

**NUMERICAL METHODS FOR THE REGULARIZATION  
OF DESCRIPTOR SYSTEMS BY OUTPUT FEEDBACK**

By

**Angelika Bunse-Gerstner**

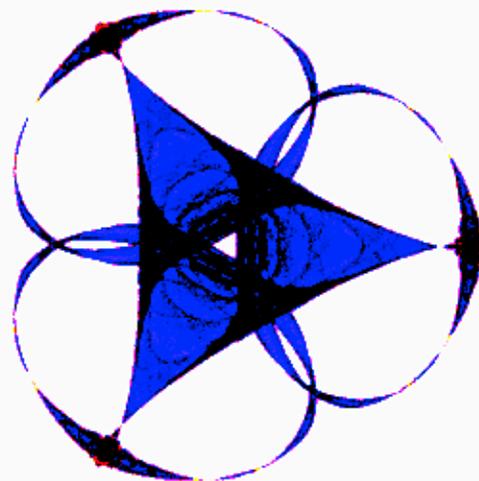
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# Numerical Methods for the Regularization of Descriptor Systems by Output Feedback

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### Abstract

Conditions are given under which a descriptor, or generalized state-space, system can be regularized by *output* feedback. It is shown that under these conditions proportional and derivative output feedback controls can be constructed such that the closed loop system is regular and has index at most 1. This property ensures the solvability of the resulting system of dynamic-algebraic equations. A canonical form is given that allows the system properties as well as the feedback to be determined. The construction procedures used to establish the theory are based only on orthogonal matrix decompositions and can therefore be implemented in a numerically stable way. A computational algorithm for improving the 'conditioning' of the regularized closed loop system is derived.

# 1 Introduction

We examine linear time-invariant (continuous or discrete) dynamical systems of the form

$$E\dot{x} := E dx/dt = Ax(t) + Bu(t), \quad Ex(t_0) = Ex_0 \quad (1)$$

$$y(t) = Cx(t), \quad (2)$$

or

$$Ex_{k+1} = Ax_k + Bu_k, \quad Ex_0 \text{ given} \quad (3)$$

$$y_k = Cx_k, \quad (4)$$

where  $E, A \in \mathbb{C}^{n,n}$ ,  $B \in \mathbb{C}^{n,m}$ ,  $C \in \mathbb{C}^{p,n}$ . Here  $x(t)$  or  $x_k \in \mathbb{C}^n$  is the state,  $y(t)$  or  $y_k \in \mathbb{C}^p$  is the output, and  $u(t)$  or  $u_k \in \mathbb{C}^m$  is the input or control of the system. Such systems are called *descriptor* or *generalized state-space* systems. In contrast to *standard* systems, where  $E = I$ , the response of a descriptor system can have a complicated structure and can even have impulsive modes [26,12,13].

The behaviour of a descriptor system depends critically upon the properties of the matrix pencil

$$\alpha E - \beta A. \quad (5)$$

The pencil (5) and the corresponding system (1)-(2) or (3)-(4) are said to be *regular* if

$$\det(\alpha E - \beta A) \neq 0 \text{ for some } (\alpha, \beta) \in \mathbb{C}^2. \quad (6)$$

Regularity of the system guarantees the existence and uniqueness of (classical) solutions to (1) and (3) [4,28]. Most known results on the behaviour of descriptor control systems depend explicitly on the assumption of regularity (see, for example, [5,6,7,16,17,18,19,20,22,23,24,26,27,28,29]). This assumption is unnecessarily strong, however, and it rules out the analysis of a number of practical physical systems (see [20]). Many systems which are not regular can, moreover, be *regularized* by proportional and/or derivative feedback. Conversely, systems which are regular can easily be transformed by linear feedback into closed loop systems which are *not* regular. It is important, therefore, to establish conditions which ensure the regularity of systems under feedback and to develop numerically reliable techniques for constructing regular closed loop systems.

The pencil (5) and the corresponding system (1)-(2) or (3)-(4) are said to have *index at most one* if the dimension of the largest nilpotent block

(which corresponds to an *infinite* pole) in the Kronecker canonical form (KCF) of the pencil  $\alpha E - \beta A$  is less than or equal to 1 (see [11,25] or [1]). A regular system which has index less than or equal to one can be transformed and separated into a purely dynamical and a purely algebraic part. The algebraic variables can be eliminated to give a *standard* system of (possibly) reduced order. Higher index descriptor systems cannot be reduced to standard systems, and impulses can arise in the response if the control is not sufficiently smooth. The system can even lose causality (see [26] or [1,12,13]). In practice, therefore, it is desirable to find feedback controls that ensure that the closed loop system is not only regular, but also of index at most one.

Early results on regularization of descriptor systems using *proportional state* feedback are presented in [10] and [15]. Algebraic conditions are given that ensure that a feedback gain matrix exists such that the closed loop system pencil is regular and of index at most one. Regularity of the open loop pencil is *not* assumed. Under these conditions it is shown, furthermore, that precisely  $r = \text{rank}(E)$  finite poles can be assigned arbitrarily to the system. Reliable numerical methods for assigning the entire eigenstructure to give a *robust* closed loop system are also given in [15].

In [1,2] and [3] we have extended the results of [10] and [15] to regularization by derivative and proportional-plus-derivative *state* feedback. Algebraic conditions are presented that ensure that the system can be transformed into a regular system of index at most one. If the open loop system is regular, these conditions correspond to *strong controllability*, that is, to controllability of both the finite and infinite poles (impulse controllability) of the system. Regularity of the open loop system is *not* needed, however, to establish the results. The same conclusions are expressed in geometric terms in [20]. In [3] numerically reliable algorithms, based only on unitary matrix transformations, are also given for constructing the required feedback matrices.

In [3], furthermore, we identify and exploit the additional freedom provided by derivative state feedback in order to ensure the optimal 'conditioning' of the closed loop system. Here optimal conditioning is achieved via singular value assignment. If a regular system of index at most one is 'well-conditioned', then the system can be reduced to a *standard* system by a computationally reliable procedure. Optimally conditioned closed loop systems also have improved performance characteristics. In [21], for example, derivative feedback is used to obtain a 'well-conditioned' discrete-time observer which gives improved state estimates.

In this paper we establish conditions for regularization by *output* feedback. We give preliminary results in [1] and [2]. The results presented here indicate precisely what can be achieved by proportional and/or derivative output feedback and how closed loop systems with improved conditioning can be obtained. A canonical form that displays the controllability/observability properties of the open loop system is constructed using numerically stable matrix transformations. No assumption of open loop regularity is required. Algebraic conditions are derived that ensure that a feedback can be selected to give a closed loop system that is regular, is strongly controllable and observable, and has index at most one. A stable computational procedure is established for constructing the required output feedback so that the closed loop system is, in addition, 'well-conditioned.'

In the next section of the paper conditions for regularizability of the systems (1)–(2) and (3)–(4) are given. In Section 3 we derive the 'canonical form' for the control system using unitary matrix transformations, and in Section 4 we establish the main results. Optimal conditioning is discussed in Section 5, numerical results are presented in Section 6 and concluding remarks are given in Section 7.

## 2 Regularizability Conditions

We denote descriptor systems (1) and (3) by the triple  $(E, A, B)$  and consider the following conditions.

$$\begin{aligned}
 \text{C0} &: \text{rank}[\alpha E - \beta A, B] = n, \quad \forall (\alpha, \beta) \in \mathbb{C}^2 \setminus \{(0, 0)\}; \\
 \text{C1} &: \text{rank}[\lambda E - A, B] = n, \quad \forall \lambda \in \mathbb{C}; \\
 \text{C2} &: \text{rank}[E, AS_\infty, B] = n, \quad \text{where the columns of } S_\infty \text{ span } \mathcal{N}(E), \\
 &\quad \text{the nullspace of } E.
 \end{aligned} \tag{7}$$

For systems that are *regular*, these conditions characterize the controllability of the system. If  $\alpha E - \beta A$  is a regular pencil, then the triple  $(E, A, B)$  and the corresponding descriptor system are said to be **completely controllable** (C-controllable) if and only if Condition C0 holds. A descriptor system satisfies Condition C0, i.e. is completely controllable, only if

$$\text{rank}[E, B] = n \tag{8}$$

Complete controllability ensures that for any given initial and final states  $x_0, x_f \in \mathbb{R}^n$  of the system, there exists an admissible control that transfers the system from  $x_0$  to  $x_f$  in finite time [28]. Hence, descriptor systems that

are completely controllable can be expected to have similar properties to standard systems.

If  $\alpha E - \beta A$  is a regular pencil, then the triple  $(E, A, B)$  and the corresponding descriptor system are said to be *strongly controllable* (S-controllable) if and only if Conditions **C1** and **C2** hold. Strong controllability ensures that the system is completely controllable in a subspace of  $\mathbb{R}^n$  of dimension equal to  $\text{rank}[E, B]$ .

We remark that C-controllability implies S-controllability. Clearly Condition **C1** follows from Condition **C0** for  $\beta \neq 0$  and  $\lambda = \alpha/\beta$ . Condition **C2** follows from (8), but is weaker. In the literature, regular systems which satisfy Condition **C2** are often described as 'controllable at infinity' or 'impulse controllable' [6,15,26]. For these systems 'impulsive modes' can be excluded. A descriptor system which has a regular pencil of index less than or equal to 1 is always controllable at infinity.

