Technical Report

Department of Computer Science
and Engineering
University of Minnesota
4-192 Keller Hall
200 Union Street SE
Minneapolis, MN 55455-0159 USA

TR 13-026

The Lion and Man Game on Convex Terrains

Narges Noori and Volkan Isler

September 13, 2013

The Lion and Man Game on Convex Terrains

***Abstract*— We study the well-known lion-and-man game in which a lion (the pursuer) tries to capture a man (the evader). The players have equal speeds and they can observe each other at all times. While the game is well-studied in two dimensional domains such as polygons, very little is known about its properties in higher dimensions. In this paper, we study the lion and man game on the surface of convex terrains. We show that the lion can capture the man in a finite number of steps which is a function of the terrain geometry.**

I. Introduction

Many robotics applications such as tracking and search can be modeled as pursuit-evasion games. In this paper, we study a fundamental pursuit-evasion game known as the lion-and-man game. In the original version of this game, a lion tries to capture a man in a circular arena. The players have equal speeds. Various variants of the lion-and-man game have been studied to model robotics applications. An overview of these results can be found in the survey paper by Chung et al. [1]. Briefly, assuming that the players move in turns, and they can observe each other at all times, the lion wins the game in the circular arena [2], [3] and in simply-connected polygons [4]. Three lions are sufficient and sometimes necessary in polygons with obstacles [5].

The lion-and-man game has been studied in higher dimensions as well. However, our understanding of the properties of the game in higher dimensions is far from complete. Kopparty and Ravishankar showed that in $\mathbb{R}^d$, $d+1$ lions can capture the man if and only if the man starts inside their convex hull [6]. Alexander et al. [7] study pursuit in more general environments. They showed that if the environment has non-positive curvature (i.e. it is CAT(0)), by greedily moving toward the evader, the pursuer can eventually capture it. The CAT(0) condition is not necessary for capture. This is noted in [7] where they show that on a hemisphere $H \subset S^2 \subset \mathbb{R}^3$ defined as $H = \{(x, y, z) : x^2 + y^2 + z^2 = 1, z \geq 0\}$, the pursuer wins. The hemisphere is not CAT(0).

In this paper, we further our understanding of the lion-and-man game in higher dimensions. We study the game on terrains and show that the lion wins the game on any convex terrain.

A terrain is characterized by unique height values for points in the two-dimensional plane. Our pursuit strategy on convex terrains is based on guarding *frontiers* given by set of points on the terrain that are on the same height. The pursuer starts from the highest frontier and pushes the frontier downwards while preventing the evader from entering any previously guarded frontier. Intuitively, the perimeter of the frontier is increased in this downward sweep. This allows the pursuer to use the difference between the perimeter of two consecutive frontiers in order to make progress. In this paper, we formalize this idea and analyze its correctness.

The paper is organized as follows. In Section II we present the preliminary definitions we use throughout the paper. Section III presents the general idea of the capture strategy. In Section IV we discuss the properties of convex terrains that are crucial for our strategy. Details of the pursuer strategy in different configurations is presented in Section V, Section VI and Section VII.

II. Definitions

In this section, we present concepts and definitions that will be useful throughout the paper. The game will take place on a terrain whose lowest point is on the $z = 0$ plane. Throughout the paper, we refer to this plane as the *base XY-plane*. We occasionally refer to this plane as the XY-plane. Moreover, we use the coordinate frame XYZ with its origin placed on any arbitrary point in the base XY-plane. Fig. 7(a) depicts our coordinate frame convention.

We are now ready to formalize the definition of a terrain.

Definition 1: A terrain is a polyhedral surface in three-dimensional space. It is represented as a finite set of vertices in the base plane, together with a unique height value associated with each vertex [8]. The vertices are triangulated which implies that the faces of the terrain are triangles in $\mathbb{R}^3$. In this paper, without loss of generality, we assume that all terrain vertices are at different heights. However, our strategy can be applied to those cases as well.

Definition 2: A convex terrain is a terrain such that every point on its surface is also a point on the boundary of the convex hull of the vertices of the terrain. In other words, a convex terrain does not have any valleys.

Definition 3 (The Perpendicular Image): For a point $p = (x, y, z)$ on the terrain, the point $q = (x, y)$ is called the *perpendicular image of* p onto the base XY-plane. Similarly, one can define the image of a path on the surface of the terrain onto the base XY-plane.

We will reserve the term *projection* for an important ingredient of our strategy.

We denote the pursuer and the evader by $\mathcal{P}$ and $\mathcal{E}$ respectively. We assume that the game is played on the surface of a convex terrain excluding the XY-plane at the base. Let us denote this surface by T. We also denote the perimeter of the boundary of T by $|T|$. We use the following game model. The players move in turns. Each turn takes a unit time step. In each turn, the players can move along any arbitrary path of length less than or equal to one (the step-size). The pursuer and also the evader both have full-visibility: they can observe the location of the other player. The pursuer captures the evader if at any time, which includes the whole time interval in one step associated with one turn, the distance between them becomes less than or equal to one (the step-size). The justification for this capture condition is that if we assume a radius around the players, the pursuer captures the evader if they collide.

Definition 4: Let p_1 and p_2 be two distinct points on the same face f of T. Consider the straight line segment that connects them on T, and denote its length by L. Also, let l be the length of its perpendicular image (Definition 3) onto the XY-plane. We refer to the ratio $\alpha = \frac{l}{L}$ as the length coefficient associated with p_1 and p_2.

For a specific face f, the minimum value of the length coefficient is the cosine of the angle between f and the XY-plane. This is because faces with vertical edges are not allowed, and also the two points p_1 and p_2 are distinct. Therefore, we have the following proposition.

Proposition 1: The length coefficients are less than or equal to 1 i.e. $\alpha \leq 1$. Also, the minimum length coefficient is well defined in the sense that it is a finite positive number. We refer to the minimum possible length coefficient on T as α_{min}.

Definition 5 (Wavefronts): We refer to the set of points on T which are on the same height z as the *wavefront* at z. Note that since T is a convex polyhedron, the wavefront at height z is also a convex polygon contained in the plane $Z = z$. Moreover, the image of a wavefront has the same shape as the wavefront itself.

Definition 6 (Wedge Regions and Edge Regions): Let W^i be the perpendicular image of a wavefront W onto the XY-plane. Suppose that the vertices of W^i are labeled as $\{w_1, w_2, ..., w_n\}$ in the clockwise order. See Fig. 1(a) for an illustration. We partition the region outside W^i (in the base XY-plane) into regions of two types: the *edge regions* and the *wedge regions*. For an edge $w_i w_{i+1}$, its corresponding edge region is the region in between the two perpendicular lines to the segment $w_i w_{i+1}$ which pass through w_i and w_{i+1} respectively. For a vertex $w_i \in W^i$, its corresponding wedge region is the region in between the two perpendicular lines to $w_{i-1} w_i$ and $w_i w_{i+1}$ which pass through w_i.

