ISOLATED SINGULARITIES OF THE SOLUTIONS OF THE SCHRÖDINGER EQUATION WITH A RADIAL POTENTIAL

By

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ABSTRACT

We study the behaviour near an isolated singularity, say $x=0$, of the solutions of the equation (1) $\Delta u(x) = V(x)u(x)$, defined in a neighbourhood of 0, in terms of the potential V that we assume to be radially symmetric.
For a class of potentials $\underline{P}_1$ that we completely characterize there is only a unique singularity with constant sign and it behaves like the fundamental solution of the Laplace operator. For a larger class $\underline{P}_2$ we yet obtain a unique type of nonnegative singularity that is radially symmetric. If $V(r) = cr^{\theta}$, with $c>0$, $\theta \in \mathbb{R}$, $\underline{P}_1$ means $\theta > -2$ and $\underline{P}_2$ means $\theta \geq -2$. The results fail for $\theta<-2$.

Using the Kelvin transformation our results give an equivalent description of the asymptotic behaviour of solutions of (1) defined for $|x| > R > 0$.

INTRODUCTION

In this paper we study the behaviour near an isolated singularity of the solutions of the time-independent Schrödinger equation

$$(0.1) \qquad \Delta u = V(x)u$$

defined in a domain of $\mathbb{R}^N$, $N \geq 2$. We shall assume that the potential V is radially symmetric, $V = V(|x|)$, and not very negative. For convenience we shall also assume that V is continuous except possibly at $x = 0$. We study in terms of V the occurence of a unique type of isolated singularity at $x = 0$ for solutions of (0.1) that have constant sign near the origin.

The behaviour of solutions of (0.1) around an isolated singularity is well-known in the case $V = 0$. In fact for every nonnegative harmonic function defined in a punctured ball $B_R^* = B_R(0) - \{0\}$ there exists $c \geq 0$ such that

$$(0.2) \qquad u(x) = cE_N(x) + g(x)$$

where E_N is the fundamental solution of the Laplace operator, see (1.1), and g is a C^∞ harmonic function in B_R.

In Section 1 we prove that this behaviour persists if V is not very singular. The class $\underline{P}_1$ of potentials for which an isolated singularity with constant sign behaves like $E_N(x)$ is shown to coincide (for $N \geq 3$) with the Kato class K_N (Theorem A). The proof relies heavily on the study of the corresponding ODE (Section 2). For potentials of the form $V(r) = cr^\theta$ with $c, \theta \in \mathbb{R}$ $V \in \underline{P}_1$ if and only if $c=0$ or $\theta > -2$.

It follows in particular from this study that when V is negative and $V \notin \underline{P}_1$ there are no radial singularities with constant sign. On the other side Willett [W] proves that the condition $V^- \equiv \max(-V,0) \in \underline{P}_1$ is sufficient to exclude the appearance of radial solutions that oscillate infinitely many times near $r = 0$. We shall also need V^- $\underline{P}_1$ to transform unilateral bounds into absolute bounds in Theorems A and C. Therefore we shall make in the sequel the assumption $V^- \in \underline{P}_1$ unless mention to the contrary.

We also study the conditions under which a singularity is removable both in the case $V \in \underline{P}_1$ (Corollary 1.1) and in the case $V \notin \underline{P}_1$ (Theorem B). All these results are proved in Section 3.

Then we consider a larger class of potentials $\underline{P}_2$ for which there is only one type of singular behaviour for the nonnegative solutions of (0.1). Moreover this singularity is isotropic, i.e. the singular solution $u(x)$ becomes radially symmetric as $r \to 0$. In fact we prove (Theorem C) that for every not very negative singular solution of (0.1) with $V \in \underline{P}_2$ we have

(0.3) $$u(x) = \overline{u}(r) + o(1),$$

where $\overline{u}$ is the angular average of u, i.e.

(0.4) $$\overline{u}(r) = |S^{N-1}|^{-1} \int_{S^{N-1}} u(r,\sigma)d\sigma .$$

Here $(r,\sigma) \in (0,\infty) \times S^{N-1}$ are spherical coordinates in $\mathbb{R}^N$, $d\sigma$ is the standard measure on $S^{N-1} = \{x \in \mathbb{R}^N : |x| = 1\}$ and $|S^{N-1}|$ is the area of S^{N-1}. We also have for a singular u

(0.5) $$\lim_{r\to\infty} \overline{u}(r)/E_N(r) \neq 0$$

and $|u|$ grows at most like a power of r as $r\to\infty$. The class $\underline{P}_2$ contains, apart from $\underline{P}_1$, the inverse square potentials $V(r) = cr^{-2}$, that play an important role in Quantum Mechanics, if $c>0$. $\underline{P}_2$ is also closed under addition. Complete details are given in Section 4 where we also prove Theorem C.

The sharpness of this theorem is shown in Section 5, where we construct infinitely many positive singular solutions of (0.1) exhibiting different singular behaviours for the potentials of a class $\underline{P}_3$ which is nearly the complement of $\underline{P}_2$ in the set of positive radial potentials. $\underline{P}_3$ contains in particular the potentials of the form $V(r) = cr^{\theta}$ with $c>0$ and $\theta < -2$. For these latter potentials there are radial solutions $u_0(r)$, $u_1(r)$ of (0.1) such that, as $r \to 0$,

(0.6.a) $$u_1(r) \simeq r^k \exp\left(\frac{c^{1/2}}{pr^p}\right), \text{ and}$$

(0.6.b) $$u_0(r) \simeq r^k \exp\left(-\frac{c^{1/2}}{pr^p}\right),$$

where $p = -\frac{1}{2}(\theta+2)>0$ and $k = \frac{1}{2}(p+2-N)$, see end of Section 2.

The study of the behaviour of the solutions of (0.1) near an isolated singularity can be transformed into the study of the asymptotic properties of the solutions of (0.1) as $|x| \to \infty$ by means of the Kelvin transformation. In fact, let $\Omega = \mathbb{R}^N - B_R(0)$ for some $R>0$ and consider the solutions $u(x)$ of (0.1) in Ω, where now $V \in C((R,\infty): \mathbb{R})$. If we set

(0.7) $$w(y) = |y|^{2-N} u(y|y|^{-2})$$

then w is a solution of the equation

(0.8.a) $$\Delta w(y) = W(y)w(y) \quad \text{for} \quad 0 < |y| < 1/R$$

with

(0.8.b) $$W(y) = |y|^{-4} V(|y|^{-1}).$$

In that way we obtain in Section 6 a description of the asymptotic behaviour of u when V belongs to the class $\underline{P}_1^\infty$, $\underline{P}_2^\infty$ or $\underline{P}_3^\infty$, which corresponds to $W \in \underline{P}_1$, $\underline{P}_2$, or $\underline{P}_3$. In particular we prove that a nonnegative solution of (0.1) behaves as $r = |x| \to \infty$ either like a constant or like r^{2-N} if $N \geq 3$ (resp. like $\log(r)$ or like a constant if $N=2$) if and only if $V \in \underline{P}_1^\infty$, cf. Theorem A'. For instance if $|V(r)| \simeq r^\theta$ this means $\theta < -2$ and in this case we have

(0.9.a) $$u(x) = C_1(1 + O(r^{\theta+2})) + C_2r^{2-N} + o(r^{2-N})$$

if $N \geq 3$ and

(0.9.b) $$u(x) = c_1(\log(r) + O(r^{\theta+2})) + C_2 + o(1)$$

if $N=2$, for some constants $c_1, c_2 \in \mathbb{R}$.

Finally we use these results to study in Section 7 the conditions under which the solutions of (0.1) in $\Omega = R^N - \{0\}$ are radially symmetric.

Two main restrictions are made in this paper. First, both the potentials and the solutions are assumed not to be very negative. This is made so as to eliminate all oscillatory solutions from the present discussion. Second, the potentials are radially symmetric, $V(x) = V(|x|)$. Though many of our results have natural counterparts for nonradial potentials we only have a complete picture of the situation in the case of radial potentials. Moreover our treatment relies on the study of the radial solutions of (0.1), where ODE techniques are used. We shall consider the case of nonradial K_N - potentials in an upcoming paper using different techniques.

The study of isolated singularities and asymptotic behaviour of the solutions of elliptic equations, among them (0.1), is a classical subject that has been pursued by many authors, in particular Serrin, cf. [GS], [S1], [S2], [SW]. Recently Brezis and Lions [BL] proved that nonnegative solutions of $\Delta u \leq cu$ with $c>0$ in B^* either have a removable or a weak singularity in the notation of Section 1. This is related to our Theorem 1. Caffarelli and Littman [CL] studied the positive solutions of (0.1) in R^N with $V(x) = 1$ (therefore $V \in \underline{P}_3^\infty$ in our notation). Murata [M] studies the behaviour of positive solutions as $|x| \to \infty$ under conditions on V different from ours. The fact that the inverse-square potentials, $V(r) = cr^{-2}$, are a sort of borderline case for different phenomena is well-known, cf. for instance [RS], [BG]. Recently Garofalo and Lin [GL] have also encountered this threshold behaviour in the study of the property of unique continuation of solutions of (0.1). Finally dal Maso and Mosco [MM] study equation (0.1) using a variational approach, obtaining in particular moduli of continuity for nonsingular solutions of (0.1).

Isolated singularities of nonlinear equations of the form $\Delta u = \phi(u)$, with ϕ continuous and increasing, have been studied by Brezis, Véron and the first author, among others, cf. [BV], [VV], [Ve]. Even in the case $\phi(u) = u^p$ with $p>1$ and $u \geq 0$ the results depart strongly from the linear case.

An announcement of results by the authors on this problem has appeared in [VY]. The present paper contains not only a full account but also a substantial improvement on that note. The authors are grateful to J. Serrin for his interest in this work.

1. BEHAVIOUR AT THE ORIGIN FOR SMALL POTENTIALS

Let $B_R = B_R(0)$ be the ball of radius $R>0$ centered at the origin in $\mathbb{R}^N$, $N\geq 2$, and let $B_R^* = B_R - \{0\}$. We consider solutions u $C^2(B_R^*)$ of equation (0.1) for $0 < |x| < R$ and describe their behaviour as $|x| \to 0$ depending on the singular character of V at 0. We shall use indistinctly the notations $V = V(x) = V(r)$, $r = |x|$, and so on, whenever no confusion arises.

Let us first briefly examine the possible behaviours of u near 0. In case u, and $V\cdot u \in L^1_{loc}(B_R)$ and (0.1) is satisfied in $\underline{D}'(B_R)$ we say that u is a <u>solution of</u> (0.1) <u>in</u> B_R or that u is <u>nonsingular</u> at 0 or also that the singularity at 0 is <u>removable</u>.

Actually the nonsingular solutions that appear in our main results are continuous functions in B_R. This is not the case for some very negative potentials (cf. Prop. 2.2).

If the above does not hold u is <u>singular</u> at 0. We are interested in describing singularities with <u>constant sign</u>. In the simplest case $V(x) = 0$, i.e. when u is a harmonic function, it is well-known that every singular solution with constant sign behaves as $|x| \to 0$ like the <u>fundamental</u> solution $E_N(x)$ of the Laplace operator,

$$
(1.1)\qquad
\begin{aligned}
E_N(x) &= k(N)\,|x|^{2-N} \quad \text{if } N \geq 3\,, \\
E_2(x) &= (2\pi)^{-1}\log(1/|x|)\,,
\end{aligned}
$$

in the sense that there exists a constant $c\in\mathbb{R}$ such that

$$
(1.2)\qquad \lim_{|x|\to 0} \frac{u(x)}{E_N(x)} = c\ .
$$

Moreover u satisfies the equation $-\Delta u = c\delta$ in $\underline{D}'(B_R)$, where δ is Dirac's mass at the origin. If $c = 0$ the singularity is removable.

