for $t_{j+1} < t \leq t_{j+2}$. We will denote by u_h the approximate solution obtained on $\mathbb{R} \times [0,\infty)$ by setting $u_h(x,t) = u_h^j(x,t)$ for $x \in \mathbb{R}$ and $t_j \leq t < t_{j+1}$. The algorithm is started by taking $u_h^0(\cdot)$ to be a piecewise constant approximation of $u_0(\cdot)$.

The piecewise constant approximations are obtained in the following manner. In the Godunov scheme one sets

$$u_i^{j+1} = u_h^{j+1}(x) = \frac{1}{h}\int_{x_i}^{x_{i+1}} u_h^j(\xi, t_{j+1})\,d\xi \qquad \text{for} \qquad x \in [x_i, x_{i+1}) \tag{3.4}$$

in order to obtain the piecewise constant initial states, see Godunov [GO]. In the Glimm scheme [GLI] a value $\theta_j \in [0,1)$ is chosen randomly and one takes

$$u_i^{j+1} = u_h^{j+1}(x) = u_h^j(x_i + \theta_j h, t_{j+1}) \text{ for } x \in [x_i, x_{i+1}). \tag{3.5}$$

For later reference we now recall the solutions to the Riemann problem in the scalar convex case. For simplicity we suppose in (3.2) that $t_j = 0,\ x_i = 0$. The are only two cases to consider. Suppose we are given two states $u_0, u_1 \in \mathbb{R}$. $u_0 > u_1$. Then the solution to the Riemann problem is given by

$$u(x,t) = \begin{cases} u_0 & x < st \\ u_1 & x > st. \end{cases} \tag{3.6}$$

This solution contains a **shock** travelling at the speed $s = \frac{f(u_0)-f(u_1)}{u_0-u_1}$. Discontinuities having the local speed s not given by this formula, where u_0, u_1 would be the respective local left and right states w.r.t the x-axis, will not be called shocks.

On the other hand if $u_0 < u_1$ the solution is given by

$$u(x,t) = \begin{cases} u_0 & x < f'(u_0)t \\ g\left(\dfrac{x}{t}\right) & f'(u_0)t < x < f'(u_1)t \\ u_1 & x > f'(u_1)t\ . \end{cases} \tag{3.7}$$

The function $g : \mathbb{R} \to \mathbb{R}$ is the inverse of the strictly monotone function $f' : \mathbb{R} \to \mathbb{R}$, i.e. $f'\left(g\left(\frac{x}{t}\right)\right) = \frac{x}{t}$. The non-constant part of (3.7) is a **rarefaction wave**. Both solutions satisfy the weak entropy inequality (2.2).

4. Interaction estimates. We will now proceed to estimate the measures $\eta(u_h) = f(u_h)_t + F(u_h)_x$ on $Y_{h,i,j}$ when they are applied to functions that are constant on $Y_{h,i,j}$. The measure consists of singular parts coming from shocks as well as discontinuities and absolutely continuous parts coming from modified rarefaction waves.

We assume the Courant number to be less than 1. Then we will only have waves originating at the points (x_i, t_j) and (x_{i+1}, t_j) within $Y_{h,i,j}$. So we only have to study the interaction of two waves.

Suppose we have a rarefaction wave at (x_i, t_j) with $f'(u^j_{i-1}) < 0 < f'(u^j_i)$. Then we will think of this wave as split into two parts, one belonging to $Y_{h,i-1,j}$ and one belonging to $Y_{h,i,j}$ in the obvious way.

Due to the fact that $f' = f'(u^i_j)$ is constant between (x_i, t_j) and (x_{i+1}, t_j) we cannot have an interaction between two rarefaction waves within $Y_{h,i,j}$. It remains to study the interaction between two shocks and between a shock and a rarefaction wave.

We now show that the interaction between two shocks produces two discontinuities such that the singular measure $\eta(u_h) = f(u_h)_t + F(u_h)_x$ is negative when applied to functions constant on $Y_{h,i,j}$.

LEMMA 4.1. *Let $u_h \in L^\infty(\mathbb{R} \times (0,\infty)) \cap BV(\mathbb{R} \times [0,\infty))$ be an approximate solution to problem (P) obtained using either scheme. Suppose that a rectangle $Y_{h,i,j}$ contains one or two shocks. Then the measure $\eta(u_h) = f(u_h)_t + (u_h)_x$ is singular and negative when applied to functions that are constant on $Y_{h,i,j}$.*

Proof. In case of only one shock we are done since Lemma 2.2 implies that $\eta(u_h)$ is negative. Now suppose that $Y_{h,i,j}$ contains two shocks. If the shocks do not meet we again obtain the result by Lemma 2.2.

We now assume that the shocks do meet at a time $t' \in [t_j + \frac{\tau}{2}\ , t_{j+1})$. For $t \in [t_j, t')$ the measure η is negative as above. It remains to look at the measure on the set $S = \{(x,t) \in Y_{h,i,j} \mid t \in [t', t_{j+1})\}$. Let Δ' denote the measure supported on the discontinuity that originated as a shock in (x_{i+1}, t_j) and Δ'' is the one originating at (x_i, t_j). We set

$u_{-1} = u^j_{i-1}$, $u_0 = u^j_i, u_1 = u^j_{i+1}$ as well as $\tilde{u} = u_{-1} - (u_0 - u_1)$. Then

$$
\begin{aligned}
\langle \eta, 1\rangle_S &= \langle \Delta', 1\rangle_S + \langle \Delta'', 1\rangle_S \\
&= \left([f(u_{-1}) - f(\tilde{u})] \frac{f(u_0) - f(u_1)}{u_0 - u_1} - [F(u_{-1}) - F(\tilde{u})] \right) \cdot (t_{j+1} - t') \qquad (4.1)\\
&+ \left([f(\tilde{u}) - f(u_1)] \frac{f(u_{-1}) - f(u_0)}{u_{-1} - u_0} - [F(\tilde{u}) - F(u_1)] \right) \cdot (t_{j+1} - t').
\end{aligned}
$$

