CHEMICALLY REACTING FLUID FLOWS: STRONG SOLUTIONS AND GLOBAL ATTRACTORS

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ABSTRACT. In this paper we consider the existence and properties of strong solutions for a model of incompressible chemically reacting flows where reactants enter the domain, react and then leave the domain. We show results which exactly parallel those of the Navier-Stokes equations, i.e., in two dimensions strong solutions exist for all time, and in three dimension we show existence only for small times. In two dimensions, we also show the existence of global attractors which are compact in L^2. Rather than considering a specific set of boundary conditions, we instead state our results based on a series of assumptions, which would be proved using the boundary conditions. This allows our results to be applied directly to the two sets of boundary conditions which appear in the literature.

1. INTRODUCTION

In this paper we will study the following model for chemically reacting fluid flows in nondimensional form

$$\partial_t u - Pr\Delta u + (u \cdot \nabla)u + \nabla p = f_0(T), \tag{1.1a}$$

$$\nabla \cdot u = 0, \tag{1.1b}$$

$$\partial_t T - \Delta T + (u \cdot \nabla)T = -\sum_{i=1}^{N} h_i W_i(Y_1, \ldots, Y_N, T), \tag{1.1c}$$

$$\partial_t Y_i - \frac{1}{Le}\Delta Y_i + (u \cdot \nabla)Y_i = W_i(Y_1, \ldots, Y_N, T), \tag{1.1d}$$

where Pr is the Prandl number, Le is the Lewis number, the terms with $(u \cdot \nabla)$ are fluid transport terms, $f_0(T)$ is the forcing term from buoyancy, $W_i(Y_1, \ldots, Y_N, T)$ describes the change in mass fractions due to the reaction, and h_i is the enthalpy of species i divided by its molecular weight, i.e., a measure of the amount of heat contained in species i. The first two equations are the usual Navier-Stokes equations, and the second two are reaction-diffusion equations with a transport term added. We assume that the equations hold in a C^2 bounded domain $\Omega \subset \mathbb{R}^d, d = 2$ or 3. We will assume that the W_i are bounded, Lipschitz functions and that $f_0(T)$ is the usual Boussinesq model for buoyancy

$$f_0(T) = -c_0\vec{g}(T - T_l) \tag{1.2}$$

where $c_0 > 0$ is a constant and $\vec{g}$ is a unit vector pointing in the direction of gravity.

Date: March 12, 1998.

1991 *Mathematics Subject Classification.* Primary 80A25; Secondary 35Q99, 35B40, 58F39.

Key words and phrases. Chemically Reacting Flow, Combustion, Arrhenius Kinetics, Incompressible Navier-Stokes Equations, Reaction Diffusion Equations, Boussinesq Models, Strong Solution .

This research was supported in part by the Army Research Office and the IMA at the University of Minnesota.

In this paper we will show the existence of strong solutions to this model which parallel the existence results for the Navier-Stokes equations. In particular, we will show that in two space dimensions strong solutions exist for all time, while in three dimensions we only show that strong solutions exist for a short time. The key step in the proof is to show that if the Navier-Stokes portion of the system has a strong solution, then the chemistry and temperature equations also have strong solutions.

Our method of proof is based on the Bubnov-Galerkin method of projecting the system onto a finite dimensional subspace of the infinite dimensional phase space, solving the resulting ODE, then taking the limit of the approximate solutions as the projection approaches the identity to get the final result. Such a proof involves two steps. First, one proves certain *a priori* estimate for the approximate solutions which show that they are bounded in the appropriate spaces, then one uses a compactness lemma to get the existence of the limiting solution.

One advantage of this method of showing the existence of strong solutions is that the boundary conditions play no direct role in the analysis. In particular, the H^1 and H^2 *a priori* estimates and the compactness depend only on certain properties of the Laplacian with the given boundary conditions and certain L^2 *a priori* estimates and elliptic regularity for the Laplacian with the associated boundary conditions. This allows us to state the hypotheses of our existence theorem without reference to the specific boundary conditions. Of course, in any specific case the verification of the hypotheses will depend on the boundary conditions, however they are typically straightforward extensions of the estimates needed for the existence of weak solutions.

One novel feature of our presentation of the existence results is that we structure our argument around a general existence theorem, which is in turn a wrapper around a well known compactness lemma. This method of presentation makes clear exactly what properties of the equations are needed for existence, along with how the properties are used.

In the final section of the paper we show the existence of global attractors for the reacting flow system in two dimensional domains. Our proof is based on the classical technique of showing that the system is both dissipative and compact. However, since showing dissipativity is typically done using energy estimates which depend heavily on the boundary conditions, we do not give a complete global attractor existence proof. Instead, our result essentially says that if the system is dissipative, then it has a global attractor. This is sufficient to get existence of global attractors for the two sets of boundary conditions considered in the literature, since in both cases the systems are known to be dissipative, see Norman [7] and Manley, Marion and Temam [5]. Our global attractor existence results are basically the same as those of Manley, Marion and Temam [5]. However they also give estimates of the dimension of the global attractor, an issue which we do not study.

As an example of how one can use the results of this paper, we would like to mention Norman [6] where these solutions are used in comparing the dynamics of the reacting flow model to the dynamics of a continuous flow stirred tank reactor (CSTR). A CSTR is an ODE approximation to a reacting flow based on the assumption that the chemical concentrations and temperature are spatially homogeneous. In Norman [6] it is shown that, for large chemical and thermal diffusivities, the CSTR ODE is a good approximation to the full reacting flow PDE, and in particular that the global attractor for the reacting flow converges to the global attractor for the CSTR in a suitable sense.

An outline of the paper is as follows. In section 2 we give an example of the type of boundary conditions which one might consider for the reacting flow system. In section 3 we give some notation and state some standing assumptions. In section 4 we state the main assumptions for the theorems, then state the theorems themselves. In section 5 we state certain preliminary lemmas which will be used in the subsequent proofs. In section 6 we state our general existence theorem and the compactness lemma on which it is based. In section 7 we analyze the equations in order to show that the existence theorem can be applied to them, and in section 8 we derive the needed *a priori* estimates. Then in section 9 we give proofs of the main existence theorems in the paper. Finally, in section 10 we give our global attractor existence proof, along with some background on global attractors.

2. Boundary Condition Example

In this section we want to give one example of the types of boundary conditions to which the theorems in this paper could be applied. The example we give was originally studied in Norman [7]. Its main feature is that the amount of the various chemicals and heat entering the system are specified as parameters of the system, which is an advantage when comparing the reacting flow model to certain reduced models, i.e., continuously stirred tank reactors (CSTRs), see Norman [6].

We begin by using Dirichlet boundary data for the velocity

$$u(x,t) = \phi_u(x) \qquad\qquad \text{for all } x \in \partial\Omega,$$

where ϕ_u is not identically zero. We then define the partition of the boundary

$$\Gamma_I \cup \Gamma_O \cup \Gamma_W = \partial\Omega$$

into inflow, outflow and wall portions, respectively. If we define

$$q(x) = \begin{cases} -u(x)\cdot n & \text{if } x \in \Gamma_I, \\ 0 & \text{otherwise,} \end{cases}$$

where $n(x)$ is the unit outward normal to $\partial\Omega$, and note that $q(x) \geq 0$. Now we can define the chemistry and temperature boundary conditions as

$$\begin{aligned} \frac{\partial T}{\partial n} + qT &= \phi_T, \\ \frac{1}{Le}\frac{\partial Y_i}{\partial n} + qY_i &= \phi_{Y_i}, \end{aligned}$$

where $q(x) = 0$ implies $\phi_T(x) = \phi_{Y_i}(x) = 0$. In the inflow portion of the boundary, i.e., where $u(x)\cdot n < 0$, these boundary conditions are of the Robin type, which has the effect of fixing the flux of the temperature and chemical species to be ϕ_T and ϕ_{Y_I}, respectively, see Norman [6]. On the outflow portion of the boundary, they are of Neumann type, which implies that the only flux across the boundary is due to fluid flow with none due to diffusion. Finally, there is no flux across the wall portion of the boundary.

3. Notation

In this section we want to give some notation that will be used in the rest of the paper. We will also state some of the assumptions which we make about the structure of the equations.

We begin by giving some notation. We let $\mathcal{B}_u$, $\mathcal{B}_T$ and $\mathcal{B}_Y$ be the linear boundary operators associated with the velocity, temperature and chemistry equations, respectively. Then, since we allow the possibility of inhomogeneous boundary conditions, we have that the solutions satisfy

$$\begin{aligned}
\mathcal{B}_u u(x,t) &= \phi_u(x) && \text{for } x \in \partial\Omega \text{ and } t \geq 0,\\
\mathcal{B}_T T(x,t) &= \phi_T(x) && \text{for } x \in \partial\Omega \text{ and } t \geq 0,\\
\mathcal{B}_Y Y_i(x,t) &= \phi_{Y_i}(x) && \text{for } x \in \partial\Omega \text{ and } t \geq 0.
\end{aligned}$$

We handle the inhomogeneous boundary conditions in the usual way by decomposing the solution into a time dependent part which satisfies homogeneous boundary conditions and a part constant in time which satisfies the inhomogeneous boundary conditions, i.e.,

$$\begin{aligned}
u(x,t) &= v(x,t) + w(x),\\
T(x,t) &= \theta(x,t) + \theta_0(x),\\
Y_i(x,t) &= \eta_i(x,t) + \eta_{i,0}(x),
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{B}_u w(x) &= \phi_u(x) && \text{for } x \in \partial\Omega,\\
\mathcal{B}_T \theta_0(x) &= \phi_T(x) && \text{for } x \in \partial\Omega,\\
\mathcal{B}_Y \eta_{i,0}(x) &= \phi_{Y_i}(x) && \text{for } x \in \partial\Omega,
\end{aligned}$$

hence

$$\begin{aligned}
\mathcal{B}_u v(x) &= 0 && \text{for } x \in \partial\Omega,\\
\mathcal{B}_T \theta(x) &= 0 && \text{for } x \in \partial\Omega,\\
\mathcal{B}_Y \eta_i(x) &= 0 && \text{for } x \in \partial\Omega.
\end{aligned}$$

We will assume that w, θ_0 and $\eta_{i.0}$ are fixed throughout the paper.