We say that a singular solution of (0.1) that satisfies (1.1) has a <u>weak singularity</u> at 0. If (1.1) is true with $c = +\infty$ or $c = -\infty$ we say that u has a <u>strong singularity</u> at 0.

Strong singularities do not exist if $V \equiv 0$. In fact all singularities with constant sign are of weak type if $V \in L^q(B_R)$ with $q>N/2$ according to the results of Serrin [S2].

We want to describe the class of radial potentials for which every singular solution with constant sign exhibits a weak singularity. We shall prove that this is precisely the class of $\underline{P}_1$ of potentials $V \in C((0,R):\mathbb{R})$ such that

$$V(|x|)\, E_N(x) \in L^1(B_R). \tag{1.3}$$

For radial functions $\underline{P}_1$ coincides with the class K_N introduced by Kato [K], see [AS, Proposition 4.10]. An important property of the class K_N, proved by Aizenman and Simon [AS], is the Harnack inequality, that we shall use in the proof of Theorems 1 and 3. Remark that $K_N \supset L^q(B_R)$ if and only if $q>N/2$. In particular if $V(r) = cr^\theta$ with $c \neq 0$, $\theta \in \mathbb{R}$, $V \in \underline{P}_1$ if $\theta > -2$.

We obtain the following result

THEOREM A. <u>Let</u> $V \in \underline{P}_1$ <u>and let</u> $u \in C^2(B_R^*)$ <u>be a solution of</u> (0.1) <u>such that</u>

$$u(x) \geq o(|x|^{1-N}) \quad \underline{\text{or}} \quad u(x) \leq o(|x|^{1-N}), \quad |x| \to 0. \tag{1.4}$$

<u>then there exists a constant</u> c <u>such that</u>

$$u(x) = cu_1(|x|) + u_2(x), \tag{1.5}$$

<u>where</u> $u_1(r)$ <u>is a radial solution of</u> (0.1) <u>such that</u> $u_1(r)/E_N(r) \to 1$ <u>as</u> $r = |x| \to 0$ <u>and</u> $u_2 \in C(B_R)$ <u>is a nonsingular solution of</u> (0.1). <u>Moreover</u> u <u>and</u> $Vu \in L^1_{loc}(B_R)$ <u>and</u>

$$(1.6)\qquad -\Delta u + Vu = c\delta \quad \text{in} \quad \underline{D}'(B_R) \, .$$

The proof of Theorem 1 relies on the study of the radially symmetric solutions of (0.1) that is done in Section 2. In particular it follows from these results that $\underline{P}_1$ can be considered as the class of "small" perturbations of the null potential. These results also allow to show that the two hypotheses made in the theorem are optimal. Consider for instance the unilateral condition (1.4). We can always construct oscillatory solutions of (0.1) that behave at 0 like $O(|x|^{1-N})$ by separation of variables. Thus if we set

$$(1.7)\qquad u(x) = H_n(\sigma)U_n(r),$$

where (r,σ) are spherical coordinates in $\mathbb{R}^N$ and $H_n(\sigma)$ is an eigenfunction of the Laplace-Beltrami operator in $S^{N-1} = \{x \in \mathbb{R}^N : |x| = 1\}$ with eigenvalue $\lambda_n = n(n+N-2)$, is a solution of (0.1) if U_n satisfies the equation

$$(1.8)\qquad U''(r) + \frac{N-1}{r} U'(r) = (\frac{\lambda_n}{r^2} + V)U(r) \, .$$

According to Proposition 2.3, the behaviour of U_n as $r \to 0$ is equivalent to the behaviour of the solutions of (0.1) with potential $\lambda_n r^{-2}$ (again $V \in \underline{P}_1$ is a small perturbation). But for potentials $V = cr^{-2}$ we find the radially symmetric solutions

$$(1.9.a)\qquad u_1(r) = r^{-p} \quad , \quad p = \frac{N-2}{2} + ((\frac{N-2}{2})^2 + c)^{1/2} \, ,$$

$$(1.9.b)\qquad u_0(r) = r^{q} \quad , \quad q = -\frac{N-2}{2} + ((\frac{N-2}{2})^2 + c)^{1/2} \, ,$$

if $c > c_* = -(\frac{N-2}{2})^2$. Observe that if $c > 0$ $u_1(r)$ is a solution with a strong singularity and $u_0(r)$ is a continuous nonsingular solution. In particular for the first eigenvalue $\lambda_1 = N-1$ we obtain a solution $U_1(r) = r^{-p}$, $p = N-1$, which is the oscillatory solution of the form (1.7) with slowest growth at 0.

Consider now that we have a weakly singular solution of (0.1). Since $Vu \in L^1_{loc}(B_R)$ and $u \sim E_N(x)$ near $x = 0$ we conclude that $VE_N \in L^1_{loc}(B_R)$, i.e. $V \in \underline{P}_1$.

Theorem 1 implies in particular the following removability result

COROLLARY 1.1. *If* u *is a solution of* (0.1) *with* $V \in \underline{P}_1$ *and* $u(x) = o(E_N(x))$ *as* $x \to 0$ *then* u *is a continuous solution of* (0.1) *in* $B_R(0)$.

Removability results are somewhat different if $V \notin \underline{P}_1$. Assume for instance that $V^+ = \max(V,0) \notin \underline{P}_1$ and $V^- \in \underline{P}_1$ and let u_0 and u_1 be two radially symmetric solutions of (0.1) such that $u_0(r) \to 0$ and $u_1(r)/E_N(r) \to \infty$ as $r \to 0$. Such solutions are constructed in Section 2. Then we have

THEOREM 1. *Let* $u \in C^2(B_R^*)$ *be a solution of* (0.1) *with* $V^+ \notin \underline{P}_1$, $V^- \in \underline{P}_1$. *The following conditions are equivalent*

i) $u(x) = o(u_1(r))$, $r = |x| \to 0$,

ii) $Vu \in L^1_{loc}(B_R)$,

iii) $u(x) = O(u_0(r))$, $r \to 0$,

iv) u *is a continuous solution of* (0.1) *in* B_R.

In contrast to these conditions every weak singularity satisfies $Vu \in L^1_{loc}(B_R)$ if if $V \in \underline{P}_1$. Moreover by the Harnack inequality $u(0) > 0$ for an nonnegative solution of (0.1) while $u(0) = 0$ characterizes the nonsingular solutions in Theorem B. Theorems A and B will be proved in Section 3 after the basic properties of the radial solutions are investigated in Section 2.

2. SOME RESULTS FOR RADIAL SOLUTIONS

2.1 This section is devoted to study behaviour near $r = 0$ of the solutions $u(r)$ of the ODE

$$u''(r) + \frac{N-1}{r} u'(r) = V(r)u(r), \quad 0 < r < R. \tag{2.1}$$

We are especially interested in the consequences of condition (1.3), i.e.

$$\int_0^R rV(r)dr < \infty \quad \text{if} \quad N \geq 3 , \tag{2.2.a}$$

$$\int_0^R r|\log(r)|\, V(r)\, dr < \infty \quad \text{if} \quad N = 2. \tag{2.2.6}$$

By means of the change of variables

$$u(r) = V(s)/s, \qquad s = r^{N-2} \tag{2.3}$$

we transform the equation (2.1) if $N \geq 3$ into the $(N=1)$-version

$$\ddot{v}(s) = X(s)\, v(s) \;, \quad 0 < s < S = R^{N-2} \;, \tag{2.4}$$

where dot means d/ds and the new potential $X(s)$ is defined by

$$X(s) = (N-2)^{-2}\; V(r)(r/s)^2. \tag{2.5}$$

Note that for $N=3$ we have $s = r$, $X = V$. For $N=2$ the appropriate change of variables is

$$u(r) = v(s)/s \quad , \quad s = \{\log(1/r)\}^{-1} \tag{2.3'}$$

and then (2.4) holds with

$$X(s) = V(r)r^2 s^{-4} \;. \tag{2.5'}$$

In every case condition (2.2) becomes

$$(2.6) \qquad \int_0^S sX(s)ds < \infty ,$$

i.e. $V \in \underline{P}_1$ if and only if $X \in \underline{P}_1$.

It is known, cf. [N, Corollaries 2.3 and 3.3] that whenever $X \in \underline{P}_1$ there exist two solutions, $v_0(s)$ and $v_1(s)$, of (2.4) such that as $s \to 0$ we have

$$(2.7.a) \qquad v_0(s) = s + o(s), \qquad v_0'(s) = 1 + o(1),$$

$$(2.7.b) \qquad v_1(s) = 1 + o(1), \qquad v_1'(s) = o(1/s).$$

It follows that for every $V \in \underline{P}_1$ there exist two solutions $u_0(r)$ and $u_1(r)$, of (2.1) such that

$$(2.8.a) \qquad \lim_{r\to 0} u_0(r) = 1, \qquad \lim_{r\to 0} ru_0'(r) = 0 ,$$

$$(2.8.b) \qquad \lim_{r\to 0} \frac{u_1(r)}{E_N(r)} = s , \qquad \lim_{r\to 0} r^{N-1}u'(r) = -(n-2)k(N) .$$

The fact that the condition $V \in \underline{P}_1$ is essential in this result is shown by the following converse result

PROPOSTION 2.1 Let V be a nonnegative potential that does not belong to P_1. Then there exist two positive solutions of (2.1), u_0 and u_1, such that

$$(2.9.a) \qquad \lim_{r\to 0} u_0(r) = \lim_{r\to 0} ru_0'(r) = 0$$

$$(2.9.b) \qquad \lim_{r\to 0} \frac{u_1(r)}{E_N(r)} = \lim (-r^{N-1}u_0'(r)) = \infty .$$

Moreover $u_0(r)u_1(r) \leq E_N(r)$ and

$$(2.10) \qquad V(r)\, u_0(r)\, E_N(r) \in L^1(B_R)$$

REMARKS. 1) u_0 is a regular solution of (0.1) at $x = 0$.

2) (2.10) is not true for $u = u_1$. On the contrary if $V \in \underline{P}_1$ it holds for all solutions of (2.1).

<u>Proof of the Proposition</u>. Using the transformation (2.3) we are reduced to construct positive solutions of (2.4) with $X \geq 0$, $X \in \underline{P}_1$, such that as $s \to 0$ $v_0(s)/s \to 0$ and $v_1(s) \to \infty$.

Let us begin with $v_1(s)$: we take the solution of (2.4) with initial condition

$$v_1(s) \geq 0 \ , \ \dot{v}_1(s) = -1$$

with $S = R^{N-2}$. By (1.4) v is convex, namely $\dot{v}$ is an increasing function in $(0,S)$, hence

$$\dot{v}_1(s) \leq -1 \quad \text{in} \quad 0 < s < S \ .$$

We conclude that the function v_1 is positive and decreasing in $(0,S)$ and that there exists the limit $\lim_{s\to 0} v_1(s) = p$, $0 < p \leq \infty$. We have to prove that $p = \infty$.