Observability conditions for descriptor systems can be defined as the dual of the controllability conditions (7). The dual conditions for systems (1)-(2) and (3)-(4), denoted by the the triple  $(E, A, C)$ , are given by

$$\begin{aligned} \text{O0} : \text{rank} \begin{bmatrix} \alpha E - \beta A \\ C \end{bmatrix} &= n, \forall (\alpha, \beta) \in \mathbb{C}^2 \setminus \{(0, 0)\}; \\ \text{O1} : \text{rank} \begin{bmatrix} \lambda E - A \\ C \end{bmatrix} &= n, \forall \lambda \in \mathbb{C}; \\ \text{O2} : \text{rank} \begin{bmatrix} E \\ T_\infty^H A, C \end{bmatrix} &= n, \end{aligned} \quad (9)$$

where the columns of  $T_\infty$  span  $\mathcal{N}(E^H)$ .

If  $\alpha E - \beta A$  is a regular pencil, then the triple  $(E, A, C)$  and the corresponding descriptor system are said to be **completely observable** (C-observable) if and only if Condition **O0** holds. Analogously to (8), a descriptor system satisfies Condition **O0**, i.e. is completely observable, only if

$$\text{rank} \begin{bmatrix} E \\ C \end{bmatrix} = n. \quad (10)$$

If  $\alpha E - \beta A$  is a regular pencil, then the triple  $(E, A, C)$  and the corresponding descriptor system are said to be *strongly observable* (S-observable) if and only if Conditions **O1** and **O2** hold. Clearly C-observability implies S-observability.

The controllability and observability conditions are preserved under certain transformations of the system. Specifically, Conditions **C0**, **C1**, **C2**,

**O0**, **O1**, **O2** are all preserved under non-singular 'equivalence' transformations of the system and under proportional state and output feedback. With the exception of conditions **C2**, **O2**, these same conditions are also preserved under derivative feedback [1]. In particular, if  $(E, A, B)$  satisfies the condition **C0** or **C1** or **C2**, then for any non-singular  $P$  and  $Q \in \mathbb{R}^{n,n}$  and for any  $F \in \mathbb{R}^{m,n}$ , the system  $(\tilde{E}, \tilde{A}, \tilde{B})$ , where

$$\tilde{E} = PEQ, \quad \tilde{A} = PAQ, \quad \tilde{B} = PB \quad (11)$$

or

$$\tilde{E} = E, \quad \tilde{A} = A + BFC, \quad \tilde{B} = B \quad (12)$$

also satisfies these conditions. Furthermore, for any matrix  $G \in \mathbb{R}^{m,n}$ , the system  $(\tilde{E}, \tilde{A}, \tilde{B})$ , where

$$\tilde{E} = E + BGC, \quad \tilde{A} = A, \quad \tilde{B} = B \quad (13)$$

also satisfies these conditions with the exception of **C2**.

By duality the analogous results hold if  $(E, A, C)$  satisfies the corresponding conditions **O0** or **O1** or **O2**.

In [1] an example is given demonstrating that Condition **C2** (and analogously Condition **O2**) is not necessarily preserved under derivative feedback. If derivative feedback is used to change the system dynamics, it is therefore necessary to be careful not to lose controllability or observability at infinity.

In Section 4 we investigate the use of proportional and derivative output feedback to *regularize* a descriptor system. It is shown that Conditions **C2** and **O2** are sufficient to achieve a closed loop system that is regular and of index at most 1; the resulting system is, therefore, always controllable at infinity. It is shown, furthermore, that if Conditions **C1** and **O1** also hold, then a strongly controllable and strongly observable closed loop system of index at most 1 can be attained. Finally it is proved that a completely controllable and observable *standard* system can be achieved if and only if Conditions **C0** and **O0** hold. *Regularity of the original open loop system is not needed to establish these results.*

### 3 A Canonical Form

In this section we present our first main theorem, which gives a canonical form for systems (1)–(2) or (3)–(4) that can be computed using only unitary equivalence transformations. This canonical form is used to establish the main results in Section 4.

**Theorem 1** If  $E, A \in \mathbb{C}^{n,n}$ ,  $B \in \mathbb{C}^{m,n}$ ,  $C \in \mathbb{C}^{p,n}$  then there exist unitary matrices  $U, V \in \mathbb{C}^{n,n}$ ,  $W \in \mathbb{C}^{m,m}$ ,  $Y \in \mathbb{C}^{p,p}$  such that

$$\begin{aligned} U^H E V &= \begin{bmatrix} \Sigma_E & 0 \\ 0 & 0 \end{bmatrix} \begin{matrix} t_1 \\ n - t_1 \end{matrix}, & U^H B W &= \begin{bmatrix} B_{11} & B_{12} & 0 \\ \hat{B}_{21} & 0 & 0 \end{bmatrix} \begin{matrix} t_1 \\ n - t_1 \end{matrix}, \\ Y^H C V &= \begin{bmatrix} C_{11} & \hat{C}_{12} \\ C_{21} & 0 \\ 0 & 0 \end{bmatrix} \begin{matrix} \ell_1 \\ \ell_2 \\ \ell_3 \end{matrix}, & U^H A V &= \begin{bmatrix} A_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix} \begin{matrix} t_1 \\ n - t_1 \end{matrix}, \end{aligned} \quad (14)$$

where

$$\begin{aligned} \hat{A}_{22} &= \begin{bmatrix} A_{22} & A_{23} & A_{24} & 0 & 0 \\ A_{32} & A_{33} & A_{34} & \Sigma_{35} & 0 \\ A_{42} & A_{43} & \Sigma_{44} & 0 & 0 \\ 0 & \Sigma_{53} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} t_2 \\ t_3 \\ t_4 \\ t_5 \\ t_6 \end{matrix}, & \hat{B}_{21} &= \begin{bmatrix} B_{21} \\ B_{31} \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{matrix} t_2 \\ t_3 \\ t_4 \\ t_5 \\ t_6 \end{matrix}, \\ \hat{C}_{12} &= \begin{bmatrix} C_{12} & C_{13} & 0 & 0 & 0 \end{bmatrix} \ell_1, & & \\ \hat{A}_{21} &= \begin{bmatrix} A_{21} \\ A_{31} \\ A_{41} \\ A_{51} \\ A_{61} \end{bmatrix} \begin{matrix} t_2 \\ t_3 \\ t_4 \\ t_5 \\ t_6 \end{matrix}, & \hat{A}_{12} &= \begin{bmatrix} A_{12} & A_{13} & A_{14} & A_{15} & A_{16} \end{bmatrix} t_1. \end{aligned} \quad (15)$$

Here  $A_{11}, \Sigma_E$  are  $t_1 \times t_1$  matrices, the row dimensions of the matrices are as depicted, and the column dimensions are  $[t_1, s_2, t_5, t_4, t_3, s_6]$  for  $U^H E V$ ,  $U^H A V$ ,  $Y^H C V$  and  $[k_1, k_2, k_3]$  for  $U^H B W$ .

The matrices  $\Sigma_E, \Sigma_{35}, \Sigma_{44}, \Sigma_{53}$  are non-singular diagonal matrices,  $B_{12}$  has full column rank,  $C_{21}$  has full row rank and the matrices

$$\begin{bmatrix} B_{21} \\ B_{31} \end{bmatrix} \in \mathbb{C}^{k_1, k_1}, \quad [C_{12}, C_{13}] \in \mathbb{C}^{\ell_1, \ell_1}$$

with  $k_1 = t_2 + t_3$  and  $\ell_1 = s_2 + t_5$  are non-singular.

*Proof.* The proof is given by construction via Algorithm 1 in Appendix A.  $\square$

Note that Algorithm 1 uses only unitary transformations and hence gives a numerically stable procedure for computing this form. The dimensions  $t_1, t_2, t_3, t_4, t_5, t_6, s_2, s_6, k_1, k_2, k_3, \ell_1, \ell_2, \ell_3$  are determined via rank decisions based on singular value decompositions [14]. This is certainly the

most stable way to perform these decisions, but the results may still be misleading, since by arbitrarily small perturbations a matrix can change its rank drastically. The usual procedure for determining the rank of a diagonal matrix  $\Sigma = \text{diag}(\sigma_1, \dots, \sigma_n)$  with  $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_n \geq 0$  is to neglect all those  $\sigma_i$  that are smaller than  $\sigma_1 \text{eps}$ , where eps is the machine precision. Precautions have to be taken, however, if the first neglected and last nonneglected  $\sigma_i$  are close together, (see [8]).

The canonical form nevertheless gives us the ability to check numerically the important properties of the system (1)-(2) or (3)-(4). (Most of the following results are given in [1,2] but with different proofs.) First, we establish from the canonical form equivalent conditions for the system to be regular and of index at most 1.

