N-dimensional Versions of Some Symmetric Univariate Distributions

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Abstract

Let $\mathcal{F}_1$ be the class of distributions on R^1 which are symmetric and have a finite variance. $\mathcal{L}(Z)$ denotes the distribution of a random variable Z. For $\mathcal{L}(Z_i) \in \mathcal{F}_1$, write $Z_1 \cong Z_2$ if there exists a $c > 0$ such that $\mathcal{L}(Z_1) = \mathcal{L}(cZ_2)$. Given n, $\mathcal{L}(Z)$ has an <u>n-dimensional version</u> iff there exists a random vector $X \in R^n$ such that $\sum_1^n b_i X_i \cong Z$ for all $b_1, \ldots, b_n$ not all zero. A representation theorem is proved which gives a necessary and sufficient condition that $\mathcal{L}(Z)$ have an n-dimensional version. It is also proved that $\mathcal{L}(Z)$ has an n-dimensional version for all n iff $\mathcal{L}(Z)$ is a scale mixture of univariate normal distributions with mean 0.

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§1. Introduction:

Let R^n denote Euclidean n-space and let $O(n)$ be the group of $n \times n$ orthogonal matrices. If X is a random vector in R^n, $\mathcal{L}(X)$ denotes the distribution of X. The problem discussed in this paper is motivated by the following considerations. If Z_0 is a random variable on R^1 and $\mathcal{L}(Z_0) = N(0,1)$, then for each n, there is a random vector $X \in R^n$ such that $\mathcal{L}(a'X) = \mathcal{L}(\|a\| Z_0)$ for all vectors $a \in R^n$ (a' is the transpose of the column vector a, $\|a\|$ is the norm of a). Of course, X must have a multivariate normal distribution with mean 0 and identity covariance matrix. Thus, we can say that $\mathcal{L}(X)$ is an n-dimensional version of $\mathcal{L}(Z_0)$ in the sense that the marginal distribution of X on the line generated by a is the distribution of Z_0 (up to a scale factor), and this holds for every a. Now, given a symmetric distribution $\mathcal{L}(Z)$ on R^1 and an integer $n > 1$, under what conditions on $\mathcal{L}(Z)$ does there exist a random vector $X \in R^n$ such that for all $a \in R^n$, $a'X$ has the same distribution as $c(a)Z$ where $c(a)$ is a constant? It is this and related questions which are discussed here.

Let S_n denote the class of all probability distributions (measures) P on R^n which satisfy $P(\{0\}) = 0$ and $P(\Gamma B) = P(B)$ for all Borel sets and $\Gamma \in O(n)$. Thus, S_1 is the class of symmetric distributions on R^1 with no mass at 0.

Definition 1: Given $\mathcal{L}(Z) \in S_1$, $\mathcal{L}(Z)$ has an <u>n-dimensional version</u> (n.v.) if there exists a random vector $X \in R^n$ and a function $c: R^n \to [0,\infty)$ which satisfy

(i) $\mathcal{L}(a'X) = \mathcal{L}(c(a)Z)$ for $a \in R^n$

(ii) $c(a) = 0$ iff $a = 0$

Remark: The assumption that $c(a) = 0$ iff $a = 0$ is to insure that X is actually n-dimensional, i.e.: $\Pr\{X \in M\} < 1$ for all proper linear subspaces $M \subseteq R^n$.

Definition 2: Given $\mathcal{L}(Z) \in S_1$, $\mathcal{L}(Z)$ has an n-dimensional isotropic version (n.i.v.) if there exists a random vector $X \in R^n$ and a function $c: R^n \to [0,\infty)$ which satisfy

(i) $\mathcal{L}(X) \in S_n$

(ii) $\mathcal{L}(a'X) = \mathcal{L}(c(a)Z)$ for $a \in R^n$

(iii) $c(a) = 0$ iff $a = 0$.