We now add and subtract the terms $[f(u_{-1}) - f(\tilde{u})]\frac{f(u_{-1})-f(\tilde{u})}{u_{-1}-\tilde{u}}$ $(t_{j+1} - t')$ and $[f(\tilde{u}) - f(u_1)]\frac{f(\tilde{u})-f(u_1)}{\tilde{u}-u_1}$ $(t_{j+1} - t')$. Using (2.8) we may then delete the terms that are negative. We obtain

$$
\begin{aligned}
\langle \eta, 1\rangle_S &\leq [f(u_{-1}) - f(\tilde{u})] \left(\frac{f(u_0) - f(u_1)}{u_0 - u_1} - \frac{f(u_{-1}) - f(\tilde{u})}{u_{-1} - \tilde{u}} \right) (t_{j+1} - t') \qquad (4.2)\\
&+ [f(\tilde{u}) - f(u_1)] \left(\frac{f(u_{-1}) - f(v_0)}{u_{-1} - u_0} - \frac{f(\tilde{u}) - f(u_1)}{\tilde{u} - u_1} \right) (t_{j+1} - t') \\
&=: D \cdot (t_{j+1} - t').
\end{aligned}
$$

We will now show that D is negative. Observe that $u_{-1} - \tilde{u} = u_0 - u_1$ and $\tilde{u} - u_1 = u_{-1} - u_0$. The interaction between the two shocks can only take place if $a := \frac{f(u_0)-f(u_1)}{u_0-u_1} < 0$. and $c = \frac{f(u_{-1})-f(u_0)}{u_{-1}-u_0} > 0$. Set $b := \frac{f(\tilde{u})-f(u_1)}{\tilde{u}-u_1}$ and $d := \frac{f(u_{-1})-f(\tilde{u})}{u_{-1}-\tilde{u}}$. The entropy inequality implies $u_{-1} > u_0 > u_1$. Since $\tilde{u} \in (u_1, u_{-1})$ we must consider two cases. First suppose $u_{-1} > \tilde{u} > u_0$. Then we have $a < b < c < d$, i.e. $d > 0$. We may rewrite D in the following way

$$D = (u_0 - u_1)(-d^2 + da) + (u_{-1} - u_0)(-b^2 + bc). \qquad (4.3)$$

In case $b < 0$ we obviously have $D < 0$. Now suppose that $b > 0$. We set $\epsilon = u_0 - u_1$, $\phi = u_{-1} - u_0$ and $\alpha = a\epsilon, \beta = b\phi,\ \gamma = c\phi, \delta = d\epsilon$. Then

$$D = \frac{-\delta^2 + \delta\alpha}{\epsilon} + \frac{-\beta^2 + \beta\gamma}{\phi}.$$

Note that $a < 0,\ 0 < b < c < d, \epsilon > 0, \phi > 0$ implies $\alpha < 0, \delta > 0, 0 < \beta < \gamma$ and $\beta < \frac{\phi}{\epsilon}\ \delta$. Also $\tilde{u} > u_0$ implies $\phi > \epsilon$. So we have

$$D = \frac{1}{\phi} \left(\frac{\phi}{\epsilon}\ \delta(\alpha - \delta) + \beta(\gamma - \beta) \right) \leq \frac{1}{\phi} \left(\beta(\alpha - \delta) + \beta(\gamma - \beta) \right)$$

Now observe that $(\alpha - \delta) = f(u_0) - f(u_1) - (f(u_{-1}) - f(\tilde{u})) = -(\gamma - \beta)$. Therefore $D \leq 0$.

In case $u_0 > \tilde{u} > u_1$ we have $b < a < d < c$. It follows analogously that $D \leq 0$. We obtain $\langle \eta, 1\rangle_{Y_{h,i,j}} < 0$ as was to be shown.

$\square$

Let us denote by $TV_{[a,b]}(u_h^j(\cdot))$ the total variation of $u_h^j(\cdot)$ on the interval [a,b]. By $TV(u_h^j(\cdot))$ we denote the total variation over the line $\{(x,t_j)|x \in \mathbb{R}\}$. Since both schemes are TVD note that $TV(u_h^{j+1}(\cdot))\ T \leq TV(u_h^j(\cdot))$ and

$$TV(u_h^j(\cdot,t)) \leq TV(u_h^j(\cdot,t_j)) = TV(u_h^j(\cdot)) = \sum_{i\in\mathbb{Z}} TV(u_{h,i}^j(\cdot)) = \sum_{i\in\mathbb{Z}} |u_i^j - u_{i-1}^j| \tag{4.4}$$

for $t \geq t_j$ and any $j \in \mathbb{N}_0$. So all these total variations are bounded by $TV(u_0(\cdot))$, u_0 being the initial data given in (1.2).

Lemma 4.2. *Let $u_h \in L^\infty(\mathbb{R}\times(0,\infty)) \cap BV(\mathbb{R}\times[0,\infty))$ be an approximate solution to problem (P) obtained using either scheme. Suppose that a rectangle $Y_{h,i,j}$ contains only shocks and the discontinuities that arise after their interaction. Then we may estimate*

$$| < \eta, \varphi >_{Y_{h,i,j}} | \leq C\cdot h\cdot TV_{[x_{i-1},x_{i+1}]}(u_h^j(\cdot))\cdot\|\varphi\|_\infty \tag{4.5}$$

with C independent of h and $\varphi \in C^0(Y_{h,i,j})$

Proof. In case of one shock with left state u_0 and right state u_1 in $Y_{h,i,j}$ we have by Lemma 2.1

$$\begin{aligned} |\langle\eta,\varphi\rangle| &\leq c_2|u_0-u_1|^3\|\varphi\|_\infty\cdot\tau \\ &= |u_0-u_1|\ h\ c_2\lambda\ \|\varphi\|_\infty\ |u_0-u_1|^2 \end{aligned} \tag{4.6}$$

Since u_h is bounded in $L^\infty(\mathbb{R}\times(0,\infty))$ we may bound $c_2\lambda\|\varphi\|_\infty|u_0-u_1|^2$ independently of h. Further, $|u_0-u_1| \leq TV_{[x_{i-1},x_{i+1}]}\ (u_h^j(\cdot))$. So we are finished in this case.