Definition 7 (Projection onto a Wavefront): Let $\mathcal{E}^i$ and W^i be the perpendicular images of $\mathcal{E}$ and a wavefront W onto the XY-plane respectively. Also, suppose that $\mathcal{E}^i$ is outside the region enclosed by W^i. The *projection* of $\mathcal{E}$ onto the wavefront W is defined as follows. See Fig. 1(a). Consider the partitioning of the exterior region of W into wedge regions and edge regions. There will be two cases based on the location of $\mathcal{E}^i$ (Fig. 1(a)): 1) $\mathcal{E}^i$ is inside the edge region associated with an edge $w_i w_{i+1}$; 2) $\mathcal{E}^i$ is inside the wedge region of a vertex w_i. In the first case, let $\pi^i(\mathcal{E})$ denote the intersection of the edge $w_i w_{i+1}$ and the perpendicular line to the edge $w_i w_{i+1}$ passing through $\mathcal{E}^i$. In the second case, $\pi^i(\mathcal{E})$ refers to the vertex w_i. Then, the projection of $\mathcal{E}$ onto W is the point on T that its image on the base plane is $\pi^i(\mathcal{E})$ (Fig. 1(b)). We denote this point on T as $\pi(\mathcal{E}, W)$. We will omit W in $\pi(\mathcal{E}, W)$ when W can be inferred from the context.

Remark 1: Notice that the perpendicular image in Definition 3 is different than the projection onto a wavefront in Definition 7. The image of a point $p \in T$ is denoted by p^i while its projection onto W is referred as $\pi(p, W)$.

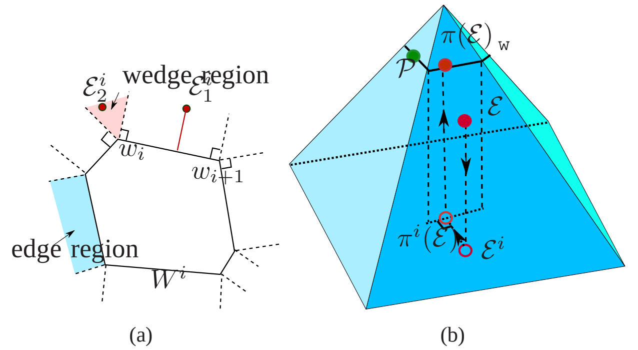


Fig. 1. (a) The partitioning of the exterior of W^i into wedge regions and edge regions. (b) The projection of $\mathcal{E}$ onto the wavefront W.

III. The Capture Strategy

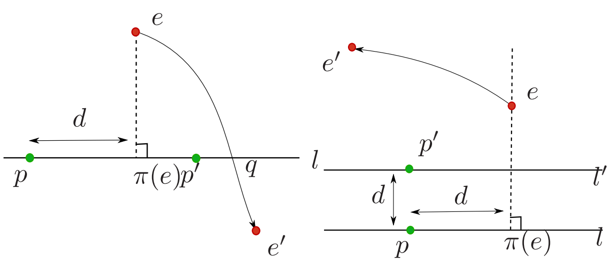
The idea of our pursuit strategy is the following. We first discretize the surface of T by a set of wavefronts as shown in Fig. 7(a). Initially, the pursuer goes to the highest point of T. (For now, let us assume that this point is unique. In Appendix, we address the case where this assumption does not hold.) The highest point is the first wavefront to be guarded by $\mathcal{P}$. Starting from this first wavefront, the images of the wavefronts in the XY-plane grow in perimeter as shown in Fig. 6. The pursuer then locates itself "close" to the projection of the evader onto the current wavefront. Shortly, we will specify what we mean by close. We will show that by staying close to the projection of the evader onto the wavefront W, the pursuer can prevent the evader from crossing W and re-contaminating the region enclosed by W without being captured. After finite number of steps, the pursuer makes progress towards the next wavefront. While the pursuer is guarding the current wavefront, and also when it makes progress to the next wavefront, it maintains the invariant that it is still *close* to the projection of the evader.

Definition 8: We refer to the wavefront that the pursuer is currently guarding as the *frontier* wavefront.

Throughout the paper, we denote the frontier wavefront by W and the wavefront next to it by W_n. The pursuer's goal is to make progress to W_n.

Definition 9: Let $p_1, p_2 \in W$ be two points on the wavefront W. We denote the shorter path from p_1 to p_2 along W by $W(p_1, p_2)$, and its length by $d_W(p_1, p_2)$. We also denote the length of the segment $p_1^i p_2^i$ in the XY-plane by $d_{XY}(p_1^i, p_2^i)$.

Our proposed pursuit strategy for guarding a wavefront and making progress towards the next wavefront is inspired from the following pursuer strategy in the two-dimensional plane which we call the *rook move*. To illustrate the idea, for now, suppose that the game is played on the XY-plane. Let l be a line lying on this plane that the pursuer is trying to guard. See Fig. 2. Suppose that the evader is at the point e and let $\pi(e)$ be the projection of the evader onto l. Suppose that the pursuer is at the point p on l such that $|\pi(e) - p| = d \leq 1$. It is not difficult to see that the evader cannot cross l without being captured as long as $d \leq h$ where h is the length of the segment between e and $\pi(e)$[1] (Fig. 2(a)). The pursuer can make progress by pushing l forward if the evader moves to the left: the pursuer moves along the y-axis for d by going to p' such that $|p' - p = d|$. In other words, it pushes the *frontier* line l to the new line l' which is closer to the evader. See Fig. 2(b). We generalize the rook move idea to the surface of a convex terrain as follows.



(a) (b)

Fig. 2. The illustration of the rook move in the XY-plane. (a) If the evader crosses l, the pursuer moves to the crossing point q. The distance between q and e' is less than one. Thus, $\mathcal{E}$ will be captured. (b) If the evader moves to the left, the pursuer can push l forward.

Definition 10 (The Guarding Configuration): Suppose that the evader is outside the region enclosed by W (i.e. points that are higher than w), and that the pursuer is on the wavefront W. Consider the projection of $\mathcal{E}$ onto W, $\pi(\mathcal{E})$, and the two paths on W from $\mathcal{P}$ to $\pi(\mathcal{E})$. The pursuer is in guarding configuration on W if $d_W(\mathcal{P}, \pi(\mathcal{E})) < d_{XY}(\mathcal{E}^i, \pi^i(\mathcal{E}))$. See Fig. 1(b) for an example.

Remark 2: Throughout the paper, for the ease of notation, we use $d = d_W(\mathcal{P}, \pi(\mathcal{E}))$.