Now, let $\mathcal{F}_n$ (respectively $\mathcal{F}_{i,n}$) be all distributions in S_1 which have n.v.'s (respectively n.i.v.'s). The main result of this paper, given in Section 2, provides a necessary and sufficient condition that $\mathcal{L}(Z) \in \mathcal{F}_{i,n}$. This result if then used to show that $\mathcal{L}(Z) \in \mathcal{F}_{i,n}$ for all n iff $\mathcal{L}(Z)$ is a scale mixture of normal distributions with mean 0. Previous work on scale mixtures of normals (in other contexts) include Teichroew (1957), Kelker (1971), Andrews and Mallows (1974), and Efron and Olshen (1977).

When $\mathcal{L}(Z) \in S_1$ has a finite variance, we give a necessary and sufficient condition that $\mathcal{L}(Z) \in \mathcal{F}_n$ in Section 3. The case when $\mathrm{Var}(Z) = +\infty$ is briefly discussed, but no positive results are presented.

§2: Main Results:

The first result in this section describes the structure of distributions $\mathcal{L}(X) \in S_n$. Let $C_n = \{x \mid x \in R^n, \|x\| = 1\}$ so $O(n)$ acts transitively on C_n. Thus the uniform distribution on C_n, say Q_0, is the unique invariant (under $O(n)$) probability measure on C_n.

<u>Theorem 1</u>: For $\mathcal{L}(X) \in S_n$, let $U = U(X) = X/\|X\|$ and let $V = \|X\|$. Then U and V are independent and $\mathcal{L}(U) = Q_0$. In addition, the first coordinate of U, say U_1, has a density given by

$$(1) \quad \Psi_n(u) = [B(\tfrac{1}{2}, (n-1)/2]^{-1}(1-u^2)^{\frac{n-3}{2}} I_{[0,1)}(u^2)$$

where $B(\cdot,\cdot)$ is the Beta function and $I_{[0,1)}$ is the indicator function of $[0,1)$.

<u>Proof</u>: That the distribution of U is given by Q_0 is well known. To show U and V are independent, consider Borel sets $B_1 \subseteq C_n$ and $B_2 \subseteq (0,\infty)$. If $P\{V \in B_2\} = 0$, then we have $0 = P\{U \in B_1, V \in B_2\} = P\{U \in B_1\}P\{V \in B_2\}$. If $P\{V \in B_2\} > 0$, consider the probability measure μ on C_n defined by $\mu(B) = P\{U \in B, V \in B_2\}/P\{V \in B_2\}$. For $\Gamma \in O(n)$,

$$\begin{aligned}(2) \quad \mu(\Gamma B) &= P\{U \in \Gamma B, V \in B_2\}/P\{V \in B_2\} \\ &= P\{\Gamma' U \in B, V \in B_2\}/P\{V \in B_2\} \\ &= P\{U(\Gamma' X) \in B, V(\Gamma' X) \in B_2\}/P\{V \in B_2\} \\ &= P\{U \in B, V \in B_2\}/P\{V \in B_2\} = \mu(B).\end{aligned}$$

Thus, μ is invariant under $O(n)$ so $\mu = Q_0$. Therefore,

$$P\{U \in B, V \in B_2\} = P\{U \in B\}P\{V \in B_2\}$$

so U and V are independent.

The density of U_1 is easily derived by taking X to be n-dimensional normal with mean 0 and identity covariance matrix and noting that $U_1^2 = X_1^2/\|X\|^2$ has a Beta $(\frac{1}{2}, \frac{n-1}{2})$ distribution.

<u>Remark</u>: The results of Theorem 1 are well known when X has a denisty on R^n (e.g. A. Kudo (1963), Lemma 3.2) and are undoubtedly known in the present generality. However, a proof for the general case does not seem to exist in the literature.

Remark: The above result is equivalent to the assertion that every distribution $\mathcal{L}(X) \in S_n$ has the representation $\mathcal{L}(X) = \mathcal{L}(VU)$ where U and V are independent, $\mathcal{L}(U) = Q_0$ and $V \in (0,\infty)$ has an arbitrary distribution. The converse is obvious--that is, $\mathcal{L}(VU) \in S_n$ when $\mathcal{L}(U) = Q_0$ and $V \in (0,\infty)$ is independent of U. Also, $\mathcal{L}(a'X) = \mathcal{L}(\|a\|X_1)$ since $\mathcal{L}(X) = \mathcal{L}(\Gamma X)$ for all $\Gamma \in O(n)$ when $\mathcal{L}(X) \in S_n$ and X_1 is the first coordinate of X.