**Corollary 2** *Let  $E, A$  be given as in the canonical form (14) with partitioning:*

$$E = \begin{bmatrix} \Sigma_E & 0 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} A_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix}. \quad (16)$$

*Then, the following are equivalent:*

- (i)  $\alpha E - \beta A$  is regular and has index at most 1;
- (ii)  $\hat{A}_{22}$  is non-singular;
- (iii)  $s_6 = t_6 = 0$ , i.e. the last block rows of  $\hat{A}_{21}, \hat{A}_{22}$  and the last block columns of  $\hat{A}_{12}, \hat{A}_{22}$  are void, (so that  $t_2 = s_2$ ) and  $A_{22}$  is non-singular;
- (iv)  $\text{rank} [E, AS_\infty] = n$ ;
- (v)  $\text{rank} \begin{bmatrix} E \\ T_\infty^H A \end{bmatrix} = n$ .

*Proof.* The equivalence of (ii), (iii), (iv) and (v) follows directly from the form of  $\hat{A}_{22}$  and the fact that

$$[E, AS_\infty] = \begin{bmatrix} \Sigma_E & 0 & \hat{A}_{12} \\ 0 & 0 & \hat{A}_{22} \end{bmatrix}, \quad \begin{bmatrix} E \\ T_\infty^H A \end{bmatrix} = \begin{bmatrix} \Sigma_E & 0 \\ 0 & 0 \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix}. \quad (17)$$

We finish the proof by showing that  $\hat{A}_{22}$  non-singular implies (i) and that (i) implies (iv).

If  $\hat{A}_{22}$  is non-singular, then  $\alpha E - \beta A$  is equivalent to

$$\alpha \begin{bmatrix} \Sigma_E & 0 \\ 0 & 0 \end{bmatrix} - \beta \begin{bmatrix} \hat{A}_{11} - \hat{A}_{12} \hat{A}_{22}^{-1} \hat{A}_{21} & 0 \\ 0 & I \end{bmatrix}, \quad (18)$$

which is clearly regular and of index at most 1.

Conversely assume (i). Then  $\alpha E - \beta A$  has the KCF

$$\alpha \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} - \beta \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix}, \quad (19)$$

and it directly follows that  $\text{rank}[E, AS_\infty] = n$ .  $\square$

If  $E, A$  are given in the form (16) with

$$B = \begin{bmatrix} \hat{B}_1 \\ \hat{B}_2 \end{bmatrix} := \begin{bmatrix} B_{11} & B_{12} & 0 \\ \hat{B}_{21} & 0 & 0 \end{bmatrix}, \quad C = [\hat{C}_1, \hat{C}_2] := \begin{bmatrix} C_{11} & \hat{C}_{12} \\ C_{21} & 0 \\ 0 & 0 \end{bmatrix} \quad (20)$$

partitioned analogously and if  $\alpha E - \beta A$  is regular and of index at most 1, then we see immediately that the system (1)-(2) can be transformed to an equivalent standard system of reduced order, given by

$$\begin{aligned} \dot{x}_1(t) &= \Sigma_E^{-1}(\hat{A}_{11} - \hat{A}_{12}\hat{A}_{22}^{-1}\hat{A}_{21})x_1(t) + \Sigma_E^{-1}(\hat{B}_1 - \hat{A}_{12}\hat{A}_{22}^{-1}\hat{B}_2)u(t), \\ y(t) &= (\hat{C}_1 - \hat{C}_2\hat{A}_{22}^{-1}\hat{A}_{21})x_1(t) - \hat{C}_2\hat{A}_{22}^{-1}\hat{B}_2u(t), \end{aligned} \quad (21)$$

where  $x_1(t_0) = x_{10}$ . Observe that in this transformed version the output and the input are now explicitly related, which was not the case in the original problem (1)-(2). Corresponding results hold for systems (3)-(4).

From (21) we can see directly that the reduction to standard form can be carried out in a numerically reliable way only if  $\Sigma_E, \hat{A}_{22}$  are well-conditioned with respect to inversion. In Section 5, we discuss the use of proportional and derivative output feedback to improve the conditioning of these matrices.

From the canonical form, we can now also establish conditions under which a descriptor system can be *regularized*. It can be seen from (14)-(15) that a regular closed loop system of index at most 1 can be attained only if the last zero rows and columns of  $\hat{A}_{22}$  are void. This must be the case if Conditions C2 and O2 hold. We have the following Corollary

**Corollary 3** *Let  $E, A, B, C$  be given as in the canonical form (14).*

(i) *Condition C2 holds, that is,  $\text{rank}[E, AS_\infty, B] = n$  if and only if  $t_6 = 0$ , i.e. the last block rows of  $\hat{A}_{21}, \hat{A}_{22}$  are void;*

(ii) *Condition O2 holds, that is,  $\text{rank} \begin{bmatrix} E \\ T_\infty^H A \\ C \end{bmatrix} = n$  if and only if  $s_6 = 0$ ,*

*i.e. the last block columns of  $\hat{A}_{12}, \hat{A}_{22}$  are void.*

*Proof.* Clear from canonical form (14).  $\square$

The stronger conditions **C0** and **O0** hold only if  $\text{rank } [E, B] = n$  and  $\text{rank} \begin{bmatrix} E \\ C \end{bmatrix} = n$ . Equivalent conditions for these properties to hold can also be derived from the canonical form. We have the following :

**Corollary 4** Let  $E, A, B, C$  be given as in the canonical form (14).

(i)  $\text{rank}[E, B] = n$  if and only if  $t_4 = t_5 = t_6 = 0$ , i.e.

$$\hat{A}_{22} = \begin{bmatrix} A_{22} & 0 & 0 \\ A_{32} & \Sigma_{35} & 0 \end{bmatrix};$$

(ii)  $\text{rank} \begin{bmatrix} E \\ C \end{bmatrix} = n$  if and only if  $t_4 = t_3 = s_6 = 0$ , i.e.

$$\hat{A}_{22} = \begin{bmatrix} A_{22} & A_{23} \\ 0 & \Sigma_{53} \\ 0 & 0 \end{bmatrix}.$$

*Proof.* Clear from canonical form (14).  $\square$

In theory it is frequently assumed that  $B, C$  have full rank. From the canonical form (14) it follows immediately that  $\text{rank } B = m$  if and only if  $k_3 = 0$  and  $\text{rank } C = p$  if and only if  $l_3 = 0$ , i.e. the column of zero blocks in  $B$  and the row of zero blocks in  $C$  are void. In general we can determine the rank deficiency of  $B$  and  $C$  by Algorithm 1 and eliminate them from the system by introducing new input and output vectors.

We assume, therefore, in the following that  $B, C$  are full rank, and  $k_3 = l_3 = 0$ .

In this section we have shown how many of the important indices that determine the properties of system (1)-(2) or (3)-(4) can be obtained directly from the canonical form (14).

## 4 Main Theorem

We now come to our main Theorem, which describes the properties that can be achieved by proportional and derivative output feedback:

**Theorem 5** Let  $E, A, B, C$  be as in (1)-(2) or (3)-(4) and let  $t_1, t_2, s_2$  be as in the canonical form (14). If Conditions **C2** and **O2** hold, that is,

if  $\text{rank}[E, AS_\infty, B] = n$  and  $\text{rank} \begin{bmatrix} E \\ T_\infty^H A \\ C \end{bmatrix} = n$ , then for all integers  $s$ ,  $0 \leq s \leq t_2 = s_2$ , there exist  $F, G \in \mathbb{C}^{m,p}$  such that  $\alpha(E+BGC) - \beta(A+BFC)$  is regular, has index at most 1 and  $\text{rank}(E+BGC) = t_1 + s$ .

(i) If  $s = t_2$ , then this is achieved by derivative feedback alone with  $F = 0$ .

(ii) If  $s = 0$ , then this is achieved by proportional feedback alone with  $G = 0$ . In this case the converse also holds, that is, if there exists  $F \in \mathbb{C}^{m,p}$  such that  $\alpha E - \beta(A+BFC)$  is regular and has index at most 1, then conditions C2 and O2 must hold.

*Proof.* We may assume that  $E, A, B, C$  are already in the canonical form (14) with  $k_3 = 0 = \ell_3$  and that  $s_6 = t_6 = 0$ ,  $s_2 = t_2$ . Let

$$G = \begin{bmatrix} G_{11} & 0 \\ 0 & 0 \end{bmatrix}, F = \begin{bmatrix} F_{11} & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{C}^{m,p} \quad (22)$$

with

$$G_{11} = \begin{bmatrix} B_{21} \\ B_{31} \end{bmatrix}^{-1} \begin{bmatrix} I_s & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} C_{12} & C_{13} \end{bmatrix}^{-1} \in \mathbb{C}^{t_2+t_3, s_2+t_5}, \quad (23)$$

$$F_{11} = \begin{bmatrix} B_{21} \\ B_{31} \end{bmatrix}^{-1} \begin{bmatrix} \mathcal{O}_s & 0 & 0 \\ 0 & \Phi & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} C_{12} & C_{13} \end{bmatrix}^{-1} \in \mathbb{C}^{t_2+t_3, s_2+t_5}, \quad (24)$$

and

$$\Phi = I_{t_2-s} - \begin{bmatrix} 0 & I_{t_2-s} \end{bmatrix} A_{22} \begin{bmatrix} 0 \\ I_{t_2-s} \end{bmatrix}. \quad (25)$$

