A FREE BOUNDARY PROBLEM ARISING IN SMOULDER COMBUSTION

By

Bei Hu

IMA Preprint Series # 972

May 1992

A FREE BOUNDARY PROBLEM ARISING IN SMOULDER COMBUSTION

BEI HU*

1. Introduction. There are many flame propagations models, see [1, 2, 6] and the references therein. The smouldering combustion is a slow burning process such as burning of a paper.

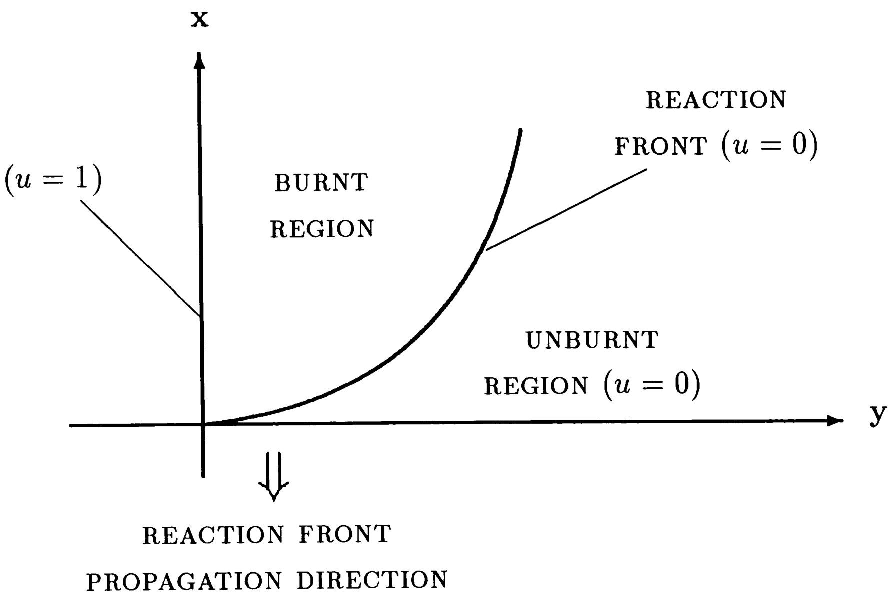

Figure 1.

Assuming that the thickness of the reaction zone is negligible, a two dimensional flame propagation model is derived in [1] as follows:

$$\frac{\partial u}{\partial t} = \varepsilon \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \qquad \text{for } y < f(x,t),\ x > 0,\ t > 0, \tag{1.1}$$

with the boundary conditions

$$u = 1 \qquad \text{on } y = 0, \tag{1.2}$$

* Department of mathematics, University of Notre Dame, Notre Dame, Indiana 46556. The author is partially supported by National Science Foundation Grant DMS–9024986.

$$u = 0 \qquad \text{on } y = f(x,t) \tag{1.3}$$

and the free boundary conditions:

$$f(0,t) = 0 \tag{1.4}$$

$$-\frac{\partial f}{\partial x}(x,t) = \frac{\partial u}{\partial y}(x, f(x,t)) - \varepsilon \frac{\partial u}{\partial x}(x, f(x,t)) \frac{\partial f}{\partial x}(x,t) \tag{1.5}$$

where u is the oxidizer concentration and $y = f(x,t)$ is the reaction front (as in figure 1); $\sqrt{\varepsilon}$ is proportional to ρ_0/ρ_f, ρ_0 is the mean density of the oxidizer, and ρ_f is the density of the solid reactant. All the quantities have been nondimensionalized and $\sqrt{\varepsilon}$ is typically small.

In §2, we study the travelling wave solutions. The Duvaut's transform reduces the problem to a variational inequality, and existence and uniqueness are therefore obtained. For this variational inequality, however, the solution will not solve the original problem (1.1)–(1.5), which will be shown in §3. In fact, the system (1.1)–(1.5) has no solutions if $\varepsilon > 0$; this is due to the assumption that the oxidizer concentration is discontinuous at the reaction front on the x-axis. After modifying the assumption so that the oxidizer at the reaction front is continuous, we obtain travelling wave solutions. In §3, we also establish the monotonicity of the free boundary. In §4, we show that the free boundary will converge to the corresponding free boundary of the Stefan problem as $\varepsilon \to 0$.

Finally in §6, we study the time dependent problem and show that the solution will converge to the travelling wave solution as $t \to \infty$ under certain assumptions.

2. Travelling wave solutions. We look for travelling wave solutions, namely, the solution which is a function of the variable $(x+t, y)$ only. As in [1], by making a change of variable $x + t = \xi$, and renaming ξ as x, the system (1.1)–(1.5) becomes

$$\frac{\partial u}{\partial x} = \varepsilon \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \qquad \text{for } y < f(x),\ x > 0, \tag{2.1}$$

with the boundary conditions

$$u = 1 \qquad \text{on } y = 0, \tag{2.2}$$

$$u = 0 \qquad \text{on } y = f(x), \tag{2.3}$$

and the free boundary conditions:

$$f(0) = 0, \tag{2.4}$$

$$-\frac{df}{dx}(x) = \frac{\partial u}{\partial y}(x, f(x)) - \varepsilon \frac{\partial u}{\partial x}(x, f(x)) \frac{df}{dx}(x). \tag{2.5}$$

When $\varepsilon = 0$, this problem is the Stefan problem with the explicit solution

$$u_0(x,y) = 1 - \int_0^{y/\sqrt{x}} \exp\left(-\sigma^2/4\right) d\sigma \Big/ \int_0^M \exp\left(-\sigma^2/4\right) d\sigma\,, \tag{2.6}$$

$$f_0(x) = M\sqrt{x} \qquad \text{for } x \geq 0 \tag{2.7}$$

where the constant $M > 0$ is the unique solution of

$$\frac{1}{2} M \int_0^M \exp\left(-\sigma^2/4\right) d\sigma = \exp\left(-M^2/4\right). \tag{2.8}$$

When $\varepsilon > 0$ and small, a similarity solution is found in [1] for the first order expansion in ε. Actually, it is more convenient to write the free boundary as $x = g(y)$. Then (2.5) becomes

$$-1 = g'(y)\frac{\partial u}{\partial y}(g(y),y) - \varepsilon\frac{\partial u}{\partial x}(g(y),y). \tag{2.9}$$

The Duvaut's transform

$$\begin{aligned} w &= \int_{g(y)}^{x} u(\sigma,y)d\sigma \qquad \text{for } x > g(y) \\ &= 0 \qquad \text{for } -K \leq x \leq g(y) \end{aligned}$$

reduces (2.1)–(2.5) to an obstacle problem:

$$\begin{aligned} -\varepsilon w_{xx} - w_{yy} + w_x = -1 \qquad &\text{for } x > g(y), \\ w = 0 \qquad &\text{for } -K \leq x \leq g(y), \end{aligned} \tag{2.10}$$

$w \in C^1(\{x > -K, y > 0\})$ and satisfies the boundary condition

$$w = (x - g(0))^+ \qquad \text{on } y = 0. \tag{2.11}$$

Here $g(0)$ should be solved as part of the solution and should not be predetermined. In fact, we can show later that if $g(0) = 0$ then $g(0) < 0$ (Theorem 3.1) and therefore we get a contradiction. On the other hand, if we do not assume $g(0) = 0$, we can make a change of variable $x - g(0) \to x$ and get $g(0) < g(0)$, which is still a contradiction. This means that there is no solution if we assume (2.11). This result is not surprising since (2.11) implies that there is a jump on the oxidizer concentration u on the reaction front $x = g(y)$ near $y = 0$. We shall modify the boundary condition (2.11) to the following condition

$$w = x^+ \qquad \text{on } y = 0, \tag{2.12}$$

which means that we have assumed $u(x,0) = 0$ in $g(0) \leq x \leq 0$ instead of $u(x,0) = 1$ in $g(0) \leq x \leq 0$. It also makes sense if we replace x^+ by a monotone decreasing, concave-up, $C^{1,1}$ function vanishing on $x < 0$ (which means that u is continuous on the entire x-axis); we shall discuss this case in §5.

We expect that for the solution (w, g), we have $g(y) > -K$ if K is large enough. Since $0 \le u(x,y) \le 1$, we have $0 \le w(x,y) \le x + K$. So the obstacle problem can be formulated as follows:

$$-\varepsilon w_{xx} - w_{yy} + w_x \ge -1 \quad \text{in} \quad \Omega, \tag{2.13}$$

$$w \ge 0 \quad \text{in} \quad \Omega, \tag{2.14}$$

$$w\,(-\varepsilon w_{xx} - w_{yy} + w_x + 1) = 0 \quad \text{in} \quad \Omega, \tag{2.15}$$

$$w(x,0) = x, \quad w(-K,y) = 0, \tag{2.16}$$

$$0 \le w(x,y) \le x + K \quad \text{in} \quad \Omega \tag{2.17}$$

where $\Omega = \{(x,y);\ x > -K, y > 0\}$.