If $Y_{h,i,j}$ contains two noninteracting shocks we obtain two terms of the above type. Using the same notation as in the proof of Lemma 4.1 we only have to note that

$$\begin{aligned} |u_0-u_{-1}| + |u_0-u_1| &= TV_{[x_{i-1},x_i]}(u_h^j(\cdot)) + TV_{[x_i,x_{i+1}]}(u_h^j(\cdot)) \\ &= TV_{[x_{i-1},x_{i+1}]}(u_h^j(\cdot)). \end{aligned}$$

In case the shocks are interacting we may proceed, using the same notation as in the proof of Lemma 4.1, to estimate the measure on the set $Y_{h,i,j}\backslash S$ as above. On the set S we proceed analogously as from (4.1) again by adding and subtracting the same terms as in that proof. We obtain one part that we can estimate using Lemma 2.2 and proceed as above, noting that $\tilde{u}-u_1 = u_{-1}-u_0$ and $u_{-1}-\tilde{u} = u_0-u_1$. There remains the term $D\cdot(t_{j+1}-t')\|\varphi\|_\infty$. This can be estimated by using (4.3)

$$|D|(t_{j-1}-t')\|\varphi\|_\infty \leq (|u_0-u_1|(d^2+|da|) + |u_{-1}-u_0|(b^2+|bc|))\lambda h\|\varphi\|_\infty.$$

Since we have a global L^∞ bound on u_h which is independent of h we may estimate $|a|,|b|,|c|,|d|$ by $A = \sup_{(x,t)\in\mathbb{R}\times[0,\infty)} |f'(u_h(x,t))|$ independently of h.

□

LEMMA 4.3. *Let $u_h \in L^\infty(\mathbb{R}\times(0,\infty)) \cap BV(\mathbb{R}\times[0,\infty))$ be an approximate solution to problem (P) obtained using either scheme. Suppose that a rectangle $Y_{h,i,j}$ contains a shock and a rerefaction wave or only a rarefaction wave. Then the dissipation measure $\eta(u_h) = f(u_h)_t + F(u_h)_x$ may be decomposed into $\eta = \Delta_{\text{sing}} + \Delta_{\text{abs}}$ where Δ_{sing} is the singular part and Δ_{abs} is the part absolutely continuous with respect to 2-dimensional Lebesgue measure. The singular part Δ_{sing} is negative. The absolutely continuous is positive if the rarefaction wave meets the shock in $Y_{h,i,j}$.*

Proof. If the two waves do not meet then $\eta(u_h) = \Delta_{\text{sing}}$ because $\eta(u_h)$ vanishes on the rarefaction wave. The measure $\eta(u_h)$ is then negative by Lemma 2.2.

Now suppose that both waves meet. Without loss of generality we may assume that the rarefaction wave is centered at (x_i, t_j) and that the shock originates at (x_{i+1}, t_j). For simplicity assume for the moment that after the "interaction" the waves become separated at time $t' \in (t_j, t_{j+1})$ by an intermediate constant state $\tilde{u} = u_{-1} - (u_0 - u_1)$, where we again use the notation $u_{-1} = u_j^{i-1}$, $u_0 = u_j^i, u_1 = u_j^{i+1}$, cp. Figure 4.1. We set $\widetilde{S} = \{(x,t) \in Y_{h,i,j} \mid t \geq t'\}$. Then we have

$$
\begin{aligned}
\langle \Delta_{\text{sing}}, 1\rangle_{\widetilde{S}} &= \left([f(u_{-1}) - f(\tilde{u})]\frac{f(u_0) - f(u_1)}{u_0 - u_1} - [F(u_{-1}) - F(\tilde{u})]\right)(t_{j+1} - t') \\
&= \left([f(u_{-1}) - f(\tilde{u})]\frac{f(u_{-1}) - f(\tilde{u})}{u_{-1} - \tilde{u}} - [F(u_{-1}) - F(\tilde{u})]\right)(t_{j+1} - t') \\
&\quad +[f(u_{-1}) - f(\tilde{u})]\left(\frac{f(u_0) - f(u_1)}{u_0 - u_1} - \frac{f(u_{-1}) - f(\tilde{u})}{u_{-1} - \tilde{u}}\right)(t_{j+1} - t').
\end{aligned}
\tag{4.7}
$$

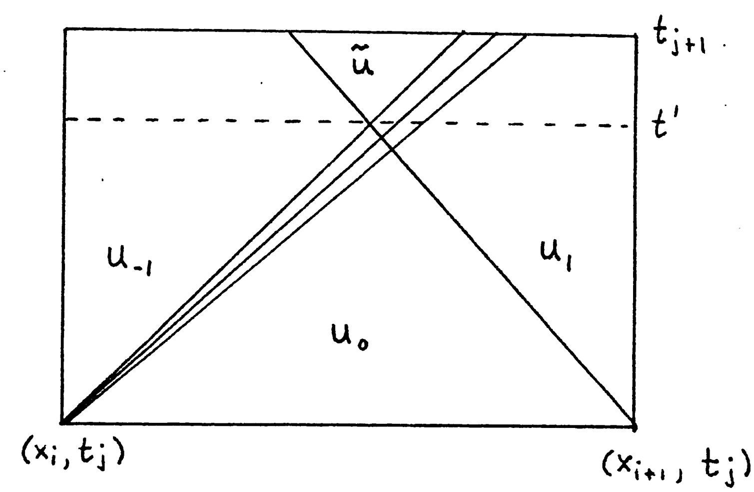

FIGURE 4.1

The first part is negative due to Lemma 2.2. By our assumptions we have $0 > \frac{f(u_0)-f(u_1)}{u_0-u_1} \geq \frac{f(u_{-1})-f(\tilde{u})}{u_{-1}-\tilde{u}}$ since f is convex. An interaction of the type being discussed can only occur if $f(u)$ has a minimum at some point $u = y$ with $u_1 < y < u_0$, see Figure 4.2. Since we assume $f'(u_{-1}) \geq 0$ we must have $u_{-1} \geq y$. Actually we must have $u_{-1} > y$ in order for the waves to separate. This implies that we must have $f(u_{-1}) < f(\tilde{u})$, whereby we have shown $\langle \Delta_{\text{sing}}, 1 \rangle_{\widetilde{S}} \leq 0$.