Now, let $\mathcal{G}$ be the class of all distributions on the open interval $(0,\infty)$. If T_1 and T_2 are two random vectors, the notation $T_1 \perp\!\!\!\perp T_2$ means that T_1 and T_2 are independent. Most of the results in this paper are consequences of the following.

Theorem 2: $\mathcal{L}(Z) \in \mathcal{F}_{1,n}$ iff $\mathcal{L}(Z)$ has a density given by

$$(3) \quad f(u) = \int_0^\infty \frac{1}{v}\, \Psi_n\left(\frac{u}{v}\right) G(dv)$$

for some $G \in \mathcal{G}$, where Ψ_n is given by (1).

Proof: If $\mathcal{L}(Z) \in \mathcal{F}_{1,n}$, then there is an $X \in R^n$ with $\mathcal{L}(X) \in S_n$ and $\mathcal{L}(a'X) = \mathcal{L}(c(a)Z)$ where $c(a) \neq 0$ iff $a \neq 0$. Setting $a' = \epsilon_1' = (1,0,\ldots,0)$ $\mathcal{L}(X_1) = \mathcal{L}(c_0 Z)$ where $c_0 = c(\epsilon_1) > 0$. Let $\tilde{X} = (1/c_0)X$ so $\mathcal{L}(\tilde{X}) \in S_n$ and $\mathcal{L}(Z) = \mathcal{L}(\tilde{X}_1) = \mathcal{L}(U_1 V)$ where $U_1 = \tilde{X}_1/\|\tilde{X}\|$ and $V = \|\tilde{X}\|$. By Theorem 1, $U_1 \perp\!\!\!\perp V$ and U_1 has density Ψ_n, so $U_1 V$ has the density (3) where $\mathcal{L}(V) = G \in \mathcal{G}$.

Conversely, if Z has a density given by (3), then $\mathcal{L}(Z) = \mathcal{L}(U_1 V)$ where $U_1 \perp\!\!\!\perp V$, $\mathcal{L}(V) = G$ and U_1 has density Ψ_n. Consider $U \in C_n$ with $\mathcal{L}(U) = Q_0$. Then $X = VU$ satisfies $\mathcal{L}(X) \in S_n$ and $\mathcal{L}(a'X) = \mathcal{L}(\|a\|X_1) = \mathcal{L}(\|a\|\, U_1 V)$ $= \mathcal{L}(\|a\|Z)$ so $\mathcal{L}(Z) \in \mathcal{F}_{1,n}$. This completes the proof.

Remark: The result of Theorem 2 may be restated as: $\mathcal{L}(Z) \in \mathcal{F}_{1,n}$ iff $\mathcal{L}(Z)$ has a density in the convex hull of the set $\{\frac{1}{v}\Psi_n(\frac{\cdot}{v}) \mid v > 0\}$. For $n = 3$, this implies that every symmetric unimodal distribution on R^1 is in $\mathcal{F}_{1,3}$. This follows because $\Psi_3(u) = \frac{1}{2} I_{[0,1)}(u^2)$ and every symmetric unimodal density is a mixture of $\frac{1}{v}\Psi_3(\frac{\cdot}{v})$, $v > 0$.

It is clear that $\mathfrak{F}_{i,n} \supseteq \mathfrak{F}_{i,n+1}$ and it is not too hard to show that the distribution defined by Ψ_n is not in $\mathfrak{F}_{i,n+1}$. Thus, $\mathfrak{F}_{i,n} \neq \mathfrak{F}_{i,n+1}$. Let

$$\mathfrak{F}_{i,\infty} = \bigcap_{n=1}^{\infty} \mathfrak{F}_{i,n}.$$

Theorem 3: $\mathcal{L}(Z) \in \mathfrak{F}_{i,\infty}$ iff Z has a density given by (up to a set of Lebesgue measure zero)

$$(4) \quad f(z) = \int_0^\infty \frac{1}{v}\varphi(z/v)G(dv)$$

for some $G \in \mathcal{G}$ where $\varphi(u) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2}$.