THEOREM 2.1. *For each K, the system (2.13)–(2.17) has a unique solution $w \in C(\overline{\Omega}) \cap W^{2,\infty}_{loc}(\Omega)$. Furthermore,*

$$0 \le w_x \le 1, \quad w_{xx} \ge 0 \quad \textit{in } \Omega. \tag{2.18}$$

Proof. The proof follows from the penalization method for the obstacle problem. For completeness, we present the proof here. The elliptic estimates used in this section can be found, for example, [4].

We first prove the uniqueness. Suppose that w_i $(i = 1,2)$ are two solutions to the system (2.13)–(2.17). Let $\widetilde{w}_i = w_i e^{-\eta(x+y)}$, then we have

$$\mathcal{L}[\widetilde{w}_i] + e^{-\eta(x+y)} \ge 0, \quad \widetilde{w}_i \ge 0, \quad \widetilde{w}_i\left(\mathcal{L}[\widetilde{w}_i] + e^{-\eta(x+y)}\right) = 0 \quad \text{in} \quad \Omega, \tag{2.19}$$

where

$$\mathcal{L} = -\varepsilon\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + (1 - 2\varepsilon\eta)\frac{\partial}{\partial x} - 2\eta\frac{\partial}{\partial y} + (\eta - \varepsilon\eta^2 - \eta^2).$$

We take $\eta > 0$ to be small enough so that $\eta - \varepsilon\eta^2 - \eta^2 > 0$. The functions $\widetilde{w}_i$ are uniformly bounded and converges to 0 when $(x,y) \to \infty$. Therefore if the supremum of $\widetilde{w}_1 - \widetilde{w}_2$ is positive, it must be obtained at a finite point (x_0, y_0), i.e.,

$$\sup_{(x,y)\in\Omega} (\widetilde{w}_1(x,y) - \widetilde{w}_2(x,y)) = \widetilde{w}_1(x_0,y_0) - \widetilde{w}_2(x_0,y_0) > 0.$$

The point (x_0, y_0) can not be on the boundary by the boundary conditions. Therefore by the maximum principle (see [5], for example),

$$\operatorname*{ess\,lim\,inf}_{(x,y)\to(x_0,y_0)} \mathcal{L}[\widetilde{w}_1 - \widetilde{w}_2] \ge (\eta - \varepsilon\eta^2 - \eta^2)(\widetilde{w}_1(x_0,y_0) - \widetilde{w}_2(x_0,y_0)) > 0. \tag{2.20}$$

Since $\widetilde{w}_1(x_0,y_0) - \widetilde{w}_2(x_0,y_0) > 0$, there is a neighborhood $B_\delta(x_0,y_0)$ in which $\widetilde{w}_1(x,y) - \widetilde{w}_2(x,y) > 0$, and therefore $\widetilde{w}_1(x,y) > \widetilde{w}_2(x,y) \ge 0$ on $B_\delta(x_0,y_0)$. It follows that

$$\mathcal{L}[\widetilde{w}_1 - \widetilde{w}_2] = -\left(\mathcal{L}[\widetilde{w}_2] + e^{-\eta(x+y)}\right) \le 0 \qquad \text{in } B_\delta(x_0,y_0),$$

which contradicts (2.20).

For the existence, we can approximate the domain Ω with $\Omega_R = \{(x,y);\ -K < x < R, 0 < y < R\}$. Take functions $\beta_\delta \in C^\infty$ such that

$$\begin{gathered} \beta_\delta'(s) \geq 0, \quad \beta_\delta''(s) \geq 0 \quad \text{for all} \quad s \in (-\infty, \infty), \\ \beta_\delta(s) = 0, \quad \text{for} \quad s < -\delta, \quad \beta_\delta(0) = 1, \\ \beta_\delta(s) \to \infty \quad \text{as } \delta \to 0 \text{ for each } s > 0, \end{gathered} \tag{2.21}$$

and consider the corresponding penalization problem on Ω_R:

$$-\varepsilon w_{xx} - w_{yy} + w_x = -1 + \beta_\delta(-w) \qquad \text{in } \Omega_R, \tag{2.22}$$

with boundary condition:

$$w(x,y) = p_R(x)(1 - y/R) \qquad \text{on } \partial\Omega_R, \tag{2.23}$$

where

$$p_R(x) = \begin{cases} x & \text{for } x > 1/R \\ 0 & \text{for } x < -1/R \end{cases}, \qquad p_R'(x) \geq 0, \quad p_R''(x) \geq 0, \quad p_R \in C^\infty(R).$$

This system has a unique solution $w_{R,\delta}$. By the maximum principle $0 \leq w_{R,\delta} \leq \dfrac{x+K}{R+K} R\left(1 - \dfrac{y}{R}\right)$ on Ω_R. Hence

$$0 \leq \frac{\partial w_{R,\delta}}{\partial x} \leq 1 \quad \text{on } \partial\Omega_R. \tag{2.24}$$

Differentiating the equation (2.22) once in x and using the maximum principle, we get

$$0 \leq \frac{\partial w_{R,\delta}}{\partial x} \leq 1 \quad \text{in } \Omega_R. \tag{2.25}$$

Since $w_{R,\delta}(x,y) \geq 0$ in Ω_R,

$$\sup_{(x,y)\in\Omega_R} \beta_\delta(-w_{R,\delta}(x,y)) = 1.$$

Next,

$$\frac{\partial^2 w_{R,\delta}}{\partial x^2} \geq 0 \qquad \text{on } y = 0 \text{ and } y = R,$$

and, by the equation,

$$\begin{aligned} \varepsilon \frac{\partial^2 w_{R,\delta}}{\partial x^2} &= \frac{\partial w_{R,\delta}}{\partial x} + 1 - \beta_\delta(-w_{R,\delta}) \\ &\geq 0 \qquad \text{on } x = 0 \text{ and } x = R. \end{aligned}$$

After differentiating (2.22) twice in x and recalling (2.21), we conclude, by the maximum principle,

$$\frac{\partial^2 w_{R,\delta}}{\partial x^2} \geq 0 \qquad \text{in } \Omega_R.$$

Now by the standard estimates for the penalization problem, we have the $W^{2,\infty}$ interior estimates. Thus by compactness, we can take the limit for $R \to \infty$ and then $\delta \to 0$ along subsequences. This completes the proof of existence. □

The equation (2.18) implies that the free boundary is a curve $x = g(y)$.

LEMMA 2.2.

$$g(y) \leq \frac{y^2}{M^2} \tag{2.26}$$

where the the constant M is defined in (2.8).

Proof. Let

$$\begin{aligned} \varphi &= \int_{y^2/M^2}^{x} u_0(\sigma, y)d\sigma \qquad \text{for } x > y^2/M^2 \\ &= 0 \qquad \text{for } -K \leq x \leq y^2/M^2 \end{aligned} \tag{2.27}$$

where u_0 is the solution for the Stefan problem defined by (2.6). Then

$$\frac{\partial^2 \varphi}{\partial x^2} = \frac{\partial u_0}{\partial x} \geq 0 \qquad \text{for } x > y^2/M^2; \tag{2.28}$$

therefore

$$-\varepsilon\varphi_{xx} - \varphi_{yy} + \varphi_x = -1 - \varepsilon\varphi_{xx} \leq -1 \qquad \text{for } x > y^2/M^2. \tag{2.29}$$

Notice that $\varphi \in C^{1,1}(\Omega) \cap C(\overline{\Omega})$ and satisfy the same boundary conditions as that of w. Although we have an unbounded region here, we still have the comparison principle for this variational inequality (the proof is essentially the same as the uniqueness proof in Lemma 2.1). Thus $\varphi(x,y) \leq w(x,y)$ in Ω and hence $g(y) \leq y^2/M^2$. □

LEMMA 2.3. *If $K > \varepsilon$, then*

$$g(y) \geq -\frac{1}{2}\varepsilon \qquad \textit{for all } y > 0. \tag{2.30}$$

Proof. Construct the one dimensional auxiliary function

$$\varphi(x,y) = \varphi(x) = \begin{cases} x & \text{for } x \geq \dfrac{\varepsilon}{2} \\ \dfrac{1}{2\varepsilon}\left(x + \dfrac{\varepsilon}{2}\right)^2 & \text{for } |x| \leq \dfrac{\varepsilon}{2} \\ 0 & \text{for } x \leq -\dfrac{\varepsilon}{2} \end{cases}.$$

Then $\varphi \in C^{1,1}(\Omega_R)$. Clearly $-\varepsilon\varphi_{xx} - \varphi_{yy} + \varphi_x \geq -1$ whenever $\varphi > 0$. On the boundary $y = 0$,

$$\varphi(x) \geq \frac{1}{2\varepsilon}\left(x^2 + \varepsilon x + \frac{\varepsilon^2}{4}\right) \geq x \geq w(0, x).$$

It follows form the comparison principle for the variational inequality that $w(x, y) \leq \varphi(x)$, and therefore $g(y) \geq -\varepsilon/2$. □

Remark 2.1. Since $w \in W^{2,\infty}_{loc}(\Omega)$, the curve $x = g(y)$ is Lipschitz continuous for all $y > 0$, by [3, Theorem 6.1, page 177].

Remark 2.2. The curve $x = g(y)$ has a local analytic representation, by the general theory of free boundary problems. Thus $g(y)$ is analytic for all $y > 0$.