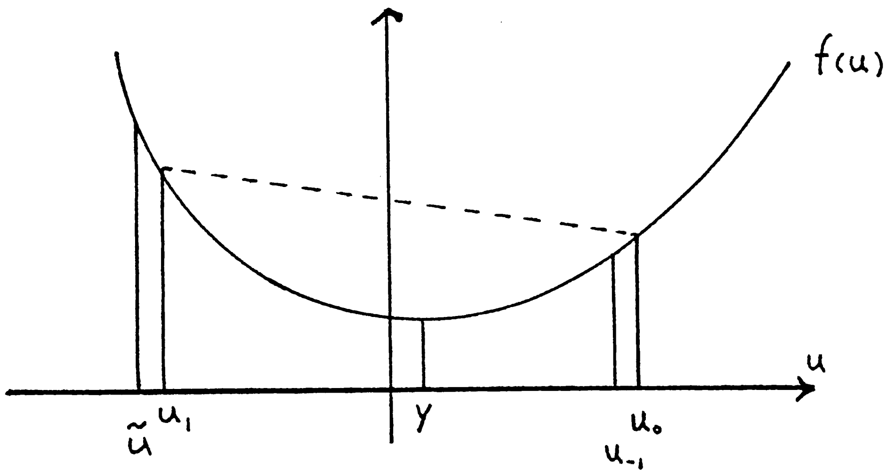

FIGURE 4.2

Note that we have actually shown that Δ_{sing} is negative at each point of its support in $\widetilde{S}$. Now observe that in the above arguments we may replace $u_{-1}, \tilde{u}$ by any of the respective left and right states that occur while the waves are intersecting. During the intersection the left state varies from u_0 to u_{-1}. The right state is obtained by subtracting $u_0 - u_1$ from the left state. The right state varies from u_1 to $\tilde{u}$. As above the measure is negative at each point.

Now let us look at the absolutely continuous part. Its' support is the region of the rarefaction wave behind the shock. There we have

$$u_h(x,t) = g\left(\frac{x - x_i}{t - tj}\right) - (u_0 - u_1). \tag{4.8}$$

Note that $w(x,t) = g\left(\frac{x-x_i}{t-tj}\right)$ satisfies the equation $w_t + f(w)_x = 0$. We write $u_h = w + \overline{u}$, $\overline{u} = u_1 - u_0$. The mean value theorem gives a value $\theta \in (w + \overline{u}, w)$ such that $f'(u_h) = f'(w + \overline{u}) = f'(w) + f''(\theta)\overline{u}$. Using the fact that $(u_h)_t = w_t$, $(u_h)_x = w_x$ and $F' = (f')^2$ we obtain

$$\begin{aligned} f(u_h)_t + F(u_h)_x =& f'(u_h)(u_h)_t + F'(u_h)(u_h)_x = f'(u_h)(w_t + f'(w)w_x) \\ &+ f'(u_h)f''(\theta)\,\overline{u}\,(u_h)_x = f'(u_h)f''(\theta)\,\overline{u}\,(u_h)_x. \end{aligned} \tag{4.9}$$

The last term is positive since $\overline{u} < 0, f'' > 0$ and $f'(u_h)w_x = f'(u_h)\frac{g'}{t-t_j} < 0$ for the values that u_h takes in this case.

□

LEMMA 4.4. *Under the assumptions of Lemma 4.3 we have the estimates*

$$|\langle \Delta_{\text{sing}}, \varphi \rangle_{Y_{h,i,j}}| \leq C \cdot h \cdot TV_{[x_{i-1},x_{i+1}]}\big(u_h^j(\cdot)\big) \cdot \|\varphi\|_\infty \tag{4.10}$$

and

$$|\langle \Delta_{\text{abs}}, \varphi \rangle_{Y_{h,i,j}}| \leq C \cdot h \cdot TV_{[x_{i-1},x_{i+1}]}\big(u_h^j(\cdot)\big) \cdot \|\varphi\|_\infty \tag{4.11}$$

for any $\varphi \in C^0(Y_{h,i,j})$. The constants $C > 0$ are independent of h.

Proof. The estimate (4.10) is obtained analogously as in Lemma 4.2.

We had shown in the proof of the previous lemma that on its support $\Delta_{\text{abs}} = f'(u_h)f''(\theta)\,\overline{u}\,(u_h)_x$. The quantities $f'(u_h), f''(\theta)$ are bounded due to the uniform bound on u_h. Further, $(u_h)_x = g'\left(\frac{x-x_i}{t-t_j}\right)\,\frac{1}{t-tj} \leq C \cdot \frac{1}{h}$. For a suitable constant $C > 0$ we obtain using $|\overline{u}| \leq TV_{[x_{i-1},x_{i+1}]}\big(u_h^j(\cdot)\big)$ the estimate

$$|\langle \Delta_{\text{abs}}, \varphi \rangle_{Y_{h,i,j}}| \leq C \cdot TV_{[x_{i-1},x_{i+1}]}\big(u_h^j(\cdot)\big)\|\varphi\|_\infty \frac{1}{h} \int\limits_{Y_{h,i,j}} dx$$

which implies (4.11).

□

5. Entropy consistency. We will now use the interaction estimates from the previous section to prove an entropy inequality for limits of sequences of approximate solutions found by LeVeque [LEV1] and Wang [WAN1] for the large time step schemes.