Lemma 1: Suppose that the pursuer is on the wavefront W such that $d_W(\mathcal{P}, \pi(\mathcal{E})) < d_{XY}(\mathcal{E}^i, \pi^i(\mathcal{E}))$. Then, the evader cannot cross W without being captured. See Definition 9 for d_W and d_{XY}.

Proof: Suppose that the evader crosses W by moving to $\mathcal{E}'$. Fig. 3 depicts the images of W, $\mathcal{P}$ and $\mathcal{E}$ on the XY-plane. Let q be the crossing point. Consider the evader's path on T from $\mathcal{E}$ to q. Let a_T and a denote the length of this path and its image respectively.

Now, there will be two cases: $d_W(\mathcal{P}, q) \leq a$ or $a < d_W(\mathcal{P}, q)$. If $d_W(\mathcal{P}, q) \leq a$, the pursuer moves to q along W. Since the length of the evader path from q to $\mathcal{E}'$ is less than 1, the pursuer will capture the evader.

Otherwise, the pursuer moves along W to the point $\mathcal{P}' \in W$ such that $\mathcal{P}\mathcal{P}' = a$. We show that there exists a path on T from $\mathcal{P}'$ to $\mathcal{E}'$ that its length is less than 1. Therefore, the pursuer captures the evader according to our capture condition by moving to $\mathcal{P}'$.

Consider the path Π from $\mathcal{P}'$ to $\mathcal{E}'$ which is composed of $W(\mathcal{P}', q)$ followed by the evader path from q to $\mathcal{E}'$. Note

[1]We will shortly present the proof of this claim for our generalization of the rook move on XY-plane to T.

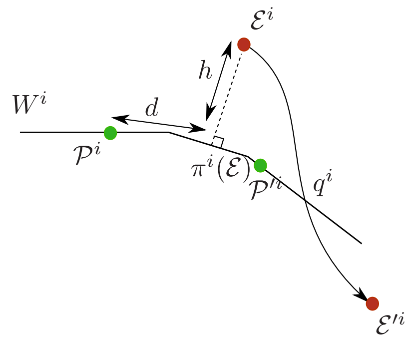


Fig. 3. Image of $\mathcal{P}$, $\mathcal{E}$ and W onto the XY-plane. If $d_W(\mathcal{P}, \pi(\mathcal{E})) < d_{XY}(\mathcal{E}^i, \pi^i(\mathcal{E}))$, the evader cannot cross W without being captured.

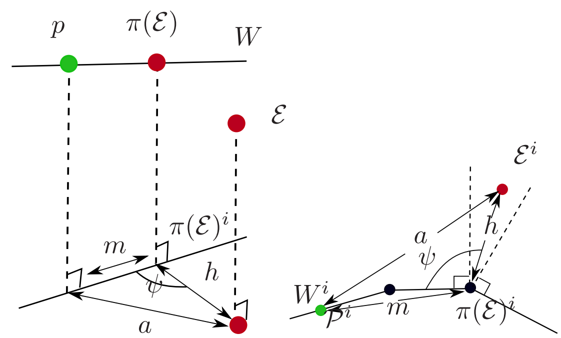
that we have $d_W(\mathcal{P}, \pi(\mathcal{E})) < d_{XY}(\mathcal{E}^i, \pi^i(\mathcal{E})) \leq a \leq a_T$. Here, the first inequality is our assumption and the second one is because $\pi^i(\mathcal{E})$ is the closest point to $\mathcal{E}^i$. Therefore, the length of Π is less than the length of the evader's path from $\mathcal{E}$ to $\mathcal{E}'$ which is one. Thus, the pursuer captures the evader by moving to $\mathcal{P}'$. ■

Lemma 2 (Condition on $d_W(\mathcal{P}, \pi(\mathcal{E}))$): On the convex terrain T, if $d_W(\mathcal{P}, \pi(\mathcal{E})) < \frac{\alpha_{min}}{2}$, then the condition $d_W(\mathcal{P}, \pi(\mathcal{E})) < d_{XY}(\mathcal{E}^i, \pi^i(\mathcal{E}))$ will be satisfied.

Proof: See Fig. 4(b) which shows the images of W, $\mathcal{P}$ and $\mathcal{E}$ in the XY-plane. First, consider the segment $\mathcal{P}^i\mathcal{E}^i$ in the base plane. We use the following proposition:

Proposition 2: The pre-image of any continuous path in the XY-plane is a continuous path in T.

Since T is convex, $\mathcal{P}^i\mathcal{E}^i$ is the image of a valid path on T from $\mathcal{P}$ to $\mathcal{E}$. Denote the length of $\mathcal{P}^i\mathcal{E}^i$ by a and its corresponding path on T by a_T. Since $\mathcal{E}$ is not captured yet we have $1 < a_T$. Thus, $\alpha_{min} < a$. We will use this property in our proof.



(a) (b)

Fig. 4. Proof of Lemma 2. Left) The images of the players and the wavefront is considered in XY -plane. Right) The XY-plane is shown with the images.

For ease of notation, let $h = d_{XY}(\mathcal{E}^i, \pi^i(\mathcal{E}))$ and $d = d_W(\mathcal{P}, \pi(\mathcal{E}))$. Moreover, let $m = d_{XY}(\mathcal{P}^i, \pi^i(\mathcal{E}))$. Finally, denote the angle between the two segments $\mathcal{E}^i\pi^i(\mathcal{E})$ and $\mathcal{P}^i\pi^i(\mathcal{E})$ by ψ. It can be easily verified that since W^i is convex, we have $\frac{\pi}{2} \leq \psi$ (Fig. 5). Thus, $c_\psi < 0$. Consequently, $0 \leq -c_\psi < 1$.

$$
\begin{aligned}
m \le d \Rightarrow -2mhc_\psi \le -2dhc_\psi \\
\alpha_{min}^2 < a^2 \le d^2 + h^2 - 2dhc_\psi \Rightarrow \\
0 < d^2 + h^2 - 2dhc_\psi - \alpha_{min}^2 \Rightarrow \\
0 < h^2 + 2hd + d^2 - \alpha_{min}^2 \\
0 < (h - (-d - \alpha_{min}))(h - (-d + \alpha_{min}))
\end{aligned}
$$

Since $0 < h$, we must have $\alpha_{min} - d < h$. Since $d < \frac{\alpha_{min}}{2}$ (according to our assumption), we conclude that $d < \alpha_{min} - d < h$. Thus $d < h$ and the condition in Definition 10 is satisfied. ■

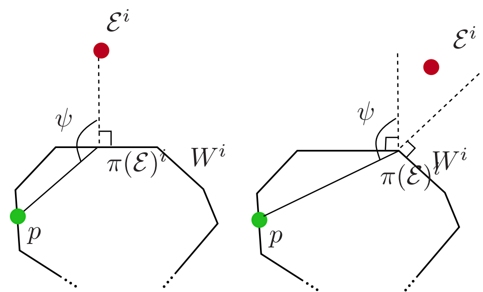


Fig. 5. Since W is convex, we have $\frac{\pi}{2} \le \psi$.

