THE COUPLING METHOD OF FINITE ELEMENTS AND BOUNDARY ELEMENTS FOR RADIATION PROBLEMS

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THE COUPLING METHOD OF FINITE ELEMENTS AND BOUNDARY ELEMENTS FOR RADIATION PROBLEMS

He Yinnian* and Li Kaitai*

Abstract The paper presents the variational formulation and well posedness of the coupling method of finite elements and boundary elements for radiation problem. The convergence and optimal error estimates for the approximation solution and numerical experiment are provided.

1. Introduction

Many scattering and radiation problems in mathematical physics are reduced to the problem of Helmholtz equation

$$\Delta u + K^2 u = f \qquad x \in R^2 \quad (\text{or } R^3)$$

It is important in many areas of applications e.g. the design of waveguide, exploring for mineral material and the study of the biological effects of microwave radiation, and so on. The difficulty is that the domain is whole space. Many people present various method which are used for this problem, e.g. FEM. BE and infinite element ([2]-[5], 29], [10]). Each method is of its characterization. However, the coupling method of FE and BE most attracts one' attention.

2. Variational Formulation

Let us consider the following radiation problem.

$$\begin{aligned} &\Delta K + K^2 u = f && \text{in } R^2 \\ (2.1) \qquad &u = 0(r^{-1}) && r = |x| \to 1^{\infty} \\ &\frac{\partial u}{\partial n} + iKu = 0(r^{-1/2}) && r \to + \infty \end{aligned}$$

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where $f(x)$ is a function on R^2 with supp $f \subset \Omega_1$, $\Omega_1 \subset R^2$ is a bounded domain with smooth boundary Γ and $\Omega_2 = R^2 \setminus \overline{\Omega}_1$ is the exterior domain (see Figure 1.).

The complex function k is defined by

(2.2) $$k(x) = \begin{cases} k_1(x) & x \in \Omega_1 , \\ k_0 & x \in \Omega_2 , \end{cases}$$

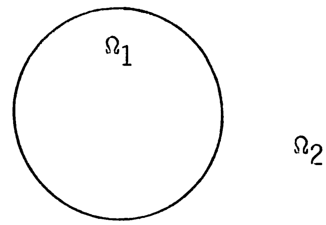

Figure 1.

where k_0 is a constant, $k_1(x)|_\Gamma = k_0$, $|k^2| \leq M$, $-C_1 \leq I_m\, k^2 \leq -C_2$, M, C_1, C_2 are positive constant.

Let

(2.3) $$H^1_*(R^2) = \{u : u \in H^1(R^2), \quad u = O(r^{-1}),$$
$$\frac{\partial u}{\partial r} + ik_0 u = o(r^{-1/2}),\ r \to \infty\}$$

with the norm $\|\cdot\|_{1,\ R^2}$ of $H^1(R^2)$, which is also a Hilbert space.

In fact, let $\{v_k\}$ $H^1_*(R^2)$ which converge to an element $v \in H^1_*(R^2)$, that is

$$\lim_{n \to \infty} \left(\int_{R^2} |\nabla v_n - \nabla v|^2\, dx + \int_{R^2} |v_n - v|^2\, dx\right) = 0$$

Recalling F. Riesz theorem, $\{v_n\}$ contains a subsequence $\{v_{nk}\}$ such that

$$\lim_{k \to \infty} v_{nk}(x) = v(x) \quad \text{a.e.} \quad x \in R^2$$

Since $v_{nk} \in H^1(R^2)$ and $v \in H^1(R^2)$, thus

$$\lim_{k \to \infty} \frac{\partial}{\partial r} V_{nk}(x) = \frac{\partial}{\partial r} V(x) \qquad \text{a.e.} \quad x \in R^2$$

This implies

$$v = O(r^{-1}) \qquad r \to \infty$$
$$\frac{\partial v}{\partial r} + ik_0 v = O(r^{-1/2}) \qquad r \to \infty$$

by

$$v_n = O(r^{-1}) \qquad r \to \infty$$

$$-\frac{\partial}{\partial r} V_n + ik_0 v_n = O(r^{-1/2}) \qquad r \to \infty$$

Hence $v \varepsilon H^1_*(R^2)$

Then, variation problem associated with (2.1) is

find $u \varepsilon H^1_*(R^2)$ such that

(2.4)
$$A(u,v) = (f,v) \qquad \forall\, v \;\varepsilon\; H^1_*(R^2).$$

by means of Green's formula, where

(2.5)
$$A(u,v) = -\int_{R^2} \nabla u\ \overline{\nabla v}\ dx + \int_{R^2} k^2\ u\overline{v}\ dx\ ,$$

(2.6)
$$(f,v) = \int_{R^2} f\ \overline{v}\ dx.$$

$\overline{v}$ denotes complex conjugate function of v.

Theorem 2.1. Let $f \varepsilon H^{-1}(R^2)$ with supp$f \subset \Omega_1$. Then, variational problem (2.4) has a unique solution $u \varepsilon H^1_*(R^2)$ and

(2.7)
$$\|u\|_{1,R_2} \leqslant C\|f\|_{-1,R^2}\ ,$$

where $C > 0$ is constant.

Proof. It is obvious that $A(\cdot,,) : H^1_*(R^2) \times H^1_*(R^2) \to C$ is continuous sesquilinear form. It is sufficient to prove that $A(\cdot,\cdot)$ is $H^1_*(R^2)$ - coercive by Lax-Milgram theorem.

In fact

(2.8)
$$A(v,v) = (-|v|^2_{1,R^2} + \int_{\Omega_1} k_1^2\ |v|^2\ dx + k_0^2\ \|v\|^2_{0,\Omega_2}\ )$$

so

$$|A(v,v)|^2 = (-|v|^2_{1,R^2} + \mathrm{Re}k_0^2 \|v\|^2_{0,\Omega_2} + \int_{\Omega_1} R_e k_1^2 \, |v|^2 dx)^2$$

$$+(\mathrm{Im}k_0^2 \, \|v\|^2_{0,\Omega_2} + \int_{\Omega_1} \mathrm{Im}k_1^2 \, |v|^2 dx)^2.$$

Set $0 < \varepsilon < 1$, using young inequality, we obtain

$$|A(v,v)|^2 \geqslant (1-\varepsilon)|v|^4_{1,R^2} + (1-\varepsilon^{-1})(R_e k_0^2 \|v\|^2_{0,\Omega_2}$$

$$+ \int_{\Omega_1} R_e k_1^2 \, |v|^2 \, dx)^2 + (\mathrm{Im}k_0^2 \, \|v\|^2_{0,\Omega_2} + \int_{\Omega_1} \mathrm{Im}k_1^2 \, |v|^2 dx)^2$$

$$\geqslant (1-\varepsilon)|v|^4_{1,R^2} + ((1 - \varepsilon^{-1})M^2 + C_2^2)|v|^4_{0,R^2}$$

Taking $\varepsilon > 0$ such that $\varepsilon^{-1} \in (1, 1 + C_2^2/M^2)$, then, there exists $C_3 > 0$, $C_4 > 0$ such that

$$1 + C_2^2/M^2 - \frac{1}{\varepsilon} > C_3 \, , \frac{1}{\varepsilon} - 1 > C_4 \, .$$

Therefore, we have

$$(2.9) \qquad |A(v,v)|^2 \geqslant C_4 \varepsilon \, |v|^4_{1,R^2} + C_3 M^2 \, \|v\|^4_{0,R^2} \geqslant C_5 \, \|u\|^4_{1,R^2}$$

where C_5 depends on C_3 and C_4. Hence, it is proved that $A(.,.)$ is $H^1_*(R^2)$ - coercive.