THEOREM 1. *Let $u \in L^\infty(\mathbb{R} \times (0,\infty)) \cap BV(\mathbb{R} \times [0,\infty))$ be a solution to problem (P). Suppose u is the local L^1 limit of a sequence of uniformly L^∞-bounded approximate solutions u_h for $h \to 0$, the u_h being calculated by the large time step Godunov scheme with Courant number $c \leq 1$. Then u satisfies the entropy inequality (2.7) i.e.*

$$\begin{aligned}\int\limits_{x_1}^{x_2} f\big(u(x,t_2-0)\big) - f\big(u(x,t_1+0)\big)dx + \int\limits_{t_1}^{t_2} F(u(x_2-0,t)\big) - F(u(x_1+0,t)\big)dt \\ \leq K(x_2-x_1)(t_2-t_1)\end{aligned} \tag{5.1}$$

where the rectangle $Q = \{x | x_1 < x < x_2, t_1 < t_2\}$ is arbitrary in $\mathbb{R} \times [0,\infty)$.

Proof. In case Q lies in a strip $S_{h,j} = \{(x,t) | x \in \mathbb{R}, t_j \leq t < t_{j+1}\}$ for some $j \in \mathbb{N}_0$ then (5.1) holds for u_h by applying Green's formula to the inequality

$$\langle \eta(u_h),\ 1 \rangle_Q \leq K\,\text{vol}(Q) \tag{5.2}$$

with an appropriate constant $K > 0$. The inequality is a consequence of Lemmas 4.1 and 4.3.

We fix Q and assume h small. We denote $Q_h = \bigcup_{Y_{h,i,j} \subset Q} \overline{Y}_{h,i,j}$ the union of all closed mesh rectangles contained in Q. We use the values x_a, x_b with $a, b \in \mathbb{Z}$ and t_c, t_d with $c, d \in \mathbb{N}_0$ to denote the corner points of Q_h. Let us set

$$\mathcal{L}_h(u_h) = \int_{x_a}^{x_b} f(u_h(x,t_d - 0)) - f(u_h(x,t_c + 0))dx + \int_{t_c}^{t_d} F(u_h(x_b - 0,t)) - F(u_h(x_a + 0,t))dt$$

and $\quad S_{h,j} = \{(x,t) | x_a \le x \le x_b, t_j \le t < t_{j+1} \quad \text{for} \quad c \le j < d.$

Then as in (5.2) we get

$$\begin{aligned} \mathcal{L}_h(u_h) &= \sum_{j=c}^{d-1} \left(\langle \eta(u_h), 1 \rangle_{S_{h,j}} + \int_{x_a}^{x_b} f(u_h(x,t_j + 0)) - f(u_h(x,t_j - 0))dx \right) \\ &\le K \ \operatorname{vol}(Q_h) + \sum_{j=c}^{d-1} \int_{x_a}^{x_b} f(u_h(x,t_j + 0)) - f(u_h(x,t_j - 0))dx. \end{aligned}$$

We have

$$\int_{x_a}^{x_b} f(u_h(x,t_j + 0)) - f(u_h(x,t_j - 0))dx = \int_{x_a}^{x_b} f(u_h^j(x)) - f(u_h^{j-1}(x,t_j))dx.$$

Take $[x_i, x_{i+1}] \subset [x_a, x_b]$ then Jensen's inequality gives

$$\begin{aligned} \int_{x_i}^{x_{i+1}} f(u_h^j(x))dx &= \int_{x_i}^{x_{i+1}} f\left(\frac{1}{h} \int_{x_i}^{x_{i+1}} u_h^{j-1}(\xi,t_j)d\xi \right) dx \\ &\le \int_{x_i}^{x_{i+1}} \frac{1}{h} \int_{x_i}^{x_{i+1}} f(u_h^{j-1}(\xi,t))d\xi \ dx \qquad (5.3) \\ &= \int_{x_i}^{x_{i+1}} f(u_h^{j-1}(\xi,t_j))d\xi. \end{aligned}$$

Now (5.3) and (3.4) give

$$\int_{x_i}^{x_{i+1}} f(u_h(x,t_j + 0)) - f(u_h(x,t_j - 0))dx \le 0$$

whereby

$$\mathcal{L}_h(u_h) \le K \ \operatorname{vol}(Q_h). \qquad (5.4)$$

In the limit $h \to 0$ this gives (5.1).

$\square$

THEOREM 2. *Let $u \in L^\infty(\mathbb{R} \times (0,\infty)) \cap BV(\mathbb{R} \times [0,\infty))$ be a solution to a problem (P). Suppose that u is the local L^1 limit for $h \to 0$ of a sequence of uniformly L^∞-bounded approximate solutions u_h, the u_h being calculated by the large time step Glimm scheme with Courant number $c \leq 1$. Then u satisfies the entropy inequality (2.3), i.e.*

$$-\iint\limits_{\mathbb{R}\times(0,\infty)} f(u)\varphi_t + F(u)\varphi_x dxdt \leq K \iint\limits_{\mathbb{R}\times(0,\infty)} \varphi \; dxdt \tag{5.5}$$

for some $K > 0$ and all $\varphi \in C_0^\infty(\mathbb{R}^2), \varphi \geq 0$.