(Here  $\mathcal{O}_s$  is the  $s \times s$  zero matrix.) Then we have that

$$BGC = \begin{bmatrix} \Delta_{11} & \Delta_{12} & 0 \\ \Delta_{21} & \Delta_{22} & 0 \\ 0 & 0 & \mathcal{O}_{n-t_1-s} \end{bmatrix} \in \mathbb{C}^{n,n} \quad (26)$$

with

$$\begin{aligned} \Delta_{11} &= B_{11} G_{11} C_{11} \in \mathbb{C}^{t_1, t_1}, \quad \Delta_{12} = B_{11} \begin{bmatrix} B_{21} \\ B_{31} \end{bmatrix}^{-1} \begin{bmatrix} I_s \\ 0 \end{bmatrix} \in \mathbb{C}^{t_1, s}, \\ \Delta_{21} &= \begin{bmatrix} I_s & 0 \end{bmatrix} \begin{bmatrix} C_{12} & C_{13} \end{bmatrix}^{-1} C_{11} \in \mathbb{C}^{s, t_1}, \quad \Delta_{22} = I_s \end{aligned} \quad (27)$$

and

$$BFC = \begin{bmatrix} \Phi_{11} & 0 & \Phi_{13} & 0 \\ 0 & \mathcal{O}_s & 0 & 0 \\ \Phi_{31} & 0 & \Phi_{33} & 0 \\ 0 & 0 & 0 & \mathcal{O}_{n-t_1-t_2} \end{bmatrix} \in \mathbb{C}^{n,n} \quad (28)$$

with

$$\begin{aligned} \Phi_{11} &= B_{11}F_{11}C_{11} \in \mathbb{C}^{t_1,t_1}, \\ \begin{bmatrix} 0 & \Phi_{13} & 0 \end{bmatrix} &= B_{11} \begin{bmatrix} B_{21} \\ B_{31} \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 & 0 \\ 0 & \Phi & 0 \\ 0 & 0 & 0 \end{bmatrix} \in \mathbb{C}^{t_1,n-t_1}, \\ \begin{bmatrix} 0 \\ \Phi_{31} \\ 0 \end{bmatrix} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & \Phi & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} C_{12} & C_{13} \end{bmatrix}^{-1} C_{11} \in \mathbb{C}^{n-t_1,t_1}, \\ \begin{bmatrix} \mathcal{O}_s & 0 & 0 \\ 0 & \Phi_{33} & 0 \\ 0 & 0 & \mathcal{O}_{n-t_1-t_2} \end{bmatrix} &= \begin{bmatrix} \mathcal{O}_s & 0 & 0 \\ 0 & \Phi & 0 \\ 0 & 0 & \mathcal{O}_{n-t_1-t_2} \end{bmatrix} \in \mathbb{C}^{n-t_1,n-t_1}. \end{aligned} \quad (29)$$

It follows that

$$E + BGC = \begin{bmatrix} \Sigma_E + \Delta_{11} & \Delta_{12} & 0 \\ \Delta_{21} & I_s & 0 \\ 0 & 0 & \mathcal{O}_{n-t_1-s} \end{bmatrix}. \quad (30)$$

Computing the Schur-complement, we obtain that

$$\text{rank}(E + BGC) = s + \text{rank}(\Sigma_E + \Delta_{11} - \Delta_{12}\Delta_{21}). \quad (31)$$

But

$$B_{11} \begin{bmatrix} B_{21} \\ B_{31} \end{bmatrix}^{-1} \begin{bmatrix} I_s \\ 0 \end{bmatrix} \begin{bmatrix} I_s & 0 \end{bmatrix} \begin{bmatrix} C_{12} & C_{13} \end{bmatrix}^{-1} C_{11} = 0. \quad (32)$$

Hence,  $\text{rank}(E + BGC) = s + t_1$ .

Due to the form of  $BFC$  it follows that

$$A + BFC = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} & A_{13} & A_{14} & A_{15} \\ \tilde{A}_{21} & \begin{bmatrix} \mathcal{O}_s & 0 \\ 0 & I_{t_2-s} \end{bmatrix} & A_{23} & A_{24} & 0 \\ A_{31} & A_{32} & A_{33} & A_{34} & \Sigma_{35} \\ A_{41} & A_{42} & A_{43} & \Sigma_{44} & 0 \\ A_{51} & 0 & \Sigma_{53} & 0 & 0 \end{bmatrix}, \quad (33)$$

with row dimensions  $[t_1, t_2, t_3, t_4, t_5]$  and column dimensions  $[t_1, t_2, t_5, t_4, t_3]$ .

It then follows immediately that the lower right  $(n - t_1 - t_2 + s) \times (n - t_1 - t_2 + s)$  principal submatrix of  $A + BFC$  is non-singular. By Corollary 2 we therefore have that  $\alpha(E + BGC) - \beta(A + BFC)$  is regular and of index at most 1. Hence, we have proved the main result.

For (i) and (ii), it is clear from the construction that if  $s = t_2$ , then  $F$  is empty, while if  $s = 0$ , then  $G$  is void. In the case  $s = 0$ , only proportional feedback is used, and hence the right and left null spaces  $S_\infty, T_\infty$  of  $E$  are not changed. Therefore,

$$\begin{aligned} \text{rank}[E, AS_\infty, B] &= \text{rank}[E, (A + BFC)S_\infty, B] = \\ \text{rank} \begin{bmatrix} E \\ T_\infty^H A \\ C \end{bmatrix} &= \text{rank} \begin{bmatrix} E \\ T_\infty^H (A + BFC) \\ C \end{bmatrix} = n. \end{aligned} \quad (34)$$

This proves the converse in (ii).  $\square$

An immediate consequence of Theorem 5 is the following Corollary.

**Corollary 6** *Let  $E, A, B, C$  be as in Theorem 5. If conditions C1 and O1, as well as conditions C2 and O2, hold, then there exist  $F, G \in \mathbb{C}^{m \times p}$  and a feedback control*

$$u = Fy - Gy + v \quad \text{or} \quad u_k = Fy_k - Gy_{k+1} + v_k \quad (35)$$

*such that the closed loop system, denoted by  $(E + BGC, A + BFC, B, C)$  is strongly controllable and strongly observable, with index at most one and  $\text{rank}(E + BGC) = s + t_1$ , where  $s$  is given such that  $0 \leq s \leq t_2$ .*

*Proof.* By Theorem 5, Conditions C2 and O2 imply that there exist  $F, G \in \mathbb{C}^{m \times p}$  such that the pencil  $\alpha(E + BGC) - \beta(A + BFC)$  is regular with index at most 1. Hence the closed loop system is controllable and observable at infinity. Then, since the Conditions C1 and O1 are preserved under output feedback, it follows that the closed loop system is S-controllable and S-observable.  $\square$

If Conditions C0 and O0 hold, then we can obtain stronger results. We have the following:

**Corollary 7** *Let  $E, A, B, C$  be as in Theorem 5. Then there exists  $G \in \mathbb{C}^{p \times m}$  such that  $E + BGC$  is non-singular if and only if*

$$\text{rank} \begin{bmatrix} E & B \end{bmatrix} = \text{rank} \begin{bmatrix} E \\ C \end{bmatrix} = n.$$

Furthermore, there exists  $G \in \mathbb{C}^{p,m}$  and a feedback control

$$u = -G\dot{y} + v \quad \text{or} \quad u_k = -Gy_{k+1} + v_k \quad (36)$$

such that the closed loop system, denoted by  $(E + BGC, A, B, C)$ , is completely controllable and completely observable with  $\text{rank}(E + BGC) = n$  if and only if Conditions C0 and O0 hold.

*Proof.* The existence of  $G \in \mathbb{C}^{p,m}$  such that  $E + BGC$  is non-singular implies that in the canonical form (14) we have  $t_3 = t_4 = t_5 = t_6 = s_6 = 0$  which is equivalent to  $\text{rank} [E, B] = \text{rank} \begin{bmatrix} E \\ C \end{bmatrix} = n$ . The converse follows directly from Theorem 5 taking  $s = t_2$ .

The second part follows, since C-controllability and C-observability are preserved by output feedback.  $\square$

In this section we have presented our major new results. We have established simple algebraic conditions that guarantee that a descriptor system can be made *regular with index at most 1*:

- (i) by *proportional output* feedback; the dimension of the 'slow' dynamical part of the closed system (equivalent to the dimension of the reduced order standard system derived as in (21) ) is then equal to  $\text{rank}(E) = t_1$ ;
- (ii) by *derivative output* feedback; the dimension of the 'slow' dynamical part of the closed loop system is then equal to  $\text{rank}(E + BGC) = t_1 + t_2$ ;
- (iii) by *proportional-plus-derivative output* feedback; the dimension of the slow system is then equal to  $t_1 + s$ , where  $s$  can be any value in the range  $0 \leq s \leq t_2$ .

Furthermore, we have established *stable numerical methods* for constructing the feedback controls.

In the next section we give stable procedures for constructing feedback matrices that not only regularize the system, but also improve the conditioning.

## 5 Improving Conditioning

In the previous section we have shown how we can use output feedback to obtain a descriptor system that is regular and of index at most 1. As discussed before, the system can then be transformed to the reduced order standard system (21). To make this reduction numerically reliable, we

need that  $\Sigma_E, \hat{A}_{22}$  are well-conditioned with respect to inversion. (A matrix  $M$  is well-conditioned with respect to inversion if its condition number  $\text{cond}(M) := \|M\| \|M^{-1}\|$  is small. Here and in the following,  $\|\cdot\|$  is the spectral norm.) If the conditioning of these matrices is improved, then the closed system is less sensitive to perturbations and also has better performance characteristics (see for example [21]).