Lemma 3: Once the pursuer establishes the guarding configuration on the wavefront W, it can maintain the configuration as the evader moves. In other words, if the pursuer establishes $d_W(\mathcal{P}, \pi(\mathcal{E})) < d_{XY}(\mathcal{E}^i, \pi^i(\mathcal{E}))$, it can maintain this condition as $\mathcal{E}$ is moving on T.

The guarding configuration allows the pursuer to prevent the evader from crossing the frontier wavefront. We now present the pursuer's strategy that is based on the evader's motion. Without loss of generality, we assume that initially the pursuer locates itself to the left of $\pi(\mathcal{E}, W)$. We then use the clockwise direction, as the base direction that the pursuer uses for making progress.

If the evader does not move in its current turn, or if it changes its direction and moves backward, we show that the pursuer's distance from $\pi(\mathcal{E}, W_n)$ is less than 1. Thus, the pursuer can move to W_n while it is on $\pi(\mathcal{E}, W_n)$ in only one step. On the other hand, if the evader moves in the same direction, we show that the pursuer can make progress to W_n after at most $O(N)$ steps where N is finite and we will specify it later in Section VII. In summary, the events that the pursuer makes progress are the following: 1) the evader does not move in its current turn, 2) the evader moves in the counter clock-wise direction in its current turn, 3) the evader moves in the clock-wise direction for the next $O(N)$ steps. In all of these events, we show that the pursuer can move onto $\pi(\mathcal{E}, W_n)$. Afterwards, the evader cannot cross W_n without being captured. Therefore, the evader is being squeezed between W_n and the boundary of T, and it will be captured in finite number of steps.

Observation 1: Observe that after moving to $\pi(\mathcal{E}, W_n)$, the pursuer needs only one extra turn to locate itself in the guarding configuration i.e. at distance d from $\pi(\mathcal{E}, W_n)$.

Theorem 1 (Capture on Convex Terrains): Suppose that the lion and man game is played on the surface of a convex terrain T. The proposed pursuit strategy, guarantees captures in at most $O(\frac{D_T}{D} \cdot \frac{|T|}{1-d-D})$ steps. Here $|T|$ denotes the perimeter of T, D_T is the diagonal of T. Also $D = \max d(W, W_n)$ is the designed distance between two consecutive wavefronts, and $d = d_W(\mathcal{P}, \pi(\mathcal{E}, W))$ is the distance that the pursuer maintains from the projection of $\mathcal{E}$ onto W (also a design parameter of the strategy).

IV. Properties of the Wavefronts

We first study the discrete events as the frontier wavefront is moved downwards. We show that the combinatorial changes to the frontier occur at vertices of T which we refer to them as the *vertex events*. Consider the wavefront W at height z and let $F(z)$ denote the set of faces that W intersects with them. We interchangeably use the notation $F(W)$ to denote this set. Also, suppose that $w_i w_{i+1}$ is an edge in W which lies on $f \in F(z)$. We refer to $w_i w_{i+1}$ as the edge *associated with* the face f.

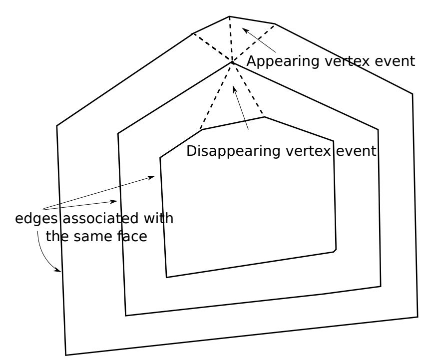
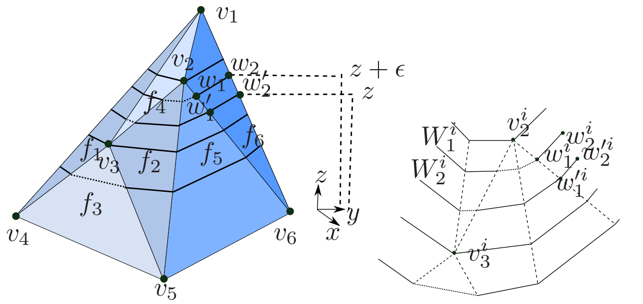


Fig. 6. The image of the wavefronts is shown. Edges associated with the same face are parallel. As the wavefront reaches a terrain vertex, some edges disappear. Afterwards, new edges appear.

Lemma 4 (Properties of the Wavefronts):

1) Suppose W_1 and W_2 are two wavefronts at heights z and $z + \epsilon$ respectively such that $F(z) = F(z + \epsilon) = F$. Let f be a face in F, and let $w_1 w_2$ and $w'_1 w'_2$ be the two edges on W_1 and W_2 respectively that are on f. Then, their perpendicular images are parallel to each other. See Fig. 7 for an illustration.
2) Let v be a vertex of T which is at height z, and let W be the wavefront at z. Also, suppose that W_1 is the wavefront at $z - \epsilon_1$ such that for $z - \epsilon_1 \le h < z$ we have $F(z - \epsilon_1) = F(h) = L$. Similarly, suppose that W_2 is the wavefront at $z + \epsilon_2$ such that for $z < h \le z + \epsilon_2$ we have $F(z + \epsilon_2) = F(h) = U$. Next, denote the two faces adjacent to v in W by f_1 and f_2. Let $L' \subset L - \{f_1, f_2\}$ be faces adjacent to v that are below v. Similarly, define U'. Then, as the frontier wavefront moves from $z + \epsilon_2$ to z, the edges associated with U' disappear. On the other hand, as the frontier moves from z to $z - \epsilon_1$ new edges associated with L' appear. In the rest of the paper, we refer to these events as the

vertex events: the former one as *disappearing events*, and the latter one as *appearing events*. See Fig. 7.



(a) (b)

Fig. 7. The pursuer starts from v_1. There are two vertex events: at v_2 and v_3. The image of edges associated with the same face are parallel e.g. $w_1^i w_2^i || w_1'^i w_2'^i$. Note the appearing edge on f_3 below v_3 which is shown in dots (left), and also the disappearing edge on f_4 above v_3 (left).