Proof. In order to show (5.5) it is sufficient to show that

$$\begin{aligned} L_\varphi(u_h) := &-\iint\limits_{\mathbb{R}\times(0,\infty)} f(u_h)\varphi_t + F(u_h)\varphi_x \; dxdt + \int\limits_{-\infty}^{\infty} u_h^0(x,0)\varphi(x,0)dx \\ &\leq K \iint\limits_{\mathbb{R}\times(0,\infty)} \varphi \; dxdt + R_h \end{aligned} \tag{5.6}$$

for any $\varphi \in C_0^\infty(\mathbb{R}^2), \varphi \geq 0$, where $R_h \to 0$ for $h \to 0$. Since u_h is by construction piecewise C^1 on $\mathbb{R} \times (0,\infty)$ we may integrate by parts on all the regions where u_h is smooth to obtain

$$\begin{aligned} L_\varphi(u_h) = &\sum_{j=1}^{\infty} \int\limits_{-\infty}^{\infty} \varphi(x,t_j)\big[f\big(u_h(x,t_j-0)\big) - f\big(u_h(x,t_j+0)\big)\big] \\ &+ \sum_{(i,j)\in\mathbb{Z}\times\mathbb{N}_0} \langle \eta, \varphi\rangle_{Y_{h,i,j}} \\ &= L(\varphi,h) + \sum_{(i,j)\in\mathbb{Z}\times\mathbb{N}_0} \big(\langle\Delta_{\text{sing}},\varphi\rangle_{Y_{h,i,j}} + \langle\Delta_{\text{abs}},\varphi\rangle_{Y_{h,i,j}}\big) \end{aligned} \tag{5.7}$$

Note that $L(\varphi,h) \to 0$ for $h \to 0$ for almost all random choice sequences (cp. Glimm [GLI] Lemma 5.1). Therefore, it remains to study the terms $\langle\eta,\varphi\rangle_{Y_{h,i,j}}$. We make a Taylor expansion of φ at the center of $Y_{h,i,j}$. This gives for $x_i = ih, \; t_j = j\tau$

$$\varphi(x,t) = \varphi(x_j + \frac{h}{2}, \; t_j + \frac{\tau}{2}) + \varphi_x(\xi)\big(x - (x_i + \frac{h}{2})\big) + \varphi_t(\xi)\big(t - (x_j + \frac{\tau}{2})\big)$$

for any $(x,t) \in Y_{h,i,j}$ and an appropriate $\xi \in Y_{h,i,j}$. We set $\varphi_{h,i,j} = \varphi\left(x_i + \frac{h}{2}\;, t_j + \frac{\tau}{2}\right)$ then the Lemmas 4.1, 4.3 give $\langle\Delta_{\text{sing}},\varphi_{h,i,j}\rangle \leq 0$ and $\langle\Delta_{\text{abs}},\varphi_{h,i,j}\rangle \leq K \iint\limits_{Y_{h,i,j}} \varphi_{h,i,j}dxdt$. Further, we apply Lemmas 4.2 and 4.4 and use the notation $i,j \in \operatorname{supp}\varphi$ to denote the i,j such that $Y_{h,i,j} \cap \operatorname{supp}\varphi \neq \emptyset$. This gives

$$\Big|\sum_{i,j\in\operatorname{supp}\varphi} \Big\langle \Delta_{\text{sing}}, \varphi_x(\xi)\big(x - (x_i + \frac{h}{2})\big) + \varphi_t(\xi)\big(t - (t_j + \frac{\tau}{2})\big)\Big\rangle\Big|$$

$$
\begin{aligned}
(5.8) \qquad & \leq C h^2 \|\nabla \varphi\|_\infty \sum_{i,j \in \operatorname{supp} \varphi} TV_{[x_{i-1}, x_{i+1}]}(u_h^j(\cdot)) \\
& \leq \widetilde{C}\, h^2 \sum_{j \in \operatorname{supp} \varphi} TV(u_h^j(\cdot)) \leq \widetilde{C} h^2 TV(u_0) \cdot \sum_{j \in \operatorname{supp} \varphi}
\end{aligned}
$$

for suitable constants $C, \widetilde{C} > 0$. For h small enough the number of $j \in \mathbb{N}_0$ such that $Y_{h,i,j} \cap \operatorname{supp} \varphi \neq \emptyset$ for any $i \in \mathbb{Z}$ may be bounded by $\frac{N}{h}$ for a suitable constant N. Therefore, the right hand side in (5.8) tends to zero as $h \to 0$.

Similarly we apply Lemma 4.4 to obtain

$$
\begin{aligned}
& \left| \sum_{i,j \in \operatorname{supp} \varphi} \left\langle \Delta_{\text{abs}}, \varphi_x(\xi) \left(x - (x_i + \frac{h}{2}) \right) + \varphi_t(\xi) \left(t - (t_j + \frac{\tau}{2}) \right) \right\rangle \right| \\
& \qquad \leq C \|\nabla \varphi\|_\infty\, h^2 \sum_{i,j \in \operatorname{supp} \varphi} TV_{[x_{i-1}, x_{i+1}]}(u_h^j(\cdot)) \\
& \qquad \leq C \|\nabla \varphi\|_\infty \lambda N h^2 TV(u_0) \cdot \sum_{j \in \operatorname{supp} \varphi} \; .
\end{aligned}
$$

So both terms involving the derivatives of φ vanish as $h \to 0$.

It remains to note that the simple functions $\varphi_h = \sum_{i,j \in \operatorname{supp} \varphi} \varphi_{h,i,j} \chi_{Y_{h,i,j}}(x,t)$, χ_Ω denotes the characteristic function of the set $\Omega \subset \mathbb{R}^2$, converge to φ in L^1 for $h \to 0$.

□

Appendix. Fluxes with almost constant curvature.

The entropy inequality (2.3) was used because of the positivity of the absolute continuous part of the dissipation measure $\eta(u_h)$ shown in Lemma 4.3.

We want to demonstrate how the negative contribution of the dissipation measure before interactions can be used to compensate the positive contribution of the dissipation measure after a shock interacts with a rarefaction wave. Then one can work with the entropy inequality (2.2) instead of (2.3). We will use a continuity argument to extend this interaction estimate to flux functions with overall curvature close to a constant.

It should be noted though, that the compensation of positive and negative parts of $\eta(u_h)$ only works for a very limited range of the Courant number. For the above proofs the use of (2.3) was more convenient and for larger Courant numbers it seems indispensable.

LEMMA 6.1. *Suppose that $f'' \equiv c_1 > 0$ and the Courant number $c \leq 1$. Let $u_h \in L^\infty(\mathbb{R} \times (0,\infty)) \cap BV(\mathbb{R} \times [0,\infty))$ be an approximate solution to problem (P) obtained using either scheme. Suppose that a rectangle $Y_{h,i,j}$ contains a shock and a rarefaction wave that meet. Then*

$$
(6.1) \qquad \langle \eta(u_h), 1 \rangle_{Y_{h,i,j}} \leq -\frac{c_1^2}{24} |u^+ - u^-|^3 \cdot \frac{\tau}{2}
$$

where u^-, u^+ are the resp. left and right state of the shock.