We take a number $c \in (0,S)$. Since v_1 is strictly decreasing we have $v(s) > v(c) > 0$ for $0 < s < c$. Integrating twice (2.4) between s and c with $0 < s < c/2$ we have

$$\dot{v}_1(c)(c-s) - v_1(c) + v_1(s) \geq v_1(c)I(s), \tag{2.11}$$

where

$$I(s) = \int_s^c dy \int_y^c dt\, X(t) = \int_s^c (t-s)\, X(t)dt \geq \int_{2s}^c \frac{t}{2}\, X(t)dt.$$

Since $X \notin \underline{P}_1$ it follows that $I(s) \to \infty$ if we let $s \to 0$. Going back to (2.11) we have $v_1(s) \to \infty$ as $s \to 0$. Since

$$v_1(c) - v_1(s) = \int_s^c X(t)v_1(t)dt > \frac{v_1(c)}{c} \int_s^c X(t)t\, dt$$

we have $\dot{v}_1(s) \to \infty$ as $s \to \infty$.

The solution v_0 is obtained by the method of variation of constants:

(2.12) $$v_0(s) = v_1(s) \int_0^s \frac{dt}{v_1^2(t)} .$$

Since v_1 is decreasing in $(0,S)$ we have

(2.13) $$v_0(s)v_1(s) \leq s ,$$

i.e. $u_0(r)u_1(r) \leq E_N(r)$. (2.10) is equivalent to prove that the integral $\int X(t)v_0(t)dt$ is convergent at 0 and this follows from

(2.14) $$\dot{v}_0(S) - \dot{v}_0(s) = \int_s^S X(t)v_0(t)dt ,$$

letting $s \to 0$. In fact since v_0 is convex and $v_0(s)/s \to 0$ we have $\dot{v}_0(s) \to 0$ as $s \to 0$.

The conclusions for u_0' and u_1' in (2.9) follow now from the formulas

(2.15a) $$ru'(r) = (N-2)[\dot{v}(s) - \frac{v(s)}{s}] \quad \text{if} \quad N \geq 3 ,$$

(2.15.b) $$ru'(r) = \dot{v}(s)s - v(s) \quad \text{if} \quad N=2 .$$ □

The situation is different for negative potentials:

PROPOSITION 2.2. <u>Let</u> V <u>a nonpositive potential that does not belong to</u> $\underline{P_1}$. <u>Then either all the solutions to</u> (2.1) <u>oscillate infinitely many times near</u> $r = 0$ <u>or they satisfy</u>

(2.16.a) $$\lim u(r) = +\infty ,$$

(2.16.b) $$\lim_{r\to 0} \frac{u(r)}{E_N(r)} = 0,$$

<u>and</u>

(2.16.c) $$\lim_{r\to 0} r^{N-1} u'(r) = 0 .$$

Proof. Using the change of variables (1.3), (1.5) (or (1.3'), (1.5') if $N=2$) we can consider solutions $v(s)$ of (2.4) in $0 < s < S$ with $X \leq 0$, $X \notin \underline{P}_1$. It is standard that either all the solutions oscillate an infinite number of times around the 0-level as $s \to 0$ or every solution is either positive or negative in a small neighbourhood of 0. Let us assume to be specific that v is a positive solution of $\ddot{v} = X(s)v$ for $0 < s < S$. Since $X \leq 0$, v is a concave function and there exists the limit

$$p = \lim_{s\to 0} v(s).$$

We have $0 \leq p < 0$. We want to prove that $p = 0$, i.e. (2.16.a). In fact if $p>0$ we integrate the equation as in Proposition 2.1 to obtain for $0 < s \leq c < S$

$$v(s) = v(c) + \dot{v}(c)(s-c) + \int_s^c dy \left(\int_y^c X(t)v(t)dt\right).$$

For c small enough we have $v(t) \geq p/2$ if $0 < s < c$. Hence

$$v(s) \geq v(c) + \dot{v}(c)(s-c) + \frac{p}{2}\int_s^c dy \int_y^c X(t)dt$$

As in Proposition 2.1 the last integral is infinite, which leads to contradiction. Therefore $p = 0$ and v is increasing in a certain neighbourhood of $s=0$.

By concavity there also exists

$$\dot{v}(0) = \lim_{s\to 0} \dot{v}(s)$$

and $0 < \dot{v}(0) \leq \infty$. It is then clear that $v(s) \geq a.s$ in a neighbourhood of 0 for a certain $a>0$. Since for $0 <s \leq c < S$ we have

$$\dot{v}(s) = \dot{v}(c) + \int_s^c (-X)(t)v(t)dt$$

we easily conclude from $X \notin \underline{P}_1$ that $\dot{v}(0) = \infty$ so that

$$\lim_{s\to 0} \frac{v(s)}{s} = \infty ,$$

i.e. (2.16.a). Finally from (2.15) and the fact that $0 < v(s) - \dot{v}(s)s < v(s)$ we get (2.16.c). □

Proposition 2.2 implies that for negative potentials that do not belong to $\underline{P}_1$ the radial solutions of (0.1) belong to one of those two types: i) all of them oscillate infinitely many times as $r \to 0$, ii) all of them are nonsingular (by (2.15.c)) and discontinuous (by 2.15.a). For potentials of the form

$$V(r) = cr^{-2} \quad , 0 < c$$

the first situation occurs if $c < c_* = -((N-2)/2)^2$ while the second one happens if $0 > c \geq c_*$, see formulas (1.9).

Our next result shows how a perturbation of the potential affects the nature of the singularity at 0 of a radially symmetric solution of (0.1).

PROPOSITION 2.3. _Let_ $V = \hat{V} + \overline{V}$ _be a potential in_ $(0,R)$ _with_ $\hat{V} \geq 0$ _and let_ $\hat{u}_0$, $\hat{u}_1$ _be the solutions of_ $\hat{u}''(r) + ((N-1)/r)\hat{u}'(r) = \hat{V}(r)\hat{u}(r)$ _constructed in Proposition_ 2.1. _Then_ i) _if_ V _satisfies_

$$\int_0^R |\overline{V}(R)|\hat{u}_o(r)\hat{u}_1(r)r^{N-1}\, dr < \infty \tag{2.17}$$

for $i = 0,1$ _there exist solutions_ u_0,u_1 _of_ (2.1) _such that_

$$\lim_{r\to 0} \frac{u_i(r)}{\hat{u}_i(r)} = 1, \quad i = 0,1. \tag{2.18}$$

ii) Assume now that

$$\int \overline{V}^{+}(r)\,\hat{u}_o(r)\hat{u}_1(r)r^{N-1}dr = \infty \,, \tag{2.19.a}$$

$$\int \overline{V}^{-}(r)\,\hat{u}_o(r)\hat{u}_1(r)r^{N-1}\,dr < \infty \,. \tag{2.19.b}$$

Then there exist solutions u_o, u_1 of (2.1) such that

$$\lim_{r\to 0} \frac{u_0(r)}{\hat{u}_0(r)} = 0 \,, \quad \lim_{r\to 0} \frac{u_1(r)}{\hat{u}_1(r)} = \infty \,. \tag{2.20}$$

REMARK 2.1. Condition (2.17) is in particular implied by $\overline{V}\in \underline{P}_1$ by virtue of Proposition 2.1. In fact (2.17) is equivalent to $\overline{V} \in \underline{P}_1$ if $\hat{V} \in \underline{P}_1$ (see (2.8)) or if $\hat{V}(r) = cr^{-2}$ with $c > 0$ (see (1.9)). We shall see later that the equivalence holds if $\hat{V} \in \underline{P}_2$. On the contrary if $\hat{V}(r) = r^{\theta}$, $\theta < -2$, there are solutions $\hat{u}_0$, $\hat{u}_1$ as in Proposition 2.1 such that

$$\hat{u}_0(r)\hat{u}_1(r) \simeq r^{P+2-N} \quad \text{if} \quad r \simeq 0 \,,$$

where $p = -\frac{1}{2}(\theta+2) > 0$, see (0.6). Therefore (2.17) means

$$\int_0^R |\overline{V}(r)|\,r^{-\theta/2}\,dr < \infty \,, \tag{2.21}$$

a class larger than $\underline{P}_1$.

Proof of the Proposition. We assume that $N \geqslant 3$, the case $N = 2$ being similar (cf. Prop. 2.1).

Part i) We perform the change of variables $u(r) = v(s)/s$, $\hat{u}(r) = y(s)/s$, $s = r^{N-2}$. As in Proposition 2.1 we obtain the equations

$$\ddot{v}(s) = X(s)\,v(s), \quad \ddot{y}(s) = X(s)\,y(s) \quad \text{for} \quad 0 < s < R^{N-2} \,,$$

with X, $\hat{X}$ defined as in (2.5). Let $y_0(s)$, $y_1(s)$, be the two solutions of the last equation obtained in Proposition 2.1. We have to prove that there exist solutions v_0 , v_1 of $\ddot{v}(s) = X(s)v(S)$ such that $v_i(s)/y_i(s) \to 1$ as $s \to 0$, $i = 0,1$.

This is done as follows. Consider the function

$$h(t) = \frac{v(s)}{y_1(s)} \quad \text{with} \quad t = \frac{y_0(s)}{y_1(s)} \tag{2.22}$$

for $0 < s \leq c < R$, c being sufficiently small so that $y(s)$ is positive in $(0,c)$. $h(t)$ satisfies the equation

$$\frac{d^2h(t)}{dt^2} = \overline{X}(s(t))y_1^4(s(t))h(t), \quad 0 < t \leq t_0 = t(c). \tag{2.33}$$

If the potential $W(t) = \overline{X}(s(t))y_1^4(s(t))$ belongs to the class $\underline{P}_1$ there exist two solutions $h_0(t)$, $h_1(t)$ of (2.23) such that as in (2.7) we have

$$h_0(t) = t + o(t), \; h_0'(t) = 1 + o(1)$$

$$h_1(t) = 1 + o(t), \; h_1'(t) = o(1/t)$$

as $t \to 0$. Setting for $i = 0,1$, $v_i(s) = h_i(t)/y_1(s)$ and $u_i(s) = v_i(s)/s$ the result (2.18) for u_0, u_1 would follow. Therefore we have to check that $W \; \underline{P}_1$.

$$\begin{aligned} \int_0^{t_0} t|W(t)|dt &= \int_0^c t|\overline{X}(s(t)|y_1^4 (s)dt = \\ &= \int_0^c |X(s)| \; y_0(s)y_1(s)ds = \int_0^{R(c)} |\overline{V}(r)|\hat{u}_0(r)\hat{u}_1(r)r^{N-1} \, dr < \infty . \end{aligned} \tag{2.24}$$

<u>Part ii)</u>. Assume first that $V = 0$. Then, with the notations of part i) we have

$$\int_0^{t_0} tW(t)dt = \int_0^{R(c)} \overline{V}(r)\hat{u}_0(r)\hat{u}_1(r)r^{N-1} \, dr = + \infty .$$

Therefore by Proposition 2.1 there exist two solutions, h_0 and h_1 of (2.23) such that as $t \to 0$ $h_0(t)/t \to 0$ and $h_1(t) \to \infty$. Reversing the changes of variables $y_1(s)h_i(t) = v_i(s)$ and $v_i(s) = su_i(r)$, $i = 0,1$, we obtain (2.19). Remark also that $h_0(t)h_1(t) < 1$, i.e. $v_1(s)v_0(s) < y_0(s)y_1(s)$ or even $u_0(r)u_1(r) < \hat{u}_0(r)\hat{u}_1(r)$.

In case $\overline{V} \neq 0$ we proceed in two steps. First we consider the potential $V_1 = \hat{V} + V^+$ and argue as above. From (2.19.b) and the remark of the last paragraph it follows that $\overline{V}$ satisfies condition (2.17) with respect to a pair of solutions of (2.1) with potential V_1. Therefore by part i) of this proof we conclude that no change of behaviour occurs upon passing from $V_1 = \hat{V} + \overline{V}^+$ to $V = V_1 - \overline{V}^-$. $\square$

We give next a simple comparison result for singular solutions corresponding to different potentials.