However, the problem (2.1) can alternatively be formulated as follows:

$$(2.10a) \qquad \Delta u_1 + k_1^2 \, u_1 = f \qquad \text{in } \Omega_1.$$

$$(2.10b) \qquad \Delta u_2 + k_0^2 u_2 = 0 \qquad \text{in } \Omega_2,$$

$$(2.10c) \qquad u_1 = u_2 \qquad \text{on } \Gamma,$$

(2.10d) $$\frac{\partial u_1}{\partial n} = \frac{\partial u_2}{\partial n} = \lambda \qquad \text{on} \quad \Gamma,$$

(2.10e) $$u_2 = O(r^{-1}),\ \frac{\partial u_2}{\partial r} + ik_0 u_2 = O(r^{-1/2}),\ r \to \infty,$$

where $u_1 = u|\Omega_i$, $i = 1,2$, and $\partial/\partial n$ denotes the outward normal derivative to Γ. The equations (2.10a) and (2.10b) signify a decomposition into two problems in the separate domains Ω_1 and Ω_2 , while (2.10c) and (2.10d) reflect the appropriate coupling of these two problems.

Let us now give a variational formulation of (2.10). Since $\Delta u + k_1^2 u = f$ in Ω_1, we find, using Green's formula, that

(2.11) $$a(u,v) + \langle\gamma_0 v,\lambda\rangle = (f,v) \qquad \forall\, v \in H^1(\Omega_1)$$

where ν_0 is 0-order trace operator on Γ and

$$\lambda = -\frac{\partial u}{\partial n}\Big|_\Gamma\ ,\ \langle\gamma_0,v,\lambda\rangle = \int_\Gamma \bar{V}\lambda ds,$$

$$a(u,v) = -\int_{\Omega_1} \nabla u \nabla\bar{v} dx + \int_{\Omega_1} k_1^2\, u\bar{v} dx.$$

Moreover, since $\Delta u + k_0^2 u = 0$ in Ω_2, we find, using Green's formula and (2.10e), that (cf[8]),

(2.12) $$\frac{1}{2}u(x) = \int_\Gamma u(y)G_n(x,y)ds_y - \int_\Gamma \lambda(y)G(x,y)dSy \qquad x \in \Gamma,$$

(2.13) $$u(x) = \int_\Gamma u(y)G_n(x,y)dsy - \int_\Gamma \lambda(y)G(x,y)dSy \qquad x \in \Omega_2.$$

where

(2.14) $$G(x,y) = -\frac{1}{4i} H_0^{(2)}(k_0\ |x-y|) \qquad x\neq y$$

is the fundamental solution associated with the two-dimensional Helmholtz equation and

(2.15) $$G_n(x,y) = \frac{\partial}{\partial n_y} G(x,y) \qquad x \neq y, \qquad y \in \Gamma$$

with n_y being the outward unit normal to Γ at $y \in \Gamma$. $H_0^{(2)}(z)$ is the second kind and 0-order Hankel function.

Now, taking complex conjugate for (2.12) and formally multiplying (2.12) by the function $\mu(x) \in H^{-1/2}(\Gamma)$ and integrating over Γ, we obtain

(2.16) $$2b(\lambda,\mu) - \langle\gamma_0 u,\mu\rangle + 2\langle G_n u,\mu\rangle = 0 \quad \forall\, \mu \in H^{-1/2}(\Gamma).$$

where

(2.17) $$b(\lambda,\mu) = -\int_\Gamma \int_\Gamma \overline{\lambda}(y)\overline{G}(x,y)\, \mu(x) ds_x\, dS_y,$$

(2.18) $$G_n u = \int_\Gamma G_n(x,y)u(y)dsy \qquad x \in \Gamma.$$

Combining (2.11) and (2.16), variational formulation for the couling method of FE and BE can be rewritten as following: Find $(u,\lambda) \in H^1(\Omega_1) \times H^{-1/2}(\Gamma)$ such that

(2.19) $$a(u,v) + \langle\gamma_0 v,\lambda\rangle = (f,v) \qquad \forall\, v \in H^1(\Omega_1),$$

$$2b(\lambda,\mu) - \langle\gamma_0 u,\mu\rangle + 2\langle G_n u,\mu\rangle = 0 \quad \forall\, \mu \in H^{-1/2}(\Gamma).$$

3. The Well-Posedness of (2.19)

First of all, we prove that sesquilinear form $b(\cdot,\cdot)$ is continuous and coercive on $H^{-1/2}(\Gamma) \times H^{-1/2}(\Gamma)$, and $\langle\gamma_0 u,\mu\rangle - 2\langle G_n u,\mu\rangle$ is a continuous linear functional on $H^{-1/2}(\Gamma)$ for $u \in H^1(\Omega_1)$. Hence the variational problem:

$$2b(\lambda,\mu) - \langle\gamma_0 u,\mu\rangle + 2\langle G_n u,\mu\rangle = 0 \quad \forall\, \mu \in H^{-1/2}(\Gamma),$$

has a unique solution $\lambda = \lambda(\gamma_0 u)$.

Next, we consider problem : Find $u \in H^1(\Omega_1)$ such that

$$a(u,v) + \langle \nu_0 v,\ \lambda(\nu_0 u)\rangle = (f,v) \quad \forall\, v \in H^1(\Omega_1). \tag{3.1}$$

Let

$$B(u,v) = a(u,v) + \langle \nu_0 v,\ \lambda(\nu_0 u)\rangle \tag{3.2}$$

Fortunately, sesquilinear form $B(\cdot,\nu)$ on $H^1(\Omega_1) \times H^1(\Omega_1)$ is continuous and $H^1(\Omega_1)$ - coercive. Therefore, the coupling variational problem (2.19) is well-posedness.

1° The Properties of $b(\cdot, \cdot)$

First, we consider an auxiliary problem:

$$\begin{aligned} &\Delta u + k^2 u = 0 && \text{in } R^2 \quad \Gamma, \\ &u = u_0 && \text{on } \Gamma. \\ &u = O(r^{-1}) && r \to \infty. \end{aligned} \tag{3.3}$$

$$-\frac{\partial u}{\partial r} + ik_0 u = 0\,(r^{-1/2}) \qquad r \to \infty.$$

where $u_0 \in H^{1/2}(\Gamma)$.

From section 2 and reference [12]. We can prove

Lemma 3.1. Let $u_0 \in H^{1/2}(\Gamma)$, then there exists a unique solution $u \in H^1_*(R^2)$ of (3.3) and

$$C_1 \|u\|_{1,R^2} \leq \|u_0\|_{1/2,\Gamma} \leq C_2 \|u\|_{1,R^2} \tag{3.4}$$

where C_1 and C_2 are positive constants.

Assume

$$K = \{v;\ v \in H^1_*\,(R^2),\ \operatorname{supp}\,(\Delta v + k_0^2 v) \quad \Gamma\}. \tag{3.5}$$

Thus, (3.3) defines a extensional operator E of $H^{1/2}(\Gamma)$ onto K such that

$$u = Eu_0 \tag{3.6}$$

Obviously, $E : H^{1/2}(\Gamma) \to K$ is an isomorphism and

(3.7) $$u_0 = E^{-1}u = \gamma_0 u.$$

Next, we define an operator L on $H^{-1/2}(\Gamma)$ by

(3.8) $$L(q) = \int_\Gamma G(x,y)q(y)ds_y. \qquad x \in R^2$$

We have the following result for the operator L.