Proof: If (4) holds, then $\mathcal{L}(Z) = \mathcal{L}(WV)$ where $W \perp\!\!\!\perp V$, $\mathcal{L}(V) = G$ and $\mathcal{L}(W) = N(0,1)$. Given n, consider $\tilde{X}$ with $\mathcal{L}(\tilde{X}) = N_n(0,I_n)$ and $\tilde{X} \perp\!\!\!\perp V$, and set $X = V\tilde{X}$. Then $\mathcal{L}(X) \in S_n$ and $\mathcal{L}(a'X) = \mathcal{L}(a'\tilde{X}V) = \mathcal{L}(\|a\|WV) = \mathcal{L}(\|a\|Z)$ so $\mathcal{L}(Z) \in \mathfrak{F}_{i,n}$ for all n.

Conversely, suppose $\mathcal{L}(Z) \in \mathfrak{F}_{i,n}$ for all n. Thus, for each n, the density of Z has the form

$$(5) \quad f(z) = \int_0^\infty \frac{1}{v}\Psi_n\left(\frac{z}{v}\right) \tilde{G}_n(dv) \quad \text{for some } \tilde{G}_n \in \mathcal{G}.$$

In terms of random variables, this means that $\mathcal{L}(Z) = \mathcal{L}(U_1^{(n)}V^{(n)}) = \mathcal{L}(\sqrt{n}\, U_1^{(n)}V^{(n)}/\sqrt{n})$ where $U_1^{(n)}$ has density Ψ_n and $\mathcal{L}(V^{(n)}) = \tilde{G}_n$. The density of $\sqrt{n}\, U_1^{(n)}$ is obviously $\frac{1}{\sqrt{n}}\Psi_n(u/\sqrt{n})$ and it is easy to show that $\lim_{n\to\infty} \frac{1}{\sqrt{n}}\Psi_n(u/\sqrt{n}) = \varphi(u)$ for each u. By Scheffe's Theorem (see Billingsley (1968), p. 224),

$$(6) \quad J_n(t) \equiv \int_{-\infty}^{\infty} e^{itu}\frac{1}{\sqrt{n}}\Psi_n\left(\frac{u}{\sqrt{n}}\right)du$$

converges uniformly to $e^{-\frac{1}{2}t^2}$. Let G_n be the distribution function of $V^{(n)}/\sqrt{n}$. Then, for each t,

$$(7) \quad h(t) \equiv \int_{-\infty}^{\infty} e^{itu}f(u)du =$$

$$\int_0^\infty \int_{-\infty}^{\infty} e^{itu}\frac{1}{\sqrt{n}\,v}\Psi_n\left(\frac{u}{\sqrt{n}\,v}\right) du\, G_n(dv) =$$

$$\int_0^\infty [J_n(tv) - e^{-\frac{1}{2}t^2v^2}]\, G_n(dv) + \int_0^\infty e^{-\frac{1}{2}t^2v^2} G_n(dv) \equiv$$

$$A_n(G_n) + \int_0^\infty e^{-\frac{1}{2}t^2v^2} G_n(dt).$$

Since $J_n(\cdot)$ converges to $e^{-\frac{1}{2}(\cdot)^2}$ uniformly, $\lim_{n\to\infty} |A_n(G_n)| = 0$ for any sequence $\{G_n\}_{n=1}^{\infty}$. Let G_∞ be a weak limit point of the sequence $\{G_n\}$ (G_∞ exists by Helly's Theorem--see Chung (1974), p. 83). Since $e^{-\frac{1}{2}t^2v^2}$ is a bounded continuous function of $v \in R^1$, we can take the limit of the right hand side of (8) to obtain

$$(8) \quad h(t) = \int_0^\infty e^{-\frac{1}{2}t^2v^2} G_\infty(dv) \ ,$$

for each $t \in R^1$. Since $h(0) = 1$, G_∞ is a proper distribution function on $[0,\infty)$.