Definition 11 (Distance Between Two Wavefronts): The distance between two consecutive wavefronts W and W_n, denoted by $d(W, W_n)$, is defined as follows. First, suppose that there is not vertex event between W and W_n. Consider the vertex wavefronts $w \in W$ and $w_n \in W_n$ such that the segment $ww_n \in T$. In other words, ww_n is part of an edge of T (Fig. 7). Then $d(W, W_n) = d_{XY}(W, W_n) = \max d_{XY}(w^i, w_n^i)$ where the maximum is taken over all such segments ww_n. Finally, let us consider the case that there is a vertex wavefront between W and W_n. In this case, we assume that W_n' is the wavefront if there were no vertex event. We then consider the distance between W and W_n' as $d(W, W_n)$. See Fig. 7 for an illustration.

Definition 12 (Discretization of T by the Wavefronts): The wavefronts are chosen such that the maximum distance between any two consecutive wavefronts is D. In fact, D quantifies the progress that the pursuer achieves by moving from W to W_n. Later as we present the strategy, we also present the conditions that we need for D (Eq. 1, Eq. 2, Eq. 3).

V. The Evader Stays Still in its Turn

In this section, we consider the case that the evader remains still in its turn. The pursuer uses this extra turn in order to make progress towards the next wavefront. The key idea is that $d(W, W_n)$ is small enough such that the pursuer can move to the projection of $\mathcal{E}$ onto W_n.

Lemma 5 (When There is No Vertex-Event): Consider the case that $\mathcal{E}$ did not move in its move. Suppose that there is no vertex event in between W and W_n. Then, the pursuer can locate itself on $\pi(\mathcal{E}, W_n)$ in one step.

Proof: There will be two cases based on whether the evader is in a wedge region or an edge region.

1) $\mathcal{E}^i$ is in the wedge region of a vertex w: Let w_n be the vertex in W_n which is associated with the same faces as the faces adjacent to w. Also, let m^1 and m^2 be the images of the two wavefront edges that are adjacent to w (Fig. 8). Moreover, let l and l' denote the two perpendicular lines to m^1 and m^2 respectively. Finally, let m_n^1 and m_n^2 be the lines parallel to m^1 and m^2 drawn from w_n. The wavefront vertex w_n can be inside the wedge region of w or outside it. In the following, without loss of generality, we assume that $\mathcal{E}^i$ is to the left of the line $w^i w_n^i$.
   a) w_n is inside the wedge region of w: In this case, since W and W_n are convex polygons, $\pi(\mathcal{E}, W_n)$ will be inside the region formed by l, l', m_n^1 and m_n^2 (The shaded region in Fig. 8). It is not difficult to show that the distance between $\pi(\mathcal{E}, W_n)$ and w is less than $w^i w_n^i$. Since $w^i w_n^i \leq D$, the pursuer can move to w and then move to $\pi(\mathcal{E}, W_n)$.
   b) w_n is outside the wedge region of w: Similarly, we can show that $\pi(\mathcal{E}, W_n)$ is inside the region formed by l_1, m_n^1 and ww_n. The rest of the proof is similar to the previous case.

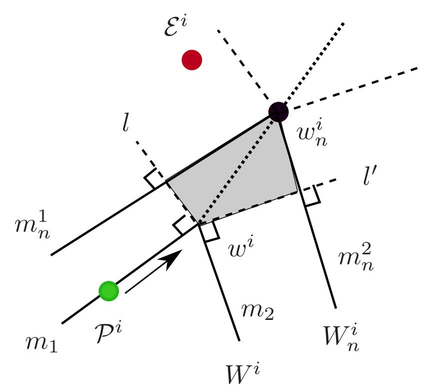


Fig. 8. The evader does not move in its turn. The case that $\mathcal{E}^i$ is in the wedge region of w. We show that $\pi(\mathcal{E}, W_n)$ is inside the shaded region. Since $w^i w_n^i \leq D$, the pursuer can move to $\pi(\mathcal{E}, W_n)$.

2) $\mathcal{E}^i$ is in the edge region of an edge m: This case is illustrated in Fig. 9. Let $m_n \in W_n$ be the edge in W_n which is parallel to $m \in W$. There will be two case: 1) $\pi(\mathcal{E}, W_n) \in m_n$, 2) $\pi(\mathcal{E}, W_n) \notin m_n$. The former case is straightforward: the pursuer simply moves to $\pi(\mathcal{E}, W)$ and then it moves to $\pi(\mathcal{E}, W_n)$ along the perpendicular line (Fig. 9). In the latter case, it can be shown $\pi^i(\mathcal{E}, W_n)$ is inside the region formed by the perpendicular line to m, m_n and the segment between $\pi^i(\mathcal{E}, W)$ and w_n^i. Since $w^i w_n^i \leq D$, the pursuer can move to $\pi(\mathcal{E}, W_n)$ in only one step.

■

Lemma 6 (Appearing Vertex Event): Consider the case that $\mathcal{E}$ did not move in its move. Suppose that there is a disappearing vertex event in the transition from W to W_n. Then, the pursuer can locate itself on $\pi(\mathcal{E}, W_n)$ in one step.
Proof:

Let w be the corresponding wavefront vertex in W. In the following, we consider the base XY-plane. See Fig. 10 for an illustration. Let m^1 and m^2 be the images of the two wavefront edges that are adjacent to w. Also, let m_n^1 and m_n^2 be the lines parallel to m^1 and m^2 in W_n. Let w_n^1

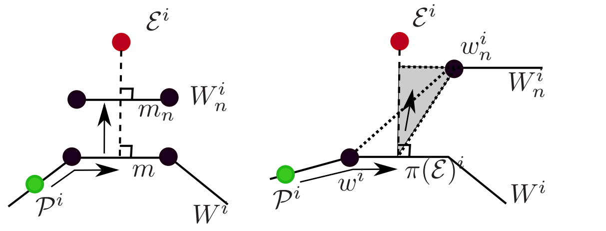


Fig. 9. The evader does not move in its turn. The case that $\mathcal{E}^i$ is in the edge region of m. Left) $\pi(\mathcal{E}, W_n) \in m_n$. Right) $\pi(\mathcal{E}, W_n) \notin m_n$. We show that $\pi(\mathcal{E}, W_n)$ is in the shaded region.

and w_n^2 be the vertices in W_n that are adjacent to m_n^1 and m_n^2 respectively. Finally, let l_1 and l_2 be the two lines that are perpendicular to m^1 and m^2 drawn from w_n^1 and w_n^2 respectively.

Now, $\mathcal{E}^i$ can be inside the region formed by l_1 and l_2 or outside it. If it is outside, we can use the result of Lemma 5.