Lemma 3.2. The operator L is an isomorphism from $H^{-1/2}(\Gamma)$ onto K and the correspondence of $q \in H^{-1/2}(\Gamma)$ and $u \in K$ is defined by

(3.9) $$u = L(q),$$
$$q = [\frac{\partial u}{\partial n}] = -\frac{\partial u}{\partial n^-} - \frac{\partial u}{\partial n+},$$

and

(3.10) $$C_1 \|u\|_{1,R^2} \leq \|q\|_{-1/2,\Gamma} \leq C_2 \|u\|_{1,R^2},$$

where $[\frac{\partial u}{\partial n}]$ denotes the jump crossing Γ and

$$\frac{\partial n}{\partial n^-} = \frac{\partial}{\partial n}(u|_{\Omega_1})|_\Gamma, \quad \frac{\partial u}{\partial n^+} = \frac{\partial}{\partial n}(u|_{\Omega_1})|_\Gamma$$

Proof For any $q \in H^{-1/2}(\Gamma)$, we consider the variational problem

(3.11) $$A_0(u,v) = -\langle\gamma_0 v,q\rangle \qquad \forall v \in H^1(R^2).$$

(3.12) $$A_0(u,v) = -\int_{R^2} \nabla u\, \nabla\bar{v}\, dx + \int_{R^2} k_0^2\, u\bar{v}\, dx.$$

So, $A_0(\cdot,\cdot)$ has the properties as $A(\cdot,\cdot)$ and $-\langle\gamma_0 v,q\rangle$ is linear bounded functional on $H^1(R^2)$ by trace theorem. Thus (3.11) has a unique solution $u \in H^1_*(R^2)$ such that

(3.13) $$C_1 \|u\|_{1,R^2} \leq \|q\|_{-1/2,\Gamma} \leq C_2 \|u\|_{1,R^2}.$$

However, using abstract Green's formula, we have

(3.14) $$A_0(u,v) = -\int_{\Omega_1} \nabla u \nabla \bar{v} dx + \int_{\Omega_1} k_0^2 \, u\bar{v} dx - \int_{\Omega_2} \nabla u \; \nabla \bar{v} dx + \int_{\Omega_1} k_0^2 u\bar{v} dx$$

$$= -\int_{\Gamma} [\frac{\partial u}{\partial n}] \bar{v} \, dx + \int_{\Omega_1} (\Delta u + k_0^2 u)\bar{v} dx + \int_{\Omega_2} (\Delta u + k_0^2 u)\bar{v} dx.$$

From (3.11), we obtain

(3.15) $$\Delta u + k_0^2 u = 0 \qquad \text{in } R^2 \setminus \Gamma.$$

and

(3.16) $$q = [\frac{\partial u}{\partial n}].$$

So, $u \in K$.

Since $\Delta u + k^2 u = 0$ in $R^2 \setminus \Gamma$, we have

(3.17) $$\int_{\Gamma} \frac{\partial u}{\partial n^-}(y) \; G(x,y) \; dsy - \int_{\Gamma} G_n(x,y) u(y) \; dsy$$

$$= \begin{cases} u\,(x) & x \in \Omega_1 \,, \\ \frac{1}{2} \; u(x) & x \in \Gamma \,, \\ 0 & x \in \Omega_2 \,, \end{cases}$$

and

(3.18) $$-\int_{\Gamma} \frac{\partial u}{\partial n+}(y) \; G(x,y) dsy + \int_{\Gamma} G_n(x,y) u(y) dsy$$

$$= \begin{cases} 0 & x \in \Omega_1, \\ \frac{1}{2} \; u(x) & x \in \Gamma \,, \\ u(x) & x \in \Omega_2 \,. \end{cases}$$

Combining (3.17) and (3.18) we have

$$u(x) = \int_\Gamma \left[\frac{\partial u}{\partial n}(y)\right] G(x,y)\, ds_y = L(q). \tag{3.19}$$

Conversely, let $u(x) \in K$, using the discussion as above, we have

$$u(x) = L(q), \tag{3.20}$$

and

$$q = \left[\frac{\partial u}{\partial n}\right]. \tag{3.21}$$

Hence L is an isomorphism of $H^{-1/2}(\Gamma)$ onto K and the proof is complete.

Theorem 3.1. The sesquilinear form $b(q_1,q_2)$ is continuous on $H^{-1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ and $H^{-1/2}(\Gamma)$ - coercive, i.e.

$$b(q_1, q_2) \leq C \|q_1\|_{-1/2,\Gamma} \|q_2\|_{-1/2,\Gamma}, \forall q_1, q_2 \in H^{-1/2}(\Gamma), \tag{3.22}$$

$$b(q_1,q_2) \geq C \|q_1\|^2_{-1/2,\Gamma}, \forall q_1 \in H^{-1/2}(\Gamma). \tag{3.23}$$

Proof Let $q_1, q_2 \in H^{-1/2}(\Gamma)$, recalling (2.17) and (3.8), we obtain

$$b(q_1,q_2) = -\langle \gamma_0 L(q_1), q_2\rangle = -\langle \gamma_0 u_1, q_2\rangle = A_0(u,v) \tag{3.24}$$

Hence

$$|b(q_1,q_2)| = |\langle \gamma_0 u_1, q_2\rangle| \leq \|\gamma_0 u_1\|_{1/2,\Gamma} \|q_2\|_{-1/2,\Gamma} \leq C \|q_1\|_{-1/2,\Gamma} \|q_2\|_{-1/2,\Gamma}$$

Notice

$$b(q_1,q_1) = -\langle \gamma_0 u_1, q_1\rangle \tag{3.25}$$

and

$$A_0(u_1,v) = -\langle \gamma_0 v, q_1\rangle \quad \forall\, v \quad H^1_*(R^2), \tag{3.26}$$

It is easy to get

$$|b(q_1,q_1)| = |\langle\gamma_0 u_1,q_1\rangle| = |A_0(u_1,u_1)| \geqslant C\|u_1\|^2_{1,R^2} \geqslant C\|q_1\|^2_{-1/2,\Gamma}$$

by (3.9). Thus, theorem 3.1 is proved.

2° The well-posedness of (2.19)

Let $u \in H^1(\Omega_1)$, we prove that $2 \langle G_n u,\mu\rangle - \langle\gamma_0 u,\mu\rangle$ is a bounded linear functional on $H^{-1/2}(\Gamma)$. To this end, we first prove the following lemma.

Lemma 3.3. Let $u_0 \in H^{1/2}(\Gamma)$, then $G_n u_0 \in H^{1/2}(\Gamma)$ and

(3.7) $$\|G_n u_0\|_{1/2,\Gamma} \leqslant C\|u_0\|_{1/2,\Gamma}.$$

Proof. We consider an auxiliary problem

(3.28) $$\begin{aligned} \Delta W + k^2 w &= 0 \qquad \text{in } \Omega_1 \\ w &= u_0 \qquad \text{on } \Gamma \end{aligned}$$

where $u_0 \in H^{1/2}(\Gamma)$.

Recalling lemma 3.1, (3.28) has a unique solution $w \in H^1(\Omega_1)$. Using Green's theorem, we obtain

(3.29) $$\frac{1}{2} u_0(x) = - G_n u_0 + \gamma_0 L\left(\frac{\partial W}{\partial n} \right) \qquad x \in \Gamma.$$

Since $\omega \in H^1(\Omega_1)$, then $\frac{\partial\omega}{\partial n} \in H^{-1/2}(\Gamma)$. We know $\gamma_0 L(\frac{\partial}{\partial n}\omega) \in H^{1/2}(\Gamma)$ by Lemma 3.2 and the trace theorem. From (3.29), $G_n u_0 \in H^{1/2}(\Gamma)$ and

$$\begin{aligned} \|G_n u_0\|_{1/2,\Gamma} &\leqslant 1/2\|u_0\|_{1/2,\Gamma} + \|\gamma_0 L\left(\frac{\partial\omega}{\partial n} \right)\|_{1/2,\Gamma} \\ &\leqslant 1/2\|u_0\|_{1/2,\Gamma} + \|\frac{\partial\omega}{\partial n}\|_{-1/2,\Gamma} \leqslant c\|u_0\|_{1/2,\Gamma}. \end{aligned}$$