We now define the spaces in which we will find the solutions v, θ and η_i. For the velocity we have

$$\begin{aligned}
H &= \mathrm{Closure}_{L^2(\Omega)^3}\left\{v \in C^\infty(\Omega)^3 : \nabla\cdot v = 0, \mathcal{B}_u v = 0\right\},\\
V &= \mathrm{Closure}_{H^1(\Omega)^3}\left\{v \in C^\infty(\Omega)^3 : \nabla\cdot v = 0, \mathcal{B}_u v = 0\right\},
\end{aligned}$$

i.e., H and V are the spaces of divergence free vector fields with the proper boundary conditions in $L^2(\Omega)$ and $H^1(\Omega)$, respectively. We will also denote the dual of V by V^{-1}. We then define the projection $\mathbb{P}$ to be the orthogonal projection of $L^2(\Omega)$ onto H. In the case where we have Dirichlet boundary data $\mathbb{P}$ is the usual Leray projection. In this case using the divergence theorem it can be shown that any gradient is orthogonal to H, hence if we apply $\mathbb{P}$ to (1.1a) the pressure term will disappear, leaving us with a parabolic evolutionary equation. It is critical to our analysis that this property continue to hold, i.e., we will assume that all gradients are orthogonal to H.

We define similar concepts for the chemistry and temperature equations. For the temperature equation we have

$$\begin{aligned}
H_1 &= \mathrm{Closure}_{L^2(\Omega)}\left\{\theta \in C^\infty(\Omega) : \mathcal{B}_T\theta = 0\right\},\\
V_1 &= \mathrm{Closure}_{H^1(\Omega)}\left\{\theta \in C^\infty(\Omega) : \mathcal{B}_T\theta = 0\right\},
\end{aligned}$$

and for the chemistry equations we have

$$H_2 = \mathrm{Closure}_{L^2(\Omega)}\left\{\eta \in C^\infty(\Omega) : \mathcal{B}_Y \eta = 0\right\},$$
$$V_2 = \mathrm{Closure}_{H^1(\Omega)}\left\{\eta \in C^\infty(\Omega) : \mathcal{B}_Y \eta = 0\right\}.$$

We will in particular assume that $V_1 = \mathcal{D}(A_1^{1/2})$ and $V_2 = \mathcal{D}(A_2^{1/2})$.

We will also use the product spaces

$$\overline{H} = H \times H_1 \times \prod_{i=1}^{N} H_2,$$
$$\overline{V} = V \times V_1 \times \prod_{i=1}^{N} V_2$$

We now want to define notation for some of the terms in the equations. We begin with the linear parts of the equations and define the linear operators

$$\begin{aligned} A &= -\mathbb{P}\Delta && \text{with boundary conditions } \mathcal{B}_u = 0,\\ A_1 &= -\Delta && \text{with boundary conditions } \mathcal{B}_T = 0,\\ A_2 &= -\Delta && \text{with boundary conditions } \mathcal{B}_Y = 0,\end{aligned}$$

as the unbounded linear operators associated with the velocity, temperature and chemistry equations, respectively, and where $\mathbb{P}$ is the projection of L^2 onto the space H. We note that now the unknowns v, θ and η_i are in the domains of their respective operators.

We also define the following bilinear and trilinear forms associated with the Navier-Stokes nonlinearity and chemistry and temperature transport terms

$$\begin{aligned} B(u,v) &= \mathbb{P}(u\cdot\nabla)v,\\ B_1(u,\psi) &= (u\cdot\nabla)\psi,\\ b(u,v,w) &= \langle (u\cdot\nabla)v, w\rangle,\\ b_1(u,\psi_1,\psi_2) &= \langle (u\cdot\nabla)\psi_1, \psi_2\rangle.\end{aligned}$$

We note that if $w \in H$, then $\langle B(u,v), w\rangle = b(u,v,w)$.

Using this notation, after applying $\mathbb{P}$ to the velocity equation, the reacting flow system becomes

$$\partial_t v + PrAv + B(v+w, v+w) = f(\theta) \quad \text{in } \Omega, \tag{3.4a}$$
$$\partial_t \theta + A_1\theta + B_1(v+w, \theta+\theta_0) = g(\eta_1,\ldots,\eta_N,\theta) \quad \text{in } \Omega, \tag{3.4b}$$
$$\partial_t \eta_i + \frac{1}{Le}A_2\eta_i + B_1(v+w, \eta_i+\eta_{i,0}) = \omega_i(\eta_1,\ldots,\eta_N,\theta) \quad \text{in } \Omega, \tag{3.4c}$$
$$\mathcal{B}_u v(x) = 0 \quad \text{for } x \in \partial\Omega, \tag{3.4d}$$
$$\mathcal{B}_T \theta(x) = 0 \quad \text{for } x \in \partial\Omega, \tag{3.4e}$$
$$\mathcal{B}_Y \eta_i(x) = 0 \quad \text{for } x \in \partial\Omega, \tag{3.4f}$$

where

$$f(\theta) = \mathbb{P}f_0(\theta+\theta_0) + Pr\mathbb{P}\Delta w, \tag{3.5a}$$

$$g(\eta_1,\dots,\eta_N,\theta) = -\sum_{i=1}^{N} h_i W_i(\eta_1+\eta_{1,0},\dots,\eta_N+\eta_{N,0},\theta+\theta_0) + \Delta\theta_0, \tag{3.5b}$$

$$\omega_i(\eta_1,\dots,\eta_N,\theta) = W_i(\eta_1+\eta_{1,0},\dots,\eta_N+\eta_{N,0},\theta+\theta_0) + \frac{1}{Le}\Delta\eta_{i,0}. \tag{3.5c}$$

For simplicity later in the paper, we introduce the notation

$$\begin{aligned}
F_v(v,\theta,\eta_i) &= -PrAv - B(v+w,v+w) + f(\theta),\\
F_\theta(v,\theta,\eta_i) &= -A_1\theta - B_1(v+w,\theta+\theta_0) + g(\eta_1,\dots,\eta_N,\theta)\\
F_{\eta_i}(v,\theta,\eta_i) &= -\frac{1}{Le}A_2\eta_i - B_1(v+w,\eta_i+\eta_{i,0}) + \omega_i(\eta_1,\dots,\eta_N,\theta),
\end{aligned}$$

where f, g and ω_i are given by (3.5). Then the reacting flow system (3.4) becomes simply

$$\begin{aligned}
\partial_t v &= F_v(v,\theta,\eta_i),\\
\partial_t \theta &= F_\theta(v,\theta,\eta_i),\\
\partial_t \eta_i &= F_{\eta_i}(v,\theta,\eta_i).
\end{aligned}$$

As mentioned in the introduction we will show the existence of solutions using the Bubnov-Galerkin approach of projecting the equations onto finite dimensional subspaces. To do this we assume that P_u^n, P_T^n and P_Y^n are orthogonal projections onto n dimensional subspaces of H, H_1 and H_2, respectively. We further assume that

$$\begin{aligned}
&\lim_{n\to\infty}(I-P_u^n)u = 0 && \text{for all } u\in H,\\
&\lim_{n\to\infty}(I-P_T^n)\psi = 0 && \text{for all } \psi\in H_1,\\
&\lim_{n\to\infty}(I-P_Y^n)\psi = 0 && \text{for all } \psi\in H_2.
\end{aligned}$$

We can then define the n^{th} order Bubnov-Galerkin approximate solution (v^n,θ^n,η_i^n) as the solution of the ODE

$$\partial_t v^n = P_u^n F_v(v^n,\theta^n,\eta_i^n), \tag{3.6a}$$

$$\partial_t \theta^n = P_T^n F_\theta(v^n,\theta^n,\eta_i^n), \tag{3.6b}$$

$$\partial_t \eta_i^n = P_Y^n F_{\eta_i}(v^n,\theta^n,\eta_i^n), \tag{3.6c}$$

$$v^n(0) = P_u^n v(0), \tag{3.6d}$$

$$\theta^n(0) = P_T^n \theta(0), \tag{3.6e}$$

$$\eta_i^n(0) = P_Y^n \eta_i(0). \tag{3.6f}$$

4. Assumptions and Statements of Theorems

In this section we will state the assumptions which we make on the system. Proving these assumptions will in particular involve the specific boundary conditions of interest. We will also give a brief sketch of the properties that are required to verify the assumptions for a specific set of boundary conditions.

In particular, if we define A, A_1, A_2 as in section 3, then we assume the following.

Assumption 1. A, A_1 and A_2 are positive, self-adjoint operators with compact inverses satisfying

$$V = \mathcal{D}(A^{1/2}) \qquad V_1 = \mathcal{D}(A_1^{1/2}) \qquad V_2 = \mathcal{D}(A_2^{1/2}).$$

Typically, the positivity and self-adjointness of the A_i is shown using integration by parts arguments, while the compact inverse follows from elliptic regularity.

Assumption 2. Suppose that for all scalar functions ϕ the projection $\mathbb{P}$ satisfies

$$\mathbb{P}\nabla\phi = 0.$$

This assumption is typically a direct consequence of the boundary conditions for the velocity.

Assumption 3. One has the following equivalent norms between Sobolev spaces and fractional power spaces for $\alpha = 1, 2$.

$$C_e^1\|v\|^2_{H^\alpha(\Omega)} \le \|A^{\alpha/2}v\|^2 \le C_e^2\|v\|^2_{H^\alpha(\Omega)} \qquad \text{for all } v \in V^\alpha, \tag{4.1a}$$

$$C_e^1\|\psi\|^2_{H^\alpha(\Omega)} \le \|A_i^{\alpha/2}\psi\|^2 \le C_e^2\|\psi\|^2_{H^\alpha(\Omega)} \qquad \text{for all } \psi \in V_i^\alpha, \tag{4.1b}$$

For the case $\alpha = 1$, the equivalent norms follow from integration by parts arguments and a Poincaré inequality. For the case $\alpha = 2$, they follow from elliptic regularity.

Assumption 4. All of the homogenizing terms w, θ_0 and $\eta_{i,0}$ are in $H^2(\Omega)$.

This assumption depends on being able to extend the ϕ_i smoothly into the interior of the domain, which in turn depends on the smoothness of the ϕ_i and trace theorems.

Assumption 5. For each n the solutions of the Bubnov-Galerkin approximate system (3.6) exist for all time.

This assumption and the following assumption would all be verified using L^2 energy estimates.

Assumption 6. Each Bubnov-Galerkin approximate solution $(v^n, \theta^n, \eta_i^n)$ satisfies the estimate

$$\|\theta^n(t)\|^2 \le C_1\left(1 + \|\theta^n(0)\|^2 + \|v^n(0)\|^2 + \sum_{i=1}^N \|\eta_i^n(0)\|^2\right) \qquad \text{for all } t > 0, \tag{4.2}$$

where C_1 is independent of n.

Also, for our two dimensional existence theorem and global attractor existence theorem, we will require the following estimates.