Therefore, suppose that $\mathcal{E}^i$ is inside the region formed by l_1 and l_2. Let h_1, h_2 denote the length of $w^i w_n^1$ and $w^i w_n^2$ respectively. Moreover, let h_3 denote the length of the segment between w^i and the intersection of m_n^1 and m_n^2. We show that the pursuer can move to w and then from w it can move to $\pi(\mathcal{E}, W_n)$. Without loss of generality, assume that $h_m = \max\{h_1, h_2, h_3\}$. Refer to Fig. 10 for angles ψ_1, ψ_2 and ψ_3. Also, let x be the distance between $\pi(w_n^1, W)$ and w. Then, $x = \frac{h_m \beta s_{\psi_1}}{t_{\psi_2}}$. Therefore, the pursuer's distance to $\pi(\mathcal{E}, W_n)$ is at most $x + h_m + d = \frac{h_m \beta s_{\psi_1}}{t_{\psi_2}} + h_m + d$. Since $h_m \leq D$, this distance is less than $D(1 + \frac{\beta s_{\psi_1}}{t_{\psi_2}}) + d$. We can design $d = d_W(\mathcal{P}, \pi(\mathcal{E}, W))$ and $D = \max d(W, W_n)$ such that:

$$D(1 + \frac{\beta s_{\psi_1}}{t_{\psi_2}}) + d \leq 1. \quad (1)$$

Consequently, the pursuer can move to w and then to $\pi(\mathcal{E}, W_n)$ in one step. ■

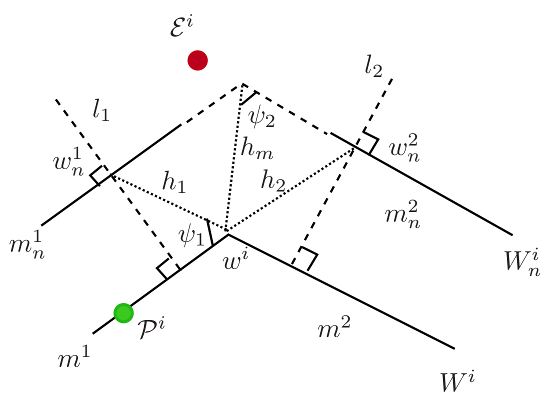


Fig. 10. The case that the evader is not moving and there an appearing vertex event.

Lemma 7 (Disappearing Vertex Event): Consider the case that $\mathcal{E}$ did not move in its move. Suppose that there is an appearing vertex event in the transition from W to W_n. Then, the pursuer can locate itself on $\pi(\mathcal{E}, W_n)$ in one step.
Proof: The proof is similar to Lemma 6. ■

VI. THE EVADER MOVES COUNTER CLOCK-WISE

We now consider the case the evader is moving in counter clock-wise direction in its current turn. Let $\mathcal{E}_n$ be the new location the evader (Fig. 11). From convexity of W, one can show that $d_W(\mathcal{P}, \pi(\mathcal{E}_n, W)) \leq (1 - d_W(\mathcal{P}, \pi(\mathcal{E}, W)) = (1 - d)$. Also, we designed $d(W, W_n)$ such that for all points p we have $d(\pi(p, W_n), \pi(p, W)) \leq D$. Therefore, if we design d and D such that:

$$1 - d + D \leq 1 \Rightarrow D \leq d \quad (2)$$

the pursuer can first move to $\pi(\mathcal{E}_n, W)$ along W and then it can move to $\pi(\mathcal{E}_n, W_n)$.

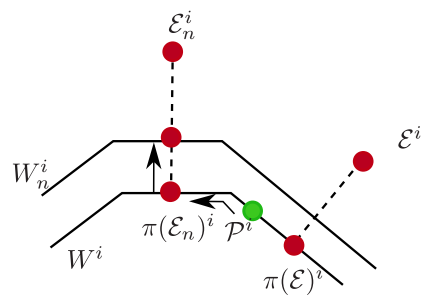


Fig. 11. The evader moves counter clockwise.

VII. THE EVADER MOVES IN CLOCK-WISE DIRECTION FOR $O(N)$ STEPS

We now consider the case that the evader is moving in the clock-wise direction for the next $O(N) = O(\frac{|T|}{1-d-D})$ steps. First, we can assume that during these $O(\frac{|T|}{1-d-D})$ it is always moving in its turn because if it stays still, then the pursuer can use the strategy in Section V for making progress. Also, if in one of these turns, the evader moves counter clock-wise the pursuer can use the strategy in Section VI for going toward W_n. We first present the case that there is no vertex event in between W and W_n.

We show that if the evader moves in the same clock-wise direction for $O(\frac{|T|}{1-d-D})$ steps, its projection onto W will circumnavigate around W for a complete round. In other words, $\pi(\mathcal{E}, W)$ will come back to the same point on W after $O(\frac{|T|}{1-d-D})$ steps. We will use this observation to show that if $\pi(\mathcal{E}, W)$ completes such round, the players will be eventually in a configuration that allows the pursuer to move toward $\pi(\mathcal{E}, W_n)$. We refer to this configuration as the *wide-turn* configuration.

Lemma 8: Consider the next $O(\frac{|T|}{1-d-D})$ time steps where $|T|$ denotes the perimeter of the boundary T. Then, we will have at least one of the following events in these $O(\frac{|T|}{1-d-D})$ steps: 1) for at least one turn $\mathcal{E}$ does not move, or 2) it moves counter clock-wise, or 3) $\mathcal{E}$ circumnavigates around W i.e. $\pi(\mathcal{E}, W)$ comes back to the same point on W.
Proof: Suppose that we don't have none of the the first and the second events. We show that for sure we will have the third event. Since the first and the second event does not occur, the evader is moving clock-wise. Let $\mathcal{E}$ and $\mathcal{E}_n$ denote the location of the evader before and after a turn.

For simplicity let us use the notations $d_e = d_W(\mathcal{E}, \mathcal{E}_n)$ and $d = d_W(\mathcal{P}, \pi(\mathcal{E}, W))$. Obviously, we have $0 \leq d_e \leq 1$. Consider the path from $\mathcal{P}$ to $\pi(\mathcal{E}_n, W_n)$ which is composed of $W(\mathcal{P}, \pi(\mathcal{E}_n, W))$ and then the shortest path from $\pi(\mathcal{E}_n, W)$ to $\pi(\mathcal{E}_n, W_n)$. The length of this path is $d_p = d + d_e + D$. Now, if $d_p \leq 1$, the pursuer can move to $\pi(\mathcal{E}_n, W_n)$ and make progress. Therefore, for contradiction, suppose that $1 < d_p = d + d_e + D$. Consequently, $1 - d - D < d_e$. In other words, after each step, the evader's projection onto W moves for at least $1 - d - D$. Therefore, after at most $\frac{|T|}{1-d-D}$ steps, $\pi(\mathcal{E}, W)$ comes back to the same point on W. Here, $|T|$ denotes the perimeter of T. ■

Now that we know $\pi(\mathcal{E}, W)$ circumnavigates around W, we show that the pursuer will be in a good configuration for making progress: the wide-turn configuration. The wide-turn configuration is the following. Fig. 12(a) shows an illustration of this configuration. Suppose that w is a vertex of W. Let ww' be the image of the edge adjacent to w in W, and f be the face that $ww' \in f$. Let $w_n w'_n$ be the image of the edge associated with the same face f in W_n. Note that ww' and $w_n w'_n$ are parallel to each other. Suppose that the perpendicular to ww' which passes from w intersects with $w_n w'_n$ inside the edge $w_n w'_n$. Let q^i be the point of intersection. The players will be in wide-turn configuration if the evader crosses the wedge region of such vertex w. In the following lemma, we prove that in this configuration the pursuer can make progress toward W_n.