In this section we discuss the use of proportional-plus-derivative output feedback to improve the conditioning of these two matrices, while in the same process making the system regular, with index  $\leq 1$ . As we have seen in Theorem 5, we have the freedom to choose the rank of the matrix  $E+BGC$  in a certain range. Typically we choose either the maximal or the minimal possible rank.

Under these assumptions we would like to solve the following problem of computing an optimally conditioned regularization.

**Problem 1** *Let the system given by  $E, A, B, C$  satisfy conditions C1, C2 and O1, O2 and be already in the canonical form (14). Given an integer  $s$ ,  $0 \leq s \leq t_2$ , choose feedback matrices  $F, G \in \mathbb{C}^{m \times p}$  such that  $\alpha(E+BGC) - \beta(A+BFC)$  is regular, has index at most 1,  $\text{rank}(E+BGC) = t_1 + s$  and such that, furthermore, the non-singular part of  $E+BGC$  and the corresponding complementary part in  $A+BFC$  are optimally conditioned with respect to inversion.*

Partial results concerning this problem are given in [9]. We do not know how to solve this problem in general and leave this as an open problem. In the following we show how we can optimize upper bounds for the condition numbers in Problem 1.

In order to obtain improved conditioning, we need all the information from the canonical form (14), to determine whether  $s_6 = t_6 = 0$ ,  $s_2 = t_2$ , to eliminate extraneous input and output variables, i.e., to achieve  $\ell_3 = k_3 = 0$ , and to find the range of integers for  $s$ ,  $0 \leq s \leq t_2 = s_2$ .

For the direct computation of the feedback, however, with this information given, we only need the matrices obtained from Steps 1, 2, 3 of Algorithm 1. In the following, therefore, we assume that  $s_6 = t_6 = \ell_3 = k_3 = 0$ ,  $s$  is given with  $0 \leq s \leq t_2 = s_2$ , and  $E, A, B, C$  are given in the form obtained after

Steps 1,2,3 of Algorithm 1. We partition the matrices as follows

$$\begin{aligned}
 E &= \begin{bmatrix} \Sigma_E & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & A &= \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{bmatrix} \\
 B &= \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ \Sigma_{B_1} & 0 & 0 \\ 0 & \Sigma_{B_2} & 0 \\ 0 & 0 & 0 \end{bmatrix} & C &= \begin{bmatrix} C_{11} & \Sigma_{C_1} & 0 & 0 \\ C_{21} & 0 & \Sigma_{C_2} & 0 \\ C_{31} & 0 & 0 & 0 \end{bmatrix}
 \end{aligned} \tag{37}$$

with row dimensions  $[t_1, s, k_1 - s, n - t_1 - k_1]$  in  $E, A, B$  and  $[s, \ell_1 - s, \ell_2]$  in  $C$  and column dimensions  $[t_1, s, \ell_1 - s, n - t_1 - \ell_1]$  in  $E, A, C$  and  $[s, k_1 - s, k_2]$  in  $B$ .

Let

$$G = \begin{bmatrix} G_{11} & G_{12} & G_{13} \\ G_{21} & G_{22} & G_{23} \\ G_{31} & G_{32} & G_{33} \end{bmatrix}, \quad F = \begin{bmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{bmatrix} \tag{38}$$

be partitioned analogously. It is our aim to choose  $F, G$  such that

$$\alpha(E + BGC) - \beta(A + BFC) = \alpha \begin{bmatrix} \mathcal{E}_{11} & 0 \\ 0 & 0 \end{bmatrix} - \beta \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \tag{39}$$

is regular and of index at most 1, and  $\mathcal{E}_{11} \in \mathbb{C}^{t_1+s, t_1+s}, A_{22} \in \mathbb{C}^{n-t_1-s, n-t_1-s}$  are as well-conditioned as possible.

Since we have assumed that  $B, C$  have full rank, it follows immediately that  $B_{13}$  has full column rank and  $C_{31}$  has full row rank. Hence we obtain from the form of  $BGC$  that  $G_{12}, G_{21}, G_{22}, G_{23}, G_{32}$  have to be zero matrices in order to achieve the required form for  $E + BGC$ . In a similar way it follows directly from the form of  $BFC$ , that the only block of  $F$  that affects the relevant part of  $A + BFC$  is the block  $F_{22}$ . To simplify the computation, we may therefore chose the other blocks of  $F$  to be zero.

For the upper left corner of  $E + BGC$  we then obtain

$$\mathcal{E}_{11} = \begin{bmatrix} \Sigma_E & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} B_{11} & B_{13} \\ \Sigma_{B_1} & 0 \end{bmatrix} \begin{bmatrix} G_{11} & G_{13} \\ G_{31} & G_{33} \end{bmatrix} \begin{bmatrix} C_{11} & \Sigma_{C_1} \\ C_{31} & 0 \end{bmatrix}. \tag{40}$$

Analogously we obtain for the lower right block of  $A + BFC$

$$\mathcal{A}_{22} = \begin{bmatrix} A_{33} + \Sigma_{B_2} F_{22} \Sigma_{C_2} & A_{34} \\ A_{43} & A_{44} \end{bmatrix}. \tag{41}$$

It remains to determine

$$\begin{bmatrix} G_{11} & G_{13} \\ G_{31} & G_{33} \end{bmatrix}, F_{22} \quad (42)$$

such that  $\mathcal{E}_{11}$ ,  $\mathcal{A}_{22}$  are as well-conditioned as possible.

Before we consider this problem, we perform a transformation with (40). Performing SVD's of the full column rank matrix

$$\begin{bmatrix} B_{11} & B_{13} \\ \Sigma_{B_1} & 0 \end{bmatrix} = \tilde{U}_B \begin{bmatrix} \tilde{\Sigma}_B \\ 0 \end{bmatrix} \tilde{V}_B^H, \quad (43)$$

and the full row rank matrix

$$\begin{bmatrix} C_{11} & \Sigma_{C_1} \\ C_{31} & 0 \end{bmatrix} = \tilde{U}_C [\tilde{\Sigma}_C, 0] \tilde{V}_C^H, \quad (44)$$

we obtain the following system, which is unitarily equivalent to (40):

$$\begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix} + \begin{bmatrix} \tilde{\Sigma}_B \\ 0 \end{bmatrix} \tilde{G} [\tilde{\Sigma}_C, 0], \quad (45)$$

where  $\begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix} = \tilde{U}_B^H \begin{bmatrix} \Sigma_B & 0 \\ 0 & 0 \end{bmatrix} \tilde{V}_C$ ,  $\tilde{G} = \tilde{V}_B^H \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \tilde{U}_C$ . Thus we have that  $\tilde{\mathcal{E}}_{11}$  is unitarily equivalent to

$$\tilde{\mathcal{E}}_{11} = \begin{bmatrix} E_{11} + \hat{G} & E_{12} \\ E_{21} & E_{22} \end{bmatrix}, \quad (46)$$

where  $\hat{G} = \tilde{\Sigma}_B \tilde{G} \tilde{\Sigma}_C$ .

The optimal choice of both feedback matrices  $F$  and  $G$  can, therefore, be determined if we can solve the following problem.

**Problem 2** For a given matrix  $M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$  determine  $\Delta$  of the same size as  $M_{11}$  such that  $\text{cond} \left( \begin{bmatrix} M_{11} + \Delta & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \right)$  is minimal.

In general this is again an open problem. Partial solutions are given in [9].

In the following we compute the optimizing feedbacks for an upper bound on the condition number. To do this we transform the problem by transformations similar to those in Steps 4, 5 of Algorithm 1, i.e. we determine unitary matrices  $U_M, V_M$  such that

$$\hat{M} = U_M^H \begin{bmatrix} M_{11} + \hat{\Delta} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} V_M = \begin{bmatrix} \Delta_{11} & \Delta_{12} & \hat{M}_{13} & 0 \\ \Delta_{21} & \Delta_{22} & \hat{M}_{23} & \Sigma_{24} \\ \hat{M}_{31} & \hat{M}_{32} & \Sigma_{33} & 0 \\ 0 & \Sigma_{42} & 0 & 0 \end{bmatrix}, \quad (47)$$

with arbitrary blocks  $\Delta_{11}, \Delta_{12}, \Delta_{21}, \Delta_{22}$  and non-singular diagonal matrices  $\Sigma_{24}, \Sigma_{42}, \Sigma_{33}$ .

For the problem of improving the conditioning of  $\mathcal{A}_{22}$  this can be directly achieved with Steps 4,5 of Algorithm 1. For the conditioning of  $\mathcal{E}_{11}$  a similar transformation is easily carried out.