PROPOSITION 2.4 <u>Let</u> $V, \overline{V}$ <u>be two potentials such that</u> $\overline{V}(r) \geq V(r)$ <u>for</u> $0 < r < R$. <u>If there is a positive singular solution</u> u_1 <u>of</u> (2.1) <u>with potential</u> V <u>there is also a singular solution</u> $\overline{u}_1$ <u>of</u> (2.1) <u>with potential</u> $\overline{V}$ <u>such that</u> $\overline{u}_1 \geq u_1$ <u>near</u> $r=0$.

<u>Proof.</u> Take a point $c \in (0,R)$ and let $\overline{u}_1$ be the solution of (2.1) with potential $\overline{V}$ and initial data

$$\overline{u}_1(c) = u_1(c), \quad \overline{u}_1'(c) = u_1'(c).$$

For every $r \in (0,c)$ we have

$$\frac{d}{dr}\{r^{N-1}(\overline{u}_1'(r)u_1(r) - \overline{u}_1(r)u'(r))\} = r^{N-1}\{\overline{V}(r)-V(r)\}\overline{u}_1(r)u_1(r) \,. \tag{2.25}$$

Therefore on every interval $I = (a,c) \subset (0,c)$ where $\overline{u}_1$ is positive the function $r^{N-1}\{\overline{u}_1'(r)u_1(r) - \overline{u}_1(r)u_1'(r)\}$ is nondecreasing. Since it vanishes at $r = c$ we conclude that

$$\frac{\bar{u}_1'(r)}{\bar{u}_1(r)} \leq \frac{u_1'(r)}{u_1(r)} \tag{2.26}$$

in (a,c). We conclude that $a = 0$ and $\bar{u}_1(r)/u_1(r)$ is nonincreasing in (0,c). Since $\bar{u}_1(c) = u_1(c)$ we get $\bar{u}_1(r) \geq u_1(r)$ for $0 < r < c$. $\square$

In the study of the asymptotic behaviour in Chapter 6 it will be interesting to know how much the solutions corresponding to a potential V $\underline{P}_1$ depart from the ones for $V = 0$. The next result settles this question as $r \to 0$.

PROPOSITION 2.5. *Let* V $\underline{P}_1$ *and let* u_0, u_1 *be as above. Then as* $r \to 0$

$$u_0(r) = 1 + p_0(r)(1 + o(1)) \ , \tag{2.27}$$

$$u_1(r) = E_N(r) + p_1(r)(1 + o(1)) + C \ , \tag{2.28}$$

where

$$p_0(r) = \int_0^r s^{1-N}\left(\int_0^s t^{N-1}V(t)dt\right)ds = o(1) \tag{2.29}$$

and

$$p_1(r) = \begin{cases} k(N)\int_r^R s^{1-N}\left(\int_0^s tV(t)dt\right)ds & \text{if } N \geq 3, \\ \frac{1}{2\pi}\int_r^R s^{-1}\left(\int_0^s t|\log t|V(t)dt\right)ds & \text{if } N=2 \ , \end{cases}$$

hence $p_1(r) = o(E_N(r))$.

Proof. We know that $u_0(r) = 1 + o(1)$. Hence integrating (2.1) we get

$$u_0'(r) = \left(r^{1-N}\int_0^r t^{N-1}V(t)dt\right)(1+o(1)),$$

so that

$$u_0(r)-1 = \left(\int_0^r s^{1-N}\int_0^s t^{N-1}V(t)dt\right)(1+o(1)),$$

i.e. (2.27), $p_0(r)$ being well-defined by (2.29) since $V \in \underline{P}_1$.

Let us now prove (2.28) when $N \geq 3$. As before, if $E_N(r) = k(N)r^{2-N}$, we get for a constant C:

$$u_1'(r) = Cr^{1-N} + (kr^{1-N} \int_0^r tV(t)dt)(1+0(1)).$$

Hence for another constant C

$$u_1(r) = k(N)r^{2-N} - \{k(N) \int_r^R s^{1-N}(\int_0^s tV(t)dt)ds\} (1+o(1)) + C,$$

since we know that $u_1(r) \simeq E_N(r)$. Finally $p_1(r) = o(E_N(r))$ follows easily from the fact that $V \in P_1$.

The case $N=2$ is similar. $\square$

COROLLARY 2.6. If $V(r) = 0(r^\theta)$ _for some_ $\theta > -2$ _we have as_ $r \to 0$

(2.31) $$u_0(r) = 1 + 0(r^{\theta+2}) ,$$

(2.32) $$u_1(r) = E_N(r)(1 + 0(r^{\theta+2})) + c,$$

for some $c \in \mathbb{R}$.

2.2. Using the Kelvin transformation

$$w(s) = s^{2-N}u(1/s), W(s) = s^{-4}V(1/s) ,$$

We can easily apply the perturbation result, Proposition 2.3, to the behaviour at infinity of the solutions of

$$w'' + \frac{N-1}{s} w' = W(s)w \quad \text{in} \quad S < s < \infty .$$

In particular condition (2.17) transforms into

(2.17') $$\int_S^\infty |\overline{W}(s)|\hat{w}_0(s)\hat{w}_1(s)s^{N-1} ds < \infty .$$

with obvious notation. Many perturbation results can be found for the case $N=1$, $W=1$ in [B].

As an application we obtain the behaviour of the solutions of (2.1) as $r \to 0$ when $V(r) = r^\theta$, $\theta < -2$. The change of variables

$$(2.33) \qquad u(r) = r^k w(s), \quad s = \frac{1}{p} r^{-p}$$

with $p = -\frac{\theta+2}{2}$ and $k = \frac{p+2-N}{2}$ transforms (2.1) into

$$(2.34) \qquad \ddot{w}(s) = (1+\overline{W}(s))w(s), \qquad S < s < \infty$$

with $\overline{W}(S) = Cs^{-2}$, $C = C(N,\theta) > 0$. Since two solutions of $\ddot{w}(s) = w(s)$ are e^s and e^{-s} and (2.17') is satisfied, Prop. 2.3 at infinity implies that there exist two solutions of (2.34), w_0 and w_1, such that as $s \to \infty$

$$(2.35) \qquad w_0(s) \simeq e^{-s}, \ w_1(s) \simeq e^s,$$

(cf. [B], pg. 125). Therefore we obtain two solutions of (2.1), u_0, u_1, which behave as $r \to 0$ like (0.6).

The estimates (0.6) are yet obtained if $V = r^\theta + V$ and $\overline{V}$ satisfies (2.21). This is the case for instance if

$$(2.36) \qquad V = r^\theta + O(r^\mu), \quad \mu > -1 + \theta/2.$$

REMARK. An explicit formula for the solutions u_0, u_1 when $V(r) = r^\theta$, $\theta < -2$, can be found for instance in [GL].

3. PROOF OF THEOREMS A AND B

In the sequel u_0 and u_1 are the radial solutions of (2.1) constructed in the previous section and $U_1(r)$ is a positive singular solution of (1.8) with $\lambda = N-1$, so that $U_1(r)$ grows like r^{1-N} as $r \to 0$ if $V \in \underline{P}_1$. Given a solution $u(x)$ of (0.1) defined in B_R^* we denote by $\bar{u}(r)$ the angular mean of u as defined in (0.4).

In order to prove Theorem A we need to control the nonradial part of u, $u(x)-\bar{u}(r)$. This is done in the next lemma that has an independent interest and exploits ideas introduced by Véron [Ve] and then used by Véron and one of the authors [VV] in dealing with semilinear elliptic equations.

LEMMA 3.1 *Let* $u \in C^2(B^*)$ *be a solution of* (0.1) *such that* $u(x) = o(U_1(r))$ *as* $r = |x| \to 0$. *Then*

$$u(x) - \bar{u}(r) = O(u_0(r)). \tag{3.2}$$

Proof: 1) The first step consists in obtaining a bound for $u(r,\sigma) - \bar{u}(r)$ in $L^2(S_{N-1})$. Let

$$g(r) = \left(\int_{S_{N-1}} (u(r,\sigma) - \bar{u}(r))^2 d\sigma\right)^{1/2} . \tag{3.3}$$

Let $A = \{r \in (0,R_0): g(r) > 0\}$, where $R_0 \in (0,R)$ is so small as to have $U_1(r) > 0$ in $(0,R_0]$. On every connected component of A, say (α,β), we have (see [Ve] or [VV])

$$g''(r) + \frac{N-1}{r} g'(r) \geq \left(\frac{N-1}{r^2} + V(r)\right)g(r). \tag{3.4}$$

Consider now the function

$$h(s) = g(r)/U_1(r) , \tag{3.5}$$

where $s = U_0(r)/U_1(r)$, $U_0(r) = U_1(r)\int_0^r t^{1-N}\, U_1^{-2}(t)dt$. $s = s(r)$ in an increasing bijection from $(0,R_0)$ onto $(0,s(R_0))$. (3.4) implies that

$$\ddot{h}(s) \geq 0 \qquad \text{for } s(\alpha) < s < s(\beta) \;\; (\cdot = \frac{d}{ds}), \tag{3.6}$$

i.e. h is convex, therefore either $s(\alpha) = 0$ and $\alpha = 0$ or $s(\beta) = s(R_0)$, i.e. $\beta = R_0$. We also now that $h(s) = o(1)$ as $s \to 0$ by the assumption on u. Therefore if $A \supset (0,\beta)$ then necessarily $\beta = R_0$. Summing up, either $A = (0,R_0)$ or $A = (\alpha,R_0)$ with $0 < \alpha < R_0$. In this last case we have $g(r) = 0$ and $u = \overline{u}$ for $0 < r \leq \alpha$. In the other case $A = (0,R_0)$, $h(0) = 0$ and h is convex in $(0,s(R_0))$, therefore $h(s) \leq Cs$ for a certain $C>0$. This means that

$$g(r) \leq CU_0(r), \qquad 0 < r < R_0 \tag{3.7}$$

2) We next obtain a bound in $L^\infty(S_{N-1})$: Let $C > 0$ be a constant such that

$$|u(x) - u(x)| \leq Cu_0(R_0) \quad \text{for} \quad |x| = R_0. \tag{3.8}$$

Let now

$$w(x) = \{(u(x) - \overline{u}(r)) - Cu_0(r)\}^+. \tag{3.9}$$

Clearly $w \in H^1(B_R^*)$. By Kato's inequality [K] w satisfies

$$\Delta w \geq Vw \quad \text{in } D_R'(B^*),$$

hence its angular mean $\overline{w}$ satisfies

$$\overline{w}'' + \frac{N-1}{r}\overline{w}' \geq V\overline{w} \quad \text{in } \underline{D}'(0,R_0).$$

As above if we set

$$h_1(s) = \overline{w}(r)/u_1(r), \quad s(r) = u_0(r)/u_1(r).$$

then h_1 turns out to be convex and there exists the limit

$$\lim_{s\to 0} h_1(s) = \lim_{r\to} \frac{\overline{w}(r)}{u_1(r)} = a \quad [0,\infty].$$

Since $\overline{w}(r) = 0\,(u_0(r) + U_0(r))$ (by (3.7), (3.9)) and $U_0(r) \leq u_0(r)$ for r small we conclude that $a = 0$, i.e. $h_1(0) = 0$. Also $h_1(R_0) = 0$, therefore $h_1 \equiv 0$ and $\overline{w} \equiv 0$ in $(0,R_0)$. w being nonnegative, it follows that $w \equiv 0$, i.e.

$$u(x) \leq \overline{u}(r) + Cu_0(r)$$

Since the same argument can be applied to $-u$ (3.2) holds. $\square$

<u>Proof of Theorem</u> A. By Propositions 2.1 and 2.3 we may assume that $\overline{u} \equiv 0$ by studying $u-\overline{u}$ instead of u.