Assumption 7. Assume that for each $\tau > 0$, there exists a constant $\overline{C}_1$, independent of n but possibly depending on the initial conditions and τ and a constant C_2 independent of n and τ and the initial conditions such that for all $0 \le t_0 \le t \le \tau$ one has

$$\|v^n(t)\|^2 \le \overline{C}_1 \tag{4.3}$$

(4.4)

$$\int_{t_0}^t \|A^{1/2}v^n(s)\|^2\,ds \le C_2 \max\{1, t - t_0\}\left(\|v^n(t_0)\|^2 + \|\theta^n(t_0)\|^2 + \sum_{i=1}^N \|\eta_i^n(t_0)\|^2 + 1\right)$$

We will also need the following norm bounds for our uniqueness theorem. Typically, such bounds will be proved using integration by parts, a Poincaré inequality and the equivalent norms (4.1).

Assumption 8. Assume that there exists a constant C_3 such that

$$C_3\|A^{1/2}v\|^2 \leq \Pr\langle Av, v\rangle + b(u, v, v), \tag{4.5a}$$

$$C_3\|A_1^{1/2}\theta\|^2 \leq \langle A_1\theta, \theta\rangle + b_1(u, \theta, \theta), \tag{4.5b}$$

$$C_3\|A_2^{1/2}\eta\|^2 \leq \frac{1}{Le}\langle A_2\eta, \eta\rangle + b_1(u, \eta, \eta), \tag{4.5c}$$

for all u, v, θ, and η satisfying

$$\begin{aligned} \nabla\cdot u &= 0, \\ \mathcal{B}_u u &= \phi_u, \\ \nabla\cdot v &= 0, \\ v &\in \mathcal{D}(A), \\ \theta &\in \mathcal{D}(A_1), \\ \eta &\in \mathcal{D}(A_2). \end{aligned}$$

This assumption would be shown using integration by parts and a Poincaré inequality.

Finally, for our global attractor existence theorem, we will require the following estimate.

Assumption 9. Assume that for each $\tau > 0$, there exists a constant C_4, independent of n but depending on the initial conditions, such that for all $0 \leq t_0 \leq t \leq \tau$ one has

$$\int_{t_0}^{t} \|A^{1/2}\theta^n(s)\|^2\, ds \leq C_4 \max\{1, t - t_0\}\left(\|\theta^n(t_0)\| + \|v^n(t_0)\| + \sum_{i=1}^{N}\|\eta_i^n(t_0)\| + 1\right) \tag{4.6}$$

$$\int_{t_0}^{t} \|A^{1/2}\eta_i^n(s)\|^2\, ds \leq C_4 \max\{1, t - t_0\}\left(\|\theta^n(t_0)\| + \|v^n(t_0)\| + \sum_{i=1}^{N}\|\eta_i^n(t_0)\| + 1\right) \tag{4.7}$$

This assumption would follow from an L^2 energy estimate.

We can now state our existence theorem.

Theorem 1. *Let $\Omega \subset \mathbb{R}^d, d = 2, 3$ be a C^2 bounded domain. Assume that f_0 and W satisfy the assumptions given in the introduction, and that assumptions 1 through 6 above are satisfied. Then for each initial condition*

$$\begin{aligned} v(0) &\in V \subset H^1(\Omega), \\ \theta(0) &\in V_1 \subset H^1(\Omega), \\ \eta_i(0) &\in V_2 \subset H^1(\Omega), \end{aligned}$$

there exists a time $\tau > 0$ such that on $[0, \tau]$ there exists a strong solution to the homogenized chemically reacting flow system (3.4). Furthermore, if $d = 2$ and assumption 7 is satisfied, then the result is true for all $\tau > 0$.

We also have that the solution satisfies the following properties.

1. *The components of solution satisfy*

$$v \in L^2(0,T;\mathcal{D}(A)) \cap L^\infty(0,T;V), \tag{4.8a}$$

$$\theta \in L^2(0,T;\mathcal{D}(A_1)) \cap L^\infty(0,T;V_1), \tag{4.8b}$$

$$\eta_i \in L^2(0,T;\mathcal{D}(A_2)) \cap L^\infty(0,T;V_2). \tag{4.8c}$$

2. *The time derivatives of the solution satisfy*

$$\partial_t v \in L^2(0,T;H), \tag{4.9a}$$

$$\partial_t \theta \in L^2(0,T;H_1), \tag{4.9b}$$

$$\partial_t \eta_i \in L^2(0,T;H_2). \tag{4.9c}$$

3. *The equations are satisfied in a weak integrated form in $L^2(\Omega)$, i.e.,*

$$\langle v(t) - v(t_0), \overline{v}\rangle = \int_{t_0}^{t} \langle F_v(v,\theta,\eta_i), \overline{v}\rangle \, ds, \tag{4.10a}$$

$$\langle \theta(t) - \theta(t_0), \overline{\theta}\rangle = \int_{t_0}^{t} \langle F_\theta(v,\theta,\eta_i), \overline{\theta}\rangle \, ds, \tag{4.10b}$$

$$\langle \eta_i(t) - \eta_i(t_0), \overline{\eta}\rangle = \int_{t_0}^{t} \langle F_{\eta_i}(v,\theta,\eta_i), \overline{\eta}\rangle \, ds. \tag{4.10c}$$

for each $\overline{v} \in H, \overline{\theta} \in H_1, \overline{\eta} \in H_2$ and $0 \le t_0 < t \le T$.

We also have the following uniqueness theorems. We should point out that the stated properties of weak solutions are typical for weak solutions in two dimensions.

Theorem 2. *Suppose that assumptions 1 through 6 and 8 above are satisfied. Let $(v^1, \theta^1, \eta_i^1)$ be a strong solution of the reacting flow system* (3.4) *on $[0,\tau]$, and let $(v^2, \theta^2, \eta_i^2)$ be a weak solution of the equations on $[0,\tau]$ satisfying*

$$v^2 \in L^2(0,\tau;V) \cap L^\infty(0,\tau;H) \tag{4.11a}$$

$$\theta^2 \in L^2(0,\tau;V_1) \cap L^\infty(0,\tau;H_1) \tag{4.11b}$$

$$\eta_i^2 \in L^2(0,\tau;V_2) \cap L^\infty(0,\tau;H_1) \tag{4.11c}$$

and

$$\partial_t v^2 \in L^2(0,\tau;V^{-1}) \tag{4.12a}$$

$$\partial_t \theta^2 \in L^2(0,\tau;V_1^{-1}) \tag{4.12b}$$

$$\partial_t \eta_i^2 \in L^2(0,\tau;V_2^{-1}). \tag{4.12c}$$

Suppose that both solutions have the same initial condition. Then they coincide on $[0,\tau]$.

We will give proofs of theorems 1 and 2 in section 9.

Corollary 3. *Let $\Omega \subset \mathbb{R}^2$ and suppose that assumptions 1 through 6 and 8 are satisfied. Then strong solutions to the reacting flow system are unique in the class of weak solutions.*

Proof. In two dimensions, the Sobolev estimates imply that the derivative properties (4.12) in the statement of theorem 2 are satisfied for all weak solutions, hence theorem 2 gives the result. ∎

Corollary 4. *Let $\Omega \subset \mathbb{R}^3$ and suppose that assumptions 1 through 6 and 8 are satisfied. Then strong solutions are unique.*

Proof. The corollary follows from theorem 2 and the fact that strong solutions satisfy the properties (4.12) in the statement of theorem 2 □

Assuming that suitable types of weak solutions exist in two dimensions, then we can show that the weak solutions immediately become strong solutions. This theorem becomes important in our global attractor existence theorem, since there we will want to allow for initial conditions in $L^2(\Omega)$.

Corollary 5. *Let $\Omega \subset \mathbb{R}^2$ and suppose that the assumptions of theorem 1 are satisfied. Suppose further that (v, θ, η_i) is a weak solution satisfying the conditions in the statement of theorem 2. Then the weak solution is a strong solution on any interval $[t_1, t_2]$ satisfying $0 < t_1 < t_2 < \infty$.*

Proof. Fix $t_1 > 0$. Then by the condition (4.11) we have that there exists a $\tau \in (0, t_1)$ such that

$$v(\tau) \in V, \theta(\tau) \qquad \in V_1, \eta_i(\tau) \in V_2.$$

Therefore, we can apply theorem 1 and corollary 3 to get the result. □

We also have the following global attractor existence theorem.

Theorem 6. *Let $\Omega \subset \mathbb{R}^2$. Suppose that assumptions 1 through 4 and assumption 7 are satisfied. Suppose also that for each initial condition in $L^2(\Omega)$ there exists a weak solution satisfying the properties in the statement of theorem 2 and that the system is dissipative in $L^2(\Omega)$, in particular that for all $t > 0$ the estimate*

$$\|v(t)\|^2 + \|\theta(t)\|^2 + \sum_{i=1}^{N} \|\eta_i(t)\|^2 \leq C_5 e^{-\sigma t}(\|\theta(0)\|^2 + \|v(0)\|^2 + \sum_{i=1}^{N} \|\eta_i(0)\|^2) + C_6, \tag{4.13}$$

holds, where σ, C_5 and C_6 are independent of the initial conditions and t. Then the reacting flow system (3.4) has a global attractor which is compact in $L^2(\Omega)$ and which attracts all sets bounded in $L^2(\Omega)$.

We will give the proof for these theorems in section 9.

We would also like to mention the issue of showing that the solutions conserve mass and that the mass fractions remain between zero and one. In Norman [7] solutions possessing these properties were said to be physically reasonable. When the solutions have enough smoothness, the proof of physical reasonableness follows directly from the maximum principle. In fact, with the degree of smoothness that these strong solutions posses, it is possible to use maximum principle arguments to show physical reasonableness. The usual trick of multiplying the solution by

$$Y^-(x,t) = \begin{cases} Y(x,t) & \text{if } Y(x,t) < 0, \\ 0 & \text{otherwise.} \end{cases}$$

and integrating in space is sufficient to prove the result, see Manley, Marion and Temam [5] for details.

5. Preliminaries

In this section we will state several preliminary lemmas which we will be using later in the paper. We begin with the following lemma which will be used in proving *a priori* estimates.

Lemma 7. *Let V, H, V' be three Hilbert spaces, each being included and dense in the following and with V' the dual of V. Suppose further that $\psi \in L^2(0,\tau;V)$ and $\partial_t\psi \in L^2(0,\tau;V')$. Then one has*

$$\frac{1}{2}\partial_t\|\psi\|_H^2 = \langle \partial_t\psi, \psi\rangle \qquad \textit{almost everywhere in } [0,\tau]. \tag{5.1}$$

For a proof, see Sell and You [9] or Temam [10].

We will also need the following uniform Gronwall inequality.