On the other hand, we find, applying Lemma 3.3, that

$$|2\langle G_n u,\mu\rangle - \langle\gamma_0 u,u\rangle| \leq |2\langle G_n u,\mu\rangle + |\langle\gamma_0 u,\mu\rangle|$$

$$\leq 2\|G_n u\|_{1/2,\Gamma}\|\mu\|_{-1/2,\Gamma} + \|\nu_0 u\|_{1/2,\Gamma}\|\mu\|_{-1/2,\Gamma} \leq C\|\gamma_0 u\|_{1/2,\Gamma}\|\mu\|_{-1/2,\Gamma} \leq C\|u\|_{1,\Omega},\|\mu\|_{-1/2,\Gamma}.$$

Thus, $2 \langle G_n u,\mu\rangle - \langle\gamma_0 u,\mu\rangle$ is a linear bounded functional on $H^{-1/2}(\Gamma)$. By virtue of properties of $b(\cdot,\cdot)$, we have

Theorem 3.2. Let $u \in H^1(\Omega_1)$, then, the variational problem: Find $\lambda \in H^{-1/2}(\Gamma)$ such that

$$b(\lambda,\mu) = \frac{1}{2}\langle\gamma_0 u,\mu\rangle - \langle G_n u,\mu\rangle, \ \forall \mu \in H^{-1/2}(\Gamma) \tag{3.30}$$

has a unique solution $\lambda = \lambda(\gamma_0 u)$ and

$$\|\lambda\|_{-1/2\cdot\Gamma} \leq C \|\gamma_0 u\|_{1/2,p} \leq C\|u\|_{1,\Omega_1} \tag{3.31}$$

Moreover, we need consider the variation problem:

Find $u \in H^1(\Omega_1)$ such that

$$B(u,v) = (f,v) \qquad \forall v \in H^1(\Omega_1). \tag{3.32}$$

To this end, we introduce an auxiliary exterior problem:

$$\begin{aligned} \Delta\omega + k_0^2\omega &= 0 && \text{in } \Omega_2 \\ \omega &= g && \text{on } \Gamma \\ \omega &= O(r^{-1}) && r \to \infty \\ \frac{\partial\omega}{\partial r} + ik_0\omega &= O(r^{-1/2}) && r \to \infty \end{aligned} \tag{3.33}$$

It is well-known (see [1]) that (3.33) has a unique solution and the normal derivative $\frac{\partial\omega}{\partial n}$ is well defined. If $g \in H^{1/2}(\Gamma)$, then $\frac{\partial\omega}{\partial n} \in H^{-1/2}(\Gamma)$, we let F be the mapping defined by $Fg = \frac{\partial\omega}{\partial n}$. Then, F is a bounded mapping from $H^{1/2}(\Gamma)$ into $H^{-1/2}(\Gamma)$.

Recalling the equation (2.10) of section 2, u_2 satisfies problem (3.33) when $g = \gamma_0 u_1$. Set $u_1 = u$ in Ω_1, then, we have

$$\lambda(\gamma_0 u) = \frac{\partial u_2}{\partial n} = F(\gamma_0 u). \tag{3.34}$$

Thus

$$B(u,v) = a(u,v) + \langle\gamma_0 v, F(\gamma_0 u)\rangle. \tag{3.35}$$

Set $u,v \in C^\infty(\Omega_1)$, extend u,v to $\hat{u}, \hat{v} \in C^\infty(R^2)$ which are defined by

$$\hat{u}|_{\Omega_1} = u, \quad \hat{u}|_{\Omega_2} = \omega,$$

and $\hat{u}|_{\Omega_2}$, $\hat{v}|_{\Omega_2}$ satisfy problem (3.33) when $g = \gamma_0 u, \gamma_0 v$. Using Green's formula in Ω_2, we have

$$\langle\gamma_0 v, F(\gamma_0 u)\rangle = -\int_{\Omega_2} \nabla\hat{u} \cdot \vec{\nabla v}\,dx + \int_{\Omega_2} k_0^2\, \hat{u}\, \vec{v}\, dx. \tag{3.36}$$

By (3.35) and (3.36), it is easy to see

$$B(u,v) = a(u,v) + \langle\gamma_0 v, F(\gamma_0 u)\rangle = A(\hat{u},\hat{v}). \tag{3.37}$$

Since $C^\infty(\Omega_1)$ and $C^\infty(R^2)$ are dense in $H^1(\Omega_1)$ and $H^1(R^2)$ respectively, thus, we obtain

$$B(u,v) = A(\hat{u},\hat{v}) \qquad \forall u, v \in H^1(\Omega_1). \tag{3.38}$$

Therefore,

$$|B(v,v)| = |A(\hat{v},\hat{v})| \geq c\|\hat{v}\|^2_{1,R^2} \geq c\|v\|^2_{1,\Omega_1}$$

$$|B(v,v)| = |a\ (u,v) + \langle \gamma_0 v,\ F(\gamma_0 u) \rangle\ |$$

$$\leq C\|u\|_{1,\Omega_1}\|v\|_{1,\Omega_1} + \|\gamma_0 v\|_{1/2,\Gamma}\|F(\gamma_0 u)\|_{-1/2,\Gamma} \leq C\|u\|_{1,\Omega_1}\|v\|_{1,\Omega_1}.$$

Hence the sesquilinear form B $(\cdot,\cdot)$ is continuous on $H^1(\Omega_1) \times H^1(\Omega_1)$ and $H^1(\Omega_1)$ - coercive. Thus, we have

Theorem 3.3. If $f \in H^{-1}(R^2)$ and supp $f \subset \Omega_1$. Then, problem (3.32) has a unique solution $u \in H^1(\Omega_1)$ and

(3.39) $$\|u\|_{1,\Omega_1} \leq c\|f\|_{-1,R^2}.$$

Further, problem (2.19) has a unique solution $(u,\lambda) \in H^1(\Omega_1) \times H^{-1/2}(\Gamma)$ and

(3.40) $$\|(u,\lambda)\|_{H^1(\Omega_1)\times H^{-1/2}(\Gamma)} = \|u\|_{1,\Omega_1} + \|\lambda\|_{-1/2,\Gamma} \leq c\|u\|_{1,\Omega_1} \leq c\|f\|_{-1,R^2}.$$

4. The Error Estimate of the Approximation Solution

We let $V_h \subset H^1(\Omega_1)$ and $H_h \subset H^{-1/2}(\Gamma)$ be finite element subspaces depending on parameter $h>0$. We also assume that V_h and H_h satisfy the following approximate hypothesis:

(H_1) For any $v \in H^m(\Omega_1)$, there exists a constant C and a function $\pi_h v \in V_h$ such that

(4.1) $$\|v-\Pi_h v\|_{1,\Omega_1} \leq ch^s\|v\|_{s+1,\Omega_1} \qquad (0 \leq s \leq m-1)$$

(H_2) For any $\mu \in H^{m-3/2}(\Gamma)$, there exists a constant c_1 and a function $P_h\mu \in H_h$ such that

(4.2) $$\|\mu-P_h\mu\|_{-1/2,p} \leq c_1 h^s\|\mu\|_{s-1/2,r} \qquad (0 \leq s \leq m-1)$$

where Π_h: $H'(\Omega_1) \to V_h$ and $P_h : H^{-1/2}(\Gamma) \to H_h$ are projectors.

The approximate space Vh and Hh can be constructed by classical method. Let $T_h = \{T\}$ be a regular triangulation of Ω_1 with maximal diameter at most h and $S_h = \{s\}$ be the sides of Γ with maximal side length at most h. V_h is the set of all continuous functions which are the polynomials in each T, the order of which is less than $m-1$ and H_h is the set of all continuous functions which are the polynomials in each s, the order of which is less than $m-2$. So, V_h and H_h satisfy hypothesis (H_1) and (H_2) (see reference [12]).