To show $G_\infty(0) = 0$, consider random variables W and V_0 with $W \perp\!\!\!\perp V_0$, $\mathcal{L}(W) = N(0,1)$ and $\mathcal{L}(V_0) = G_\infty$. Then

$$(9) \quad \mathcal{E}\, e^{itWV_0} = \int_0^\infty e^{-\frac{1}{2}t^2v^2}\, G_\infty(dv) = h(t)$$

which implies that $\mathcal{L}(WV_0) = \mathcal{L}(Z)$. Since $P\{Z = 0\} = 0$, it follows that $P\{V_0 = 0\} = 0$ and hence $G_\infty \in \mathcal{Q}$. By the uniqueness of characteristic functions,

$$(10) \quad f(z) = \int_0^\infty \frac{1}{v}\, \varphi(\frac{z}{v})\, G_\infty(dv) \qquad \text{(a.e.)}$$

and this completes the proof.

<u>Remark</u>: Theorem 2 is a formal statement of "$\mathcal{L}(Z) \in \mathfrak{F}_{i,\infty}$ iff $\mathcal{L}(Z)$ is a scale mixture of normal distributions with mean 0". Also, Theorem 2 shows that $\mathfrak{F}_{i,\infty}$ is a convex set whose set of extreme points is $\{\frac{1}{v}\,\varphi(\frac{\cdot}{v}) |\ v > 0\}$.

As pointed out in Andrews and Mallows (1974), the symmetric stable laws, the double exponential distribution and the logistic distribution are all scale mixtures of normals. Thus, these distributions are in $\mathfrak{F}_{i,\infty}$. The following proposition allows us to give some other examples of distributions in $\mathfrak{F}_{i,\infty}$.

Proposition 1: Consider a density f on R^1 given by

$$(11) \quad f(z) = \int_0^\infty \frac{1}{v} \varphi(z/v) G(dv)$$

where $G \in \mathcal{G}$ and assume that

$$(12) \quad f(0) = \left(\frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{1}{v} G(dv)\right) < +\infty.$$

Then $\hat{f}(t) = \int_{-\infty}^\infty e^{itz} f(z)dz$ is integrable on R^1, $\int_{-\infty}^\infty \hat{f}(t)dt = 2\pi f(0)$ and $\hat{f}/2\pi f(0)$ is a density whose distribution is in $\mathfrak{F}_{i,\infty}$.

Proof: From (11),

$$(13) \quad \hat{f}(t) = \int_0^\infty e^{-\frac{1}{2}t^2v^2} G(dv) \geq 0$$

so

$$(14) \quad \int_{-\infty}^\infty \hat{f}(t)\, dt = \sqrt{2\pi} \int_0^\infty \frac{1}{v} G(dv) = 2\pi f(0).$$

Thus,

$$(15) \quad \frac{\hat{f}(t)}{2\pi f(0)} = \int_0^\infty v\varphi(tv) \frac{G(dv)}{\sqrt{2\pi}\, v f(0)}$$

so $\hat{f}/2\pi f(0)$ is a scale mixture of normal distributions with mixing distribution $G(dv)/\sqrt{2\pi}\, v\, f(0)$. The proof is complete.

As an application, consider f to be the density of a symmetric stable law of order α. Then $\hat{f}(t) = e^{-|t|^\alpha}$ and we see that $k(\alpha)e^{-|t|^\alpha}$ is a density (for proper choice of $k(\alpha)$). Thus $k(\alpha)e^{-|t|^\alpha}$ defines a distribution in $\mathfrak{F}_{i,\infty}$ for $0 < \alpha \leq 2$.