Lemma 8. *Let $y(t)$ be absolutely continuous and $h(t)$ and $g(t)$ be locally integrable functions, with $h(t)$ positive. Then for $0 < t$ one has*

$$y(t) \le \left(\frac{1}{t-\tau}\int_\tau^t y(s)\,ds + \int_\tau^t h(s)\,ds\right)\exp\left(\int_\tau^t g(s)\,ds\right), \tag{5.2}$$

where $\tau = \max\{0, t-1\}$.

For a proof see Sell and You [9].

We will also need the usual Sobolev estimates for the bilinear and trilinear forms, see Sell and You [9] or Constantine and Foias [2], namely that there is a constant C_s depending only on $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ such that

$$\|B(u,v)\| \le C_s\|u\|_{H^1}^{3/4}\|u\|_{H^2}^{1/4}\|v\|_{H^1}^{3/4}\|v\|_{H^2}^{1/4} \tag{5.3a}$$

$$\|B_1(u,\psi)\| \le C_s\|u\|_{H^1}^{3/4}\|u\|_{H^2}^{1/4}\|\psi\|_{H^1}^{3/4}\|\psi\|_{H^2}^{1/4} \tag{5.3b}$$

$$|b(u,v,w)| \le C_s\|u\|_{H^1}\|v\|_{H^1}^{1/2}\|v\|_{H^2}^{1/2}\|w\|, \tag{5.3c}$$

$$|b(u,v,w)| \le C_s\|u\|_{H^2}\|v\|_{H^1}\|w\|, \tag{5.3d}$$

$$|b(u,v,w)| \le C_s\|u\|^{1/2}\|u\|_{H^1}^{1/2}\|v\|_{H^2}\|w\|, \tag{5.3e}$$

$$|b(u,v,w)| \le C_s\|u\|_{H^1}\|v\|_{H^1}^{1/2}\|v\|_{H^2}^{1/2}\|w\|, \tag{5.3f}$$

$$|b_1(u,\phi,\psi)| \le C_s\|u\|_{H^1}\|\phi\|_{H^1}^{1/2}\|\phi\|_{H^2}^{1/2}\|\psi\|, \tag{5.3g}$$

for u, v, w vector fields and ϕ, ψ scalars which are in the appropriate Sobolev spaces. We also have the estimates for $\Omega \subset \mathbb{R}^2$

$$|b(u,v,w)| \le C_s\|u\|^{1/2}\|u\|_{H^1}^{1/2}\|v\|_{H^1}^{1/2}\|v\|_{H^2}^{1/2}\|w\|, \tag{5.4}$$

(5.5)

We will also use the estimates

$$|b(u,v,w)| \le C_s\|u\|_{H^1}\|v\|_{H^2}\|w\|, \tag{5.6a}$$

$$|b_1(u,\phi,\psi)| \le C_s\|u\|_{H^1}\|\phi\|_{H^2}\|\psi\|, \tag{5.6b}$$

which follow from (5.3f) and (5.3g). Furthermore, we will freely use the equivalent norms in (4.1) in these estimates instead of the corresponding Sobolev norms, possibly by picking a larger C_s.

6. General Existence Theorem

For ease of exposition of our existence proofs, we will use the following general existence theorem, which is basically just a wrapper around a well known compactness lemma. The main advantage of this approach is that it makes clear exactly what properties of the equations we are using and how they are used.

Our general existence theorem is set in the following situation. For each equation, we assume the existence of three Hilbert spaces

$$X_i^1 \hookrightarrow X_i \hookrightarrow X_i^{-1} \qquad i = 1, \dots, N,$$

where in each case the embedding is dense and compact. Although we do not assume that X_i^{-1} is the dual of X_i^1, we do assume that it is contained in the dual. We also define

$$\overline{W}^1 = \prod_{i=1}^N X_i^1,$$

$$\overline{W} = \prod_{i=1}^N X_i,$$

$$\overline{W}^{-1} = \prod_{i=1}^N X_i^{-1}.$$

We further assume that we have a system of equations

$$\partial_t \phi_i = F_i(\phi_1, \dots \phi_N) \qquad \text{for } i = 1, \dots N.$$

If we let $P_i^n, i = 1, \dots N$ be an orthogonal projection onto an n dimensional subspace of X_1^i consisting of smooth functions, then we can define the n^{th} order Bubnov-Galerkin system for approximate solutions to the real solution with initial condition $(\phi_i(0))$ as

$$\partial_t \phi_i^n = P_i^n F_i(\phi_1^n, \dots \phi_N^n) \qquad \text{for } i = 1, \dots N. \tag{6.1a}$$

$$\phi_i^n(0) = P_i^n \phi_i(0) \qquad \text{for } i = 1, \dots N. \tag{6.1b}$$

We note that (3.6) is a special case of this construction.

Our existence theorem is basically a wrapper around the following compactness lemma.

Lemma 9 (Compactness Lemma). *Fix $\tau > 0$. Assume that, for $i = 1, \dots, N$ we have three Hilbert spaces X_i^1, X_i and X_i^{-1} as above. Let ϕ_i^n be a sequence in $L^2(0, \tau; X_i^1)$ such that ϕ_i^n is bounded in $L^2(0, \tau; X_i^1)$ and the sequence $\partial_t \phi_i^n$ is bounded in $L^p(0, \tau; X_i^{-1})$, where $1 < p \leq \infty$. Then for each $i = 1, \dots, N$, there exists a subsequence, which we also denote by ϕ_i^n, and a function $\phi_i \in L^2(0, \tau; X_i^1)$, with $\partial_t \phi_i \in L^p(0, \tau; X_i^{-1})$, such that the following properties hold:*

1. *One has $\phi^n \xrightarrow{w} \phi$ weakly in $L^2(0, \tau; X_i^1)$*
2. *One has $\partial_t \phi^n \xrightarrow{w} \partial_t \phi$ weakly in $L^p(0, \tau; X_i^{-1})$*
3. *One has $\phi^n \to \phi$ strongly in $L^2(0, \tau; X_i)$*
4. *For each $t \in [0, \infty)$, one has $\phi^n(t) \to \phi(t)$ strongly in X_i^{-1}.*
5. *There is a set E in $(0, \infty)$ having measure zero, such that for $t \in \mathbb{R}^+ - E$, one has $\phi_i^n(t) \to \phi_i(t)$ strongly in X_i.*

This lemma has been used many times before, see for example Constantine and Foias [2] and Lions [4]. In particular Sell [8] gives an outline of the proof.

The key property that the right hand sides of the equations need to satisfy when applying the compactness lemma is the following.

Property 1 (Continuity Property). *A function $F_i(\phi_1,\ldots,\phi_N)$ is said to satisfy the continuity property if, for each sequence $\phi^j = (\phi_1^j,\ldots,\phi_N^j)$ in $L^2(0,\tau;\overline{W}^1)$ and limit function $\phi^0 = (\phi_1^0,\ldots,\phi_N^0)$ in $L^2(0,\tau;\overline{W}^1)$ satisfying the conclusions of the compactness lemma 9 for each $i = 1,\ldots N$ we have*

$$\lim_{j\to\infty} F_i(\phi_i^j) \xrightarrow{w} F_i(\phi_i^0) \qquad \text{weakly in } L^1(0,\tau;X_i^{-1}) \text{ for } 1\le i\le N.$$

Theorem 10 (General Existence Theorem). *Fix $\tau > 0$. Let X_i^1, X_i and X_i^{-1} be given as above and suppose that each of the $F_i, i = 1,\ldots N$ satisfies the continuity property 1 in the corresponding spaces. Further assume that for each $i = \ldots, N$, each of the Bubnov-Galerkin approximate solutions $\phi^n = (\phi_1^n,\ldots,\phi_N^n)$ exists on $[0,\tau]$ and that $(\phi_i^n, \partial_t\phi_i^n)$ is uniformly bounded in $L^2(0,\tau;X_i^1)\times L^p(0,\tau;X_i^{-1})$ independent of n for some $p > 1$.*

Then for each initial condition $\phi(0) = (\phi_1(0),\ldots,\phi_N(0))$ there exists a solution $\phi^n = (\phi_1^n,\ldots,\phi_N^n)$ satisfying the following properties.

1. *For each i, one has that ϕ_i is the limit of a subsequence of the ϕ_i^n in the sense of the compactness lemma. As a consequence one has that if the ϕ_i^n satisfy a property which is preserved under the convergence of the compactness lemma, then ϕ_i satisfies the same property.*
2. *For each $i = 1,\ldots N$ and for every $\overline{\phi}\in X_i^{-1}$ and $t > t_0$ one has*

$$\langle \phi_i(t)-\phi_i(t_0), \overline{\phi}\rangle_{X_i^{-1}} = \int_{t_0}^t \langle F_i(\phi_1,\ldots,\phi_N), \overline{\phi}\rangle\, ds \tag{6.2}$$

for every $0\le t_0 < t\le\tau$.

Proof. Because of the boundedness assumption on the $(\phi_i^n, \partial_t\phi_i^n)$ we can apply the compactness lemma and take a subsequence N times to get limiting functions $\phi_i, i = 1,\ldots,N$. If we continue to denote the subsequence with n, then we have that

$$\phi_i^n \to \phi_i,$$

where the convergence is in the sense of the compactness lemma. Also because of the continuity assumption on the F_i we have that

$$F_i(\phi_1^n,\ldots\phi_N^n) \xrightarrow{w} F_i(\phi_1,\ldots\phi_N) \qquad \text{weakly in } L^1(0,\tau;X_i^{-1}) \text{ for } 1\le i\le N.$$

We now just need to check property 2, i.e., that the limiting solutions satisfy the equations. Because the Bubnov-Galerkin approximations satisfy an ODE, we have that, for $0 < t < \tau$,

$$\phi_i^n(t)-\phi_i^n(t_0) = \int_{t_0}^t P_i^n F_i(\phi_1,\ldots\phi_N)\, ds \qquad \text{for } i = 1,\ldots N,$$

or we can write, for each $\overline{\phi}\in\overline{W}^{-1}$,

$$\langle\phi_i^n(t)-\phi_i^n(t_0), \overline{\phi}\rangle = \int_{t_0}^t P_i^n F_i(\phi_1,\ldots\phi_N)\, ds \qquad \text{for } i = 1,\ldots N. \tag{6.3}$$

We want to take the limit of this equation as $n\to\infty$ to get (6.2). We note that by property 4 of the compactness lemma, the left hand side of (6.3) converges to the left hand side of (6.2) for every t and t_0.