Now, let us consider the approximate problem of (2.19):

Find $(u_h, \lambda_h) \in V_h \times H_h$ such that

$$\text{(4.3)} \quad \begin{aligned} &a(u_h, v) + \langle\gamma_0 v, \lambda_h\rangle = (f,v) && \forall\, v \in V_h, \\ &2b(\lambda_h, \mu) + 2\langle G_n u_h, \mu\rangle - \langle\gamma_0 u_h, \mu\rangle = 0 && \forall\, \mu \in H_h. \end{aligned}$$

Due to $V_h \subset H^1(\Omega_1)$ and $H_h \subset H^{-1/2}(\Gamma)$, the approximate problem (4.3) also has a unique solution $(u_h, \lambda_h) \in V_h \times H_h$ by the discussion of section 3.

Subtracting (4.3) from (2.19), we have

$$\text{(4.4)} \quad \begin{aligned} &a(u-h_h,v) + \langle\gamma_0 v,\lambda-\lambda_h\rangle = 0 && \forall\, v \in V_h, \\ &2b(\lambda-\lambda_h,\mu) + 2\langle G_n(u-u_h),\mu\rangle - \langle\gamma_0(u-u_h),\mu\rangle = 0 && \forall\, \mu \in H_h. \end{aligned}$$

Let $\mu_h \in H_h$, from (4.4), we have

$$\text{(4.5)} \quad \begin{aligned} 2b(\mu_h - \lambda_h,\mu) = {} & \langle\gamma_0(u-u_h), \mu\rangle - 2\langle G_n(u-u_h),\mu\rangle \\ & -2b(\lambda-\lambda_h,\mu) \qquad \forall\, \mu \in H_h. \end{aligned}$$

Taking $\mu = \mu_h - \lambda_h \in H_h$ in (4.5), in view of coerciveness of $b(\cdot,\cdot)$, we obtain

$$\begin{aligned} c_1\|\mu_h - \lambda_h\|^2_{1/2,\Gamma} &\le |b(\mu_h-\lambda_h, \mu_h - \lambda_h)| \\ &\le c_2(\|\gamma_0(u-u_h)\|_{1/2,\Gamma} + \|\lambda-\mu_h\|_{-1/2,\Gamma})\|\mu_h-\lambda_h\|_{-1/2,\Gamma}. \end{aligned}$$

Hence

$$(4.6) \qquad \|\mu_h - \lambda_h\|_{-1/2,\Gamma} \le c(\|u-u_h\|_{1,\Omega_1} + \|\lambda-\mu_h\|_{-1/2,\Gamma}).$$

By triangle inequality, final result is obtained

$$(4.7) \qquad \|\lambda-\lambda_h\|_{-1/2,\Gamma} \le c(\|u - u_h\|_{1,\Omega_1} + \|\lambda - \mu_h\|_{-1/2,\Gamma}).$$

On the other hand, from (4.4) and (3.34), we have

$$(4.8) \qquad B(u-u_h, v) = a(u-u_h,v) + \langle\gamma_0 v, F(\gamma_0(u-u_h))\rangle = 0.$$

Taking $v = v_h - u_h$, (4.8) tends to that

$$(4.9) \qquad B(v_h-u_h, v_h-u_h) = - B(u-v_h, V_h - u_h).$$

Using the properties of $B(\cdot,\cdot)$, there exist $C_1 > 0$ and $C_2 > 0$ such that

$$(4.10) \qquad C_1\|v_h - U_h\|^2_{1,\Omega_1} \le |B(v_h - u_h, v_h - u_h)|$$

$$= |B(u-u_h, v_h - u_h)| \le C_2\|u-u_h\|_{1,\Omega_1}\|v_h-u_h\|_{1,\Omega_1}.$$

Then

$$(4.11) \qquad \|v_h - u_h\|_{1,\Omega_1} \le C\|u-v_h\|_{1,\Omega_1}.$$

Thus

$$(4.12) \qquad \|u-u_h\|_{1,\Omega_1} \le \|u-vh\|_{1,\Omega_1} + \|v_h-u_h\|_{1,\Omega_1} \le c\|u-v_h\|_{1,\Omega_1}.$$

Since (4.7) and (4.12) hold for any $\mu_h \varepsilon H_h$ and $v_h \varepsilon V_h$, we have

$$(4.13) \qquad \|u-u_h\|_{1,\Omega_1} + \|\lambda-\lambda_h\|_{-1/2,\Gamma} \le C(\inf_{V_h\varepsilon V_h} \|u-v_h\|_{1,\Omega_1} + \inf_{\mu_h\varepsilon H_h} \|\lambda-\mu_h\|_{-1/2,\Gamma}).$$

By the regularity properties of elliptic equation in subdomain Ω_1 of Ω (see 18), then, $u \varepsilon H^m(\Omega_1)$ if $f \varepsilon H^{m-2}(R^2)$. Hence, recalling the hypothesis (H_1) and (H_2) and (4.13), we have

Theorem 4.1. Let $f \varepsilon H^{m-2}(R^2)$ with supp $f \varepsilon \Omega_1$, V_h and H_h are constructed as above, (u,λ) and (u_h, λ_h) are solutions of (2.19) and (4.3) respectively. Then, the error estimate (u_h, λ_h) and (u,λ) is that

$$\| u - u_h \|_{1,\Omega_1} + \| \lambda - \lambda_h \|_{-1/2,\Gamma} \leq ch^{m-1}(\| u \|_{m,\Omega_1} + \| \lambda \|_{m-3/2,\Gamma}). \tag{4.14}$$

5. The Example of Numerical Computation

The numerical example is given for testing the correctness of theoretical analysis. Let $f(x,y) = \delta(x)\ \delta(y)$, $k^2 = 1-i$, $k = 2^{1/2}\exp(-i\pi/8)$, thus, the solution of (2.1) is

$$u_1 = \frac{-1}{4i} H_0^{(2)}(k_0 r), \quad r = (x^2 + y^2)^{1/2}.$$

In this section, we find the approximate solution u_h by means of the coupling method and compute the error of u_1 and u_h in Ω_1.

Let $\Omega_1 = \{(x,y) \in R^2;\ (x^2+y^2)^{1/2} < R_0\}$ with $R_0 = 5$ and $T_h = \{e\}$ be a regular triangulation of Ω_h and $s_h = \{s_i\}$ be the sides of $\Gamma_h = \partial\Omega_h$, where Ω_h is a approximate domain of Ω, see Figure 2.

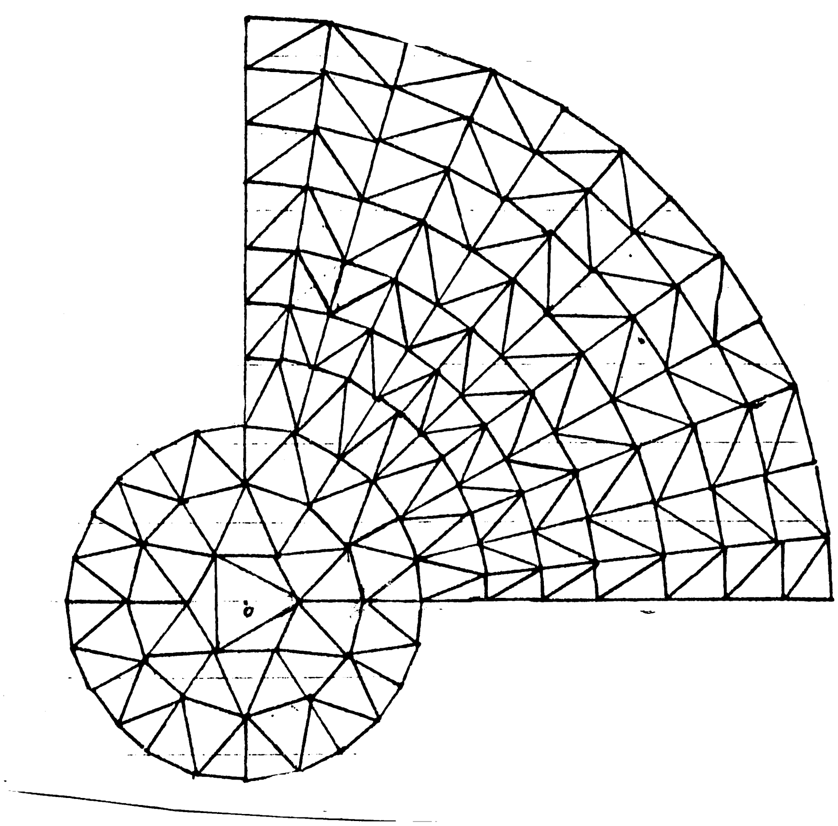

Figure 2.