Proof. We divide $Y_{h,i,j}$ into two parts

$$Y_1 = \left\{(x,t) \in Y_{h,i,j} \;\middle|\; t < t_j + \frac{\tau}{2}\right\} \quad \text{and} \quad Y_2 = Y_{h,i,j} \backslash Y_1.$$

On Y_1 we have by Lemma 2.2

$$\langle \eta(u_h), 1 \rangle_{Y_{h,i,j}} = \frac{-c_1^2}{12} \, |u^+ - u^-|^3 \cdot \frac{\tau}{2} \, . \tag{6.2}$$

where u^-, u^+ denote the left resp. right states of u at the shock.

Let us again, as in the proof of Lemma 4.3, assume without loss of generality that the rarefaction wave is centered at (x_i, t_j) and the shock originates at (x_{i+1}, t_j). Thereby $u^- = u_j^i$ and $u^+ = u_j^{i+1}$. We also assume again that there is a time $t' \in (t_j, t_{j+1})$ where the waves become separated by an intermediate constant state $\tilde{u} = u_{-1} - (u_0 - u_1)$, with $u_{-1} = u_j^{i-1}, u_0 = u_j^i, u_1 = u_j^{i+1}$, see Figure 4.1.

Let us fix a time $\tilde{t} \in (t', t_{j+1})$ and study the measure $\eta(u_h)$ on the line $\mathcal{L} = \{(x,t) | x_i < x < x_{i+1}, t = \tilde{t}\}$. The contribution of the discontinuity is given by

$$\begin{aligned} \langle \Delta_{\text{sing}}, 1 \rangle_{\mathcal{L}} &= [f(u_{-1}) - f(\tilde{u})] \frac{f(u_0) - f(u_1)}{u_0 - u_1} - [F(u_{-1}) - F(\tilde{u})] \\ &= \frac{-c_1^2}{12} \, (u_{-1} - \tilde{u})^3 + [f(u_{-1}) - f(\tilde{u})] \left(\frac{f(u_0) - f(u_1)}{u_0 - u_1} - \frac{f(u_{-1}) - f(\tilde{u})}{u_{-1} - \tilde{u}} \right). \end{aligned} \tag{6.3}$$

Using the fact that $u_{-1} - \tilde{u} = u_0 - u_1$ and Taylor's theorem this gives

$$\begin{aligned} \langle \Delta_{\text{sing}}, 1 \rangle_{\mathcal{L}} &= \frac{-c_1^2}{12} \, (u_0 - u_1)^3 \\ &\quad + \left[f'(u_{-1})(u_0 - u_1) - \frac{c_1}{2} \, (u_0 - u_1)^2 \right] \Big(f'(u_1) - f'(u_{-1}) + c_1(u_0 - u_1) \Big) \\ &= \frac{-c_1}{12} \, (u_0 - u_1)^3 + \left[f'(u_{-1})(u_0 - u_1) - \frac{c_1}{2} \, (u_0 - u_1)^2 \right] \cdot c_1(u_0 - u_{-1}) \\ &= \frac{-c_1^2}{12} \, (u_0 - u_1)^3 - \frac{c_1^2}{2} \, (u_0 - u_1)^2 (u_0 - u_{-1}) + c_1 f'(u_{-1})(u_0 - u_1)(u_0 - u_{-1}). \end{aligned} \tag{6.4}$$

Now let us look at the absolute continuous part on $\mathcal{L}$. Using (4.9) we have

$$\int\limits_{f'(u_{-1})(\tilde{t}-t_j)+x_i}^{f'(u_0)(\tilde{t}-t_j)+x_i} f(u_h)_t + F(u_h)_x \, dx = -c_1(u_0 - u_1) \int\limits_{f'(u_{-1})(\tilde{t}-t_j)+x_i}^{f'(u_0)(\tilde{t}-t_j)+x_i} f'(u_h)(u_h)_x dx.$$

We have $f'(u_h) = f'(g\left(\frac{x-x_i}{\tilde{t}-t_j}\right)) - c_1(u_0-u_1) = \frac{x-x_i}{\tilde{t}-t_j} - c_1(u_0-u_1)$ and $(u_h)_x = g' \cdot \frac{1}{\tilde{t}-t_j} = \frac{1}{c_1(\tilde{t}-t_j)}$. Thereby we may evaluate the last integral and use the mean value theorem to get

$$
\begin{aligned}
&-c_1(u_0-u_1)\left[\frac{(x-x_i)^2}{2c_1(\tilde{t}-t_j)^2} - \frac{u_0-u_1}{\tilde{t}-t_j}\, x\right]_{f'(u_{-1})(\tilde{t}-t_j)+x_i}^{f'(u_0)(\tilde{t}-t_j)+x_i} \\
&\quad = -c_1(u_0-u_1)\left[\frac{f'(u_0)^2 - f'(u_{-1})^2}{2c_1} - (u_0-u_1)\big(f'(u_0)-f'(u_{-1})\big)\right] \\
&\quad = -c_1^2(u_0-u_1)(u_0-u_{-1})\left[\frac{f'(u_0)+f'(u_{-1})}{2c_1} - (u_0-u_1)\right] \qquad (6.5)\\
&\quad = -c_1^2(u_0-u_1)(u_0-u_{-1})\left[\frac{f'(u_{-1})}{c_1} + \frac{(u_0-u_{-1})}{2} - (u_0-u_1)\right].
\end{aligned}
$$

Since both (6.4) and (6.5) are independent of $\tilde{t}$ for $\tilde{t} \in [t', t_{j+1}]$, looking at the measure over the rectangle $(x_i, x_{i+1}) \times [t', t_{j+1}]$ will only introduce the common factor $(t_{j+1}-t')$. Therefore, we now look at the sum of both terms.