For the right hand side, we note, for each i,

$$\int_{t_0}^{t} \left|\langle P_n F_i(\phi_1^n,\dots,\phi_N^n) - F_i(\phi_1,\dots,\phi_N), \overline{\phi}\rangle\right| \, ds$$

$$\begin{aligned}
&\le \int_{t_0}^{t} \left|\langle P_n F_i(\phi_1^n,\dots,\phi_N^n) - F_i(\phi_1^n,\dots,\phi_N^n), \overline{\phi}\rangle\right| \, ds \\
&\quad + \int_{t_0}^{t} \left|\langle F_i(\phi_1^n,\dots,\phi_N^n) - F_i(\phi_1,\dots,\phi_N), \overline{\phi}\rangle\right| \, ds \\
&\le \int_{t_0}^{t} \left|\langle F_i(\phi_1^n,\dots,\phi_N^n), (I-P_n)\overline{\phi}\rangle\right| \, ds \\
&\quad + \int_{t_0}^{t} \left|\langle F_i(\phi_1^n,\dots,\phi_N^n) - F_i(\phi_1,\dots,\phi_N), \overline{\phi}\rangle\right| \, ds \\
&\le \int_{t_0}^{t} \left|\langle F_i(\phi_1^n,\dots,\phi_N^n), (I-P_n)\overline{\phi}\rangle\right| \, ds \\
&\quad + \int_{t_0}^{t} \left|\langle F_i(\phi_1^n,\dots,\phi_N^n) - F_i(\phi_1,\dots,\phi_N), \overline{\phi}\rangle\right| \, ds
\end{aligned}$$

The first term goes to zero, since the $F_i(\phi_1^n,\dots,\phi_N^n)$ are bounded in $L^2(0,T;X_i^{-1})$ and $(I-P_n)\overline{\phi}$ goes to zero in X^{-1} uniformly in t. The second term also goes to zero because of the continuity property of F_i. Hence we have that (6.2) is true for every t_0 and t, which imples that property 2 is satisfied. □

7. Analysis of the Equations

We now want to return to the reacting flow system and prove the properties necessary to apply the general existence theorem. One of the hypotheses of the theorem is that the derivatives of the approximate solutions are bounded in the space $L^p(0,\tau;\overline{W}^{-1})$. Since

$$\partial_t \phi_i^n = F_i(\phi_j^n),$$

we see that the boundedness of the derivative is really a question of whether F_i maps the ϕ_i^n into bounded sets. The way one typically proves this is to show that the ϕ^n themselves are bounded in various spaces, then to show that F_ϕ maps such bounded sets into bounded sets. In our case, we can summarize the boundedness properties of the F_ϕ we will use in the following way.

Property 2 (Boundedness Property)**.** *We say that the functions $F_i(\phi_1,\dots\phi_N), i = 1,\dots N$ satisfy the boundedness property for a given $p \in (1,\infty]$ in the spaces X_i^1, X_i and X_i^{-1} if, for each i, F_i maps sets which are bounded in $L^2(t_0,t;\overline{W}^1) \cap L^\infty(t_0,t;\overline{W})$ into sets which are uniformly bounded in $L^p(t_0,t;\overline{W}_i^{-1})$. In particular, we assume that for some function B_F we have for $t > t_0$*

$$\int_{t_0}^{t} \|F(\phi_1(s),\dots,\phi_n(s))\|_{X_i^{-1}}^p \, ds \le \tag{7.1}$$

$$\le B_F\left((t-t_0) + \max_i \int_{t_0}^{t} \|\phi_i(s)\|_{V_2}^2 \, ds + \max_i \operatorname{ess\,sup}_{t_0 \le s \le t} \|\phi_i(s)\|\right),$$

for all $i = 1, \ldots, N$.

For strong solutions to our chemically reacting flow we use the following spaces when we apply the existence theorem:

$$
\begin{aligned}
X_v^1 &= \mathcal{D}(A)\ X_v^1 = V X_v^{-1} = H, \\
X_\theta^1 &= \mathcal{D}(A_1) X_\theta^2 = V_1 X_\theta^{-1} = H_1, \\
X_\eta^1 &= \mathcal{D}(A_2) X_\eta^2 = V_2 X_\eta^{-1} = H_2.
\end{aligned} \tag{7.2}
$$

We recall that this implies that for each of the equations the triple of spaces is contained in the triple of Sobolev spaces $H^2(\Omega), H^1(\Omega)$ and $L^2(\Omega)$.

In the following series of lemmas we prove the continuity and boundedness properties for the various terms in the equations, then combine them to get the properties for the right hand sides.

Lemma 11. *The maps* $(u, v) \mapsto B(u, v)$ *and* $(u, \psi) \mapsto B_1(u, \psi)$ *satisfy the continuity property in the above spaces.*

Proof. We will actually show strong convergence rather than just weak convergence. We begin with $B(u, v)$. Pick $u_i \to u_0$ and $v_i \to v_0$ satisfying the conditions in the continuity property. Then one has

$$B(u_i, v_i) - B(u_0, v_0) = B(u_i - u_0, v_i) - B(u_0, v_0 - v_i).$$

Fixing $t > t_0$ and using the Hölder inequality and (5.3a), one obtains the estimates

$$
\begin{aligned}
\int_{t_0}^t \|B(u_i - u_0, v_i)\|\, ds \leq & C_7 \int_{t_0}^t \|Au_i - u_0\|^{1/4} \|A^{1/2}(u_i - u_0)\|^{3/4} \|Av_i\|^{1/4} \|A^{1/2} v_i\|^{3/4}\, ds \\
\leq & C_8 \left(\int_{t_0}^t \|Au_i - u_0\|^2 \right)^{1/8} \left(\int_{t_0}^t \|A^{1/2}(u_i - u_0)\|^2 \right)^{3/8} \times \\
& \times \left(\int_{t_0}^t \|Av_i\|^2 \right)^{1/8} \left(\int_{t_0}^t \|A^{1/2} v_i\|^2 \right)^{3/8}.
\end{aligned}
$$

The second term goes to zero by property 3 of the compactness lemma, while the other terms are bounded. The analysis of the term $B(u_0, u_0 - u_i)$ is identical. Hence the property is shown for $B(u, v)$.

The same proof using (5.3b) instead of (5.3a) gives the result for $B_1(u, \psi)$. ☐

Lemma 12. *The maps* $(u, v) \mapsto B(u, v)$ *and* $(u, \psi) \mapsto B_1(u, \psi)$ *satisfy the boundedness property where* $p = 2$.

Proof. Fix u, v satisfying the hypotheses of the boundedness property. For all $t > t_0$, using the Sobolev estimate (5.3a) and the Hölder inequality one obtains the estimate,

$$\begin{aligned}\int_{t_0}^{t} \|B(u,v)\|^2 \, ds \leq & C_9 \int_{t_0}^{t} \|Au\|^{1/2} \|A^{1/2}u\|^{3/2} \|Av\|^{1/2} \|A^{1/2}v\|^{3/2} \, ds \\ \leq & C_9 \|u\|^{3/2}_{L^\infty(0,\infty;V)} \|u\|^{3/2}_{L^\infty(0,\infty;V)} \\ & \left(\int_{t_0}^{t} \|Au\| \, ds \right)^{1/2} \left(\int_{t_0}^{t} \|Av\| \, ds \right)^{1/2} \\ \leq & C_{10} \|u\|^{3/2}_{L^\infty(0,\infty;V)} \|u\|^{3/2}_{L^\infty(0,\infty;V)} \\ & \left(\int_{t_0}^{t} \|Au\|^2 \, ds \right)^{1/2} \left(\int_{t_0}^{t} \|Av\|^2 \, ds \right)^{1/2},\end{aligned}$$

where C_{10} depends on $t - t_0$. This final estimate is bounded by the assumptions in the boundedness property.

The identical proof works for $B_1(u, \psi)$. □

Lemma 13. *The maps $u \mapsto Au$, $\psi \mapsto A_i\psi$, for $i = 1, 2$, satisfy the continuity property.*

Proof. We first consider A. By the definition of the Sobolev norms we have that A maps $\mathcal{D}(A)$ continuously into H. Hence property 1 of the compactness lemma implies the result.

The same proof works for the $A_i\psi$. □

Lemma 14. *The maps $u \mapsto Au$, $\psi \mapsto A_i\psi, i = 1, 2$ satisfy the boundedness property for $p = 2$.*

Proof. First we look at Au. Fix $t > t_0$. Then we need to consider

$$\int_{t_0}^{t} \|Au\|^2 \, ds.$$

The boundedness of this integral is one of the assumptions of the boundedness property, so the result is proved.

The same proof works for the $A_i\psi$. □

Lemma 15. *Suppose that the W_i are Lipschitz functions. Then $\omega_k(\phi_1, \dots, \phi_N)$ satisfies the continuity property for each k.*

Proof. Pick $\phi_i^j \to \phi_i$ satisfying the conditions in the continuity property for each i. We will actually show that

$$\omega_k(\phi_1^j, \dots, \phi_N^j) \to \omega_k(\phi_1, \dots, \phi_N)$$

strongly in $L^2(0, \tau; L^2(\Omega))$. We note that since ω_k is Lipschitz in each coordinate, we have

$$\left| \omega_k(\phi_1^j, \dots, \phi_N^j) - \omega_k(\phi_1, \dots, \phi_N) \right| \leq C_{11} \sum_{i=1}^{N} \left| \phi_i^j - \phi_i \right|$$

for some constant C_{11}. This, plus property 3 of the compactness lemma, gives the result after integrating in time and space. □

Lemma 16. *The functions $\omega_k(\eta_1, \dots, \eta_N, \theta)$ satisfy the boundedness property for any $p > 1$.*

Proof. This is a direct consequence of the fact that the ω_k are everywhere bounded. □

Corollary 17. *The maps F_v, F_θ and F_{η_i} all satisfy the boundedness and continuity properties.*

Proof. The statement follows directly from the above lemmas and the fact that the sets of functions satisfying the two properties are closed under scalar multiplication and addition. □

8. *A Priori* Estimates

In this section we will prove *a priori* estimates for the solutions of the Bubnov-Galerkin system (3.6), which we repeat here for convenience:

$$\partial_t v^n + PrAv^n + P_n B(v^n + w, v^n + w) = P_n f(\theta^n), \tag{8.1a}$$

$$\partial_t \theta^n + A_1 \theta^n + Q_n B_1(v^n + w, \theta^n + \theta_0) = Q_n g(\eta_1^n, \dots, \eta_N^n, \theta^n) \tag{8.1b}$$

$$\partial_t \eta_i^n + \frac{1}{Le} A_2 \eta_i^n + R_n B_1(v^n + w, \eta_i^n + \eta_{i,0}) = R_n \omega_i(\eta_1^n, \dots, \eta_N^n, \theta^n) \tag{8.1c}$$

$$v^n(0) = P_n v(0), \tag{8.1d}$$

$$\theta^n(0) = Q_n \theta(0), \tag{8.1e}$$

$$\eta_i^n(0) = R_n \eta_i(0), \tag{8.1f}$$

The estimates we will derive will, along with the boundedness property, give the bounds needed to apply the general existence theorem. The key observation is contained in the following lemma, which says that whenever the velocity is bounded in the correct spaces, so are the chemistry and temperature.