Remark 2.3. It is obvious that w is smooth up to the free boundary from the side $\{x > g(y)\}$. Hence w_x solves the problem consisting of (2.1), (2.9), $u = 0$ on $x = g(y)$, and $u(0, x) = H(x)$, where $H(x)$ is the Heaviside function.

3. Properties of the free boundary.

PROPOSITION 3.1. *There exists* $\eta = \eta_\varepsilon > 0$ *such that*

$$g(y) \leq -\eta \qquad \textit{for } 0 < y < \eta. \tag{3.1}$$

Proof. We shall prove that

$$w_y(x, y) \geq 1 \quad \text{for } |x| \leq \eta,\ 0 \leq y \leq \eta, \text{ for some } \eta > 0. \tag{3.2}$$

The theorem will then follow since $\nabla w(g(y), y) \equiv 0$. Set

$$\varphi(\xi, y) = \frac{y}{2\pi}\log\frac{(\xi - 1)^2 + y^2}{\xi^2 + y^2} + \frac{\xi}{\pi}\left(\arctan\frac{1 - \xi}{y} + \arctan\frac{\xi}{y}\right). \tag{3.3}$$

Then φ is continuous in $\widetilde{D}_s \equiv \left\{(\xi, y);\ 0 \leq y \leq \sqrt{s^2 - \xi^2}\right\}$ $(0 < s < 1)$,

$$\frac{\partial^2\varphi}{\partial\xi^2} + \frac{\partial^2\varphi}{\partial y^2} = 0 \quad \text{in } \widetilde{D}_s\,, \tag{3.4}$$

and

$$\lim_{y\to 0+}\varphi(\xi, y) = \xi^+ \quad \text{uniformly for } |\xi| \leq \frac{3}{4}. \tag{3.5}$$

We next set $\xi = x/\sqrt{\varepsilon}$ and define

$$\psi(x, y) = \sqrt{\varepsilon}\varphi\left(\frac{x}{\sqrt{\varepsilon}}, y\right),$$

then we have

$$\varepsilon\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=0 \quad \text{in } D_s\equiv\left\{(x,y);\ 0\le y\le\sqrt{s^2-\frac{x^2}{\varepsilon}}\right\},$$

and

$$\lim_{y\to 0+}\psi(x,y)=x^+ \quad \text{uniformly for } |x|\le\frac{3}{4}\sqrt{\varepsilon}.$$

We now go back to the equation (2.22) for the solution $w_{R,\delta}$. Since $0\le(w_{R,\delta})_x\le 1$ and $0\le\beta_\delta(-w_{R,\delta})\le 1$, we have

$$0\le\varepsilon\frac{\partial^2 w_{R,\delta}}{\partial x^2}+\frac{\partial^2 w_{R,\delta}}{\partial y^2}\le 2 \quad \text{in } D_s\equiv\left\{(x,y);\ 0\le y\le\sqrt{s^2-\frac{x^2}{\varepsilon}}\right\} \quad (0<s<1).$$

Therefore the limit w satisfies

$$\varepsilon\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\equiv f(x,y) \quad \text{in } D_s, \quad \|f\|_{L^\infty}\le 2, \quad f(x,y)\ge 0.$$

Clearly $w-\psi$ is continuous on $D_{3/4}$, $w-\psi=0$ on $y=0$ and

$$\varepsilon\frac{\partial^2(w-\psi)}{\partial x^2}+\frac{\partial^2(w-\psi)}{\partial y^2}\equiv f(x,y) \quad \text{in } D_s, \quad \|f\|_{L^\infty}\le 2, \quad f(x,y)\ge 0.$$

It follows by interior-boundary estimates for the elliptic equation that

$$|\nabla(w-\psi)(x,y)|\le C_\varepsilon \quad \text{for } (x,y)\in D_{1/2}. \tag{3.6}$$

A direct calculation shows that

$$\begin{aligned}\frac{\partial\psi}{\partial y} &= \sqrt{\varepsilon}\left[\frac{1}{2\pi}\log\frac{(\xi-1)^2+y^2}{\xi^2+y^2}+\frac{1}{\pi}\left(\frac{y^2}{(\xi-1)^2+y^2}-\frac{y^2}{\xi^2+y^2}\right)\right.\\ &\qquad\left.+\frac{1}{\pi}\left(\frac{\xi(\xi-1)}{(\xi-1)^2+y^2}-\frac{\xi^2}{\xi^2+y^2}\right)\right]\\ &= \frac{\sqrt{\varepsilon}}{2\pi}\left(\log\frac{1}{\xi^2+y^2}+O(1)\right) \quad \text{in } D_{1/2} \qquad (\xi=x/\sqrt{\varepsilon}).\end{aligned}$$

Combining this with (3.6), we obtain (3.2). □

This Proposition shows that, even if ε is small, it still played a dramatic role for the local behavior of the free boundary near $(0,0)$. Since $g(y)$ is away from $(0,0)$, $D^2_{x,y}w$ is uniformly bounded near the free boundary. Therefore we have

PROPOSITION 3.2. *The function $g(y)$ is continuous on $[0,\infty)$.*

Proof. We need only to show that $g(y)$ is continuous at $y=0$. It suffices to show that the oscillation of $g(y)$ converges to 0 as $y\to 0+$. Suppose that $g(y_1)=g(y_2)\equiv b$

and $a \le g(y) \le b$ for $y_1 < y < y_2$. Then the function $\frac{1}{2\varepsilon}(x-a)^2$ in the region $a \le x \le b, y_1 \le y \le y_2$ is a supersolution for the equation $-\varepsilon w_{xx} - w_{yy} + w_x = -1$. Since $|w_{yy}| \le C_\varepsilon$ and $w = w_y = 0$ at (b, y_1),

$$w(b,y) \le C_\varepsilon (y_2 - y_1)^2 = \frac{1}{2\varepsilon}(b-a)^2$$

provided $\sqrt{C_\varepsilon}(y_2 - y_1) = \frac{1}{\sqrt{2\varepsilon}}(b-a)$. Thus $|g(y) - g(y_1)| \le b - a \equiv \sqrt{2C_\varepsilon \varepsilon}\,(y_2 - y_1)$ for all $y_1 \le y \le y_2$. It follows that there are no fingers for the free boundary and therefore $g(y)$ is continuous at 0. $\square$

Now define $N = \{(x,y);\ g(y) < x < \infty,\ 0 < y < \infty\}$ and let $\Gamma = \{(g(y), y);\ 0 \le y < \infty\}$.

PROPOSITION 3.3. *There exists $y_0 > 0$ such that*

$$g'(y) < 0 \quad \text{for } 0 < y < y_0, \quad g'(y_0) = 0, \quad \text{and } g'(y) > 0 \text{ for } y_0 < y < \infty, \tag{3.7}$$

furthermore

$$\liminf_{y \to 0+} \frac{g(y) - g(0)}{y} = -\infty\ . \tag{3.8}$$

Proof. First we notice that $w_x(x,0) = 0$ for $g(0) < x < 0$ is the minimum of w_x in N. The maximum principle implies that $w_{xy}(x,0) > 0$ for $g(0) < x < 0$ and therefore $w_y(x,0)$ is monotone increasing. Since $w_y(g(0),0) = 0$, $w_y(x,0) > 0$ for $g(0) < x < 0$.

Similarly, $w_x(x,0) = 1$ for $x > 0$, which is the maximum of w_x in N. By the maximum principle $w_{xy}(x,0) < 0$ for $x > 0$. It follows that $w_y(x,0)$ is monotone decreasing along the positive x axis. From the proof of Lemma 2.3, we have already obtained that $w(x,y) \le x$ for $x \ge \varepsilon/2$. Hence $w_y(x,0) \le 0$ for $x \ge \varepsilon/2$. Note also that $w_y(x,y) > 0$ in a neighborhood of $(0,0)$, from the proof of Proposition 3.1. Therefore there exists a $\tau \in (0, \varepsilon/2]$ such that $w_y(x,0) > 0$ for $0 < x < \tau$ and $w_y(x,0) < 0$ for $\tau < x < \infty$.

Differentiating $w_x(g(y),y) = 0$ and $w_y(g(y),y) = 0$, we get

$$g'(y) w_{xx}(g(y),y) + w_{xy}(g(y),y) = 0, \quad g'(y) w_{yx}(g(y),y) + w_{yy}(g(y),y) = 0.$$

Combining these relations with the equation

$$\varepsilon w_{xx}(g(y),y) + w_{yy}(g(y),y) = 1,$$

we obtain

$$w_{xx} = \frac{1}{\varepsilon + (g'(y))^2}, \quad w_{xy} = -\frac{g'(y)}{\varepsilon + (g'(y))^2}, \quad w_{yy} = \frac{(g'(y))^2}{\varepsilon + (g'(y))^2} \quad \text{on } \Gamma. \tag{3.9}$$

Next, we claim that

$$\text{if } g'(\tilde{y}) < 0, \text{ then } g'(y) < 0 \text{ for all } 0 < y < \tilde{y}. \tag{3.10}$$

Since $g'(\tilde{y}) < 0$, $w_{yx}(g(\tilde{y}),\tilde{y}) > 0$, by (3.9). It follows that $w_y(x,\tilde{y})$ is positive and strictly monotone increasing for $g(\tilde{y}) < x \leq \delta$ for some small $\delta > 0$. We can take $h > 0$ to be sufficiently small such that h is not a critical value for w_y in N, by Sard's lemma. $w_y(x_h,\tilde{y}) = h$ for some $x_h \in (g(\tilde{y}),\delta)$ and we can initiate a curve γ_h starting from $(x_h,\tilde{y})$. By the theory of ODE, the curve γ_h must either hit the boundary or exit any bounded domain in a finite time. The curve γ_h can not hit the free boundary $x = g(y)$ since $w_y(g(y),y) = 0 \neq h$.