We first transform the unilateral bound (1.4) into an absolute bound. Let $x_0 \in B_{r/2}$ and let $\rho_0 = |x_0|/2$. We apply Kato's inequality to u in $B_{\rho_0}(x_0) \subset B_R^*$:

$$\Delta|u| \geq \Delta u\cdot \mathrm{sign}_0(u) = V|u| \geq -V^-(r)|u(x)|. \tag{3.10}$$

Therefore $|u|$ is a positive subsolution of the equation $-\Delta z - V^-(r)z = 0$. Aizenman and Simon have proved, [AS], that the Harnack inequality

$$|u(x_0)| \leq C \int_{|x-x_0| \leq \delta} |u(x)|dx \tag{3.11}$$

holds provided that the potential V^- belongs to the Kato class K_N, hence in our case $V^- \in \underline{P}_1$. C and δ are positive constants depending on the norm of V^- in K_N, cf. [AS, Theorems 3.8 and 6.1]. A careful scrutiny of their proof shows in our case we can take

$$C = k\delta^{-N}\ , \ 0 < \delta < \delta_0 \tag{3.12}$$

with k, δ_0 positive constants not depending on x_0.

Assuming now that $u^+(x) = o(r^{1-N})$ and using (3.11), (3.12), we have for all $|x_0|$ small ($\rho_0 = |x_0|/2$)

$$|u(x_0)| \leq k\rho_0^{-N} \int_{B_{\rho_0}(x_0)} |u(x)|dx \leq |S_{N-1}|k\rho_0^{-N} \int_{\rho_0}^{3\rho_0} \overline{|u(r)|}^{N-1}dr$$

$$= 2|S_{N-1}|k\rho_0^{-N} \int_{\rho_0}^{3\rho_0} u^+(r)r^{N-1}\,dr = o(|x_0|^{1-N}).$$

Since $U_1(r) \approx r^{1-N}$, Lemma 3.1 can be applied. Thus we get

$$u(x) = u(x)-\overline{u}(x) = o(u_0(r)),$$

therefore u is bounded. It follows that $Vu \in L^1(B_R)$ and that (0.1) is satisfied in $\underline{D}'(B_R)$. By Theorem 1.5 of [AS] we conclude that u is continuous.

Finally proving (1.6) is equivalent to prove that $\Delta u_1 + u_1 = \delta$ in $\underline{D}'(B_R)$ and this follows by standard arguments from the properties of u_1. $\square$

We end this section with <u>the proof of Theorem B</u>:

i) $\Rightarrow$ iii) This is a consequence of Lemma 3.1 and the observation that $u_1(r) = O(U_1(r))$ that follows easily from the equations satisfied by both of them.

iii) $\Rightarrow$ i) and iii) $\Rightarrow$ ii) See Proposition 2.1, 2.3.

iv) $\Rightarrow$ ii) By definition.

iii) $\Rightarrow$ iv) This follows from a calculus lemma: "if $u \in L^{N/N-2}(B_R)$ (resp. $u \in L^\infty(B_R)$ and $u \to 0$ as $|x| \to 0$ if N=2), $f \in L^1(B_R)$ and $\Delta u = f$ in $\underline{D}'(B_R^*)$, then $\Delta u = f$ in $\underline{D}'(B_R)$" (see for instance [BV]).

ii) $\Rightarrow$ iii) This follows from

LEMMA 3.2. *Let* $u \in C^2(B_R^*)$ *be a solution of* (0.1) *with* $V^+ \notin \underline{P}_1$ *and suppose that* $Vu \in L^1_{loc}(B_R)$. *Then*

$$u(x) = O(u_0(r)) \quad \text{as} \quad r \to 0. \tag{3.17}$$

The proof of Lemma 3.2. is parallel to Step 2 of Lemma 3.1, putting $w(x) = (u(x) - cu_0(r))^+$ and using $V^+ \notin (\underline{P}_1)$, $Vu \in L^1_{loc}(B_R)$ and $u_1(r)/E_N(r) \to \infty$ instead of (3.7). □

4. ISOTROPIC SOLUTIONS

In this section we consider a class of potentials $\underline{P}_2$ for which it can be proved that a singular solution with constant sign becomes radially symmetric as $x \to 0$. This class is defined as follows: a potential $V \in C((0,R) : \mathbb{R})$ belongs to $\underline{P}_2$ if $V^- \in \underline{P}_1$ and there exists a singular radially symmetric solution of (0.1) that satisfies the following <u>power-growth</u> <u>condition</u>: there is a constant $K>0$ such that

$$r|u_1'(r)| \leq Ku_1(r) \tag{4.1}$$

if r is small enough. It follows from (2.8) that every potential in $\underline{P}_1$ belongs to $\underline{P}_2$. Moreover u_1 satisfies in that case

$$\lim_{r \to 0} \frac{ru_1'(r)}{u_1(r)} = 2-N. \tag{4.2}$$

A second important example of potentials in $\underline{P}_2$ consists in those V's of the form $V(r) = cr^{-2}$ with $c>0$. According to (1.9) we have

$$\lim_{r \to 0} \frac{ru_1'(r)}{u_1(r)} = -p \; \varepsilon(2-N, -\infty) \tag{4.3}$$

The condition (4.1) implies that $u_1(r)$ grows at most like r^K as $r \to 0$. Therefore the potentials $V(r) = cr^{\theta}$ with $c>0$, $\theta < -2$, do not belong to $\underline{P}_2$, since then the singular solution $u_1(r)$ grows exponentially, see (0.6).

Using the results of Section 2, if $V^- \in \underline{P}_1$ we can obtain a regular radial solution of (0.1) that is positive for small r by means of the formula

$$u_0(r) = u_1(r) \int_0^r \frac{dt}{t^{n-1}u_1^2(t)} \tag{4.4}$$

and $u_0(r) \to 0$ as $r \to 0$ if and only if $V \notin \underline{P}_1$. Let U_1 be a positive singular solution of (1.8) with $\lambda_n = N-1$. By Proposition 2.4 we may assume that $U_1 > u_1$ near 0 and $U_1(r)$ grows at least like r^{1-N} as $r \to 0$. We shall see later that $U_1(r)/u_1(r) \to \infty$ as $r \to \infty$.

We can state now our isotropy result

THEOREM C. Let $V \in \underline{P}_2$ and let $u \in C^2(B_R^*)$ be a solution of (0.1) such that

(4.5) $$u(x) > o(U_1(r)) \text{ or } u(x) < o(U_1(r))$$

as $|x| = r \to 0$. Then we have

(4.6) $$u(x) = \overline{u}(r) + g(x),$$

where g is a nonsingular solution of (0.1) in B_R. Moreover $g \in C(B_R)$ and $g(x) = o(u_0(r))$ as $r \to 0$.

Since $u(r) = c_1 u_1(r) + c_0 u_0(r)$ for some constants $c_0, c_1 \in \mathbb{R}$, the non-singular part of u, $c_0 u_0(r) + g(x)$ is $O(u_0)$. u is singular if $c_1 \neq 0$ and in this case $u \approx c_1 u_1$ has a singularity at 0 of weak type if $V \in \underline{P}_1$, of strong type otherwise. Remark that if $V \notin \underline{P}_1$ any nonsingular solution vanishes at $x = 0$.

Before giving the proof of Theorem C we pursue the study of the class $\underline{P}_2$. We first derive a comparison result.

PROPOSITION 4.1. Let V and $\hat{V}$ be potentials such that $V^- \in \underline{P}_1$, $\hat{V} \in \underline{P}_2$ and $V < \hat{V}$. Then $V \in \underline{P}_2$.

Proof. Let $\overline{V} = -V^-$ and let $\overline{u}_1(r)$ be a singular positive solution of (2.1) with potential $\overline{V}$. Choose a point $R_0 > 0$ such that $\overline{u}_1'(R_0) < 0$ and $\overline{u}_1(r) > 0$ for $0 < r < R_0$. Consider the solution u_1 of (2.1) with potential V and initial values

$$u_1(R_0) = \overline{u}_1(R_0) \quad , \quad u_1'(R_0) = \overline{u}_1'(R_0) \ .$$

We use the comparison technique of Proposition 2.4 to compare u_1 and $\overline{u}_1$. In fact it follows from it that if

$$G(r) = r^{N-1}\{u_1'(r)\overline{u}_1(r) - u_1(r)\overline{u}_1'(r)\}$$

we have $G(R_0) = 0$, $G'(r) \geq 0$ for $0 < r < R$, $u_1(r) \geq \overline{u}_1(r) > 0$ in $(0,R)$ and

$$\text{(4.7)} \qquad -\frac{ru_1'(r)}{u_1(r)} \geq -\frac{r\overline{u}_1'(r)}{\overline{u}_1(r)} \ .$$

By the same argument, applied now to V and $\hat{V}$ instead of $\overline{V}$ of V, and have

$$\text{(4.8)} \qquad -\frac{ru'(r)}{u(r)} \leq -\frac{r\hat{u}_1'(r)}{\hat{u}_1(r)} \ ,$$

where $\hat{u}_1$ is a singular positive solution of (0.1) with potential $\hat{V}$ and $u = c_1u_1 + c_0u_0$ is a singular solution of (0.1) with potential V that satisfies suitable initial conditions. Since $c_1>0$ and $u_0(r)$, $ru_0'(r) = o(u_1(r))$ as $r \to 0$, we have

$$\text{(4.9)} \qquad \frac{ru'(r)}{u(r)} = \frac{ru_1'(r)}{u_1(r)}\,(1 + o(1)) \ .$$

It follows from (4.7) - (4.9) that u_1 satisfies (4.1). $\square$

In fact it follows from the above comparisons that for every potential V such that $V^- \in \underline{P}_1$ we have

$$\text{(4.10)} \qquad \liminf_{r\to 0} \frac{ru_1'(r)}{u_1(r)} \leq 2-N.$$

Also the limit

$$(4.11) \qquad K_1 = \limsup_{r\to 0} \left(-\frac{ru_1'(r)}{u_1(r)}\right)$$

is independent of the singular solution $u_1(r)$ and depends only on V, $K_1 = K_1(V)$. We prove next that the class $\underline{P}_2$ is closed under addition

PROPOSITION 4.2. $\underline{P}_2 + \underline{P}_2 \subset \underline{P}_2$.

<u>Proof</u>. Let $V = \hat{V} + \overline{V}$ with $\hat{V}, \overline{V} \in \underline{P}_2$. Choose singular solutions $\hat{u}_1, \overline{u}_1$ of (2.1) with potential $\hat{V}, \overline{V}$ resp. as in the definition of $\underline{P}_2$ and set $u(r) = \hat{u}_1(r)\overline{u}_1(r)$. We have

$$(4.12) \qquad u''(r) + \frac{N-1}{r} u'(r) = (\hat{V}(r) + \overline{V}(r))u(r) + 2\hat{u}'(r)\overline{u}'(r) .$$

Since for $N\geq 3$ $\hat{u}_1'(r)$ and $\overline{u}_1'(r) \leq 0$ for r small enough we see that u is a solution of (2.1) with potential

$$(4.13) \qquad V_1(r) = \hat{V}(r) + \overline{V}(r) + \frac{2\hat{u}_1'(r)\overline{u}_1'(r)}{\hat{u}_1(r)\overline{u}_1(r)} \geq \hat{V}(r) + \overline{V}(r)$$

Moreover since

$$\frac{ru'(r)}{u(r)} = \frac{r\hat{u}_1'(r)}{\hat{u}_1(r)} + \frac{r\overline{u}_1'(r)}{\overline{u}_1(r)} ,$$

$V_1 \in \underline{P}_2$ and $K_1(V_1) \leq K_1(V_1) + K_1(V_2)$. By Proposition 4.1 we have $V \in \underline{P}_2$ and $K_1(V) \leq K_1(V_1) + K_2(V_2)$.