Lemma 18. *Fix $\tau > 0$ and suppose that assumptions 1 through 4 are satisfied. Suppose further that for the temperature and chemistry equations one has initial conditions in $H^1(\Omega)$, i.e.,*

$$\theta(0) \in V_1$$

$$\eta_i(0) \in V_2 \qquad \textit{for } i = 1, \dots, N,$$

and that on the interval $[0, \tau]$ one has

$$\|v^n(s) + 2\|_{H^1(\Omega)}^2 \leq \overline{C}_2 \qquad \textit{for all } s \in [0, \tau],$$

where $\overline{C}_2$ may depend on the initial conditions and τ, but not on n. Then one has

$$\|A_1^{1/2} \theta^n(t)\|^2 \leq \overline{C}_3 \qquad \textit{for } 0 \leq t \leq \tau, \tag{8.2}$$

$$\int_{t_0}^{t} \|A_1 \theta^n\|^2 \, ds \leq \overline{C}_3 \max\{t - t_0, 1\} \qquad \textit{for } 0 \leq t_0 \leq t \leq \tau, \tag{8.3}$$

where $\overline{C}_3$ is independent of n, but may depend on the initial conditions.

Proof. Throughout this proof, C_{12} will denote a constant independent of n and the initial conditions, but possibly depending on τ, while $\overline{C}_3$ will denote a constant which is also independent of n, but which may depend on the initial conditions and τ. We begin by taking the inner product of the equation for θ, equation (8.1b), with $A_1\theta$, to get

$$\frac{1}{2}\partial_t\|A_1^{1/2}\theta^n\|^2 + \|A_1\theta^n\|^2 + b_1(v^n + w, \theta^n + \theta_0, A_1\theta^n) = \langle g(\eta_1^n, \dots, \eta_N^n, \theta^n), A_1\theta^n\rangle. \tag{8.4}$$

From the Sobolev estimates (5.3g) and (5.6b), the Young inequality and the fact that g is bounded, one obtains the following estimates

$$\begin{aligned}
|b_1(v^n + w, \theta^n, A_1\theta^n)| &\le \|v^n + w\|_{H^1}\|A_1^{1/2}\theta^n\|^{1/2}\|A_1\theta^n\|^{3/2} \\
&\le \frac{1}{6}\|A_1\theta^n\|^2 + C_{12}\overline{C}_2^4\|A_1^{1/2}\theta^n\|^2, \\
|b_1(v^n + w, \theta_0, A_1\theta^n)| &\le \|v^n + w\|_{H^1}\|\theta_0\|_{H^2}\|A_1\theta^n\| \\
&\le \frac{1}{6}\|A_1\theta^n\|^2 + C_{12}\overline{C}_2^2, \\
|\langle g(\eta_1^n, \dots, \eta_N^n, \theta^n), A_1\theta^n\rangle| &\le \frac{1}{6}\|A_1\theta^n\|^2 + C_{12}.
\end{aligned}$$

Using these estimates in (8.4) gives

$$\begin{aligned}
\partial_t\|A_1^{1/2}\theta^n\|^2 + \|A_1\theta^n\|^2 &\le C_{12} + C_{12}\overline{C}_2^2 + C_{12}\overline{C}_2^4\|A_1^{1/2}\theta^N\|^2 \\
\partial_t\|A_1^{1/2}\theta^n\|^2 + (\mu_1 - C_{12}\overline{C}_2^4)\|A_1^{1/2}\theta^n\|^2 &\le C_{12} + C_{12}\overline{C}_2^2,
\end{aligned} \tag{8.5}$$

where $\mu_1 > 0$ is the smallest eigenvalue of A_1. This, by the Gronwall inequality, implies that on $[0, \tau]$ there exists a C_{12} independent of n such that

$$\|A_1^{1/2}\theta^n(t)\|^2 \le C_{12} \qquad \text{for } t \in [0, T],$$

i.e., (8.2) holds.

We now turn to proving (8.3). Integrating (8.5) from t_0 to t, where $0 \le t_0 < t \le \tau$, and using (8.2) gives

$$\int_{t_0}^{t} \|A_1\theta^n\|^2\, ds \le C_{13}(t - t_0)$$

which gives (8.3).

Identical arguments give the results for the η_i equations.

□

We are now ready to consider the velocity equation. Because of the assumption we make on $\|\theta^n\|^2$ and the definition of f (3.5a), we have that

$$\|f(\theta^n)\|^2 \le \overline{C}_4,$$

where $\overline{C}_4$ is independent of the order of the approximation n. This implies that our velocity equation is just a usual Navier-Stokes equation with forcing term in $L^\infty(0, \infty; L^2(\Omega))$. Therefore we expect the same qualitative results as for the usual Navier-Stokes equations. The only difference we have from the standard literature is our inhomogeneous boundary conditions. However, we will see that they present no problem. In fact, in the following two lemmas our proofs consist of taking the inhomogeneous Navier-Stokes equations and deriving inequalities which are qualitatively identical to the inequalities one obtains in the homogeneous case. We can then appeal to the standard literature to complete our results.

Lemma 19. *Assume that assumptions 1 through 4 are satisfied. Suppose further that* $\Omega \subset \mathbb{R}^3$ *and that the initial velocity is in* $H^1(\Omega)$*, i.e.,*

$$v(0) \in V.$$

Then there exists a time τ *and a constant* $\overline{C}_5$*, depending on the initial conditions but independent of* n *such that*

$$\|A^{1/2}v^n(t)\|^2 \leq \overline{C}_5 \qquad \text{for all } 0 \leq t \leq \tau, \tag{8.6}$$

$$\int_{t_0}^{t} \|Av^n(s)\|^2 \, ds \leq \overline{C}_5 \max\{t - t_0, 1\} \qquad \text{for all } 0 \leq t_0 \leq t \leq \tau, \tag{8.7}$$

Proof. Throughout the proof we will let $\overline{C}_5$ denote a constant independent of n, but possibly depending on the initial conditions. We begin by taking the inner product of Av^n with equation (8.1a) to get

$$\frac{1}{2}\|A^{1/2}v^n\|^2 + Pr\|Av^n\|^2 \leq \langle f(\theta^n), Av^n\rangle + |b(v^n + w, v^n + w, Av^n)|\,. \tag{8.8}$$

Next, we note that (4.2) and the definition of f implies that

$$\|f(\theta^n(t))\|^2 \leq C_1(1 + \|\theta^n(0)\| + \|v^n(0)\| + \sum_{i=1}^{N}\|\eta_i^n(0)\|) \qquad \text{for all } t > 0, \tag{8.9}$$

where C_1 is independent of n and the initial conditions. Now from the Sobolev estimates (5.3c) and (5.3d), the bound on $f(\theta^n)$ (8.9) and the Young inequality one obtains the estimates

$$\begin{aligned}
|b(v^n, v^n, Av^n)| &\leq C_s\|A^{1/2}v^n\|^{3/2}\|Av^n\|^{3/2} \\
&\leq \frac{Pr}{10}\|Av^n\|^2 + \overline{C}_5\|A^{1/2}v^n\|^6 \\
|b(w, v^n, Av^n)| &\leq C_s\|w\|_{H^2}\|A^{1/2}v^n\|\|Av^n\| \\
&\leq \frac{Pr}{10}\|Av^n\|^2 + \overline{C}_5\|A^{1/2}v^n\|^6 + \overline{C}_5\|w\|_{H^2}^3 \\
|b(v^n, w, Av^n)| &\leq C_s\|A^{1/2}v^n\|\|w\|_{H^1}^{1/2}\|w\|_{H^2}^{1/2}\|Av^n\| \\
&\leq \frac{Pr}{10}\|Av^n\|^2 + \overline{C}_5\|A^{1/2}v^n\|^6 + \overline{C}_5\|w\|_{H^2}^3 \\
|b(w, w, v^n)| &\leq C_s\|w\|_{H^2}\|w\|_{H^1}\|Av^n\| \\
&\leq \frac{Pr}{10}\|Av^n\|^2 + \overline{C}_5\|w\|_{H^2}^4 \\
|\langle f(\theta^n, Av^n\rangle| &\leq \frac{Pr}{10}\|Av^n\|^2 + \overline{C}_5
\end{aligned}$$

Substituting in these estimates into (8.8) gives

$$\begin{aligned}
\partial_t\|A^{1/2}v^n\|^2 + Pr\|Av^n\|^2 &\leq \overline{C}_5\|A^{1/2}v^n\|^6 + \overline{C}_5 \\
\partial_t\|A^{1/2}v^n\|^2 &\leq \overline{C}_5\|A^{1/2}v^n\|^6 + \overline{C}_5.
\end{aligned} \tag{8.10}$$

Now this inequality is well known from the 3D Navier-Stokes equations, see Sell and You [9], Temam [10] or Constantin and Foias [2], and implies that there is a time $\tau > 0$ and a constant

$\overline{C}_5$ satisfying the conclusions of the lemma. Furthermore, by integrating (8.10) from t_0 to t one can show (8.7).