We then obtain that

$$\hat{M} = RDL =: \begin{bmatrix} I & 0 & \hat{M}_{13}\Sigma_{33}^{-1} & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & \hat{M}_{32}\Sigma_{42}^{-1} \\ 0 & 0 & 0 & I \end{bmatrix} \begin{bmatrix} \tilde{\Delta}_{11} & \Delta_{12} & 0 & 0 \\ \Delta_{21} & \Delta_{22} & 0 & \Sigma_{24} \\ 0 & 0 & \Sigma_{33} & 0 \\ 0 & \Sigma_{42} & 0 & 0 \end{bmatrix} \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ \Sigma_{33}^{-1}\hat{M}_{31} & 0 & I & 0 \\ 0 & 0 & \Sigma_{24}^{-1}\hat{M}_{23} & I \end{bmatrix}, \quad (48)$$

where  $\tilde{\Delta}_{11} := \Delta_{11} - \hat{M}_{13}\Sigma_{33}^{-1}\hat{M}_{31}$ .

From this we obtain an upper bound for the condition number. Set

$$c_L := \text{cond}(L) = \text{cond}\left( \begin{bmatrix} I & 0 & \hat{M}_{13}\Sigma_{33}^{-1} & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & \hat{M}_{32}\Sigma_{42}^{-1} \\ 0 & 0 & 0 & I \end{bmatrix} \right) \quad (49)$$

$$c_R := \text{cond}(R) = \text{cond}\left( \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ \Sigma_{33}^{-1}\hat{M}_{31} & 0 & I & 0 \\ 0 & 0 & \Sigma_{24}^{-1}\hat{M}_{23} & I \end{bmatrix} \right). \quad (50)$$

Then we obtain

$$\text{cond}(\hat{M}) \leq \text{cond}(L)\text{cond}(D)\text{cond}(R) = c_L c_R \text{cond} \left( \begin{bmatrix} \tilde{\Delta}_{11} & \Delta_{12} & 0 & 0 \\ \Delta_{21} & \Delta_{22} & 0 & \Sigma_{24} \\ 0 & 0 & \Sigma_{33} & 0 \\ 0 & \Sigma_{42} & 0 & 0 \end{bmatrix} \right). \quad (51)$$

Now only the term  $\text{cond}(D)$  depends on the free matrix  $\hat{\Delta} = \begin{bmatrix} \tilde{\Delta}_{11} & \Delta_{12} \\ \Delta_{21} & \Delta_{22} \end{bmatrix}$ . Since we have to invert the diagonal matrices  $\Sigma_{33}$ ,  $\Sigma_{24}$  and  $\Sigma_{42}$  to obtain  $D^{-1}$  and  $\hat{\Delta}$  is free, we choose  $\Delta_{12}, \Delta_{21}, \Delta_{22}$  to be zero and  $\tilde{\Delta}_{11} = \alpha I$ , where  $\alpha$  is the minimum singular value of

$$\Sigma = \begin{bmatrix} \Sigma_{24} & 0 & 0 \\ 0 & \Sigma_{42} & 0 \\ 0 & 0 & \Sigma_{33} \end{bmatrix}.$$

We then have the feedback

$$\Delta = \begin{bmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{21} & \Delta_{22} \end{bmatrix} = \begin{bmatrix} \alpha I + \hat{M}_{13} \Sigma_{33}^{-1} \hat{M}_{31} & 0 \\ 0 & 0 \end{bmatrix}. \quad (52)$$

It is possible that this choice does not improve the condition number; in this case we choose  $\Delta$  to be the zero matrix.

Performing this method for  $\mathcal{E}_{11}, \mathcal{A}_{22}$  we can improve the conditioning of the system. An Algorithm is given in Appendix B.

## 6 Numerical results

We have tested the algorithms given in the appendices on a variety of examples. The two methods were implemented in Matlab on a PC. (Matlab m-files are available from the authors).

We present here three typical test cases:

**Example 1** Let

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 1000 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1000 & 0 \\ 0 & 0 & 0 & 10 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

This quadrupel of matrices is already in canonical form with row dimensions  $[t_1, t_2, t_3, t_4, t_5, t_6] = [1, 2, 0, 1, 0, 0]$  and column dimensions  $[t_1, s_2, t_5, t_4, t_3, s_6] = [1, 2, 0, 1, 0, 0]$  for  $E$  and  $A$ , column dimensions  $[k_1, k_2, k_3] = [2, 0, 0]$  for  $B$  and row dimensions  $[\ell_1, \ell_2, \ell_3] = [2, 0, 0]$  for  $C$ . The pencil  $\alpha E - \beta A$  is singular and the condition of the matrix  $\mathcal{A}_{22}$  is infinite. Algorithm 2, described in Appendix B, with  $s = 0$  yields a pencil  $\alpha E - \beta(A + BFC)$  for which the modified block  $\mathcal{A}_{22}$  has condition number 10.

The feedback matrix is then  $F = \begin{bmatrix} 1 & 0 \\ 0 & -999 \end{bmatrix}$ . If  $s = 2$  then Algorithm 2 produces a pencil

$$\alpha(E + BGC) - \beta A = \alpha \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} - \beta \begin{bmatrix} 1000 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1000 & 0 \\ 0 & 0 & 0 & 10 \end{bmatrix}.$$

Here the feedback matrix is  $G = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$  and the condition numbers of both  $\mathcal{E}_{11}, \mathcal{A}_{22}$  are 1.

This example shows that it is possible to treat a singular pencil by our procedure, to make it regular and of index at most 1 with improved conditioning.

In the next example we make the rank of  $E$  maximal and do not change  $A$ :

**Example 2** Let

$$E = \begin{bmatrix} 1 & & & & & \\ & 0 & & & & \\ & & 0 & & & \\ & & & 0 & & \\ & & & & 0 & \\ & & & & & 0 \end{bmatrix}, B = \begin{bmatrix} 0.2408 & 0.6553 & 0.9166 \\ 0.6907 & 0.9700 & 0.1402 \\ 0.1062 & 0.0380 & 0.7054 \\ 0.2640 & 0.0988 & 0.0178 \\ 0.7034 & 0.2560 & 0.2611 \\ 0.4021 & 0.5598 & 0.1358 \end{bmatrix},$$

$$A = \begin{bmatrix} 0.2113 & 0.4524 & 0.6538 & 0.7469 & 0.1167 & 0.2260 \\ 0.0824 & 0.8075 & 0.4899 & 0.0378 & 0.6250 & 0.8159 \\ 0.7599 & 0.4832 & 0.7741 & 0.4237 & 0.5510 & 0.2284 \\ 0.0087 & 0.6135 & 0.9626 & 0.2613 & 0.3550 & 0.8553 \\ 0.8096 & 0.2749 & 0.9933 & 0.2403 & 0.4943 & 0.0621 \\ 0.8474 & 0.8807 & 0.8360 & 0.3405 & 0.0365 & 0.7075 \end{bmatrix},$$

$$C = \begin{bmatrix} 0.0503 & 0.2432 & 0.5876 & 0.2849 & 0.8642 & 0.0580 \\ 0.5782 & 0.9448 & 0.7256 & 0.6767 & 0.1943 & 0.6908 \end{bmatrix}.$$

The canonical form, obtained by Algorithm 1 in Appendix A, has row dimensions  $[t_1, t_2, t_3, t_4, t_5, t_6] = [1, 2, 1, 2, 0, 0]$ , column dimensions  $[t_1, s_2, t_5, t_4, t_3, s_6] = [1, 2, 0, 2, 1, 0]$  for  $E$  and  $A$ , column dimensions  $[k_1, k_2, k_3] = [3, 0, 0]$  for  $B$  and row dimensions  $[\ell_1, \ell_2, \ell_3] = [2, 0, 0]$  for  $C$ . The pencil  $\alpha E - \beta A$  is regular and  $\text{rank } E = 1$ . Algorithm 2, described in Appendix B, with  $s = t_2$  yields a closed loop pencil  $\alpha(E + BGC) - \beta A$  for which the modified block  $\mathcal{E}_{11}$  has rank 3 and condition number 12.57, while  $\mathcal{A}_{22}$  has condition number 3.22. The feedback matrix is

$$G = \begin{bmatrix} -0.1905 & -0.2966 \\ 0.6928 & -0.1992 \\ -0.1660 & 3.1308 \end{bmatrix}.$$

**Example 3** Let

$$E = \begin{bmatrix} 0.1504 & 0.1154 & 0.1535 & 0.0104 \\ 0.0323 & 0.0248 & 0.0330 & 0.0022 \\ 0.4662 & 0.3577 & 0.4760 & 0.0322 \\ 0.4665 & 0.3579 & 0.4763 & 0.0322 \\ 0.1779 & 0.1072 & 0.1819 & 0.0821 \\ 0.4794 & 0.2889 & 0.4903 & 0.2213 \\ 0.8524 & 0.5137 & 0.8718 & 0.3935 \\ 1.3670 & 0.8238 & 1.3981 & 0.6310 \\ 0.5890 & 0.0920 & 0.9103 & 0.7361 \\ 0.9304 & 0.6539 & 0.7622 & 0.3282 \end{bmatrix}, B = \begin{bmatrix} 0.9347 & 0.0346 \\ 0.3835 & 0.0535 \\ 0.5194 & 0.5297 \\ 0.8310 & 0.6711 \end{bmatrix},$$

$$A = \begin{bmatrix} 0.1779 & 0.1072 & 0.1819 & 0.0821 \\ 0.4794 & 0.2889 & 0.4903 & 0.2213 \\ 0.8524 & 0.5137 & 0.8718 & 0.3935 \\ 1.3670 & 0.8238 & 1.3981 & 0.6310 \\ 0.5890 & 0.0920 & 0.9103 & 0.7361 \\ 0.9304 & 0.6539 & 0.7622 & 0.3282 \end{bmatrix},$$

$$C = \begin{bmatrix} 0.5890 & 0.0920 & 0.9103 & 0.7361 \\ 0.9304 & 0.6539 & 0.7622 & 0.3282 \end{bmatrix}.$$