In case N=2 we do not necessarily have $\hat{u}_1'(r), \overline{u}_1'(r) \leq 0$ but since

$$\liminf_{r\to 0} \frac{r\hat{u}_1'(r)}{\hat{u}_1(r)} , \liminf_{r\to 0} \frac{r\overline{u}_1'(r)}{\overline{u}_1(r)} \geq 0$$

we can make a similar argument with $u(r) = u_1(r)u_2(r)r^{-p}$ for any $p>0$. $\square$

The above comparison methods allow us in particular to estimate the ratio $U_1(r)/u_1(r)$ for the functions U_1, u_1 appearing in Theorem C. We obtain first a general result.

PROPOSITION 4.3. *Let* $V \in \underline{P}_2$, $c>0$ *and* $\hat{V} = V + cr^{-2}$. *For every* $\varepsilon>0$ *there exists* $r_0>0$ *such that*

$$p_1 - \varepsilon < \frac{ru_1'(r)}{\hat{u}_1(r)} - \frac{r\hat{u}_1'(r)}{u_1(r)} < p_2 + \varepsilon \tag{4.14}$$

where $p_2 = -\frac{(N-2)}{2} + \left(\left(\frac{N-2}{2}\right)^2 + c\right)^{1/2}$ *and* p_1 *is the positive solution of*

$$p^2 + p(2K_1(V) - N+2) = c \tag{4.15}$$

Proof. For $p>0$ the function $Z(r) = u_1(r)r^{-p}$ is a solution of the Schrödinger equation with potential

$$V + \frac{p}{r^2}\left[p - (N-2) + 2\left(-\frac{ru_1'(r)}{u_1(r)}\right)\right]$$

If $p > p_2$, since $-ru_1'(r) > (N-2-\varepsilon)u_1(r)$ for r small, this potential is larger than $\hat{V}$ and the conclusion (4.14) - right follows by comparison (see Proposition 4.1). Since $-ru'(r)/u(r) < K_1(V) + \varepsilon$ for r small the potential is smaller than $\hat{V}$ if $0 < p < p_1$. □

COROLLARY. *With the notations of Theorem C we have for every small* $\varepsilon>0$

$$0 < p_1 - \varepsilon < \frac{\log(U_1(r)/u_1(r))}{\log(r)} < 1 + \varepsilon \tag{4.16}$$

if $0 < r < r_\varepsilon$. p_1 *is determined from* (4.15) *with* $c = N-1$. □

We now derive some properties of the regular solution $u_0(r)$.

PROPOSITON 4.4. <u>Let</u> $V \in \underline{P}_2$ <u>and</u> u_1 <u>satisfy</u> (4.1). <u>Then there exists a nonsingular solution</u> $u_0(r)$ <u>given by</u>

(4.17) $$u_0(r) = u_1(r) \int_0^r \frac{dt}{t^{N-1} u_1^2(t)},$$

<u>and the following properties hold:</u> u_0 <u>is bounded and there exists a constant</u> $C>0$ <u>such that for all small</u> $r>0$

(4.18.a) $$\frac{1}{C} r^{2-N} \leq u_0(r) u_1(r) \leq C r^{2-N} \quad \text{if} \quad N \geq 3,$$

(4.18.b) $$\frac{1}{C} \leq u_0(r) u_1(r) \leq C|\log r| \quad \text{if} \quad N = 2,$$

(4.19) $$r|u_0'(r)| \leq C u_0(r).$$

<u>Proof.</u> By (2.8), (2.9) and Proposition 2.3 there exists the limit

(4.20) $$L = \lim_{r \to 0} (u_1(r)/E_N(r))$$

and $0 < L < \infty$. Therefore if r is small we have

$$\frac{u_1(t)}{E_N(t)} \geq \frac{u_1(r)}{2E_N(r)}$$

if $0 < t < r$. Hence

$$u_1(r) u_0(r) \leq 4E_N^2(r) \int_0^r \frac{dt}{t^{N-1} E_N^2(t)} = C E_N(r)$$

where $C = C(N) > 0$. This proves the right-hand side of (4.8). To prove the left-hand side we observe that since $|r u_1'(r)| \leq C_1 u_1(r)$ we have for $r/2 \leq t \leq r$

$$|\log(\frac{u_1(t)}{u_1(r)})| = \int_t^r \frac{|u_1'(r)|}{u_1(r)}\, dr \leq C_1 \log(2).$$

Therefore $u_1(t) \leq 2^{C_1} u_1(r)$ and

$$u_0(r)\, u_1(r) \geq 4^{-C_1} \int_{r/2}^{r} \frac{dt}{t^{N-1}} = \frac{1}{C}\, r^{2-N} .$$

Finally (4.19) follows from the equation

$$\frac{ru_0'(r)}{u_0(r)} - \frac{ru_1'(r)}{u_1(r)} = \frac{1}{r^{N-2}\, u_0(r)u_1(r)} , \tag{4.21}$$

using (4.1) and (4.18). □

We end this study by proving that $\underline{P}_1$ is a class of small perturbations in $\underline{P}_2$ in a strong sense.

PROPOSITION 4.5. <u>Let</u> u_1, $\hat{u}_1$ <u>be radial singular solutions of</u> (2.1) <u>with potentials</u> V, $\hat{V}$ <u>resp. and assume that</u> $V, \hat{V} \in \underline{P}_2$ <u>and</u> $\bar{V} = V - \hat{V} \in \underline{P}_1$. <u>Then</u>

$$\lim_{r\to 0} \left(\frac{ru_1'(r)}{u_1(r)} - \frac{r\hat{u}_1'(r)}{r\hat{u}_1(r)} \right) = 0 \tag{4.22}$$

<u>Proof</u>. Using the notations of Proposition 2.3, if $u_1(r) = h(t)\hat{u}_1(r)$ and $t = \hat{u}_0(r)/\hat{u}_1(r)$ we have

$$\frac{ru_1'(r)}{u_1(r)} - \frac{r\hat{u}_1'(r)}{\hat{u}_1(r)} = \frac{th'(t)}{h(t)}\, \phi(r) \tag{4.23}$$

where $\phi(r) = C(N)(r^{N-2}\, u_0(r)u_1(r))^{-1}$, $C(N) = N-2$ if $N \geq 3$, $C(2) = 1$. By (4.18) ϕ is bounded. On the other hand $h(t)$ satisfies the ODE

$$h''(t) = W(t)h(t),\ 0 < t < t_0$$

where (see (2.23)) $\bar{V} \in \underline{P}_1$ implies $W \in \underline{P}_1$. Therefore, since $h(t) \simeq 1$ for $t \simeq 0$ (see Proposition 2.3), it follows from (2.7) that $h'(t) = o(1/t)$ and the second member of (4.23) vanishes as $r \to 0$. □

The class $\underline{P}_2$ has other nice properties. As an example, it is easy to prove, arguing much as above, that for every potential $V \in \underline{P}_2$ and constants $a,b>0$, the potential $\tilde{V}(r) = aV(br)$ belongs again to $\underline{P}_2$.

We return now to the Proof of Theorem C. As in Theorem A we may assume that $u = 0$ by considering $g = u-\bar{u}$ instead of u. Most of the rest of the proof is also similar. Thus we can prove that (4.5) implies

$$|u_0(x)| = o(|x_0|^{-N} \int_{\rho_0}^{3\rho_0} r^{N-1}\, U_1(r)dr) \tag{4.24}$$

for $x_0 \in B_{R/2}$ with $|x_0|$ small enough and $\rho_0 = \frac{1}{2}|x_0|$. (Here the condition $V^- \in \underline{P}_1$ is used). Since by Proposition 4.2 U_1 also satisfies an inequality like (4.1) we have for some $c>0$

$$U_1(\rho_0) \geq 3^c U_1(\rho) \quad \text{if} \quad \rho_0 \leq \rho \leq 3\rho_0 \ ,$$

and

$$U_1(\rho_0) \leq 2^c U_1(2\rho_0) \ .$$

Hence it follows from (4.24) that $u(x_0) = o(U_1(|x_0|)$.

Our next step consists in checking that Lemma 3.1 is yet true under the present circumstances. In this way we get $u(x) = O(u_0 x))$.

To prove that indeed $u(x) = o(u_0(r))$ as $r \to 0$ we observe that the function $h(x) = u(x)/u_0(r)$ satisfies in $B^*_{R_0}$, R_0 small, the equation

$$\Delta h + 2\,\frac{u_0(|x|)}{u_0(|x|)} \sum_{i=1} \frac{x_i}{|x|} \frac{\partial h}{\partial x_i} = 0 \ . \tag{4.25}$$

The limit of $h(x)$ as $|x| \to 0$ exists by lemma 4.6 below. Since $\bar{h}(r) = \bar{u}(r)/u_0(r) = 0$ we conclude that this limit is 0 and the proof is done.

$\square$

<u>LEMMA</u> 4.6. *Let* $h \in L^\infty_{loc}(B_R) \cap C^2(B_R^*)$ *be a solution of the equation*

$$\Delta h + \sum_{i=1}^{N} c_i(x) \frac{\partial h}{\partial x_i} = 0 \quad \text{for} \quad 0 < |x| < R \tag{4.26}$$

with $|c_i(x)| \leq k|x|^{-1}$. *Then there exists the limit of* $h(x)$ *as* $|x| \to 0$.

Outline of the proof. We may assume that $h \geq 0$. As in [Ve, Lemma 1.5], it can be proved that

$$\max_{|x|=r} h(x) \leq C \min_{|x|=r} h(x), \quad 0 < r \leq R/2$$

with $C>0$ independent of r, as a consequence of the Harnack inequality of [GT, pg. 189]. Then we can use the method of [GS, Theorem 2], to conclude that the limit exists. $\square$

5. ANISOTROPIC SINGULAR SOLUTIONS

5.1. In this section we use the method of separation of variables mentioned in Section 1 to construct infinitely many linearly independent solutions, both singular and regular, that are nonisotropic and yet of constant sign near $r = 0$, for the class of potentials $\underline{P}_3$ defined as follows. Let V be a non-negative potential and let $u(r)$, $u_1(r)$ be a regular and a singular nonnegative solution of

$$(5.1) \qquad u''(r) + \frac{N-1}{r} u'(r) = V(r)u(r), \qquad 0 < r < R,$$

as constructed in Proposition 2.1. Then $V \in \underline{P}_3$ if and only if

$$(5.2) \qquad \int_0^r r^{N-3} u_0(r)u_1(r)dr < +\infty .$$

(We could replace $V \geq 0$ by $V^- \in \underline{P}_1$ and nothing would change in what follows).