If the curve γ_h goes to infinity, then γ_h, the line segment $\{(x,\tilde{y});\ g(\tilde{y}) \leq x \leq x_h\}$ and $\{x = g(y);\ y \geq \tilde{y}\}$ enclose a region D. Clearly w_y is bounded by a polynomial near ∞. Thus the maximum principle implies that $w_y > 0$ in D, which implies $w_{yx}(g(y),y) > 0$ for $y > \tilde{y}$ and hence $g'(y) < 0$ for all $y > \tilde{y}$ by (3.9). This is a contradiction to the fact that $\lim_{y\to\infty} g(y) = \infty$ (this fact will be proved in §4, and it is valid for any $\varepsilon > 0$).

Therefore the curve γ_h can only hit the positive x-axis at a point $x = \tau_h < \tau$. Thus the curve γ_h, the x-axis for $g(0) \leq x \leq \tau_h$ and $\{x = g(y);\ 0 \leq y \leq \tilde{y}\}$ enclose a domain D. w_y is bounded from below in D. The maximum principle implies that $w_y > 0$ in D, which in turn implies that $w_{yx}(g(y),y) > 0$ for $0 < y < \tilde{y}$. Now by (3.9), (3.10) follows.

Similarly, we claim that

$$\text{if } g'(\tilde{y}) > 0, \text{ then } g'(y) > 0 \text{ for all } y > \tilde{y}. \tag{3.11}$$

The proof is essentially the same. Since $w_{yx}(g(\tilde{y}),\tilde{y}) < 0$, $w_y(x,\tilde{y})$ is negative and strictly monotone decreasing for $g(\tilde{y}) < x \leq \delta$ for some small $\delta > 0$. Take $h > 0$ to be sufficiently small such that $-h$ is not a critical value for w_y in N. $w_y(x_h,\tilde{y}) = -h$ for some $x_h \in (g(\tilde{y}),\delta)$ and we can initiate a curve γ_h starting from $(x_h,\tilde{y})$. The curve γ_h will either extend to infinity or hit the positive x-axis at $x = x_h > \tau$. In both cases the curve γ_h, the line segment $\{(x,\tilde{y});\ g(\tilde{y}) \leq x \leq x_h\}$ and $\{x = g(y);\ y \geq \tilde{y}\}$ enclose a region D. $w_y < 0$ in D by the maximum principle, which implies that $w_{yx}(g(y),y) < 0$ for $y > \tilde{y}$. Using (3.9), we conclude (3.11).

Note that there will be no horizontal segment for the free boundary since $g(y)$ is analytic. It follows from (3.10) and (3.11) that

$$\text{either:} \quad \begin{cases} \text{there exists } y_0 > 0 \text{ such that} \\ g'(y) < 0 \quad \text{in } (0,y_0), \quad g'(y_0) = 0, \quad g'(y) > 0 \quad \text{in } (y_0,\infty), \end{cases} \tag{3.12}$$

$$\text{or:} \quad g'(y) > 0 \quad \text{for all } y > 0. \tag{3.13}$$

To complete the proof, it remains to establish (3.8), which will then imply that (3.13) will not happen.

We claim that, there exists $C > 0$ such that

$$g(y) - g(0) \le Cy \quad \text{for small } y > 0. \tag{3.14}$$

Take a curve γ in N connecting $(g(y_1), y_1)$ $(0 < y_1 < 1)$ to a point $(x_1, 0)$ $(g(0) < x_1 < 0)$ such that γ hits the free boundary Γ in the reverse direction of the x-axis. Consider the domain D bounded by γ, the free boundary $\{x = g(y);\ 0 \le y \le y_1\}$ and x-axis for $g(0) \le x \le x_1$. In a neighborhood of $(x_1, 0)$, $w_y \ge 0$. Since $w_{xx}(g(y_1), y_1) > 0$ and all second derivatives of w are bounded in D, $Cw_x + w_y \ge 0$ on γ in a neighborhood of $(g(y_1), y_1)$ if C is large enough. Since $w_x > 0$ in N, $Cw_x + w_y \ge 0$ on the rest of the boundary of ∂D if C is taken large enough. By the maximum principle, $Cw_x + w_y > 0$ in D. From which (3.14) follows.

Suppose that (3.8) is not true. Then there exists $\theta_0 \in (\pi/2, \pi)$ such that $x = g(y)$ lies in the sector $S \equiv \{0 < \theta < \theta_0,\ 0 < \rho < \rho_0\}$ (ρ_0 is a small constant); here we used the polar coordinate defined as follows:

$$\frac{1}{\sqrt{\varepsilon}}(x - g(0)) = \rho\cos\theta,\ y = \rho\sin\theta.$$

Let $\alpha = \pi/\theta_0$ (> 1) and define $\varphi(x, y) = C\rho^\alpha \sin(\alpha\theta)$. Then $\varepsilon\varphi_{xx} + \varphi_{yy} = 0$ in S. Clearly $w_x = 0$ on $\{\theta = 0;\ g(0) \le x < g(0) + \sqrt{\varepsilon}\rho_0\}$; $w_x = 0$ on the free boundary, which lies in the sector S for $\rho < \rho_0$. Since all the second derivatives of w is bounded in S, we can choose C to be large enough so that $\varphi \ge w_x$ on $\rho = \rho_0$.

Since $w_{xx} \ge 0$, $\varepsilon w_{xxx} + w_{xyy} = w_{xx} \ge 0$ in $S \cap N$. By the maximum principle $w_x \le \varphi$ in $S \cap N$. It follows that $w_x \le C\left(\sqrt{\frac{1}{\varepsilon}(x - g(0))^2 + y^2}\right)^\alpha$ for $\rho < \rho_0$. Integrating in x, we obtain,

$$\begin{aligned} w(x,y) &\le C(x - g(y))\left(\sqrt{\frac{1}{\varepsilon}(x - g(0))^2 + y^2}\right)^\alpha \\ &\le C\left(|x - g(0)| + |\cot\theta_0|y\right)\left(\sqrt{\frac{1}{\varepsilon}(x - g(0))^2 + y^2}\right)^\alpha \quad \text{for } \rho < \frac{\rho_0}{\sqrt{2}}. \end{aligned} \tag{3.15}$$

Now we define

$$\widetilde{w}_\delta(x, y) = \frac{1}{\delta^2} w\left(\delta x + g(0), \delta y\right).$$

Then by (3.15) (notice that $\alpha > 1$),

$$\lim_{\delta \to 0} \widetilde{w}_\delta(x, y) \equiv 0.$$

On the other hand $D^2 \widetilde{w}_\delta(x, y)$ is uniformly bounded in any bounded domain, and $\widetilde{w}_\delta$ satisfies the equation

$$-\varepsilon\left(\widetilde{w}_\delta\right)_{xx} - \left(\widetilde{w}_\delta\right)_{yy} + \delta\left(\widetilde{w}_\delta\right)_x = -1 \quad \text{for } 0 < Cy < x,$$

by (3.14). Letting $\delta \to 0$ along a subsequence, we get a contradiction. $\square$

4. Asymptotic behavior for small ε. Physically, ε is a small constant. In this section we prove that the free boundary of the problem (2.1)–(2.5) converges to the free boundary corresponding to $\varepsilon = 0$, namely, $x = y^2/M^2$, where M is given in (2.8). To emphasis the dependence on ε we denote $g_\varepsilon(y) = g(y)$.