Lemma 9: Suppose that the evader and the pursuer are in the wide-turn configuration. Then, the length of the path $W(\mathcal{P}, w)$ followed by the segment wq is less than or equal to the length of the path that the evader travels when it crosses the wedge region of w.

Proof: Suppose that the evader crosses the wedge region of w by moving from $\mathcal{E}_1$ to $\mathcal{E}_2$. Consider the segment in between the images of $\mathcal{P}$ and $\mathcal{E}_1$ in the base XY-plane. Since T is convex, this segment is the image of a valid path on T. Let us denote the length of this path on T from $\mathcal{P}$ to $\mathcal{E}_1$ by a_T. Also, let a be the length of the image of this path. Since $\mathcal{E}$ is not captured at $\mathcal{E}_1$, then it must be that $1 \leq a_T$. Therefore, $\alpha_{min} \leq a$.

Next, let b be the length of the segment between $\mathcal{E}_1^i$ and w^i. Notice that $\mathcal{P}$ is in the guarding configuration. Thus, the distance between $\mathcal{P}^i$ and w^i is d. Therefore, using the Pythagorean theorem, we have $\alpha_{min}^2 \leq a^2 = d^2 + b^2$. Thus, $\sqrt{\alpha_{min}^2 - d^2} \leq b$.

Let l denote the length of the path between $\mathcal{E}_1^i$ and $\mathcal{E}_2^i$. Also, let us denote the angle of the wedge region at w by β. Then, $\sin\beta\sqrt{\alpha_{min}^2 - d^2} \leq b \sin\beta \leq l$. Therefore, the path that the evader travels on T from $\mathcal{E}_1$ to $\mathcal{E}_2$ is longer than $\sin\beta\sqrt{\alpha_{min}^2 - d^2}$. Therefore, as long as:

$$d \leq \frac{\alpha_{min} \sin\beta}{\sqrt{1 + \sin\beta^2}} \tag{3}$$

we would have $d < \sin\beta\sqrt{\alpha_{min}^2 - d^2}$. Consequently, the pursuer can move to w, and use the remaining $\sin\beta\sqrt{\alpha_{min}^2 - d^2} - d$ for moving to W_n.

Next, let us consider the pursuer's move to W_n. The pursuer moves to q such that its image moves along the segment wq^i. Let c_T denote the length of this path on T, and c denote the length of its image i.e. wq^i. Since $c = \alpha c_T$ and $\alpha_{min} \leq \alpha \leq 1$ we have $c \leq c_T \leq \frac{c}{\alpha_{min}}$. Since $c \leq \alpha_{min}(\sin\beta\sqrt{\alpha_{min}^2 - d^2} - d)$ (Definition 12), we have $c_T \leq \sin\beta\sqrt{\alpha_{min}^2 - d^2} - d$. Thus, the pursuer can move to q along the aforementioned path.

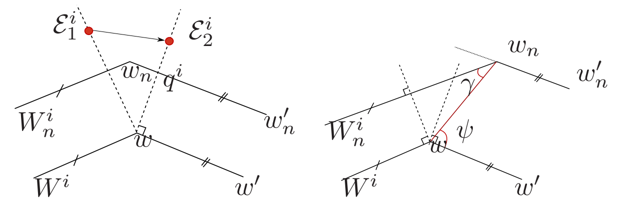


Fig. 12. Left) The wide-turn configuration. Right) The proof of Lemma 10.

■

Lemma 10 (When There is No Vertex-Event): Suppose that the pursuer is currently guarding W, and let W_n be the next wavefront such that there is no vertex event in between W and W_n. Then, if during the next $O(\frac{|T|}{1-d-D})$ steps, the evader always moves and it does not change its direction, the players will be in the wide-turn configuration. Thus, the pursuer can make wide-turn progress.

Proof: We show that the wide-turn configuration will eventually arise by contradiction. In particular, since the evader is circumnavigating around W, it crosses the wedge regions of all vertices of W. We show that it is impossible that none of these events is a wide-turn configuration.

Suppose that the evader is crossing the wedge region of w. For this configuration to not be a wide-turn configuration, the vertex w_n must lie outside the wedge region of w as shown in Fig. 12(b). Let us denote the angle between $w_n w$ and ww' by ψ. Also, let γ be the angle between ww_n and the second edge adjacent to w_n on W_n. Observe that $\gamma < \psi$.

Let h_2 be the distance between the two edges ww' and $w_n w'_n$. Note that these two edges are parallel. Similarly, let h_1 be the distance between the other two parallel edges that are adjacent to w and w_n respectively.

Observe that since $\gamma < \psi$ we have $h_1 < h_2$. We will use this property to conclude our contradiction.

Let us denote the vertices on W by $\{w_1, w_2, \cdots, w_m\}$. Also, let h_j be the distance between the edge on W and the edge on W_n that are associated with the same face. Since the evader is circumnavigating around W, it crosses the wedge regions of all w_i's. If none of these events are wide-turn configuration, then we must have $h_1 < h_2 < \cdots < h_m < h_1$. In other words, $h_1 < h_1$ which is a contradiction. ■

Lemma 11 (Appearing Vertex Event): Suppose that there is an appearing vertex event in the transition from W to W_n. Then, if during the next $O(\frac{|T|}{1-d-D})$ steps, the evader always moves and it does not change its direction, the pursuer can make progress by moving to $\pi(\mathcal{E}, W_n)$. *Proof:* Let w be the corresponding vertex on W (Fig. 13). Also, let m^1 and

m^2 be the two edges in W^i that are adjacent to w. Moreover, let m_n^1 and m_n^2 be the edges in W_n that are parallel to m^1 and m^2 respectively. Also, let w_n^1 and w_n^2 be the two vertices in W_n that are adjacent to m_n^1 and m_n^2. Let l_1 and l_2 be the perpendicular lines to m^1 and m^2 drawn from w^i.