Let us consider, as in (1.7), solutions of (0.1) of the form $u_n(x) = H_n(\sigma)U_n(r)$, where U_n satisfies

$$(5.3) \qquad U'' + \frac{N-1}{r} U' = (V(r) + \frac{\lambda_n}{r^2})U , \quad 0 < r < R .$$

It follows from Proposition 2.3 that there exist solutions $U_n^0(r)$, $U_n^1(r)$ of (4.3) that behave as $r \to 0$ like $u_0(r)$, $u_1(r)$. resp. if and only if (5.2) holds. In that case positive, non-isotropic, singular solutions of (0.1) can be constructed in the form

$$(5.4) \qquad u(x) = u_1(r) + \sum_{i=1}^{k} c_i H_i(\sigma)\, U_i^1(r)$$

provided that $\sum |c_i| \|H_i\|_\infty < 1$. Replacing the index 1 by 0 in (5.4) we obtain positive, non-isotropic, regular solutions of (0.1).

5.2 The condition (5.2) is not directly verifiable on V. For that reason the introduce a set of conditions on V under which we shall prove that $V \in \underline{P}_3$. These conditions are

$$(C_3)\begin{cases} \text{i)} & V \text{ is a } C^1 \text{ positive function in } (0,R)\,, \\ \text{ii)} & \lim\limits_{r\to 0} \frac{d}{dr}(V^{-1/2}(r)) = 0 \quad \text{and} \\ \text{iii)} & r^{-2}\, V^{-1/2} \in L^1(0,R) \ \text{ if } \ N\geq 3\,,\ (r|\log r|)^{-2}V^{-1/2} \in L^1(0,R) \text{ if } \ N=2\,. \end{cases}$$

It is clear that a potential of form $V(r) = cr^{\theta}$ satisfies (C_3) if and only if $c>0$ and $\theta<-2$.

We prove first an analogon of Proposition 4.4.

PROPOSITION 5.1 *Let V be a potential that satisfies (C_3) and let $u_0(r)$, $u_1(r)$ be solutions of (2.1) as in Section 2. Then $V(r)r^2 \to \infty$ as $r \to 0$ and for $i=0,1$ we have*

$$(5.5)\qquad \lim_{r\to 0} \frac{u_i'(r)}{V^{1/2}(r)u_i(r)} = (-1)^i$$

and there exists a constant $k>0$ such that

$$(5.6)\qquad \lim_{r\to 0} V^{1/2}(r)r^{N-1}\, u_0(r)u_1(r) = k \quad \text{if} \quad N\geq 3.$$

Proof. Let us use the notation of Section 2 and consider only the case $N\geq 3$. (5.2) can be written in terms of v_0 and v_1 as

$$(5.7)\qquad \int_0^S s^{-2}v_0(s)v_1(s)ds < \infty.$$

First we remark that if $V \in (C_3)$ then $V^{-1/2}(r) = c + o(r)$. Since $r^{-2}V^{-1/2}(r)$ is integrable we get $c = 0$ and

$$(5.8)\qquad V(r)r^2 \to \infty \quad \text{as} \quad r \to 0\,.$$

From this it follows easily that $X(s) = cV(r)(r/s)^2$, $s = r^{N-2}$, $c = c(N)$, belongs to (C_3).

We now prove that

$$(5.9)\qquad \lim_{s\to 0} X^{-1/2}(s)\,\frac{\dot v_1(s)}{v_1(s)} = -1$$

that is equivalent to (5.5) for $i = 1$ thanks to (5.8). In fact we prove (5.9) for every solution of (2.4) such that $v(s) \to \infty$ as $s \to 0$. Since $X \in (C_3)$ for every $\varepsilon>0$ there exists $s_0 = s_0(\varepsilon) > 0$ such that

$$(5.10) \qquad - \varepsilon < -X^{-3/2}(s)\dot{X}(s) < \varepsilon \quad \text{if} \quad 0 < s < s_0 .$$

Now let $v_\varepsilon(s)$ be the solution of (2.4) with $v_\varepsilon(s_0) = 1$, $\dot{v}_\varepsilon(s_0) = -kX^{1/2}(s_0)$, $k = 1 - \varepsilon > 0$. Then, arguing as above, we get $v_\varepsilon(s) > 0$, $\dot{v}_\varepsilon(s) < 0$ in $(0,s_0)$ and $v_\varepsilon(s) \to \infty$ as $s \to 0$.

We consider the functions $p(s) = \dfrac{\dot{v}_\varepsilon(s)}{v_\varepsilon(s)}$ and $H(s) = -kX^{1/2}(s)$. The former satisfies in $(0,s_0)$ the equation

$$(5.11) \qquad \dot{p}(s) + p^2(s) = X(s).$$

On the other hand $\dot{H}(s) + H^2(s) = - \frac{k}{2} X^{-1/2}\dot{X}(s) + k^2X(s)$. Using (5.10) and choosing $\varepsilon < 1$ we obtain

$$(5.12) \qquad \dot{H}(s) + H^2(s) < X(s).$$

Since $H(s_0) = p(s_0)$ it follows that $p(s) < H(s)$ for $0 < s < s_0$, therefore

$$(5.13) \qquad \limsup_{s \to 0} X^{-1/2}(s)\frac{\dot{v}_\varepsilon(s)}{v_\varepsilon(s)} < -1 + \varepsilon.$$

Now every solution $v_1(s)$ of (2.4) such that $v_1(s) \to \infty$ is of the form $v_1(s) = av_\varepsilon(s) + bv_0(s)$, $a>0$. Since $v_0(s) = o(s)$ and $\dot{v}_0(s) = o(1)$, cf. Prop. 2.1, (5.13) is true for $v_1(s)$ instead of $v_\varepsilon(s)$. Letting $\varepsilon \to 0$ we obtain the upper bound part of (5.9). A similar argument proves the other inequality.

Finally since the wronskian of v_0 and v_1, $\dot{v}_0v_1-\dot{v}_1v_0$, is constant, the function $Q(s) = v_0(s)/v_1(s)$ satisfies

(5.14) $$\dot{Q}(s) = \frac{\dot{v}_0 v_1 - \dot{v}_1 v_0}{v_1^2} = \frac{2k}{v_1^2}$$

for a certain $k > 0$. Then, using L'Hôpital's rule, we get

(5.15) $$\lim_{s\to 0} X^{1/2}(s)v_0(s)v_1(s) = \lim_{s\to 0} \frac{Q(s)}{v_1^{-2}(s)X^{-1/2}(s)} =$$

$$= \lim_{s\to 0} \frac{2k}{-2v_1^{-1}(s)\dot{v}_1(s)X^{-1/2}(s) - \frac{1}{2} X^{-3/2}(s)\dot{X}(s)} = k.$$

This proves (5.6). Finally (5.5) for $i = 0$ follows from

$$X^{-1/2}\left(\frac{\dot{v}_0}{v_0} - \frac{\dot{v}_1}{v_1}\right) = \frac{2kX^{-1/2}}{v_0 v_1},$$

using (5.9) and (5.15). □

COROLLARY 5.1 *Every V that satisfies (C_3) belongs to $\underline{P}_3$.*

Proof. It follows from (5.6) and the assumption

$$r^{-2}V^{-1/2}(r) \in L^1(0,R).$$ □

5.3. In particular cases one can find solutions that have a stronger anisotropy property, namely positive solutions $u(x)$ such that

(5.16) $$\limsup_{|x|\to 0} \frac{u(x)}{u_1(x)} = \infty, \quad \liminf_{|x|\to 0} \frac{u(x)}{u_1(x)} = 0.$$

In all the examples we take $V(r) = r^\theta$, $\theta < -2$, $p = -\frac{\theta+2}{2}$ and $u(x)$ has the form

(5.17) $$u(x) = r^{2-N}\exp(f(x)), \quad r = |x|.$$

Example 1. $N > 2$, $\theta = -4$. Take $f(x) = \dfrac{x_1}{|x|^2}$.

Example 2. $N = 2n$ even, $\theta = -6$. Take

$$f(x) = \{(x^2 + \dots + x_n^2) - (x_{n+1}^2 + \dots + x_{2n}^2)\} \frac{1}{pr^{p-2}}$$

Example 3. $N = 2$, p positive integer. Take

$$f(x) = (\cos p\alpha)\frac{1}{pr^p}\ , \quad (r,\alpha) \text{ polar coordinates in } \mathbb{R}^2.$$

6. BEHAVIOUR AT INFINITY

By means of the Kelvin transformation (0.7) we can rephrase the results obtained in previous sections as a study of the behaviour as $|x| \to \infty$ of the solutions of

$$(6.1) \qquad -\Delta u + V(|x|)u = 0, \quad |x| > R$$

where $V \in C(R,\infty)$ is a radial potential. In particular we define the classes $\underline{P}_1^\infty$, $\underline{P}_2^\infty$, $\underline{P}_3^\infty$ through the condition that the transformed potential $W(r) = r^{-4}V(r^{-1})$ belongs to $\underline{P}_1$, $\underline{P}_2$, $\underline{P}_3$ resp. In terms of V this means

$\underline{P}_1^\infty$:
$$\int_{|x|>R} |V(|x|)\, E_N(x)|dx < \infty.$$

$\underline{P}_2^\infty$: $V^- \in \underline{P}_1^\infty$ and there exists a positive radial solution of (0.1) in an interval (R,∞) such that $E_N(r)/u(r)$ is bounded and $|ru'(r)| \le Cu(r)$ for a certain constant $C>0$.

$\underline{P}_3^\infty$:
$$\int_R^\infty r^{N-3}\, u_0(r)u_1(r)dr < \infty.$$

Here u_0, u_1 are two linearly independent, positive solutions of (6.1) for $r = |x| > R$ such that

$$(6.2) \qquad \lim_{r\to\infty} r^{N-2}u_0(r) = 1 \text{ if } V \in \underline{P}_1^\infty, \quad = 0 \text{ if } V \notin \underline{P}_1^\infty$$

$$(6.3.a) \qquad N\geq 3: \lim_{r\to\infty} u_1(r) = 1 \text{ if } V \in \underline{P}_1^\infty, = \infty \text{ if } V \notin \underline{P}_1^\infty.$$

$$(6.3.b) \qquad N=2: \lim_{r\to\infty} \frac{u_1(r)}{\log(r)} = 1 \text{ if } V \in \underline{P}_1^\infty, = \infty \text{ if } V \notin \underline{P}_1^\infty.$$

All this follows from (0.7) and Sections 2, 4.