THEOREM 4.1.

$$y^2/M^2 - O(\sqrt{\varepsilon}\,)y - O(\sqrt{\varepsilon}\,) \le g_\varepsilon(y) \le y^2/M^2. \tag{4.1}$$

Proof. We have already proved that $g_\varepsilon(y) \le y^2/M^2$ in §2. What we need to establish is the other half of the inequality in (4.1). Let $\eta \in (0, 1/2]$ and let N_η be the unique solution of

$$\frac{1}{2}(1-\eta)N\int_0^N \exp\left(-\sigma^2/4\right)d\sigma = \exp\left(-N^2/4\right). \tag{4.2}$$

Then the pair

$$u_\eta(x,y) = 1 - \int_0^{y/\sqrt{x}} \exp\left(-\sigma^2/4\right)d\sigma \Big/ \int_0^{N_\eta} \exp\left(-\sigma^2/4\right)d\sigma\,, \tag{4.3}$$

$$f_\eta(x) = N_\eta\sqrt{x} \qquad \text{for } x \ge 0 \tag{4.4}$$

is the explicit solution to the Stefan problem

$$\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial y^2} \qquad \text{for } y < f(x),\ x > 0,\ t > 0, \tag{4.5}$$

$$u = 1 \qquad \text{on } y = 0, \tag{4.6}$$

$$u = 0 \qquad \text{on } y = f(x), \tag{4.7}$$

$$f(0) = 0, \tag{4.8}$$

$$(1-\eta)\frac{df}{dx}(x) = -\frac{\partial u}{\partial y}(x, f(x)). \tag{4.9}$$

It follows that

$$\begin{aligned} \varphi_\eta &= \int_{y^2/N_\eta^2}^{x} u_\eta(\sigma,y)d\sigma \qquad \text{for } x > y^2/N_\eta^2 \\ &= 0 \qquad \text{for } -K \le x \le y^2/N_\eta^2 \end{aligned} \tag{4.10}$$

is in $C^1(\Omega)$ and satisfies

$$\frac{\partial \varphi_\eta}{\partial x} - \frac{\partial^2 \varphi_\eta}{\partial y^2} = -1 + \eta \qquad \text{for } x > y^2/N_\eta^2. \tag{4.11}$$

Now let $\psi(x,y) = \varphi_\eta(x+\delta, y)$, then $\psi(-\varepsilon/2, y) \ge 0 = w(-\varepsilon/2, y)$, $\psi(x,0) = (x+\delta)^+ > w(x,0)$, and $0 \le \psi(x,y) \le x+\delta$. Clearly

$$-\varepsilon\frac{\partial^2\psi}{\partial x^2} - \frac{\partial^2\psi}{\partial y^2} + \frac{\partial\psi}{\partial x}$$

$$
\begin{aligned}
&= -1+\eta - \frac{\varepsilon}{2(x+\delta)}\frac{y}{\sqrt{x+\delta}}\exp\left(-\frac{y^2}{4(x+\delta)}\right)\Big/\int_0^{N_\eta}\exp\left(-\sigma^2/4\right)d\sigma \\
&\geq -1+\eta - \frac{\varepsilon}{2\delta-\varepsilon}\sqrt{2}e^{-1/2}\Big/\int_0^{N_\eta}\exp\left(-\sigma^2/4\right)d\sigma \geq -1,
\end{aligned}
$$

provided $\varepsilon \leq \delta$ and

$$
(4.12)\,\varepsilon \leq \frac{\sqrt{2}}{2}\sqrt{e}\,\eta\delta\int_0^{M}\exp\left(-\sigma^2/4\right)d\sigma \left(\leq \frac{\sqrt{2}}{2}\sqrt{e}\,\eta\delta\int_0^{N_\eta}\exp\left(-\sigma^2/4\right)d\sigma\right) .
$$

Notice that (4.12) is satisfied if $\varepsilon \leq \eta\delta$. We thus conclude that $w(x,y) \leq \psi(x,y)$ in Ω and hence

$$
g_\varepsilon(y) \geq y^2/N_\eta^2 - \delta \geq y^2/M^2 - C\eta y^2 - \delta, \tag{4.13}
$$

where we used (4.2) for the inequality $1/N_\eta^2 \geq 1/M^2 - C\eta$. The inequality (4.13) is valid as long as (4.12) holds. For $y \leq 1$ we can take $\delta = \sqrt{\varepsilon}$ and $\eta = \sqrt{\varepsilon}$. For $y \geq 1$ we take $\delta = \sqrt{\varepsilon}\, y$ and $\eta = \sqrt{\varepsilon}/y$. Thus the theorem follows from (4.13). □

Remark 4.1. Theorem 4.1 implies that

$$
\lim_{y\to\infty}\frac{g_\varepsilon(y)}{y^2} = \frac{1}{M^2}\ , \tag{4.14}
$$

A careful examination of the proof indicates that this equality is valid without assuming ε to be small.

Remark 4.2. Theorem 4.1 also implies that the first zero $y_1 > 0$ of the function $g(y)$ satisfies

$$
y_1 \leq O(\varepsilon^{1/4}).
$$

5. Continuous oxidizer concentration for the boundary conditions. Now let's consider the case where x^+ is replace by $p(x)$ in (2.12). Here $p(x)$ is a $C^{1,1}$ function such that

$$
p(x)\begin{cases} = 0 & \text{for } x \leq 0 \\ > 0 & \text{for } x > 0 \end{cases}, \quad p(x) = x - \frac{1}{2} \quad \text{for } x \geq 1, \quad p''(x) \geq 0\ .
$$

The existence and uniqueness results of §2 extend to this case without any difficulties. The analog of Lemmas 2.2 and 2.3 is

LEMMA 5.1. *Suppose that $x = \tilde{g}(y)$ is the free boundary, then*

$$
-\frac{1}{2}\varepsilon \leq \tilde{g}(y) \leq g^*(y)\ , \tag{5.1}
$$

where $g^*(y)$ is the free boundary of the Stefan problem

$$\begin{aligned} &\varphi_x = \varphi_{yy} \quad \text{in } g^*(y) < x < \infty, \\ &\varphi(x,0) = p'(x) \quad \text{for } x \geq 0, \\ &\varphi(x,y) = 0 \quad \text{on } x = g^*(y), \\ &g^*(0) = 0, \quad (g^*)'(y)\varphi_y(g^*(y),y) = -1. \qquad \square \end{aligned}$$

Next, we claim that the free boundary is monotone increasing if ε is small enough.

THEOREM 5.2. *If $\varepsilon \leq 1/\|p''\|_{L^\infty}$, then*

$$0 \leq w_x \leq 1, \quad w_y \leq 0, \quad 0 \leq \varepsilon w_{xx} \leq 2, \quad 0 \leq w_{yy} \leq 2, \tag{5.2}$$

and the free boundary is monotone increasing. Furthermore, $\tilde{g}(y)$ is analytic for $y > 0$ and $\tilde{g}(0) = 0$.

Proof. As before, we can derive that

$$0 \leq w_x \leq 1, \quad w_{xx} \geq 0. \tag{5.3}$$

From the proof of Theorem 2.1, we see that the estimate can be extended to $(w_{\delta,R})_{yy}$ to derive that $(w_{\delta,R})_{yy} \geq -\varepsilon\|p''\|_{L^\infty}$. Using the equation we find that both $(w_{\delta,R})_{xx}$ and $(w_{\delta,R})_{yy}$ are uniformly bounded. Hence after taking the limit,

$$\sup_{\Omega} \left| \frac{\partial^2 w}{\partial x^2}, \frac{\partial^2 w}{\partial y^2} \right| < \infty \,,$$

where $\Omega = \{-K < x < \infty,\ 0 < y < \infty\}$. By the assumption, we have $w_{yy}(x,0) = 1 + p'(x) - \varepsilon p''(x) \geq 0$ for $x > g(0)$. By the general theory of free boundary problems, $w_{yy} \geq 0$ on the free boundary $x = g(y)$, $y > 0$. Thus by the maximum principle (for the unbounded domain), we obtain, $w_{yy} \geq 0$ in Ω. In particular, $-(w_y)_y \leq 0$ on the x-axis. It follows by the maximum principle that $w_y \leq 0$ in Ω. Combining these estimates and the equation, (5.2) follows, which also implies that the free boundary is monotone increasing and $\tilde{g}(0) = 0$. The analyticity of $\tilde{g}$ is the same as before. $\square$

6. Time dependent problem. Now we go back to the time dependent problem (1.1)–(1.5). One can use the Duvaut's transform integrate either in t or in x direction. If we integrate the equation in t, we get a quasi-variational inequality with the equation itself involving the derivative of the free boundary on the right-hand-side. So, we will integrate in x direction, as before. Naturally, the burnt region should be growing. We also assume that the free boundary is given by $x = g(y,t)$, and $g(y,t)$ is differentiable. Let

$$\begin{aligned} w &= \int_{g(y,t)}^{x} u(\sigma,y,t)\,d\sigma \qquad \text{for } x > g(y,t) \\ &= 0 \qquad \text{for } -K \leq x \leq g(y,t). \end{aligned} \tag{6.1}$$

Assume that $0 \le u(x,y,0) = u_0(x,y) \le 1$ and set $Q_T = R_K \times [0,T]$. Then $w \in C^1(Q_T)$ and satisfies

$$w_t - \varepsilon w_{xx} - w_{yy} \ge -1 \quad \text{in } Q_T, \tag{6.2}$$

$$w \ge 0 \quad \text{in } Q_T, \tag{6.3}$$

$$w\,(w_t - \varepsilon w_{xx} - w_{yy} + 1) = 0 \quad \text{in } Q_T, \tag{6.4}$$

$$0 \le w(x,y,t) \le x + K \quad \text{in } Q_T, \tag{6.5}$$

where $R_K \equiv \{(x,y);\ -K < x < \infty, 0 < y < \infty\}$, with the initial and boundary conditions:

$$w(x,y,0) = w_0(x,y) \quad \text{for } (x,y) \in R_K, \tag{6.6}$$

$$\begin{cases} w(x,0,t) = p\Big(x - g(0,t)\Big) & \text{for } x \in R, t \in [0,T], \\ w(-K,y,t) = 0 \quad \text{for } y > 0,\ t \in [0,T] \end{cases} \tag{6.7}$$

$$g(y,0) = g_0(y), \tag{6.8}$$

where $p(x) = x^+$. However, like the travelling wave solution case in §2, the existence of a solution will very unlikely be valid if $p(x) = x^+$. We shall make a little modification by assuming that the oxidizer concentration is continuous, namely, we assume that $p \in C^{1,1}$ and

$$p(x) \begin{cases} = 0 & \text{for } x \le 0 \\ > 0 & \text{for } x > 0 \end{cases}, \quad p(x) = x - \frac{1}{2} \quad \text{for } x \ge 1, \quad p''(x) \ge 0\ . \tag{6.9}$$

We expect that if $K = K_T$ is large enough, then $g(y,t) > -K$ for $0 \le t \le T$. From the equation (6.1),

$$w_0(x,y) = \int_{g_0(y)}^{x} u_0(\sigma,y)d\sigma. \tag{6.10}$$

For this variational inequality, the solution is not unique. In fact, there are infinitely many solutions, as we shall see below.