Proposition 2: Consider $\mathcal{L}(Z_1) \in \mathfrak{F}_{i,\infty}$ and $\mathcal{L}(Z_2) \in \mathfrak{F}_{i,\infty}$ with $Z_1 \perp\!\!\!\perp Z_2$. Then $\mathcal{L}(Z_1 + Z_2) \in \mathfrak{F}_{i,\infty}$ and $\mathcal{L}(Z_1 Z_2) \in \mathfrak{F}_{i,\infty}$.

Proof: Consider W_j and V_j, $j = 1,2$ all independent with $\mathcal{L}(W_j) = N(0,1)$, $j = 1,2$ and $\mathcal{L}(W_j V_j) = \mathcal{L}(Z_j)$ for $j = 1,2$. Then $\mathcal{L}(Z_1 + Z_2) = \mathcal{L}(W_1V_1 + W_2V_2) = \mathcal{L}(W_3\sqrt{V_1^2 + V_2^2})$ where $W_3 \perp\!\!\!\perp \sqrt{V_1^2 + V_2^2}$ and $\mathcal{L}(W_3) = N(0,1)$. Thus, $\mathcal{L}(Z_1 + Z_2)$

is a scale mixture of normals so $\mathcal{L}(Z_1 + Z_2) \in \mathcal{F}_{i,\infty}$. Also $\mathcal{L}(Z_1Z_2) = \mathcal{L}(W_1W_2V_1V_2) = \mathcal{L}(W_1|W_2|V_1V_2) = \mathcal{L}(W_1V_3)$ where $W_1 \perp\!\!\!\perp V_3$ and $V_3 = |W_2|V_1V_2$. Hence $\mathcal{L}(Z_1Z_2) \in \mathcal{F}_{i,\infty}$.

If $\mathcal{L}(Z) \in \mathcal{F}_{i,\infty}$, then for each n there is a convenient form for an n-dimensional version of $\mathcal{L}(Z)$; namely, the random vector X with a density given by

$$(16)\quad h(x) = \int_0^\infty \frac{1}{(\sqrt{2\pi}\, v)^n} e^{-\frac{1}{2}\frac{x'x}{v^2}} G(dv) \ , \quad x \in R^n$$

where $\mathcal{L}(Z)$ has a density given by

$$(17)\quad f(z) = \int_0^\infty \frac{1}{v}\varphi\left(\frac{z}{v}\right) G(dv).$$

Thus, if the mixing distribution G is known for a particular distribution $\mathcal{L}(Z) \in \mathcal{F}_{i,\infty}$, then we have an explicit form for an n-dimensional version of $\mathcal{L}(Z)$. For the specific case of the double exponential distribution, the mixing distribution is given in Andrews and Mallows (1974).

§3. Finite variance case:

From the definition of $\mathcal{F}_n$, it is clear that $\mathcal{F}_n \supseteq \mathcal{F}_{i,n}$ and $\mathcal{F}_n \supseteq \mathcal{F}_{n+1}$.

Theorem 4: Suppose $\mathcal{L}(Z) \in S_1$ and $Var(Z) < +\infty$. Then $\mathcal{L}(Z) \in \mathcal{F}_n$ iff $\mathcal{L}(Z) \in \mathcal{F}_{i,n}$.

Proof: If $\mathcal{L}(Z) \in \mathcal{F}_{i,n}$, then $\mathcal{L}(Z) \in \mathcal{F}_n$ by definition. Conversely, if $\mathcal{L}(Z) \in \mathcal{F}_n$, then there exists an $X \in R^n$ and $c\colon R^n \to [0,\infty)$ with $c(a) = 0$ iff $a = 0$ such that $\mathcal{L}(a'X) = \mathcal{L}(c(a)Z)$. Since $Var(z) < +\infty$, each coordinate of X, say X_i, must satisfy $\mathcal{E}X_i^2 < +\infty$. Hence X has a covariance matrix Σ. Since $\mathcal{L}(a'X) = \mathcal{L}(c(a)Z)$, $a'\Sigma a = Var(a'X) = c^2(a)Var(Z)$. Setting $\Sigma_1 = \Sigma/Var(Z)$ we have $c(a) = (a'\Sigma a)^{\frac{1}{2}}$ and Σ_1 is positive definite since $c(a) > 0$ if $a \neq 0$.