□

Lemma 20. *Assume that assumptions 1 through 4 and assumption 7 are satisfied. Suppose further that $\Omega \subset \mathbb{R}^2$ and that the initial velocity is in $H^1(\Omega)$, i.e.,*

$$v(0) \in V.$$

Then for each $\tau > 0$, there exists a constant $\overline{C}_6$, depending on the initial conditions and τ but independent of n such that

$$\|A^{1/2}v^n(t)\|^2 \leq \overline{C}_6 \qquad \text{for all } 0 \leq t \leq \tau, \tag{8.11}$$

$$\int_{t_0}^{t} \|Av^n(s)\|^2\, ds \leq \overline{C}_6 \max\{t - t_0, 1\} \qquad \text{for all } 0 \leq t_0 \leq t \leq \tau, \tag{8.12}$$

Proof. Throughout this proof we let $\overline{C}_6$ denote a constant independent of n, but possibly depending on the initial conditions and τ, and let C_{14} denote a constant independent of n, the initial conditions and τ. We again begin by taking the inner product of the equation for v^n with Av^n and obtain

$$\frac{1}{2}\partial_t\|A^{1/2}v^n\|^2 + Pr\|Av^n\|^2 \leq \langle f(\theta^n), Av^n\rangle + |b(v^n + w, v^n + w, Av^n)|\,. \tag{8.13}$$

Now we use the two dimensional Sobolev estimate (5.4), the three dimensional estimates (5.3e) and (5.3d) and the Young inequality to get

$$\begin{aligned}
|b(v^n, v^n, Av^n)| &\leq C_s\|v^n\|^{1/2}\|A^{1/2}v^n\|\|Av^n\|^{3/2} \\
&\leq \frac{Pr}{10}\|Av^n\|^2 + C_{14}\|v^n\|^2\|A^{1/2}v^n\|^4 \\
|b(v^n, w, Av^n)| &\leq C_s\|v^n\|^{1/2}\|A^{1/2}v^n\|^{1/2}\|w\|_{H^2}\|Av^n\| \\
&\leq \frac{Pr}{10}\|Av^n\|^2 + C_{14}\|v^n\|^2\|A^{1/2}v^n\|^4 + C_{14}\|v^2\| \\
|b(w, v^n, Av^n)| &\leq C_s\|w\|_{H^2}\|A^{1/2}v^n\|\|Av^n\| \\
&\leq \frac{Pr}{10}\|Av^n\|^2 + C_{14}\|A^{1/2}v^n\|^2 \\
|b(w, w, Av^n)| &\leq C_s\|w\|_{H^2}\|Av^n\| \\
&\leq \frac{Pr}{10}\|Av^n\|^2 + C_{14} \\
|\langle f(\theta^n), Av^n\rangle| &\leq \frac{Pr}{10}\|Av^n\|^2 + C_{14}
\end{aligned}$$

Substituting these estimates into (8.13) and using the assumption on $\|v^n\|$ (4.3) gives

$$\begin{aligned}
\partial_t\|A^{1/2}v^n\|^2 + Pr\|Av^n\|^2 &\leq C_{14}\|v^n\|^2\|A^{1/2}v^n\|^4 + C_{14}\|A^{1/2}v^n\|^2 + C_{14}\|v\| + C_{14} \\
\partial_t\|A^{1/2}v^n\|^2 &\leq \overline{C}_6\left(\|A^{1/2}v^n\|^2 + 1\right)\|A^{1/2}v^n\|^2 + \overline{C}_6.
\end{aligned} \tag{8.14}$$

This last inequality is equivalent to the inequality one encounters when studying the 2D Navier-Stokes equations with homogeneous boundary conditions. In particular it, combined with the estimates in assumption 7 and the uniform Gronwall inequality from lemma 8 inequality, implies the bound (8.11). The key point is that integrals of coefficient of $\|A^{1/2}v^n\|^2$ on the right hand side are bounded in n, i.e.,

$$\int_{t_0}^{t} \|v^n(s)\|^2\|A^{1/2}v^n(s)\|^2 + 1\; ds \leq \overline{C}_6 \qquad\qquad \text{for } 0 \leq t_0 <\leq t \leq \tau,$$

holds with $\overline{C}_6$ independent of n. In our case, this follows from the estimates in assumption 7. For more details, we refer the reader to the proof of theorem 6 where similar calculations are done.

Finally, the bound (8.12) follows from integrating (8.14) and using the bounds for $\|v\|$ and $\|A^{1/2}v\|$ in (4.3) and (4.3), respectively. □

9. Proofs of Theorems

Proof of Theorem 1: Main Existence Theorem. We first note that from the *a priori* estimates in lemmas 18 and 19 or 20 we have that there exists a τ such that

$$\begin{aligned}
&v^n \text{ is bounded in } L^2(0,\tau;\mathcal{D}(A)) \text{ and } L^\infty(0,\tau;V),\\
&\theta^n \text{ is bounded in } L^2(0,\tau;\mathcal{D}(A_1)) \text{ and } L^\infty(0,\tau;V_1),\\
&\eta_i^n \text{ is bounded in } L^2(0,\tau;\mathcal{D}(A_2)) \text{ and } L^\infty(0,\tau;V_2),
\end{aligned}$$

where the bounds are independent of n. Furthermore, in two dimensions, $\tau > 0$ is arbitrary. The boundedness properties of the equations given in corollary 17 then imply that

$$\begin{aligned}
&\partial_t v^n \text{ is bounded in } L^2(0,T;H),\\
&\partial_t \theta^n \text{ is bounded in } L^2(0,T;H_1)\ ,\\
&\partial_t \eta_i^n \text{ is bounded in } L^2(0,T;H_2)\ ,
\end{aligned}$$

where the bounds are independent of n.

We now want to apply the general existence theorem, theorem 10, where the X_i^j are given in (7.2). We can do this in particular because the equivalent norms (4.1) imply the needed compact embeddings. This then gives the result.

□

Proof of Theorem 1: Uniqueness Theorem. We begin by defining

$$\begin{aligned}
\overline{v} &= v^2 - v^1,\\
\overline{\theta} &= \theta^2 - \theta^1,\\
\overline{\eta}_i &= \eta^2 - \eta^1.
\end{aligned}$$

Then we have that

$$\begin{aligned}
\partial_t\overline{v} + PrA\overline{v} + B(v^2+w,\overline{v}) + B(\overline{v},v^1+w) &= f(\overline{\theta}),\\
\partial_t\overline{\theta} + A_1\overline{\theta} + B_1(v^2+w,\overline{\theta}) + B_1(\overline{v},\theta^1+\theta_0) &= g(\eta_1^2,\ldots,\eta_N^2,\theta^2) - g(\eta_1^1,\ldots,\eta_N^1,\theta^1),\\
\partial_t\overline{\eta}_i + \frac{1}{Le}A_2\overline{\eta}_i + B_1(v^2+w,\overline{\eta}_i) + B_1(\overline{v},\eta_i^1+\eta_{i,0}) &= \omega_i(\eta_1^2,\ldots,\eta_N^2,\theta^2) - \omega_i(\eta_1^1,\ldots,\eta_N^1,\theta^1).
\end{aligned}$$

Most importantly, we have that

$$
\begin{aligned}
\partial_t \overline{v} &\in L^2(0,T;V^{-1}),\\
\partial_t \overline{\theta} &\in L^2(0,T;V_1^{-1}),\\
\partial_t \overline{\eta}_i &\in L^2(0,T;V_2^{-1}).
\end{aligned}
$$

These properties imply, by lemma 7, the following formal properties in fact hold

$$
\begin{aligned}
\frac{1}{2}\partial_t \|\overline{v}\|^2 &= \langle \partial_t \overline{v}, \overline{v}\rangle,\\
\frac{1}{2}\partial_t \|\overline{\theta}\|^2 &= \langle \partial_t \overline{\theta}, \overline{\theta}\rangle,\\
\frac{1}{2}\partial_t \|\overline{\eta}\|^2 &= \langle \partial_t \overline{\eta}, \overline{\eta}\rangle.
\end{aligned}
$$

From the Sobolev estimates (5.6a) and (5.6b) and the Young inequality we get the estimates

$$
\left|b(\overline{v}, v^1 + w, \overline{v})\right| \le \frac{C_3}{3}\|A^{1/2}\overline{v}\|^2 + C_{15}\|v^1 + w\|_{H^2}^2 \|\overline{v}\|^2, \tag{9.1a}
$$

$$
\left|b_1(\overline{v}, \theta^1 + \theta_0, \overline{\theta})\right| \le \frac{C_3}{3}\|A^{1/2}\overline{v}\|^2 + C_{15}\|\theta^1 + \theta_0\|_{H^2}^2 \|\overline{\theta}\|^2, \tag{9.1b}
$$

$$
\left|b_1(\overline{v}, \eta_i^1 + \eta_{i,0}, \overline{\eta}_i)\right| \le \frac{C_3}{3}\|A^{1/2}\overline{v}\|^2 + C_{15}\|\eta_i^1 + \eta_{i,0}\|_{H^2}^2 \|\overline{\eta}_i\|^2, \tag{9.1c}
$$

where C_{15} depends only on C_3 and C_s. Furthermore, using (1.2), (3.5) and the Lipschitz property of the W_i we can estimate

$$
\left|\langle f(\theta^2) - f(\theta^1), \overline{v}\rangle\right| \le c_0 \|\overline{\theta}\|\|\overline{v}\| \tag{9.2a}
$$

$$
\le c_0^2\|\overline{\theta}\|^2 + \|\overline{v}\|^2, \tag{9.2b}
$$

$$
\left|\langle g(\eta_j^2, \theta^2) - g(\eta_j^1, \theta^1), \overline{\theta}\rangle\right| \le C_{16}\left(\|\overline{\theta}\|^2 + \sum_{j=1}^{N}\|\overline{\eta}_j\|^2\right), \tag{9.2c}
$$

$$
\left|\langle W_i(\eta_j^2, \theta^2) - W_i(\eta_j^1, \theta^1), \eta_i\rangle\right| \le C_{16}\left(\|\overline{\theta}\|^2 + \sum_{j=1}^{N}\|\overline{\eta}_j\|^2\right), \tag{9.2d}
$$

where C_{16} is independent of t, but may depend on τ.

Now we can take the inner product of (3.4a), (3.4b) and (3.4c) with v, θ and η_i respectively and use the equivalent norms from (4.5) to get

(9.3a)
$$\partial_t \|v\| + C_3 \|A^{1/2} v\|^2 \leq \frac{C_3}{3} \|A^{1/2} v\|^2 + C_{15} \|v^1 + w\|^2_{H^2} \|\overline{v}\|^2 + \|\overline{v}\|^2 + c_0^2 \|\overline{\theta}\|^2,$$

(9.3b)
$$\partial_t \|\theta\| + C_3 \|A_1^{1/2} \theta\|^2 \leq \frac{C_3}{3} \|A^{1/2} v\|^2 + C_{15} \|\theta^1 + \theta_0\|^2_{H^2} \|\overline{\theta}\|^2 + C_{16} \left(\|\overline{\theta}\|^2 + \sum_{j=1}^{N} \|\overline{\eta}_j\|^2 \right),$$

(9.3c)
$$\partial_t \|\eta_i\| + C_3 \|A_1^{1/2} \theta\|^2 \leq \frac{C_3}{3} \|A^{1/2} v\|^2 + C_{15} \|\eta_i^1 + \eta_{i,0}\|^2_{H^2} \|\eta_i\|^2 + C_{16} \left(\|\overline{\theta}\|^2 + \sum_{j=1}^{N} \|\overline{\eta}_j\|^2 \right).$$

(9.3d)

Now from the equivalent norms (4.1) and (4.8) we get that $\|v^1 + w\|^2_{H^2}$, $\|\theta^1 + \theta_0\|^2_{H^2}$ and $\|\eta_i^1 + \eta_{i,0}\|^2_{H^2}$ are integrable. Therefore, since the maximum of a finite number of integrable functions is again integrable, we can sum the equations in (9.3) to get

$$\partial_t \left(\|\overline{v}\|^2 + \|\overline{\theta}\|^2 + \sum_{i=1}^{N} \|\eta_i\|^2 \right) \leq h(t) \left(\|\overline{v}\|^2 + \|\overline{\theta}\|^2 + \sum_{i=1}^{N} \|\overline{\eta}_i\|^2 \right),$$

where $h(t)$ is an integrable function. Therefore, the Gronwall inequality implies

$$\left(\|\overline{v}(t)\|^2 + \|\overline{\theta}(t)\|^2 + \sum_{i=1}^{N} \|\overline{\eta}_i(t)\|^2 \right) \leq$$
$$\leq \left(\|\overline{v}(0)\|^2 + \|\overline{\theta}(0)\|^2 + \sum_{i=1}^{N} \|\overline{\eta}_i(0)\|^2 \right) \exp\left(\int_0^t h(s)\, ds \right),$$

which implies that the solutions coincide, which is the desired result.