We assume that

$$g_0 \in C^2[0,\infty), \quad g_0(0+) = 0, \quad \text{for } y > 0, \tag{H1}$$

that u_0 satisfies

$$\begin{cases} u_0 \in C^\infty(\{x \ge g_0(y)\} \setminus (0,0)), \quad 0 \le u_0(x,y) \le 1 \quad \text{in } \{x \ge g_0(y)\}, \\ \text{and the compatibility conditions:} \\ \qquad u_0(g_0(y),y) = 0, \quad u_0(x,0) = p'(x), \\ \qquad -1 = g_0'(y)\dfrac{\partial u_0}{\partial y}(g_0(y),y) - \varepsilon\dfrac{\partial u_0}{\partial x}(g_0(y),y) \quad , \end{cases} \tag{H2}$$

that ε satisfies

(H3)
$$\varepsilon \leq 1/\|p''\|_{L^\infty} ,$$

and that

(H4)
$$\begin{cases} \|u_0\|_{W^{2,\infty}(\{x>g_0(y)\})} < \infty, \quad \dfrac{\partial u_0}{\partial y}(x,y) \leq 0, \\ 0 \leq \varepsilon \dfrac{\partial^2 u_0}{\partial x^2}(x,y) + \dfrac{\partial^2 u_0}{\partial y^2}(x,y) \leq C^* \quad \text{in } \{x > g_0(y)\}. \end{cases}$$

Existence. Let

$$\mathcal{M} = \{\eta \in C^1[0,T];\ \eta(0) = 0, \eta'(t) \leq 0 \quad \text{for } t \in [0,T]\} . \tag{6.11}$$

THEOREM 6.1. *Let the assumptions (6.9) and* (H1)–(H4) *be in force. Then for each $\eta \in \mathcal{M}$, there exists a unique solution (w,g) to the system (6.2)–(6.8), with $w \in W^{2,1,\infty}_{loc}(Q_T) \cap C(\overline{Q}_T)$, $g(y,t)$ is monotone decreasing in t, monotone increasing in y and $g(0+,t) = \eta(t)$.*

Proof. The uniqueness proof is essentially the same as in Theorem 2.1. One can establish that solution $w \in W^{2,1,n+1}_{loc}(Q_T) \cap C(\overline{Q}_T)$ ($n = 2$ here) which is bounded by a polynomial at infinity is unique. In fact, One can use the change of variable $w = e^{-\alpha t} e^{x+y} \tilde{w}$ to overcome the difficulties at ∞, and apply a parabolic version of the maximum principle for functions in $W^{2,1,n+1}$ [7, Theorem 3.1].

For the existence, we can approximate the domain Q_T with $Q_{T,R} = \Omega_R \times [0,T]$ where $\Omega_R = \{(x,y);\ -K < x < R, 0 < y < R\}$. Take functions $\beta_\delta \in C^\infty$ such that

$$\begin{gathered} \beta'_\delta(s) \geq 0, \quad \beta''_\delta(s) \geq 0 \quad \text{for all} \quad s \in (-\infty,\infty), \\ \beta_\delta(s) = 0, \quad \text{for} \quad s < -\delta, \quad \beta_\delta(0) = 1, \\ \beta_\delta(s) \to \infty \quad \text{as } \delta \to 0 \text{ for each } s > 0, \end{gathered} \tag{6.12}$$

and consider the corresponding penalization problem on $Q_{T,R}$:

$$w_t - \varepsilon w_{xx} - w_{yy} = -1 + \beta_\delta(-w) \quad \text{in } Q_{T,R}, \tag{6.13}$$

with boundary and initial conditions:

$$w(x,y,t) = p\left(x - \eta(t)\right)\left(1 - \frac{y}{R}\right) \quad \text{on } \partial\Omega_R \times [0,T], \tag{6.14}$$

$$w(x,y,0) = w_0(x,y) \quad \text{in } \Omega_R. \tag{6.15}$$

We denote the solution by $w_{R,\delta}(x,y,t)$. By (H2) and (H4),

$$\frac{\partial w_{R,\delta}}{\partial t}(x,y,0) \begin{cases} \geq \varepsilon \dfrac{\partial^2 w_0}{\partial x^2} + \dfrac{\partial^2 w_0}{\partial y^2} - 1 \geq 0 \quad \text{for } x > g_0(y), y > 0, \\ = -1 + \beta_\delta(0) = 0 \quad \text{for } x \leq g_0(y), y > 0, \end{cases}$$

It is obvious that

$$\frac{\partial w_{R,\delta}}{\partial t}(x,y,t) \geq 0 \quad \text{on } \partial\Omega_R \times [0,T] \, .$$

Differentiating the equation in t and applying the maximum principle, we get

$$\frac{\partial w_{R,\delta}}{\partial t} \geq 0 \quad \text{in } Q_T, \tag{6.16}$$

which also implies that $w_{R,\delta} \geq 0$. Similarly, we have

$$\frac{\partial w_{R,\delta}}{\partial t}(x,y,0) \begin{cases} \leq \varepsilon \dfrac{\partial^2 w_0}{\partial x^2} + \dfrac{\partial^2 w_0}{\partial y^2} \leq 1 + C^*(x+K) & \text{for } x > g_0(y), y > 0, \\ = -1 + \beta_\delta(0) = 0 & \text{for } x \leq g_0(y), y > 0, \end{cases}$$

and, notice that $0 \leq p'(x) \leq 1$,

$$\frac{\partial w_{R,\delta}}{\partial t}(x,y,t) \leq \|\eta'\|_{L^\infty[0,T]} \quad \text{on } \partial\Omega_R \times [0,T] \, .$$

It follows that

$$\frac{\partial w_{R,\delta}}{\partial t} \leq \max\left(\|\eta'\|_{L^\infty[0,T]}, 1\right) + C^*(x+K) \quad \text{in } Q_T, \tag{6.17}$$

by the maximum principle. Differentiating the equation in x twice and applying the maximum principle, we obtain

$$\frac{\partial^2 w_{R,\delta}}{\partial x^2} \geq -\left\|\left((u_0)_x\right)^-\right\|_{L^\infty(\{x>g_0(y)\})} \quad \text{in } Q_T. \tag{6.18}$$

Similarly,

$$\frac{\partial^2 w_{R,\delta}}{\partial y^2} \geq \min\left(-1, -\left\|\left(1-\varepsilon(u_0)_x\right)^-\right\|_{L^\infty(\{x>g_0(y)\})}\right) \quad \text{in } Q_T \, , \tag{6.19}$$

where (H2) and (H4) are used. Now using the equation we find that $(w_{R,\delta})_{xx}$ and $(w_{R,\delta})_{yy}$ are also bounded from above by $C(x+K)$. Now we can use the compactness to pass the limit for $R \to \infty$ and $\delta \to 0$ along a subsequence, and a solution w is therefore obtained.

Using (H3) and the fact that $\eta'(t) \geq 0$, we find that $p(x-\eta(t))$ is a supersolution. It follows that $w(x,y,t) \leq p(x-\eta(t))$ in Q_T, and thus $w_y(x,0,t) \leq 0$. By the maximum principle, $w_y \leq 0$ in Q_T. Similarly, $w_x(x,0,t) = p'(x-\eta(t)) \geq 0$, and so $w_x \geq 0$ in Q_T, by the maximum principle. We established:

$$w_t \geq 0, \quad w_x \geq 0, \quad w_y \leq 0 \quad \text{in } Q_T. \tag{6.20}$$

These inequalities implies that the free boundary is a graph $x = g(y,t)$, $g(y,t)$ is monotone decreasing in t, monotone increasing in y and $g(0+,t) = \eta(t)$. Therefore we can take $K > \|\eta\|_{L^\infty[0,T]}$, and the free boundary will not touch $x = -K$. $\square$

Theorem 6.1 asserts the existence of a solution. The solution is uniquely determined by the propagation speed of the reaction front on the x-axis $(g(0,t),0)$. Next, we shall study the asymptotic behavior of the solution when $t \to \infty$. We first study the case $g(0,t) \equiv -t$. In this case the solution should go to the travelling wave solution, as we shall show below.