In case $V \in \underline{P}_1^\infty$ we can derive from Prop. 2.6 more precise estimates on the behaviour of u_0 and u_1:

PROPOSITION 6.1. *As* $r \to \infty$ *we have*

$$u_0(r) = r^{2-N} + \overline{p}_0(r)(1+o(1)). \tag{6.4}$$

$$u_1(r) = 1 + \overline{p}_1(r)(1+o(1)) + Cr^{2-N} \qquad \text{if} \quad N \geq 3 \,, \tag{6.5.a}$$

$$u_1(r) = \log(r) + \overline{p}_1(r)(1+o(1)) + C \quad \text{if} \quad N = 2 \,, \tag{6.5.b}$$

where

$$\overline{p}_0(r) = r^{2-N} \int_r^\infty s^{N-3}\left(\int_s^\infty t^{3-N} V(t)dt\right)ds \tag{6.6}$$

and

$$\overline{p}_1(r) = \begin{cases} r^{2-N}\displaystyle\int_R^r s^{N-3} \left(\int_s^\infty t\, V(t)dt\right)ds & \text{if} \quad N \geq 3, \\ \displaystyle\int_R^r s^{-1}\left(\int_s^\infty t\, |\log t| V(t)dt\right)ds & \text{if} \quad N = 2 \,. \end{cases} \tag{6.7}$$

$\square$

Similarly (C_3^∞) is obtained from (C_3) by replacing $r \to 0$ by $r \to \infty$. In particular if $V(r) = ar^\theta$, $a > 0$, $\theta \in \mathbb{R}$ we have

$$V \in \underline{P}_1^\infty \quad \text{if} \quad \theta < -2 \quad ,$$

$$V \in \underline{P}_2^\infty \quad \text{if} \quad \theta \leq -2 \quad \text{and}$$

$$V \in (C_3^\infty) \quad \text{if} \quad \theta > -2.$$

The meaning at infinity of the theorems of Section 1 is the following. Let $\Omega = \mathbb{R}^N - B_R(0)$ and let $u \in C^2(\Omega)$ be a solution of (6.1):

<u>THEOREM</u> A'. *If* $V \in \underline{P}_1^{\infty}$ *and* u *satisfies as* $|x| \to \infty$

$$u(x) \geq o(|x|) \quad \text{or} \quad u(x) \leq o(|x|), \tag{6.8}$$

then

$$u(x) = u(r) + o(r^{2-N}) \tag{6.9}$$

Moreover there exists a constant C_1 *such that if* $C_1 \neq 0$

$$\overline{u}(r) = C_1 + O(\overline{p}_1(r) + r^{2-N}) \quad \text{if} \quad N \geq 3 \tag{6.10.a}$$

$$u(r) = C_1 \log(r) + O(p_1(r) + 1) \quad \text{if} \quad N = 2 \tag{6.10.b}$$

If $C_1 = 0$ *there exists a constant* C_o *such that*

$$\overline{u}(r) = C_o r^{2-N} + \overline{p}_o(r)(1+o(1)). \tag{6.11}$$

Finally $V(x)u(x)r^{2-N} \in L^1(\Omega)$.

As in Theorem A it can be noted that the conditions $V \ \underline{P}_1^{\infty}$ and (6.8) are optimal.

<u>THEOREM</u> B'. *Assume that* $V \notin \underline{P}_1^{\infty}$ *but* $V^- \in \underline{P}_1^{\infty}$. *The following conditions are equivalent*

i) $u(x) = o(u_1(r)), \ r = |x| \to \infty$,

ii) $r^{2-N}V(r)u(x) \in L^1(\Omega)$,

iii) $u(x) = O(u_o(r)), \ r \to \infty$.

If $\tilde{U}_1$ is the solution of (1.7) in $R < r < \infty$ constructed as the U_1 in Section 1 we have

<u>THEOREM</u> C'. *Let* $V \in \underline{P}_2^{\infty}$ *and assume that as* $r \to \infty$

$$u(x) \geq o(\tilde{U}_1(r)) \quad \text{or} \quad u(x) \leq o(\tilde{U}_1(r)). \tag{6.12}$$

Then

$$u(x) = \overline{u}(r) + g(x) \tag{6.13}$$

where $g(x) = o(u_0(r))$.

In the same way the results of Section 5 can be translated to study the existence of anisotropic solutions.

7. SOLUTIONS IN $\mathbb{R}^N-\{0\}$.

In this section we study the conditions under which a solution $u \in C^2(R^N-\{0\})$ of

$$(7.1) \qquad -\Delta u + V(x)u = 0 \quad , \quad 0 < |x| < \infty$$

is radially symmetric. For simplicity we deal only with the case $N \geq 3$. The differences for $N = 2$ are similar to those of previous sections. As above we assume that $V(x)$ is continuous, radially symmetric: $V = V(r)$, $r = |x| > 0$, and we restrict the growth of V^- as $r \to 0$ or $r \to \infty$ as follows:

$$(7.2) \qquad \int_0^\infty V^-(r)r\,dr < \infty .$$

We also need another condition on the potential, namely that the ordinary differential equation

$$(7.3) \qquad u''(r) + \frac{N-1}{r}u'(r) = V(r)u(r), \quad 0 < r < \infty$$

be disconjugate in $[0,\infty)$, namely that every solution has at most one zero in that interval, cf. [W]. This is equivalent to the existence of a positive function $y \in C^2(0,\infty)$ satisfying

$$(7.4) \qquad y''(r) + \frac{N-1}{r}y'(r) + V^-(r)y(r) \leq 0,$$

(cf. [W], Theorem 2.1). It is therefore clear that (7.3) is disconjugate if $V \geq 0$. More precisely, using the change of variables (2.5) and Corollary 2.1* of [W], it is easy to see that (7.3) is disconjugate in $[0,\infty)$ if

$$\int_0^\infty V^-(r)r\,dr < \frac{N-2}{2}, \; N\geq 3 .$$

We begin with the study of the radial solutions of (7.3)

LEMMA 7.1 *Under the above assumptions on* V^- *there exist two positive solutions of* (7.3), $u_1(r)$ *and* $u_2(r)$, *such that as* $r\to\infty$.

(7.4) $$r^{N-2}\, u_1(r) \to C_1,\ u_2(r) \to C_2$$

and as $r \to 0$

(7.5) $$r^{N-2}u_1(r) \to C_3\ ,\ u_2(r) \to C_4\ ,$$

where $C_1 = 1$ *if* $V \in \underline{P}_1^\infty$, $C_1 = 0$ *if* $V \notin \underline{P}_1^\infty$,

$C_2 =$ *positive constant depending on* V *if* $V \in \underline{P}_1^\infty$, $C_2 = \infty$ *if* $V \notin \underline{P}_1^\infty$,
$C_3 =$ *positive constant depending on* V *if* $V \in P_1$, $C_3 = \infty$ *if* $V \notin \underline{P}_1$ *and*
$C_4 = 1$ *if* $V \in \underline{P}_1$, $C_4 = 0$ *if* $V \notin \underline{P}_1$.

Proof. i) Let u_1 be a solution of (4.3) such that $r^{N-2}u_1(r) \to C_1$ as $r \to \infty$ with C_1 as given above. Now we perform the change of variables

$$v_1(s) = u_1(r),\ s = r^{2-N}.$$

Then $v_1(s) \to 0$ as $s \to 0$. Since (7.3) is disconjugate the equation satisfied by v_1 also is and it follows that $v_1(s) \neq 0$ for $s>0$, hence $u_1(r) \neq 0$ for every $r>0$. We may assume that $u_1(r) > 0$.

ii) Now we let u_2 be a solution of (7.3) satisfying $u_2(r) \to C_4$ as $r \to 0$ with c_4 as above. With the change of variables

$$v_2(s) = r^{N-2}u_2(r),\ s = 1/r,$$

then $v_2(s)$ satisfies (7.3) and $s^{N-2}v_2(s) \to C_4$ as $s \to \infty$. Using step i) we conclude that $v_2 \neq 0$ on $(0,\infty)$ and we may assume that $u_2(r) > 0$ for $0 < r < \infty$.

iii) Let u be a solution of (7.3) such that $u(r)/E_N(r) \to k_1$ as $r \to 0$, with $k_1 = 1$ if $V \in \underline{P}_1$, $k_1 = \infty$ if $V \notin \underline{P}_1$. Then we have

$$u(r) = c_1 u_1(r) + c_2 u_2(r)$$

with u_1, u_2 as above. Since u_2 is bounded near 0 it follows that the limit $\lim_{r\to 0} r^{N-2} u_1(r) = C_3$ is either positive or $+\infty$.

iv) Finally let u be a solution of (7.3) such that, as $r \to \infty$, $u(r) \to k_2$, with $k_2 = 1$ if $V \in \underline{P}_1^\infty$, $k_2 = \infty$ if $V \notin \underline{P}_1^\infty$. Arguing as in iii) we obtain the existence of the limit $\lim u_2(r) = C_2 \in (0,+\infty]$. $\square$

We are also interested in the solutions obtained by separation of variables as in Section 1. Therefore we want to study the solutions of the equation

$$U''(r) + (N-1/r)U'(r) = ((N-1/r^2) + V(r))U(r), \quad 0 < r < \infty, \tag{7.6}$$

Since the potential $\frac{N-1}{r^2} + V(r) > V(r)$ and (7.3) is disconjugate, (7.6) also is. (See 7.4)). Besides, the change of variable $w(r) = U(r)/r$ transforms solutions of (7.6) into solutions of

$$w''(r) + (N+1/2)w'(r) = V(r)u(r). \tag{7.7}$$

Therefore applying the results of Lemma 7.1 in dimension N+2 instead of N we obtain:

<u>LEMMA</u> 7.2. <u>There exist two positive solutions</u> U_1, U_2 <u>of</u> (7.6) <u>such that as</u> $r \to \infty$

$$r^{N-1}U_1(r) \to c_1, \quad r^{-1} U_2(r) \to c_2 \tag{7.8}$$

<u>and as</u> $r \to 0$

$$r^{N-1}U_2(r) \to c_3\ , \quad r^{-1}U_2(r) \to c_4 \tag{7.9}$$

<u>with</u> c_1, c_2, c_3, c_4 <u>as in Lemma</u> 7.1.

We can now state our main result

THEOREM D. *Let* $u \in C^2(R^N-\{0\})$ *be a solution of* (7.1) *such that*

(7.10) $$u(x) = o(U_1(r)) \quad \text{as} \quad r \to 0$$

(7.11) $$u(x) = o(U_2(r)) \quad \text{as} \quad r \to \infty.$$

Then u *is radial*: $u(x) = \bar{u}(r)$ *for every* $0 < |x| = r < \infty$.

Proof. Since U_2 is nonsingular at 0 and U_1 is positive we may take U_2 to be

$$U_2(r) = U_1(r) \int_0^r t^{1-N} U_1^{-2}(t)dt.$$

Now we consider the function

$$h(x) = \frac{\|u(r,\sigma)-u(r)\|_{L^2(S_{N-1})}}{U_1(r)} \quad , \; s(r) = \frac{U_2(r)}{U_1(r)} \, .$$

Repeating the argument of Lemma 3.1 we prove that $h(r)$ is a convex function in $(0,\infty)$. Since $h(s) = o(1)$ as $s \to 0$ and $h(s) = o(1)$ as $s \to \infty$ it follows that $h \equiv 0$, therefore $u(r,\sigma) = \bar{u}(r)$. □

For the potentials of the class $\underline{P}_2$ we have proved that unilateral bounds imply absolute bounds.

Therefore we have

COROLLARY 7.1. *Let* $V \in \underline{P}_2 \cap \underline{P}_2^\infty$ *and let* $u \in C^2(\mathbb{R}^N-\{0\})$ *be a solution of* (7.1) *such that*

(7.12) $$u(x) \geq o(U_1(r)) \quad \text{or} \quad u(x) \leq o(U_1(r)) \quad \text{as} \quad r \to 0$$

(7.13) $$u(x) \geq o(U_2(r)) \quad \text{or} \quad u(x) \leq o(U_2(r)) \quad \text{as} \quad r \to 0$$

then u *is radial.*

In fact under those conditions $|u(x)| = o(U_1(r))$ as $r \to 0$ cf. Theorems A and C, and $u(x) = o(U_2(r))$ as $r \to \infty$, cf. Theorems A' and C'.

Observe that in the case $V \in \underline{P}_3 \cap \underline{P}_3^\infty$, i.e. if

$$\int_0^\infty r^{N-3} u_1(r) u_2(r) dr < \infty ,$$

since $u_1 \simeq U_1$ as $r \to 0$ and $u_2 \simeq U_2$ as $r \to \infty$, Theorem D says that if a solution u is $o(u_1)$ at 0 and $o(u_2)$ at infinity it vanishes identically in $\mathbb{R}^N$, a result that follows simply from Theorems B and B'.

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