$$c_1^2 E := c_1^2\left[\frac{-1}{12}(u_0-u_1)^3 + \frac{1}{2}(u_0-u_1)^2(u_0-u_{-1}) - \frac{1}{2}(u_0-u_1)(u_0-u_{-1})^2\right].$$

It remains to estimate E. Set $a = u_0 - u_1 > 0,\ b = u_0 - u_{-1} > 0$ and $z = \frac{b}{a}$ then

$$E = \frac{-1}{12}\, a^3 + \frac{1}{2}\, a^2 b - \frac{1}{2}\, ab^2 = \frac{1}{2}\, a^3\left(-z^2 + z + \frac{1}{6}\right).$$

Since $a \geq 2b$ we are interested in $z \in [0, \frac{1}{2}]$. The polynomial $-z^2 + z - \frac{1}{6}$ has its maximum at $z = \frac{1}{2}$, therefore we may estimate $E < \frac{1}{24}\, a^3$ for $z \in [0, \frac{1}{2}]$. Setting $S = \{(x,t) \in Y_{h,i,j} \mid t > t'\}$ this gives

$$\langle \eta(u_h), 1\rangle_S \leq \frac{1}{24}\, |u_0 - u_1|^3 (t_{j+1} - t'). \tag{6.6}$$

For $t \in [t_j + \frac{\tau}{2}, t')$ the terms corresponding to E are smaller. In that case u_{-1} approaches u_0 so b becomes smaller and $\frac{\partial E}{\partial b} \geq 0$. Therefore we have

$$\langle \eta(u_h), 1\rangle_{Y_2} \leq \frac{c_1^2}{24}\, |u^+ - u^-|^3 \cdot \frac{\tau}{2}. \tag{6.7}$$

From (6.2) and (6.7) we get (6.1).

□

Now let us look at the same analysis for the case $c_1 \leq f'' \leq c_2$ and assume $t' \geq t_j + \frac{\tau}{2}$. Using the notation from the previous proof we have by Lemma 2.2

$$\langle \eta(u_h), 1\rangle_{Y_1} \leq \frac{-c_1^2}{12}\, |u^+ - u^-|^3 \cdot \frac{\tau}{2}. \tag{6.8}$$

Now to estimate $f(u_h)_t + F(u_h)_x$ we have to modify (6.5) to give

$$
\begin{aligned}
&-c_2(u_0-u_1)\left[\frac{f'(u_0)^2-f'(u_{-1})^2}{2c_1} - \frac{c_2}{c_1}(u_0-u_1)(f'(u_0)-f'(u_1))\right] \\
(6.9)\qquad &\le -c_2^2(u_0-u_1)(u_0-u_{-1})\left[\frac{f'(u_0)+f'(u_{-1})}{2c_1} - \frac{c_2}{c_1}(u_0-u_1)\right] \\
&\le -c_2^2(u_0-u_1)(u_0-u_{-1})\left[\frac{f'(u_{-1})}{c_1} + \frac{(u_0-u_{-1})}{2} - \frac{c_2}{c_1}(u_0-u_1)\right].
\end{aligned}
$$

Note that $u_0 > u_{-1} > u_1$.

Now we have to modify (6.4) appropriately. Remember that $\frac{f(u_0)-f(u_1)}{u_0-u_1} > \frac{f(u_{-1})-f(\tilde{u})}{u_{-1}-\tilde{u}}$ and $u_{-1} - \tilde{u} = u_0 - u_1$. We get

$$
\begin{aligned}
\langle \Delta_{\text{sing}}, 1\rangle_{\mathcal{L}} \le &\frac{-c_1^2}{12}(u_0-u_1)^3 + \\
&\left[f'(u_{-1})(u_0-u_1) - \frac{c_1}{2}(u_0-u_1)^2\right]\left(\frac{f(u_0)-f(u_1)}{u_0-u_1} - \frac{f(u_{-1})-f(\tilde{u})}{u_{-1}-\tilde{u}}\right). \\
\le &\frac{-c_1^2}{12}(u_0-u_1)^3 + f'(u_{-1})(u_0-u_1)\left(c_2(u_0-u_{-1}) + (c_2-c_1)(u_0-u_1)\right) \\
(6.10)\qquad &\frac{-c_1}{2}(u_0-u_1)^2\left(c_1(u_0-u_{-1}) + (c_1-c_2)(u_0-u_1)\right) \\
= &\frac{-c_1^2}{12}(u_0-u_1)^3 + c_2 f'(u_{-1})(u_0-u_1)(u_0-u_{-1}) + (c_2-c_1)f'(u_{-1})(u_0-u_1)^2 \\
&-\frac{c_1^2}{2}(u_0-u_1)^2(u_0-u_{-1}) + \frac{c_1}{2}(c_2-c_1)(u_0-u_1)^3.
\end{aligned}
$$

Now adding the right hand sides of (6.9) and (6.10) gives

$$
\begin{aligned}
F(c_1,c_2) = &-\frac{c_1^2}{12}(u_0-u_1)^3 + \left(\frac{c_2^3}{c_1} - \frac{c_1^2}{2}\right)(u_0-u_1)^2(u_0-u_{-1}) - \frac{c_2^2}{2}(u_0-u_1)(u_0-u_{-1})^2 \\
&\left(c_2 - \frac{c_2^2}{c_1}\right) f'(u_{-1})(u_0-u_1)(u_0-u_{-1}) + (c_2-c_1)f'(u_{-1})(u_0-u_1)^2. \\
&+\frac{c_1}{2}(c_2-c_1)(u_0-u_1)^3.
\end{aligned}
$$

Note that $F(c_1,c_1) = c_1^2 E$. Therefore one may argue by continuity for c_2 close to c_1 that still $\langle \eta(u_h), 1\rangle_{Y_{h,i,j}} \le 0$. Under these circumstances one does not have to use the entropy inequality (2.3). The inequality (2.2) would be sufficient for the proofs in Section 5.

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