LEMMA 6.2. *Let the assumptions of Theorem 6.1 be in force, if*

$$g(0,t) \equiv -t, \tag{6.21}$$

then

$$\lim_{t\to\infty}(g(y,t)+t) = \tilde{g}(y), \tag{6.22}$$

where we denote by $(\tilde{w},\tilde{g})$ the solution of the elliptic variational inequality in Lemma 5.1 corresponding to the boundary condition $p(x)$.

Proof. Let $\varphi(x,y,t) = w(x-t,y,t)$, then φ satisfies the variational inequality:

$$\begin{aligned}
\varphi_t - \varepsilon\varphi_{xx} - \varphi_{yy} + \varphi_x \geq -1 \quad & \text{for } 0 < x < \infty, 0 < y < \infty, t > 0 \\
\varphi \geq 0 \quad & \text{for } 0 < x < \infty, 0 < y < \infty, t > 0 \\
\varphi \cdot (\varphi_t - \varepsilon\varphi_{xx} - \varphi_{yy} + \varphi_x + 1) = 0 \quad & \text{for } 0 < x < \infty, 0 < y < \infty, t > 0 \\
\varphi(x,y,0) = w_0(x,y) \quad & \text{for } 0 < x < \infty, 0 < y < \infty, \\
\varphi(x,0,t) = p(x) \quad & \text{for } 0 < x < \infty, t > 0 \\
\varphi(0,y,t) = 0 \quad & \text{for } 0 < y < \infty, t > 0 \\
0 \leq \varphi(x,y,t) \leq x \quad & \text{for } 0 < x < \infty, 0 < y < \infty, t > 0,
\end{aligned}$$

with its free boundary given by $x = g(y,t) + t$. Like the system in Theorem 6.1, this system can also be approximated by

$$\varphi_t - \varepsilon\varphi_{xx} - \varphi_{yy} + \varphi_x = -1 + \beta_\delta(-\varphi) \quad \text{for } 0 < x < R, 0 < y < R, t > 0, \tag{6.23}$$

with the initial and boundary conditions:

$$\begin{aligned}
\varphi(x,y,0) = w_0(x,y) \quad & \text{for } 0 < x < \infty, 0 < y < \infty, \\
\varphi(x,y,t) = p(x)\left(1 - \frac{y}{R}\right) \quad & \text{for } (x,y) \in \partial([0,R]\times[0,R]), t > 0.
\end{aligned}$$

Denote the solution by $\varphi_{R,\delta}$, then by (H2) and (H4),

$$\begin{aligned}
\left|\frac{\partial\varphi_{R,\delta}}{\partial t}(x,y,0)\right| &= |-1 + \beta_\delta(-w_0) + \varepsilon(w_0)_{xx} + (w_0)_{yy} - (w_0)_x| \\
&\leq 1 + C^* x \quad \text{for } 0 \leq x \leq g_0(y), y > 0.
\end{aligned} \tag{6.24}$$

Next, we fix $\lambda > 0$ to be small enough such that $\lambda - \varepsilon\lambda^2 > \lambda/2$ and set

$$\psi = \left(t + \frac{2}{\lambda}\right) e^{-\lambda x} \frac{\partial \varphi_{R,\delta}}{\partial t} .$$

Then

$$\psi_t - \varepsilon\psi_{xx} - \psi_{yy} + (1 - 2\varepsilon\lambda)\psi_x + \beta'_\delta \cdot \psi + \left(\lambda - \varepsilon\lambda^2 - \frac{1}{t + 2/\lambda}\right)\psi = 0 .$$

By (6.24),

$$|\psi(x, y, 0)| \leq \frac{2}{\lambda}(1 + C^* x)e^{-\lambda x} \leq \tilde{C}.$$

From the boundary conditions,

$$\psi = 0 \quad \text{on } (\partial([0, R] \times [0, R])) \times [0, \infty) .$$

Thus by the maximum principle, $|\psi| \leq \tilde{C}$ for $0 < x < R$, $0 < y < R$, $t > 0$. Hence, after taking the limit as $R \to \infty$ and $\delta \to 0$, we obtain,

$$\left|\frac{\partial \varphi}{\partial t}\right| \leq \tilde{C} e^{\lambda x} \frac{1}{t + 2/\lambda} \quad \text{for } x > 0, y > 0, t > 0. \tag{6.25}$$

We can derive estimates for φ_{xx} and φ_{yy} in any bounded domain in x–y plane, as in Theorem 6.1. The estimates is independent of t. Thus, there exists a subsequence $\{t_j\}$ such that $\lim_{t_j \to \infty} \varphi(x, y, t_j) = h(x, y)$. Recalling (6.25), we derive that $h(x, y)$ satisfies the elliptic variational inequality in §5. By uniqueness, we conclude that $h(x, y) \equiv \tilde{w}(x, y)$. Since the limit is independent of the choice of the sequence $\{t_j\}$, the limit as $t \to \infty$ exists, i.e.,

$$\lim_{t \to \infty} \varphi(x, y, t) = \tilde{w}(x, y) . \tag{6.26}$$

Noticing that $\tilde{w}(x, y) > 0$ for $x > \tilde{g}(y)$, we derive

$$\limsup_{t \to \infty} (g(y, t) + t) \leq \tilde{g}(y). \tag{6.27}$$

By the nondegeneracy lemma (a parabolic version nondegeneracy lemma similar to Lemma 3.1 of [3, page 154]), $\varphi(x, y, t)$ can not be uniformly small in a neighborhood of any point (x_0, y_0, t_0) such that $\varphi(x_0, y_0, t_0) > 0$. It follows that

$$\liminf_{t \to \infty} (g(y, t) + t) \geq \tilde{g}(y), \tag{6.28}$$

and the lemma follows. □

COROLLARY 6.3. *Let the assumptions of Theorem 6.1 be in force, if*

$$\lim_{t\to\infty}(t+g(0,t))=0\ , \tag{6.29}$$

then

$$\lim_{t\to\infty}(g(y,t)+t)=\tilde{g}(y). \tag{6.30}$$

Proof. We will take T large enough so that $|t+g(0,t)|\leq\delta$ for $t>T$. Then we use the comparison function with $g(0,t)$ replaced by $-t\pm\delta$. Applying Lemma 6.2, we immediately obtain

$$\limsup_{t\to\infty}|g(y,t)+t-\tilde{g}(y)|\leq\delta. \tag{6.31}$$

Letting $\delta\to 0$, we conclude. □

REFERENCES

[1] J. ADLER AND D. M. HERBERT, *Diffusion Controlled Smoulder Propagation Parallel to a Plane Surface,* SIAM J. Appl. Math. Vol. 52, No. 1 (1992), pp. 155 – 162.

[2] H. BERESTYCKI, B. LARROUTUROU AND P. L. LIONS, *Multi-Dimensional Travelling Wave Solutions of a Flame Propagation model,* Arch. Rat. Mech. Anal., Vol. 111, No. 1 (1990), pp. 33 – 49.

[3] A. FRIEDMAN, *Variational Principles and Free Boundary Problems,* Wiley and Sons, New York, 1982.

[4] D. GILBARG AND N. S. TRUDINGER, *Elliptic Partial Differential Equation of Second Order,* 2nd edition, Springer-Verlag, New York, 1983.

[5] P. L. LIONS, *Remark on Bony Maximum Principle,* AMS Proc., 88 (1983), pp 503 – 508.

[6] N. F. SMYTH, *Propagation of Flame Fronts,* J. Austral. Math. Soc., Ser. B, Vol. 31, No 4, (1990), pp. 385 – 396.

[7] K. S. TSO, *On an Aleksandrov-Bakelman Type Maximum Principle for Second Order Parabolic Equations,* Comm. PDE, 10 (1985), pp. 543 – 553.

