ON THE SPREAD OF CONTINUOUS–TIME LINEAR SYSTEMS

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IMA Preprint Series # 412

April 1988

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Abstract. Given the impulse response h of a linear time invariant system, we consider signals $y = h * u$ with inputs u subject to $|u(t)| \le 1$ and ask: for a given $\tau > 0$ and $y(t_0)$, what is the set of all the possible values (the "spread") of $y(t_0 + \tau)$? We characterize this set, study its properties, and compute it for some functions h.

§1. Basic definitions and results. Let $h(t)$ be a prescribed continuous function defined for $0 \le t < \infty$ and belonging to $L^1(0, \infty)$; we refer to it as a *an impulse response*. Let $u(t)$ be any measurable function for $0 \le t < \infty$ satisfying $|u(t)| \le 1$; we refer to it as an *input*. The function $y(t)$, defined

$$y(t) = \int_0^t h(t-s)u(s)\, ds \ ,$$

is called the *output* or the *signal*.

Given $\alpha \in \mathbf{R}$, $t_0 \ge 0$, $\tau > 0$, we would like to estimate the range of the output $y(t)$ at time $t = t_0 + \tau$, given that $y(t_0) = \alpha$. More quantitatively, we wish to bound the numbers

$$\tilde{\sigma}^+(\alpha, \tau, t_0) = \sup_u \{y(t_0 + \tau); \quad \text{given} \quad y(t_0) = \alpha\},$$
$$\tilde{\sigma}^-(\alpha, \tau, t_0) = \inf_u \{y(t_0 + \tau); \quad \text{given} \quad y(t_0) = \alpha\}.$$

Introduce the class of control functions

$$K_{\tau,\alpha} = \{u \in L^\infty(-\infty, \tau), \quad -1 \le u(s) \le 1, \int_{-\infty}^0 h(-s)u(s)\, ds = \alpha\} \tag{1.1}$$

and the functional

$$J_\tau(u) = \int_{-\infty}^{\tau} h(\tau - s)u(s)\, ds, \tag{1.2}$$

*This work is partially supported by National Science Foundation Grant DMS–86–12880 and Air Force Grant AFOSR–86–0134

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and define

$$\sigma^+(\tau,\alpha) = \sup_{u\in K_{\tau,\alpha}} J_\tau(u) \ , \tag{1.3}$$

$$\sigma^-(\tau,\alpha) = \inf_{u\in K_{\tau,\alpha}} J_\tau(u) \ . \tag{1.4}$$

Definition 1.1. The function $\sigma(\tau,\alpha) = \sigma^+(\tau,\alpha) - \sigma^-(\tau,\alpha)$ is called the *spread of the linear system.*

The motivation comes from the following theorem.

THEOREM 1.1. *For any $\alpha \in \mathbf{R}, \tau > 0$,*

$$\sup_{t_0\geq 0} \tilde\sigma^+(\tau,\alpha,t_0) = \sigma^+(\tau,\alpha), \tag{1.5}$$

$$\inf_{t_0\geq 0} \tilde\sigma^-(\tau,\alpha,t_0) = \sigma^-(\tau,\alpha) \tag{1.6}$$

and, consequently,

$$\sup_{t_0\geq 0} \tilde\sigma^+(\tau,\alpha,t_0) - \inf_{t_0\geq 0} \tilde\sigma^-(\tau,\alpha,t_0) = \sigma(\tau,\alpha) \ . \tag{1.7}$$

Proof. The condition $y(t_0) = \alpha$ means that

$$\int_0^{t_0} h(t_0 - s)u(s)\ ds = \alpha. \tag{1.8}$$

Writing

$$y(t_0+\tau) = \int_0^{t_0+\tau} h(t_0+\tau-s)u(s)\ ds$$

and substituting $t_0 - s = s', \quad u(t_0 - s') = v(s')$ we get

$$y(t_0+\tau) = \int_{-t_0}^{\tau} h(\tau - s')v(s')\ ds' \ .$$

The same substitution applied to (1.8) gives

$$\int_{-t_0}^{0} h(-s')v(s')\ ds' = \alpha.$$

Hence

$$\tilde{\sigma}^+(\tau,\alpha,t_0) = \sup\left\{ \int_{-t_0}^{\tau} h(\tau - s')v(s')\, ds' \ ; \ v \quad \text{satisfies} \right.$$
$$\left. |v(s')| \le 1, \qquad \int_{-t_0}^{0} h(-s')v(s')\, ds' = \alpha \right\}.$$

Extending $v(s')$ to $s' < -t_0$ by zero we see that $\sigma^+(\tau,\alpha,t_0)$ is $\sup J_\tau(v)$ when v is restricted to a subset say K_{τ,α,t_0}, of $K_{\tau,\alpha}$; hence

$$\tilde{\sigma}^+(\tau,\alpha,t_0) \le \sigma^+(\tau,\alpha).$$

As $t_0 \to \infty$ the subsets K_{τ,α,t_0} increase and every $u \in K_{\tau,\alpha}$ restricted to a bounded interval is a function in $\bigcup_{t_0>0} K_{\tau,\alpha,t_0}$ restricted to the same interval; this implies the equality in (1.5). The proof of (1.6) is similar.

THEOREM 1.2. *For any $\alpha \in \mathbf{R}$, $\tau > 0$ there exist admissible functions $u^+_{\tau,\alpha}$, $u^-_{\tau,\alpha}$ in $K_{\tau,\alpha}$ such that*

$$J_\tau(u^+_{\tau,\alpha}) = \sup_{u\in K_{\tau,\alpha}} J_\tau(u) = \sigma^+(\tau,\alpha), \tag{1.9}$$

$$J_\tau(u^-_{\tau,\alpha}) = \inf_{u\in K_{\tau,\alpha}} J_\tau(u) = \sigma^-(\tau,\alpha). \tag{1.10}$$

Indeed, taking a maximizing sequence u_j, we can extract a subsequence which is weakly convergent in L^1_{loc} to a function u_0. It is easy to check that u_0 is a maximizer for J_τ, i.e., u_0 is the asserted $u^+_{\tau,\alpha}$. The proof of (1.10) is similar.