□

10. Global Attractors

In this section we want to prove the two dimensional global attractor existence theorem 6. The classical global attractor existence theorem we want to use is based on two definitions. We say that a dynamical system $S(t)$ on a metric space X is *dissipative* if there exists a bounded set $U \subset X$ such that for all $x \in X$, there exists a time τ_x such that $S(t)x \in U$ for all $t > \tau_x$. We call U an absorbing set. The second definition is that $S(t)$ is *compact* if for each bounded set V there exists a time t such that $S(t)V$ is compact. For more information on these definitions another dynamical systems theory, see Sell and You [9], Hale [3] and Temam [11]. Given these definitions, we can state the following existence theorem.

Theorem 21 (Billotti-LaSalle). *Suppose that the dynamical system $S(t)$ is compact and point dissipative. Then $S(t)$ has a global attractor. Furthermore, the attractor uniformly attracts all bounded sets.*

For a proof, see for example Billotti and LaSalle [1] or Sell and You [9]. This theorem was used, for instance, in Manley, Marion and Temam [5] to show the existence of global attractors to the reacting flow system (1.1) with boundary conditions representing a two dimensional premixed flame in a tube.

Typically the application of this theorem in situations similar to ours involves two steps. First one uses *a priori* L^2 energy estimates to show the existence of an absorbing set in $L^2(\Omega)$. Secondly, one uses smoothing properties of the equation to show compactness. The simplest way to do this is to show the existence of an absorbing set in $H^1(\Omega)$, then use the fact that $H^1(\Omega)$ is compactly embedded in $L^2(\Omega)$ to get compactness. It is this strategy that we will follow.

However, in this paper we will not show dissipativity for the reacting flow system. As mentioned above, dissipativity typically depends on *a priori* L^2 energy estimates, which in turn depend intimately on the boundary conditions. Instead, our global attractor existence theorem 6 essentially states that dissipativity implies compactness, which in turn implies the existence of global attractors. We should point out that dissipativity is known for the two sets of boundary conditions which we know of in the literature, see Manley, Marion and Temam [5] and Norman [7].

The first step in proving the global attractor existence theorem is the following theorem which states that if the system is dissipative and v is uniformly bounded in $H^1(\Omega)$, then there exists a bounded absorbing set in $H^1(\Omega)$ for the chemistry and temperature equations.

Lemma 22. *Let $\Omega \subset \mathbb{R}^d, d = 2$ or 3. Suppose that the reacting flow system (3.4) is dissipative, i.e., (4.13) is satisfied. Suppose further that for each initial condition $(v(0), \theta(0), \eta_i(0)) \in \overline{H}$ there exists a τ such that*

$$\|A^{1/2}v(t) + w\|^2 \leq C_{17} \qquad \text{for } t \geq \tau, \tag{10.1}$$

where C_{17} is independent of the initial conditions. Then there exists an absorbing set in $H^1(\Omega)$ for the chemistry and temperature equations.

Proof. Fix a set of initial conditions $(v(0), \theta(0), \eta_i(0)) \in \overline{H}$ and associated time τ. Throughout the rest of the proof we will work on the interval $(\tau + 1, \infty)$.

We begin by taking the inner product of the equation for θ (3.4b) with $A_1\theta$ and use theorem 7 to get

$$\frac{1}{2}\partial_t\|A_1^{1/2}\theta\|^2 + \|A_1\theta\|^2 + b_1(v + w, \theta + \theta_0, A_1\theta) = \langle g(\theta, \eta_j), \theta\rangle \tag{10.2}$$

Now for $t \in [\tau + 1, \infty)$, from the Sobolev estimates (5.3g) and (5.6b), the bound for g and the Young inequality one obtains the estimates

$$\begin{aligned}
|b_1(v + w, \theta, A_1\theta)| &\leq C_s\|A^{1/2}v\|\|A_1^{1/2}\theta\|^{1/2}\|A_1\theta\|^{3/2} \\
&\leq \frac{1}{6}\|A_1\theta\|^2 + \frac{729}{32}C_s^4C_{17}^2\|A_1^{1/2}\theta\|^2, \\
|b_1(v + w, \theta_0, A_1\theta)| &\leq \frac{1}{6}\|A_1\theta\|^2 + \frac{3}{2}C_s^2C_{17}\|\theta_0\|_{H^2}^2, \\
\langle g, A_1\theta\rangle &\leq \frac{1}{6}\|A_1\theta\|^2 + C_{18},
\end{aligned}$$

where C_{18} depends only on the bound for g. Substituting these estimates into (10.2) gives

$$\partial_t \|A_1^{1/2}\theta\|^2 + \mu_1 \|A_1^{1/2}\theta\|^2 \leq \frac{729}{16} C_s^4 C_{17}^2 \|A_1^{1/2}\theta\|^2 + C_{19},$$

where μ_1 is the smallest eigenvalue of A_1. Now we can apply the uniform Gronwall inequality from lemma 8 on the interval $(t-1, t)$ for $t > \tau + 1$ to get

$$\|A_1^{1/2}\theta(t)\|^2 \leq \left(C_{20} \int_{t-1}^{t} \|A_1^{1/2}\theta(s)\|^2 \, ds + C_{19} \right) e^{\mu_1}$$

where

$$C_{20} = \frac{729}{16} C_s^4 C_{17}^2 + 1.$$

We now want to apply the estimate (4.6) from assumption 9. Since (4.6) is written in terms of the Bubnov-Galerkin approximation, we need to take the limit as $n \to \infty$ to get an estimate for the solution itself. We can do this in particular because the 2D uniqueness theorem, corollary 3, implies that the specific solution we are working with is the limit of the Bubnov-Galerkin approximations. However, this is straightforward. In particular, the left hand side converges for all t and t_0 because of property 4 and the right hand side converges for all t and $t_0 > 0$ by property 3 of the compactness lemma. This gives

$$\int_{t_0}^{t} \|A^{1/2}\theta(s)\|^2 \, ds \leq C_4 \max\{1, t - t_0\} \left(\|\theta(t_0)\|^2 + \|v^n(t_0)\|^2 + \sum_{i=1}^{N} \|\eta_i(t_0)\|^2 + 1 \right), \tag{10.4}$$

for all $t > t_0 > 0$.

Now, from (10.4) and the dissipativity estimate (4.13) one obtains the inequality

$$\|A_1^{1/2}\theta(t)\|^2 \leq \left(C_{20} C_4 e^{-\sigma t} \left(\|\theta^n(0)\| + \|v^n(0)\| + \sum_{i=1}^{N} \|\eta_i^n(0)\| \right) + C_{19} + C_4 \right) e^{\mu_1},$$

which implies the existence of an absorbing set for θ.

Exactly the same argument works for the η_i equation. We omit the details. □

Proof of global attractor existence theorem 6. Throughout this proof C_{21} will denote a constant independent of the initial conditions and t, but possibly depending on t_0 defined below. We begin by showing the existence of an absorbing set for v in $H^1(\Omega)$. In particular, using the same calculations that led to (8.14) in the proof of lemma 20, but applied to the original v equation we get:

$$\partial_t \|A^{1/2}v(t)\|^2 + \mu Pr \|A^{1/2}v(t)\|^2 \leq C_{14} g(t) \|A^{1/2}v(t)\|^2 + C_{14}\|v(t)\| + C_{14},$$

where

$$g(t) = \|v(t)\|^2 \|A^{1/2}v(t)\|^2 + 1,$$

and μ is the smallest eigenvalue of A. Now, by the dissipativity, there exists a constant C_{22} and a time $t_0 > 0$, both independent of the initial conditions such that

$$\|v(t)\| \leq C_{22} \qquad \text{for all } t \geq t_0.$$

Hence one has

$$\partial_t \|A^{1/2}v(t)\|^2 + \mu Pr \|A^{1/2}v(t)\|^2 \leq C_{14} g(t) \|A^{1/2}v(t)\|^2 + C_{21} \qquad \text{for all } t \geq t_0. \tag{10.5}$$

Now we want to derive estimates for the integral of $\|A^{1/2}v(s)\|^2$ on $[t-1,t]$, where $t_0 \leq t-1$. As in the proof of the previous lemma, because of the compactness lemma convergence, we can apply (4.4) to get

$$\int_{t-1}^{t} \|A^{1/2}v(s)\|^2 \, ds \leq C_2 \left(\|v(t-1)\|^2 + \|\theta(t-1)\|^2 + \sum_{i=1}^{N} \|\eta_i(t-1)\|^2 + 1 \right).$$

Now, again by dissipativity, we can assume, possibly by increasing t_0, that

$$\int_{t-1}^{t} \|A^{1/2}v(s)\|^2 \, ds \leq C_{21}. \tag{10.6}$$

This also implies that

$$\int_{t-1}^{t} g(s) \, ds \leq C_{21}. \tag{10.7}$$

Now we apply the uniform Gronwall inequality from lemma 8 on $[t-1,t]$ with $t-1 > t_0$ to (10.5) to get

$$\|A^{1/2}v(t)\|^2 \leq C_{21} \qquad \text{for } t \geq t_0 + 1,$$

i.e., there exists an absorbing set for v in $H^1(\Omega)$.

Now, by the assumptions of the theorem, for each initial condition in $L^2(\Omega)$ there exists a unique weak solution, which immediately becomes a strong solution. This allows us to define a dynamical system $S(t)$ based on these solutions. Using a generalization to the arguments in the uniqueness theorem, theorem 2, one can show that $S(t)$ is continuous in $L^2(\Omega) \times \mathbb{R}$. By the above calculations, this semiflow has an absorbing set in $H^1(\Omega)$, which implies that $S(t)$ is compact, since $H^1(\Omega)$ is compactly embedded in $L^2(\Omega)$. Since $S(t)$ is dissipative by assumption, we can apply theorem 21 to get the result. □

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