Recent IMA Preprints

#	Author/s	Title

892 **E.G. Kalnins, Willard Miller, Jr. and Sanchita Mukherjee**, Models of q-algebra representations: the group of plane motions
893 **T.R. Hoffend Jr. and R.K. Kaul**, Relativistic theory of superpotentials for a nonhomogeneous, spatially isotropic medium
894 **Reinhold von Schwerin**, Two metal deposition on a microdisk electrode
895 **Vladimir I. Oliker and Nina N. Uraltseva**, Evolution of nonparametric surfaces with speed depending on curvature, III. Some remarks on mean curvature and anisotropic flows
896 **Wayne Barrett, Charles R. Johnson, Raphael Loewy and Tamir Shalom**, Rank incrementation via diagonal perturbations
898 **Mingxiang Chen, Xu-Yan Chen and Jack K. Hale**, Structural stability for time-periodic one-dimensional parabolic equations
899 **Hong-Ming Yin**, Global solutions of Maxwell's equations in an electromagnetic field with the temperature-dependent electrical conductivity
900 **Robert Grone, Russell Merris and William Watkins**, Laplacian unimodular equivalence of graphs
901 **Miroslav Fiedler**, Structure-ranks of matrices
902 **Miroslav Fiedler**, An estimate for the nonstochastic eigenvalues of doubly stochastic matrices
903 **Miroslav Fiedler**, Remarks on eigenvalues of Hankel matrices
904 **Charles R. Johnson, D.D. Olesky, Michael Tsatsomeros and P. van den Driessche**, Spectra with positive elementary symmetric functions
905 **Pierre-Alain Gremaud**, Thermal contraction as a free boundary problem
906 **K.L. Cooke, Janos Turi and Gregg Turner**, Stabilization of hybrid systems in the presence of feedback delays
907 **Robert P. Gilbert and Yongzhi Xu**, A numerical transmutation approach for underwater sound propagation
908 **LeRoy B. Beasley, Richard A. Brualdi and Bryan L. Shader**, Combinatorial orthogonality
909 **Richard A. Brualdi and Bryan L. Shader**, Strong hall matrices
910 **Håkan Wennerström and David M. Anderson**, Difference versus Gaussian curvature energies; monolayer versus bilayer curvature energies applications to vesicle stability
911 **Shmuel Friedland**, Eigenvalues of almost skew symmetric matrices and tournament matrices
912 **Avner Friedman, Bei Hu and J.L. Velazquez**, A Free Boundary Problem Modeling Loop Dislocations in Crystals
913 **Ezio Venturino**, The Influence of Diseases on Lotka-Volterra Systems
914 **Steve Kirkland and Bryan L. Shader**, On Multipartite Tournament Matrices with Constant Team Size
915 **Richard A. Brualdi and Jennifer J.Q. Massey**, More on Structure-Ranks of Matrices
916 **Douglas B. Meade**, Qualitative Analysis of an Epidemic Model with Directed Dispersion
917 **Kazuo Murota**, Mixed Matrices Irreducibility and Decomposition
918 **Richard A. Brualdi and Jennifer J.Q. Massey**, Some Applications of Elementary Linear Algebra in Combinations
919 **Carl D. Meyer**, Sensitivity of Markov Chains
920 **Hong-Ming Yin**, Weak and Classical Solutions of Some Nonlinear Volterra Integrodifferential Equations
921 **B. Leinkuhler and A. Ruehli**, Exploiting Symmetry and Regularity in Waveform Relaxation Convergence Estimation
922 **Xinfu Chen and Charles M. Elliott**, Asymptotics for a Parabolic Double Obstacle Problem
923 **Yongzhi Xu and Yi Yan**, An Approximate Boundary Integral Method for Acoustic Scattering in Shallow Oceans
924 **Yongzhi Xu and Yi Yan**, Source Localization Processing in Perturbed Waveguides
925 **Kenneth L. Cooke and Janos Turi**, Stability, Instability in Delay Equations Modeling Human Respiration
926 **F. Bethuel, H. Brezis, B.D. Coleman and F. Hélein**, Bifurcation Analysis of Minimizing Harmonic Maps Describing the Equilibrium of Nematic Phases Between Cylinders
927 **Frank W. Elliott, Jr.**, Signed Random Measures: Stochastic Order and Kolmogorov Consistency Conditions
928 **D.A. Gregory, S.J. Kirkland and B.L. Shader**, Pick's Inequality and Tournaments
929 **J.W. Demmel, N.J. Higham and R.S. Schreiber**, Block LU Factorization
930 **Victor A. Galaktionov and Juan L. Vazquez**, Regional Blow-Up in a Semilinear Heat Equation with Convergence to a Hamilton-Jacobi Equation
931 **Bryan L. Shader**, Convertible, Nearly Decomposable and Nearly Reducible Matrices
932 **Dianne P. O'Leary**, Iterative Methods for Finding the Stationary Vector for Markov Chains
933 **Nicholas J. Higham**, Perturbation theory and backward error for $AX - XB = C$
934 **Z. Strakos and A. Greenbaum**, Open questions in the convergence analysis of the lanczos process for the real symmetric eigenvalue problem
935 **Zhaojun Bai**, Error analysis of the lanczos algorithm for the nonsymmetric eigenvalue problem
936 **Pierre-Alain Gremaud**, On an elliptic-parabolic problem related to phase transitions in shape memory alloys
937 **Bojan Mohar and Neil Robertson**, Disjoint essential circuits in toroidal maps

939 **Bojan Mohar and Svatopluk Poljak** Eigenvalues in combinatorial optimization
940 **Richard A. Brualdi, Keith L. Chavey and Bryan L. Shader**, Conditional sign-solvability
941 **Roger Fosdick and Ying Zhang**, The torsion problem for a nonconvex stored energy function
942 **René Ferland and Gaston Giroux**, An unbounded mean-field intensity model: Propagation of the convergence of the empirical laws and compactness of the fluctuations
943 **Wei-Ming Ni and Izumi Takagi**, Spike-layers in semilinear elliptic singular Perturbation Problems
944 **Henk A. Van der Vorst and Gerard G.L. Sleijpen**, The effect of incomplete decomposition preconditioning on the convergence of conjugate gradients
945 **S.P. Hastings and L.A. Peletier**, On the decay of turbulent bursts
946 **Apostolos Hadjidimos and Robert J. Plemmons**, Analysis of p-cyclic iterations for Markov chains
947 **ÅBjörck, H. Park and L. Eldén**, Accurate downdating of least squares solutions
948 **E.G. Kalnins, Willard Miller, Jr. and G.C. Williams**, Recent advances in the use of separation of variables methods in general relativity
949 **G.W. Stewart**, On the perturbation of LU, Cholesky and QR factorizations
950 **G.W. Stewart**, Gaussian elimination, perturbation theory and Markov chains
951 **G.W. Stewart**, On a new way of solving the linear equations that arise in the method of least squares
952 **G.W. Stewart**, On the early history of the singular value decomposition
953 **G.W. STewart**, On the perturbation of Markov chains with nearly transient states
954 **Umberto Mosco**, Composite media and asymptotic dirichlet forms
955 **Walter F. Mascarenhas**, The structure of the eigenvectors of sparse matrices
956 **Walter F. Mascarenhas**, A note on Jacobi being more accurate than QR
957 **Raymond H. Chan, James G. Nagy and Robert J. Plemmons**, FFT-based preconditioners for Toeplitz-Block least squares problems
958 **Zhaojun Bai**, The CSD, GSVD, their applications and computations
959 **D.A. Gregory, S.J. Kirkland and N.J. Pullman**, A bound on the exponent of a primitive matrix using Boolean rank
960 **Richard A. Brualdi, Shmuel Friedland and Alex Pothen**, Sparse bases, elementary vectors and nonzero minors of compound matrices
961 **J.W. Demmel**, Open problems in numerical linear algebra
962 **James W. Demmel and William Gragg**, On computing accurate singular values and eigenvalues of acyclic matrices
963 **James W. Demmel**, The inherent inaccuracy of implicit tridiagonal QR
964 **J.J.L. Velázquez**, Estimates on the $(N-1)$-dimensional Hausdorff measure of the blow-up set for a semilinear heat equation
965 **David C. Dobson**, Optimal design of periodic antireflective structures for the Helmholtz equation
966 **C.J. van Duijn and Joseph D. Fehribach**, Analysis of planar model for the molten carbonate fuel cell
967 **Yongzhi Xu, T. Craig Poling and Trent Brundage**, Source localization in a waveguide with unknown large inclusions
968 **J.J.L. Velázquez**, Higher dimensional blow up for semilinear parabolic equations
969 **E.G. Kalnins and Willard Miller, Jr.**, Separable coordinates, integrability and the Niven equations
970 **John M. Chadam and Hong-Ming Yin**, A diffusion equation with localized chemical reactions
971 **A. Greenbaum and L. Gurvits**, Max-min properties of matrix factor norms
972 **Bei Hu**, A free boundary problem arising in smoulder combustion
973 **C.M. Elliott and A.M. Stuart**, The global dynamics of discrete semilinear parabolic equations
974 **Avner Friedman and Jianhua Zhang**, Swelling of a rubber ball in the presence of good solvent
975 **Avner Friedman and Juan J.L. Velázquez**, A time-dependence free boundary problem modeling the visual image in electrophotography
976 **Richard A. Brualdi, Hyung Chan Jung and William T. Trotter, Jr.**, On the poset of all posets on n elements
977 **Ricardo D. Fierro and James R. Bunch**, Multicollinearity and total least squares
978 **Adam W. Bojanczyk, James G. Nagy and Robert J. Plemmons**, Row householder transformations for rank-k Cholesky inverse modifications
979 **Chaocheng Huang**, An age-dependent population model with nonlinear diffusion in R^n
980 **Emad Fatemi and Faroukh Odeh**, Upwind finite difference solution of Boltzmann equation applied to electron transport in semiconductor devices
981 **Esmond G. Ng and Barry W. Peyton**, A tight and explicit representation of Q in sparse QR factorization
982 **Robert J. Plemmons**, A proposal for FFT-based fast recursive least-squares
983 **Anne Greenbaum and Zdenek Strakos**, Matrices that generate the same Krylov residual spaces
984 **Alan Edelman and G.W. Stewart**, Scaling for orthogonality
985 **G.W. Stewart**, Note on a generalized sylvester equation
986 **G.W. Stewart**, Updating URV decompositions in parallel