The canonical form obtained by Algorithm 1 for this quadrupel is

$$\begin{aligned}
 E &= \begin{bmatrix} 1.0995 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1.1558 & 0.8372 \\ 0.8099 & 0 \\ 0 & 0.1528 \\ 0 & 0 \end{bmatrix}, \\
 A &= \begin{bmatrix} 2.4874 & 0.6483 & 0.1702 & 0.0051 \\ 0.1550 & 0.0404 & 0.0106 & 0.0003 \\ -0.8194 & -0.2136 & -0.0561 & -0.0017 \\ -0.3185 & -0.0830 & -0.0218 & 0.0006 \end{bmatrix}, \\
 C &= \begin{bmatrix} 1.0100 & 0.8615 & 0 & 0 \\ 1.3754 & 0 & 0.2317 & 0 \end{bmatrix}
 \end{aligned}$$

with row and column dimensions  $[1, 2, 0, 1, 0, 0]$  for  $E$  and  $A$ , column dimensions  $[2, 0, 0]$  for  $B$  and row dimensions  $[2, 0, 0]$  for  $C$ . The pencil  $\alpha E - \beta A$  is regular and the condition of the matrix  $\mathcal{A}_{22}$  is  $1.51 \times 10^{16}$ . Algorithm 2 with  $s = 0$  yields a closed loop pencil  $\alpha E - \beta(A + BFC)$  for which the modified block  $\mathcal{A}_{22}$  has condition number  $1.58 \times 10^3$ . The feedback matrix is

$$F = \begin{bmatrix} 1.3753 & -0.0565 \\ 1.6220 & 29.8195 \end{bmatrix}.$$

## 7 Conclusion

We investigate here the use of proportional and derivative *output* feedback for regularizing a descriptor, or generalized state-space system. We define various algebraic conditions guaranteeing that a descriptor system can be transformed by output feedback into a closed loop system that is regular and of index at most 1.

Conditions are given under which the system can be transformed into a completely controllable and completely observable *standard* system (of full dimension) by a combination of derivative and proportional output feedback.

Conditions are also given under which the system can be transformed by proportional and derivative output feedback into a *strongly controllable* and *strongly observable* system that is regular and of index at most 1 with precisely  $r$  finite poles, where  $r$  lies in a region of integers determined from  $E, A, B, C$  via a canonical form. Such a system is 'impulse controllable' and can be transformed into a reduced-order controllable and observable *standard* system of precise dimension  $r$ .

The proofs of these results do not require regularity of the original open loop system. Furthermore, the procedure for constructing the feedback matrices which regularize the closed loop system are based on unitary matrix decompositions and are numerically stable. In practice it is desirable not only that the closed loop descriptor system is regular, but also 'well-conditioned' in the sense that the reduction to standard form is computationally reliable. We show here that the feedback matrices which regularize the system can also be chosen to improve the 'conditioning' of the closed loop system, and a computational algorithm for achieving this result is presented.

MATLAB procedures for the algorithms described in the Appendix are available from the authors.

## A Appendix

### Algorithm 1:

**Input:** Matrices  $E, A \in \mathbb{C}^{n,n}, B \in \mathbb{C}^{n,m}, C \in \mathbb{C}^{p,n}$ .

**Output:** Unitary matrices  $U, V \in \mathbb{C}^{n,n}, W \in \mathbb{C}^{m,m}, Y \in \mathbb{C}^{p,p}$  such that  $U^H E V, U^H A V, U^H B W, Y^H C V$  are in the canonical form (14).

Set  $U := I_n, V := I_n, W := I_m, Y = I_p$ .

**Step 1:** Perform an SVD  $E = U_E \begin{bmatrix} \Sigma_E & 0 \\ 0 & 0 \end{bmatrix} V_E^H$  with  $\Sigma_E$  non-singular and set

$$\begin{aligned} E &:= U_E^H E V_E = \begin{bmatrix} \Sigma_E & 0 \\ 0 & 0 \end{bmatrix}, \quad A := U_E^H A V_E = \begin{bmatrix} A_{11}^1 & A_{12}^1 \\ A_{21}^1 & A_{22}^1 \end{bmatrix} \\ B &:= U_E^H B = \begin{bmatrix} B_1^1 \\ B_2^1 \end{bmatrix}, \quad C := C V_E = \begin{bmatrix} C_1^1 & C_2^1 \end{bmatrix}, \\ U &:= U U_E, \quad V := V V_E. \end{aligned} \quad (53)$$

**Step 2:** Perform SVD's

$$B_2^1 = U_B \begin{bmatrix} \Sigma_B & 0 \\ 0 & 0 \end{bmatrix} V_B, \quad C_2^1 = U_C \begin{bmatrix} \Sigma_C & 0 \\ 0 & 0 \end{bmatrix} V_C$$

with  $\Sigma_B$  and  $\Sigma_C$  non-singular and set

$$\begin{aligned} E &:= \begin{bmatrix} I_{t_1} & 0 \\ 0 & U_B^H \end{bmatrix} E \begin{bmatrix} I_{t_1} & 0 \\ 0 & V_C \end{bmatrix} = \begin{bmatrix} \Sigma_E & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\ A &:= \begin{bmatrix} I_{t_1} & 0 \\ 0 & U_B^H \end{bmatrix} A \begin{bmatrix} I_{t_1} & 0 \\ 0 & V_C \end{bmatrix} = \begin{bmatrix} A_{11}^2 & A_{12}^2 & A_{13}^2 \\ A_{21}^2 & A_{22}^2 & A_{23}^2 \\ A_{31}^2 & A_{32}^2 & A_{33}^2 \end{bmatrix}, \\ B &:= \begin{bmatrix} I_{t_1} & 0 \\ 0 & U_B^H \end{bmatrix} B V_B = \begin{bmatrix} B_{11}^2 & B_{12}^2 \\ \Sigma_B & 0 \\ 0 & 0 \end{bmatrix}, \\ C &:= V_B^H C \begin{bmatrix} I_{t_1} & 0 \\ 0 & V_C \end{bmatrix} = \begin{bmatrix} C_{11}^2 & \Sigma_C & 0 \\ C_{21}^2 & 0 & 0 \end{bmatrix}, \quad Y := Y U_C \\ U &:= U \begin{bmatrix} I_{t_1} & 0 \\ 0 & U_B \end{bmatrix}, \quad V := V \begin{bmatrix} I_{t_1} & 0 \\ 0 & V_C \end{bmatrix}, \quad W := W V_B. \end{aligned} \quad (54)$$

**Step 3:** Perform SVD's

$$B_{12}^2 = U_{12} \begin{bmatrix} \Sigma_{12} & 0 \\ 0 & 0 \end{bmatrix} V_{12}^H, \quad C_{21}^2 = U_{21} \begin{bmatrix} \Sigma_{21} & 0 \\ 0 & 0 \end{bmatrix} V_{21}^H$$

with  $\Sigma_{21}$  and  $\Sigma_{12}$  non-singular and set

$$\begin{aligned} B &:= B \begin{bmatrix} I_{k_1} & 0 \\ 0 & V_{12} \end{bmatrix} \begin{bmatrix} B_{11}^3 & B_{12}^3 & 0 \\ \Sigma_B & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, W := W \begin{bmatrix} I_{k_1} & 0 \\ 0 & V_{12} \end{bmatrix}, \\ C &:= \begin{bmatrix} I_{t_1} & 0 \\ 0 & U_{21}^H \end{bmatrix} C = \begin{bmatrix} C_{11}^3 & \Sigma_C & 0 \\ C_{21}^3 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, Y = Y \begin{bmatrix} I_{t_1} & 0 \\ 0 & U_{21} \end{bmatrix}. \end{aligned} \quad (55)$$

**Step 4:** Perform an SVD  $A_{33}^3 = U_A \begin{bmatrix} \Sigma_{44} & 0 \\ 0 & 0 \end{bmatrix} V_A^H$  with  $\Sigma_{44}$  non-singular and set

$$\begin{aligned} E &:= \begin{bmatrix} I_{t_1} & 0 & 0 \\ 0 & I_{k_1} & 0 \\ 0 & 0 & U_A^H \end{bmatrix} E \begin{bmatrix} I_{t_1} & 0 & 0 \\ 0 & I_{k_1} & 0 \\ 0 & 0 & V_A \end{bmatrix} = \begin{bmatrix} \Sigma_E & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \\ A &:= \begin{bmatrix} I_{t_1} & 0 & 0 \\ 0 & I_{k_1} & 0 \\ 0 & 0 & U_A^H \end{bmatrix} A \begin{bmatrix} I_{t_1} & 0 & 0 \\ 0 & I_{k_1} & 0 \\ 0 & 0 & V_A \end{bmatrix} = \begin{bmatrix} A_{11}^3 & A_{12}^3 & A_{13}^3 & A_{14}^3 \\ A_{21}^3 & A_{22}^3 & A_{23}^3 & A_{24}^3 \\ A_{31}^3 & A_{32}^3 & \Sigma_{44} & 0 \\ A_{41}^3 & A_{42}^3 & 0 & 0 \end{bmatrix}, \\ B &:= \begin{bmatrix} I_{t_1} & 0 & 0 \\ 0 & I_{k_1} & 0 \\ 0 & 0 & U_A^H \end{bmatrix} B = \begin{bmatrix} B_{11}^3 & B_{12}^3 & 0 \\ \Sigma_B & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, U := U \begin{bmatrix} I_{t_1} & 0 & 0 \\ 0 & I_{k_1} & 0 \\ 0 & 0 & U_A \end{bmatrix}, \\ C &:= C \begin{bmatrix} I_{t_1} & 0 & 0 \\ 0 & I_{k_1} & 0 \\ 0 & 0 & V_A \end{bmatrix} = \begin{bmatrix} C_{11}^3 & \Sigma_C & 0 & 0 \\ C_{21}^3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, V := V \begin{bmatrix} I_{t_1} & 0 & 0 \\ 0 & I_{k_1} & 0 \\ 0 & 0 & V_A \end{bmatrix} \end{aligned} \quad (56)$$