The approximate space V_h and H_h are

$$V_h = \{v \varepsilon H^1(\Omega_h);\ v|_e \text{ is a linear function}, \forall\, e \varepsilon T_h\},$$

$$H_h = \{\mu \varepsilon L^2(\Gamma_h)\ ;\ \mu|_s \text{ is a constant}, \ \forall\, s \varepsilon S_h\}.$$

So, V_h and H_h satisfy the hypothesis (H_1) and (H_2) with $m = 2$. Thus, the convergence order of the approximate solution is $O(h)$. The base functions $\phi_1,\ldots,\phi_N$ of V_h are linear continuous functions which satisfy

$$\phi_i(p_j) = \delta_{ij},\ i,j = 1,\ldots,N$$

where p_j is the nodes in Ω_h. The base functions $\psi_1,\ldots,\psi_m$ of H_h satisfy

$$\psi_i(x) = \begin{cases} 1 & x \varepsilon S_i \\ 0 & x \overline{\varepsilon} S_i \end{cases} \qquad i = 1,\ldots, m$$

Hankel function can be computed by summing a series. That is

(6.1) $$J_0(z) = \sum_{k=0}^{NO} (-1)^k (z/2)^{2k}/(k!)^2.$$

(6.2) $$J_1(z) = \sum_{k=0}^{NO} (-1)^k\ (z/2)^{2k+1}/(K!(k+1)!),$$

(6.3) $$N_0(z) = \frac{2}{\pi}\left(\ln\left(\frac{z}{2}\right) + \nu\right)J_0(g) - \frac{2}{\pi}\sum_{k=1}^{NO}(-1)^k/(k!)^2\left(\frac{z}{2}\right)^{2k}\sum_{m=1}^{k}\frac{1}{m}.$$

(6.4) $$N_1(z) = \frac{2}{\pi}\left(\ln\left(\frac{z}{2}\right) + \nu\right)J_1(z) + \frac{2}{\pi z}$$
$$+ \frac{2}{\pi}\sum_{k=1}^{NO}(-1)^k\left(\frac{z}{2}\right)^{2k-1}\left(\sum_{m=1}^{k}\frac{1}{m} - \frac{1}{2k}\right)/(k!(k+1)!),$$

(6.5) $$H_j^{(2)}(z) = J_j(z) - i\ N_j(z),$$

where N_0 is a integer number and $J_j(z)$, $N_j(z)$, $H^{(2)}(z)$ are Bessel function, Neumann function, Hankel function of j-order, $j = 0,1$, where the Euler's constant ν is that

$$\nu = 0.5772156649$$

Further, the normal derivative of Hankel function is

(6.6) $$\frac{\partial}{\partial n} H_0^{(2)}(k_0\rho) = -k_0 H_1^{(2)}(k_0\rho)\frac{\partial\rho}{\partial n} .$$

In the approximate problem (4.3), let $u_h = \sum_{i=1}^{m} u_i \ \phi_i$ and $\lambda_h = \sum_{i=1}^{m} \lambda_i \psi_i$ and $u_h \ |\Gamma_h = \sum u_i \phi_i |\Gamma_h$ and $v = \phi_j$ and $\mu = \psi_j$. Thus, we have

(6.7) $$\sum_{i=1}^{N} U_i A\ (\phi_i,\ \phi_j) + \sum_{i=1}^{M} \lambda_i \ \langle\phi_j,\ \psi_i\rangle = (f, \phi_j) \ j = 1,\ldots,N.$$

(6.8) $$\sum_{i=1}^{N} \lambda_i 2\overline{b}(\psi_i, \psi_j) + \sum_{i=1}^{M} U_i \ 2 \ \overline{\langle G_n \phi_i,\ \psi_j\rangle} - \sum_{i=1}^{M} U_i \ \langle\phi_i, \psi_j\rangle = 0.$$

Let

$$F = ((f, \phi_i))_{N\times 1},\ U = (U_i)_{N\times 1},\ L = (\lambda_i)_{m\times 1}$$

$$B = (2b(\psi_i,\ \psi_j))_{m\times m},\ \ C = (\langle\phi_i, \psi_j\rangle)_{m\times m},$$

$$D = (2\ \langle G_n \phi_j, \psi_i\rangle - \langle\phi_j,\ \psi_i\rangle)_{m\times m},$$

$$A = (a(\phi_i,\ \phi_j))_{N\times N}.$$

From (6.7) and (6.8), we obtain algebraic equation system:

(6.9) $$AU + C^{*}L = F,$$
$$D^{*}U + BL = 0 ,$$

where $C^{*} = (C,0)^{T}$ is $N\times m$ matrix and D^{*} is $m\times N$ matrix.

The equation system (6.9) can be solved by elimination method. Recalling the define of A, we know that A is a matrix of symmetry, band type, positive-defined. From the first equation system of (6.9), we obtain

(6.10) $$U = A^{-1}F - A^{-1}C^{*}L.$$

From (6.10) and

$$D^{*}U + BL = 0,$$

we obtain

$$(D^{*}A^{-1}C^{*} - B)\ L = D^{*}A^{-1}F. \tag{6.11}$$

Then L can be solved by (6.11) when $D^{*}A^{-1}C^{*}$ and $D^{*}A^{-1}F$ are computed. To this end, we compute $D^{*}A^{-1}C^{*}$ and $D^{*}A^{-1}F$ as follows.

Assume $X_1, \ldots, X_n, X_{n+1}$ be n+1 N-dimension column vector, we solve the equation system:

$$A(X_1, \ldots, X_n, X_{n+1}) = (C^{*}, F) \tag{6.12}$$

Hence, we have

$$D^{*}A^{-1}C^{*} = D^{*}(X_1, \ldots, X_n), \tag{6.13}$$

$$D^{*}A^{-1}F = D^{*}X_{n+1} \tag{6.14}$$

Next, computing the matrix

$$B1 = D^{*}A^{-1}C^{*} - B, \tag{6.22}$$

We can obtain L by means of the subroutine GS1 of the reference [21] for (6.11). From (6.10), we obtain

$$U = X_{n+1} - (X_1, \ldots, X_n)\ L. \tag{6.23}$$

Below we shall describe the approximate solution and the analytic solution by FIG1, FIG2, FIG3 and FIG4.