Let A denote the unique positive definite square root of Σ_1. Then, setting $b = Aa$ for $a \in R^n$, $\mathcal{L}(\|b\|Z) = \mathcal{L}((a'\Sigma_1 a)^{\frac{1}{2}}Z) = \mathcal{L}(a'X) = \mathcal{L}(a'AA^{-1}X)$ $= \mathcal{L}(b'\tilde{X})$ where $\tilde{X} = A^{-1}X$. Thus, for $\Gamma \in O(n)$, $\mathcal{L}(b'\Gamma\tilde{X}) = \mathcal{L}(\|\Gamma'b\|Z) = \mathcal{L}(\|b\|Z)$ $= \mathcal{L}(b'\tilde{X})$ for all $b \in R^n$. Hence, $\mathcal{L}(\Gamma\tilde{X}) = \mathcal{L}(\tilde{X})$ for $\Gamma \in O(n)$ and it is clear that $P\{\tilde{X} = 0\} = 0$. Therefore $\mathcal{L}(\tilde{X}) \in S_n$ and $\mathcal{L}(Z) \in \mathfrak{F}_{i,n}$. This completes the proof.

Theorem 4 shows that as long as $\mathrm{Var}(Z) < +\infty$ ($\mathcal{L}(Z) \in S_1$), any n-dimensional version is equivalent (up to a linear transformation) to an n-dimensional orthogonally invariant version. Thus, nothing new is obtained by generalizing from $\mathfrak{F}_{i,n}$ to $\mathfrak{F}_n$ or $\mathfrak{F}_{i,\infty}$ to $\mathfrak{F}_\infty$ when $\mathrm{Var}(Z) < +\infty$. However, when $\mathrm{Var}(Z) = +\infty$, the situation seems to be more complicated. First, consider the following example.

Example 1: Let Z_α be a symmetric stable random variable on R^1 with characteristic function $t \to e^{-|t|^\alpha}$. Also, consider $h_\alpha: R^n \to [0,\infty)$ defined by $h_\alpha(a) = (\sum_1^n |a_i|^\alpha)^{1/\alpha}$, $0 < \alpha < 2$. If $X_1, \ldots, X_n$ are i.i.d. with the distribution of Z_α, it is not hard to show (using characteristic functions) that $\mathcal{L}(a'X) = \mathcal{L}(h_\alpha(a)Z_\alpha)$ where $X' = (X_1, X_2, \ldots, X_n)$. Thus, X is an n-dimensional version of Z_α and it is clear that no nonsingular linear transformation of X will have an orthogonally invariant distribution. However, the results of section 2 show that $\mathcal{L}(Z_\alpha)$ does have an n-dimensional isotropic version. Thus, the symmetric stable laws ($\alpha < 2$) provide examples of distributions in S_1 which have non-equivalent (up to linear transformations) n-dimensional versions. Theorem 4 shows this cannot happen when $\mathcal{L}(Z) \in S_1$ and $\mathrm{Var}(Z) < +\infty$.

Although the symmetric stable laws ($0 < \alpha < 2$) do have n-dimensional versions which are not orthogonally invariant, they do not provide an example of a distribution in $\mathfrak{F}_n$ but not in $\mathfrak{F}_{i,n}$. The equality of $\mathfrak{F}_{i,n}$ and $\mathfrak{F}_n$ (or more particularly, $\mathfrak{F}_{i,\infty}$ and $\mathfrak{F}_\infty$) is an open question as far as I know.

Now, consider $\mathcal{L}(Z) \in \mathfrak{F}_\infty$ with $\text{Var}(Z) < +\infty$. Then $\mathcal{L}(Z) \in \mathfrak{F}_{i,\infty}$ and we have an n-dimensional isotropic version for each n. Further, one particular choice for such a version is the distribution with density (16) when $\mathcal{L}(Z)$ has a density given by (17). The family of distributions with densities given by (16) (G-fixed) seem to be a reasonable family with which to begin a study of non-normal multivariate analysis. Work is currently under way on such problems.

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