Since $\pi(\mathcal{E}, W)^i$ circumnavigates around W, the evader will cross the wedge region of w. Now, there will be two cases. First, the edge m_n^2 intersects l_2 (Fig. 13(a)). In this case, we have a wide-turn configuration and the proof follows from Lemma 9.

Second, the edge m_n^2 does not intersect l_2 (Fig. 13(b)). In this case, consider the auxiliary wavefront W_n' that is obtained from W_n as follows: $W_n(w_n^1, w_n^2)$ is replaced by the m_n^1 and m_n^2 as shown in Fig. 13(b). Let w_n be the intersection of m_n^1 and m_n^2 in W_n'. It is not difficult to show that $\pi^i(\mathcal{E}_2, W_n)$ is in the region formed by l_2, m_n^2 and ww_n^2 (the shaded region in Fig. 13(b)). Therefore, the distance between w^i and $\pi^i(\mathcal{E}_2, W_n)$ is less than $w^i w_n^i$. Consider the pursuer path composed of $W(\mathcal{P}, w)$ and then the segment from w^i to $\pi^i(\mathcal{E}_2, W_n)$. The length of this path is at most $d + D$. Similar to Lemma 9, d and D can be designed such that the pursuer can move to $\pi(\mathcal{E}_2, W_n)$.

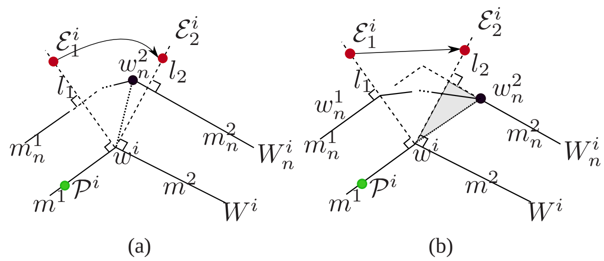


Fig. 13. The evader moves for the next $O(N)$ steps when there is an appearing vertex event. (a) the edge m_n^2 intersects with l_2. (b) the edge m_n^2 does not intersect with l_2.

■

Lemma 12 (Disappearing Vertex Event): Suppose that there is a disappearing vertex event in the transition from W to W_n. Then, if during the next $O(\frac{|T|}{1-d-D})$ steps, the evader always moves and it does not change its direction, the pursuer can make progress by moving to $\pi(\mathcal{E}, W_n)$.

Proof: Let w_n be the corresponding vertex in W_n (Fig. 14).

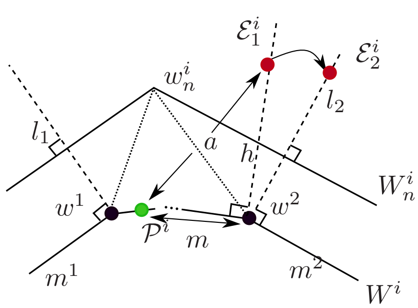


Fig. 14. The evader moves for the next $O(N)$ steps when there is a disappearing vertex event.

Let m_n^1 and m_n^2 be the two edges in W_n that are adjacent to w_n. Also, let m^1 and m^2 be the two edges in W that are parallel to m_n^1 and m_n^2. Let w^1 and w^2 be the two vertices that are adjacent to m^1 and m^2 (Fig. 14). Notice that as the evader crosses the wedge region of w_n^2, we have a wide-turn configuration. Therefore, Lemma 9 applies and thus the pursuer can make progress towards W_n. ■

VIII. Conclusion

We studied the lion and man game on convex terrains and presented a pursuit strategy which guarantees that the pursuer can reduce the distance between the players to the step size in finite time. The capture time is a function of the terrain's properties such as its height and maximum slope as well as the perimeter of its projection onto the base plane.

One of the questions left open in this work is the optimality of this strategy. A second research direction is to characterize terrains in which a single pursuer suffices for capture. Even though convexity is sufficient, it is not necessary. A related question is to compute minimum number of pursuers for a given terrain.

Appendix

A. Special Case: The Highest Point is Not Unique

In this section, we consider the case that the highest point is not unique: the highest points are on a line segment or on a face of T. Without loss of generality, let us assume that the highest points are on the face f. The idea is to first clear f which can be seen as a plane in $\mathbb{R}^2$ and then continue with the pursuit strategy for terrains.

In order to clear f and push the evader outside it, two different strategies can be employed. The first strategy is straightforward: the pursuer can use the lion's move with respect to any arbitrary center inside f. The second strategy is a special case of our wavefront strategy. Let us denote the polygonal boundary of f by ∂f. Consider the straight skeleton [9] of ∂f which is the same as the medial axis of ∂f since the ∂f is convex. This straight skeleton is obtained by shrinking ∂f as follows: move the edges of ∂f in parallel with equal speed. Continuing this shrinking process, we will end up with a tree which is the same as the medial axis. The shrunken polygon after each step can be seen as a wavefront. The pursuer can apply the proposed wavefront guarding strategy to these wavefronts as well and force the evader to exit f in order to prevent capture.

References

[1] T. Chung, G. Hollinger, and V. Isler, "Search and pursuit-evasion in mobile robotics," *Autonomous Robots*, no. 3, 2011.

[2] J. E. Littlewood, *A mathematician's miscellany / J. E. Littlewood*. Methuen, London :, 1953.

[3] L. Alonso, A. S. Goldstein, and E. M. Reingold, "Lion and Man: Upper and lower bounds," *INFORMS Journal on Computing*, vol. 4, no. 4, p. 447, 1992.

[4] V. Isler, S. Kannan, and S. Khanna, "Randomized pursuit-evasion in a polygonal environment," *IEEE Transactions on Robotics*, vol. 21, no. 5, pp. 875–884, 2005.

[5] D. Bhadauria, K. Klein, V. Isler, and S. Suri, "Capturing an evader in polygonal environments with obstacles: The full visibility case," *International Journal of Robotics Research*, 2012, in press.

[6] S. Kopparty and C. V. Ravishankar, "A framework for pursuit evasion games in rn," *Information Processing Letters*, vol. 96, no. 3, pp. 114–122, 2005.

[7] S. Alexander, R. Bishop, and R. Ghrist, “Pursuit and evasion in non-convex domains of arbitrary dimensions.” in *Robotics: Science and Systems*. Citeseer, 2006.
[8] S. Eidenbenz, C. Stamm, and P. Widmayer, “Inapproximability results for guarding polygons and terrains,” *Algorithmica*, vol. 31, no. 1, pp. 79–113, 2001.
[9] O. Aichholzer and F. Aurenhammer, “Straight skeletons for general polygonal figures in the plane,” in *Computing and Combinatorics*, ser. Lecture Notes in Computer Science, J.-Y. Cai and C. Wong, Eds. Springer Berlin Heidelberg, 1996, vol. 1090, pp. 117–126.