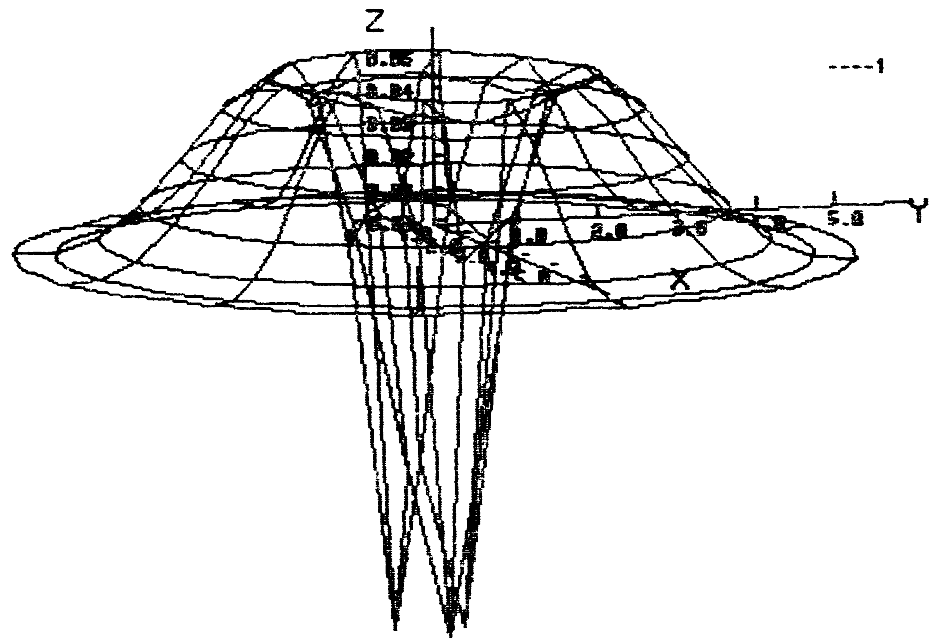

FIG 1 : REAL PART RU OF APPROXIMATE SOLUTION

TYPE 88 TO DRAW ANOTHER SURFACEI

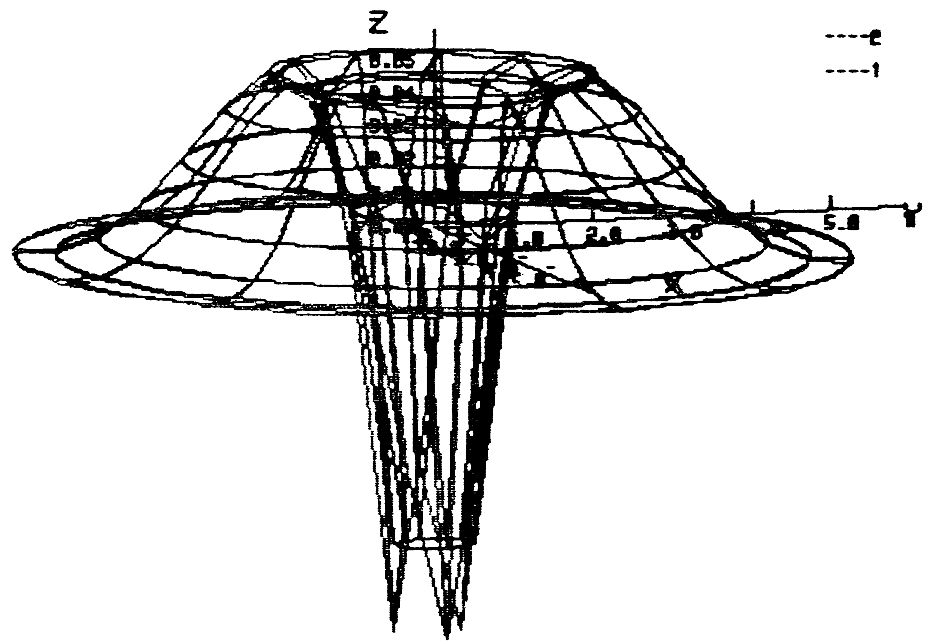

FIG 2 : REAL PART RUM OF ANALYTIC SOLUTION

FIG 1 : REAL PART RU OF APPROXIMATE SOLUTION

TYPE 88 TO DRAW ANOTHER SURFACEI_

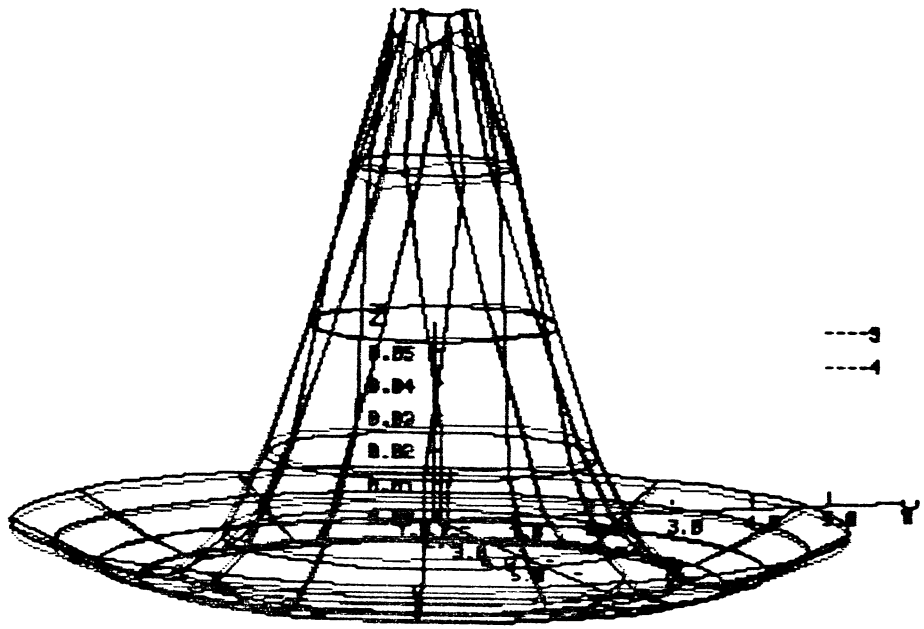

FIG 4 : IMAG PART AUM OF ANALYTIC SOLUTION
FIG 3 : IMAG PART AU OF APPROXIMATE SOLUTION

TYPE 88 TO DRAW ANOTHER SURFACEI

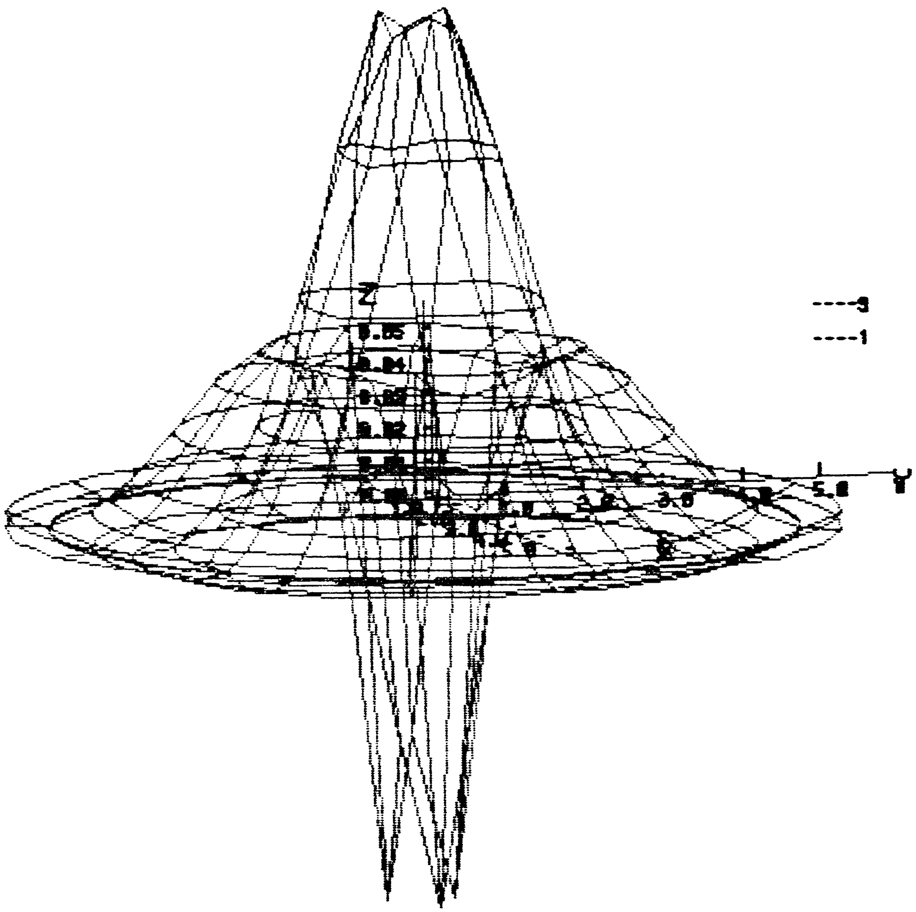

FIG 3 : IMAG PART AU OF APPROXIMATE SOLUTION

FIG 1 : REAL PART RU OF APPROXIMATE SOLUTION

TYPE 88 TO DRAW ANOTHER SURFACEI

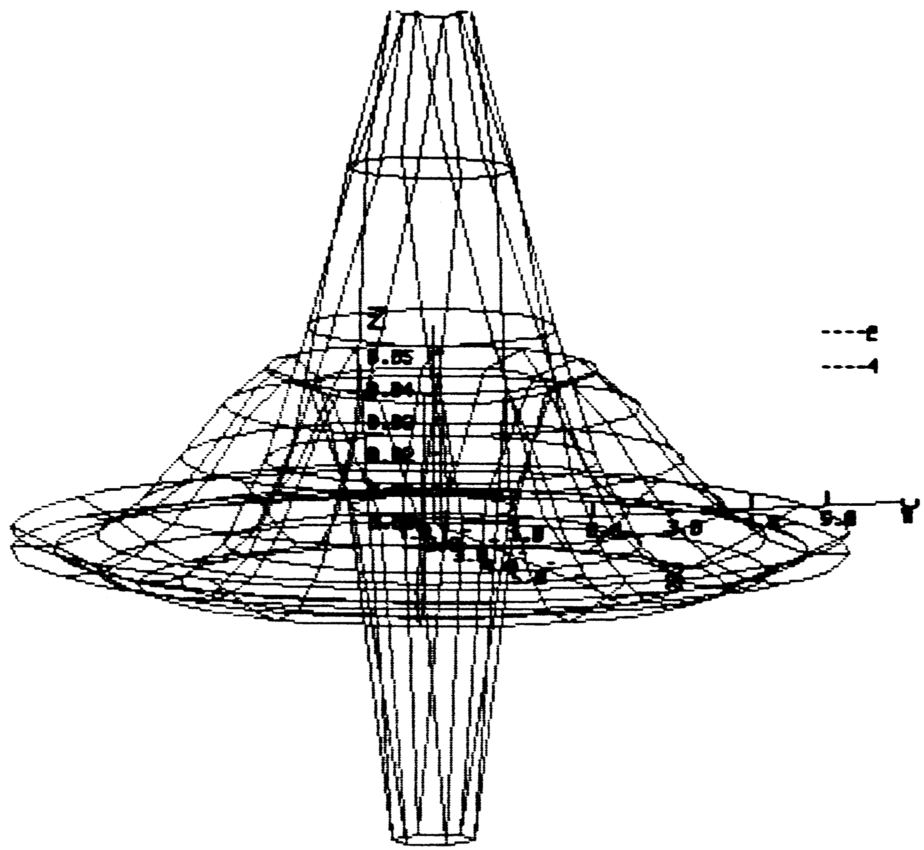