**Step 5:** Perform SVD's

$$A_{42}^3 = U_{42} \begin{bmatrix} 0 & \Sigma_{53} \\ 0 & 0 \end{bmatrix} V_{42}^H, A_{24}^3 = U_{24} \begin{bmatrix} 0 & 0 \\ \Sigma_{35} & 0 \end{bmatrix} V_{24}^H$$

with  $\Sigma_{53}$  and  $\Sigma_{35}$  non-singular and set

$$\begin{aligned}
E &:= \begin{bmatrix} I_{t_1} & 0 & 0 & 0 \\ 0 & U_{24}^H & 0 & 0 \\ 0 & 0 & I_{t_4} & 0 \\ 0 & 0 & 0 & U_{42}^H \end{bmatrix} E \begin{bmatrix} I_{t_1} & 0 & 0 & 0 \\ 0 & V_{42} & 0 & 0 \\ 0 & 0 & I_{t_4} & 0 \\ 0 & 0 & 0 & V_{24} \end{bmatrix} \\
&= \begin{bmatrix} \Sigma_E & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \\
A &:= \begin{bmatrix} I_{t_1} & 0 & 0 & 0 \\ 0 & U_{24}^H & 0 & 0 \\ 0 & 0 & I_{t_4} & 0 \\ 0 & 0 & 0 & U_{42}^H \end{bmatrix} A \begin{bmatrix} I_{t_1} & 0 & 0 & 0 \\ 0 & V_{42} & 0 & 0 \\ 0 & 0 & I_{t_4} & 0 \\ 0 & 0 & 0 & V_{24} \end{bmatrix} \\
&= \begin{bmatrix} A_{11}^4 & A_{12}^4 & A_{13}^4 & A_{14}^4 & A_{15}^4 & A_{16}^4 \\ A_{21}^4 & A_{22}^4 & A_{23}^4 & A_{24}^4 & 0 & 0 \\ A_{31}^4 & A_{32}^4 & A_{33}^4 & A_{34}^4 & \Sigma_{35} & 0 \\ A_{41}^4 & A_{42}^4 & A_{43}^4 & \Sigma_{44} & 0 & 0 \\ A_{51}^4 & 0 & \Sigma_{53} & 0 & 0 & 0 \\ A_{61}^4 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \\
B &:= \begin{bmatrix} I_{t_1} & 0 & 0 & 0 \\ 0 & U_{24}^H & 0 & 0 \\ 0 & 0 & I_{t_4} & 0 \\ 0 & 0 & 0 & U_{42}^H \end{bmatrix} B = \begin{bmatrix} B_{11}^4 & B_{12}^4 & 0 \\ B_{21}^4 & 0 & 0 \\ B_{31}^4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\
C &:= C \begin{bmatrix} I_{t_1} & 0 & 0 & 0 \\ 0 & V_{42} & 0 & 0 \\ 0 & 0 & I_{t_4} & 0 \\ 0 & 0 & 0 & V_{24} \end{bmatrix} = \begin{bmatrix} C_{11}^4 & C_{12}^4 & C_{13}^4 & 0 & 0 & 0 \\ C_{21}^4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \\
U &:= U \begin{bmatrix} I_{t_1} & 0 & 0 & 0 \\ 0 & U_{24} & 0 & 0 \\ 0 & 0 & I_{t_4} & 0 \\ 0 & 0 & 0 & U_{42} \end{bmatrix}, \quad V := V \begin{bmatrix} I_{t_1} & 0 & 0 & 0 \\ 0 & V_{42} & 0 & 0 \\ 0 & 0 & I_{t_4} & 0 \\ 0 & 0 & 0 & V_{24} \end{bmatrix}
\end{aligned} \tag{57}$$

## B Appendix

### Algorithm 2

**Input:**  $E, A, B, C$  in canonical form (14) with  $s_6 = t_6 = 0$ ,  $0 \leq s \leq t_2 = s_2$ , and matrices  $\Sigma_B, \Sigma_C$  obtained in Step 4 of Algorithm 1 and matrices  $U_{24}, V_{42}$  obtained in Step 5 of Algorithm 1.

**Output:**  $F, G \in \mathbb{C}^{m,p}$  such that  $\text{rank}(E + BGC) = t_2 + s$ ,  $\alpha(E + BGC) - \beta(A + BFC) = \alpha \begin{bmatrix} \mathcal{E}_{11} & 0 \\ 0 & 0 \end{bmatrix} - \beta \begin{bmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} \\ \mathcal{A}_{21} & \mathcal{A}_{22} \end{bmatrix}$  is regular and of index at most 1, and  $\mathcal{E}_{11} \in \mathbb{C}^{t_1+s, t_1+s}$ ,  $\mathcal{A}_{22} \in \mathbb{C}^{n-t_1-s, n-t_1-s}$  has improved condition number.

IF  $s = 0$  THEN  $G = 0$

ELSE

let  $\alpha$  be the minimal singular value of  $\Sigma_E$  and let  $\Delta_s = \alpha I_s$ .

Set  $G = \begin{bmatrix} G_{11} & 0 \\ 0 & 0 \end{bmatrix}$ , and  $F = \begin{bmatrix} F_{11} & 0 \\ 0 & 0 \end{bmatrix}$  with

$$G_{11} = \begin{bmatrix} B_{21} \\ B_{31} \end{bmatrix}^{-1} \begin{bmatrix} \Delta_s & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} C_{12} & C_{13} \end{bmatrix}^{-1} = \Sigma_B^{-1} U_{24} \begin{bmatrix} \Delta_s & 0 \\ 0 & 0 \end{bmatrix} V_{42}^T \Sigma_C^{-1},$$

$$F_{11} = \begin{bmatrix} B_{21} \\ B_{31} \end{bmatrix}^{-1} \begin{bmatrix} \mathcal{O}_s & 0 & 0 \\ 0 & \Phi & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} C_{12} & C_{13} \end{bmatrix}^{-1} =$$

$$\Sigma_B^{-1} U_{24} \begin{bmatrix} \mathcal{O}_s & 0 & 0 \\ 0 & \Phi & 0 \\ 0 & 0 & 0 \end{bmatrix} V_{42}^T \Sigma_C^{-1}$$

with

$$\Phi = \Delta - \begin{bmatrix} 0 & I_{t_2-s} \end{bmatrix} A_{22} \begin{bmatrix} 0 \\ I_{t_2-s} \end{bmatrix},$$

where  $\Delta = \beta I_{t_2-s}$  with  $\beta$  the minimum singular value of

$$\begin{bmatrix} \Sigma_{53} & & \\ & \Sigma_{44} & \\ & & \Sigma_{35} \end{bmatrix}.$$

END

Then we obtain

$$E + BGC = \begin{bmatrix} I & 0 & B_{11} & 0 \\ & I & B_{21} & 0 \\ & & I & 0 \\ & & & I \end{bmatrix} \begin{bmatrix} \Sigma_1 & & & \\ & \Sigma_2 & & \\ & & \Delta_s & \\ & & & 0 \end{bmatrix} \begin{bmatrix} I \\ 0 & I \\ C_{11} & C_{12} & I \\ 0 & 0 & 0 & I \end{bmatrix}$$

and

$$A + BFC = A + \begin{bmatrix} \Phi_{11} & 0 & \Phi_{13} & 0 \\ 0 & O_s & 0 & 0 \\ \Phi_{31} & 0 & \Phi_{33} & \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where

$$\begin{aligned} \Phi_{11} &= B_{11}F_{11}C_{11}, \Phi_{31} = \begin{bmatrix} 0 & \Phi & 0 \end{bmatrix} V_{42}^T \Sigma_C^{-1} C_{11}, \\ \Phi_{13} &= B_{11} \Sigma_B^{-1} U_{24} \begin{bmatrix} 0 \\ \Phi \\ 0 \end{bmatrix}, \Phi_{33} = \Phi. \end{aligned}$$

In  $E + BGC$  and  $A + BFC$  the rows and columns are partitioned as  $\begin{bmatrix} t_1 & s & t_2 - s & n - t_1 - t_2 \end{bmatrix}$ .

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