FIG 4 : IMAG PART AU1 OF ANALYTIC SOLUTION
FIG 2 : REAL PART RU1 OF ANALYTIC SOLUTION

TYPE 88 TO DRAW ANOTHER SURFACEI

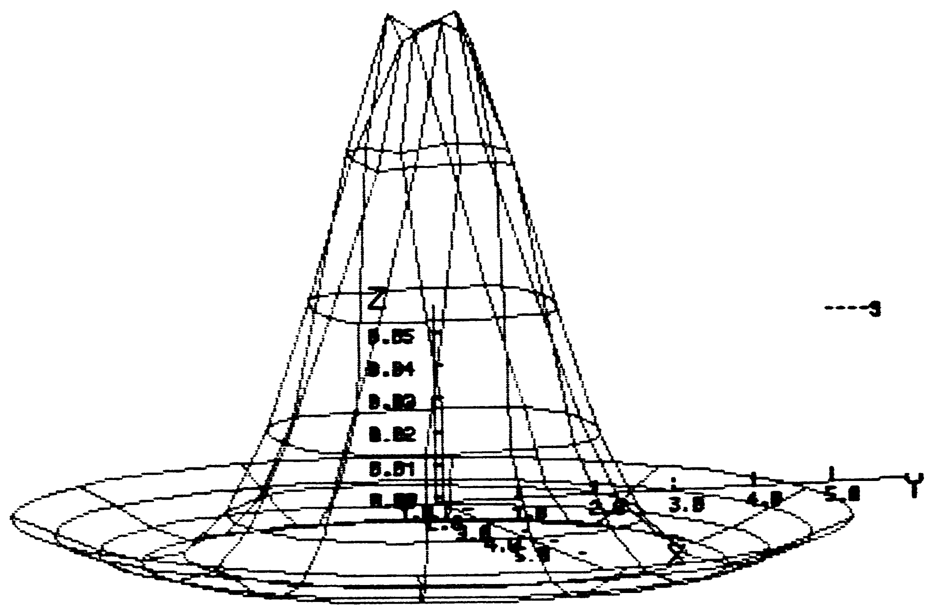

FIG 3 : IMAG PART AU OF APPROXIMATE SOLUTION

TYPE 88 TO DRAW ANOTHER SURFACEI_

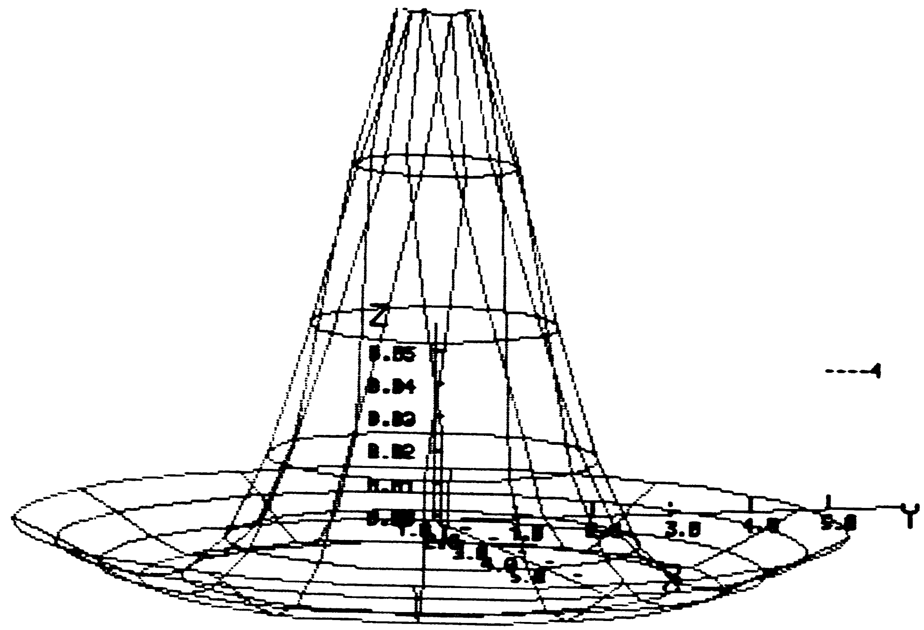

FIG 4 : IMAG PART AU1 OF ANALYTIC SOLUTION

TYPE 88 TO DRAW ANOTHER SURFACEI

The coupling method presented in this paper is correction in theoretical and can be computed in practice. The advantages of this method are little work amount and optimal error estimal and general application. The method can be generalized in the case of three-dimension.

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