In this paper we study the structure of $u^\pm_{\tau,\alpha}$ and this enables us to compute the spread of some linear systems of interest. In §2 we solve a general maximization problem, which is then used (in§3) to analyze the structure of $u^\pm_{\tau,\alpha}$. In §4 we establish various properties of $\sigma(\tau,\alpha)$ and, in §5, we compute $\sigma(\tau,\alpha)$ for some examples. Finally in §6 we show that all the results can be extended to the case where $y(t_0),\ y(t_0+\tau_1),\dots,y(t_0+\tau_{N-1})$ are prescribed and the range of $y(t_0+\tau_N)$ is sought; here $0<\tau_1<\tau_2<\cdots<\tau_N$.

Motivation for studying the function $\sigma^\pm(\tau,\alpha)$ comes from the following problem posed in [1] and [2]. For any $d>0$, $T>0$ and impulse response $h(\cdot)$, denote by $N_{\max}(T,d)$ the maximum number of inputs $u_j(s)$ such that the corresponding outputs $y_j(t)$ satisfy:

$$\max_{0<t\le T} |y_i(t) - y_j(t)| \ge d \qquad \forall\ i \ne j\ .$$

We define

$$(1.11) \qquad MCT(d) = \lim_{T\to\infty} \frac{\log\ N_{\max}(T,d)}{T} \quad \text{bits/sec},$$

and would like to obtain bounds on the $MCT(d)$ for any $h(\cdot)$. Set

$$\tau^* = \inf\{\tau| \quad \sigma(\tau,0) = d\}.$$

Work in progress [3] indicates that

$$(1.12) \qquad MCT(d) \le \frac{1}{\tau^*}$$

for any $h(\cdot)$ that satisfies

$$\int_{\{\frac{h(\tau-s)}{h(-s)} \ge 1\}} |h(-s)|\ ds \le \int_{\{\frac{h(\tau-s)}{h(-s)} \le 1\}} |h(-s)|\ ds$$

for all $\tau \le \tau^*$; the arguments used depend on results derived in this paper. Results obtained here (in §6) for the N contraint problem, in which N output values are specified, can be used to tighten the upper bound given by (1.12) (see [3]).

The problem of computing spread for a discrete–time linear system with impulse response h_i, $i = 0, 1, 2, \dots,$ is considered in [2]. This computation is equivalent to solving a linear program with bounded variables and one equality constraint. Here we show how the spread can be computed for a continuous–time linear system. Two examples of special interest are presented in which the spread can be computed by finding a solution to a transcendental equation.

A general optimization problem. Let $f(s)$, $g(s)$ be continuous functions in $\infty < s \le 0$, which belong to $L^1(-\infty, 0)$, and assume:

$$(2.1) \qquad f \ne 0 \quad \text{a.e.},$$

$$(2.2) \qquad \text{meas}\ \{\frac{g}{f} = \mu\} = 0 \quad \text{for any} \quad \mu \in \mathbf{R}\ .$$

Let

$$K = \{u(s) \quad \text{measurable for} \quad -\infty < s < 0, \qquad |u(s)| \le 1,$$
$$\int_{-\infty}^{0} f(s)u(s)\ ds = \alpha\}$$

for some fixed $\alpha \in \mathbf{R}$, and

$$J(u) = \int_{-\infty}^{0} g(s)u(s)\, ds.$$

As in the proof of Theorem 1.2 one can show that there exists a function $u_0 \in K$ such that

$$J(u_0) = \max_{u \in K} J(u)\ . \tag{2.3}$$

THEOREM 2.1. *For any solution $u_0 \in K$ of (2.3) there exists a number $\lambda \in \mathbf{R}$ such that almost everywhere*

$$u_0(s) = \begin{cases} \operatorname{sgn}\, f(s) & \text{if} \quad \dfrac{g(s)}{f(s)} > \lambda \\ -\operatorname{sgn}\, f(s) & \text{if} \quad \dfrac{g(s)}{f(s)} < \lambda\ . \end{cases} \tag{2.4}$$

Notice that (2.4) is equivalent to

$$u_0(s) = \operatorname{sgn}\, [g(s) - \lambda f(s)]\ .$$

Proof. We begin by proving that $u_0 = 1$ a.e. If the assertion is not true then the set $G_0 = \{|u_0| < 1\}$ has positive measure. Denote by G the subset of G_0 consisting of all points t of G_0-density equal to 1, such that also $f(t) \neq 0$. Then meas $G =$ meas $G_0 > 0$.

Take t_1, t_2 in G $\quad (t_1 \neq t_2)$ and let G_i be a subset of G contained in δ_0-neighborhood of t_i, such that $\sup_{G_i} |u| < 1$,

$$G_1 \cap G_2 = \emptyset, \quad \operatorname{meas} G_1 = \operatorname{meas} G_2 = \delta > 0.$$

For any real numbers A_1, A_2, if ϵ is positive and small enough then the function

$$\tilde{u} = u_0 + A_1 \frac{\epsilon}{\delta} \chi_{G_1} + A_2 \frac{\epsilon}{\delta} \chi_{G_2} \tag{2.5}$$

satisfies $|\tilde{u}| \leq 1$. Further, if

$$A_1 \int_{G_1} f(s)\, ds + A_2 \int_{G_2} f(s)\, ds = 0 \tag{2.6}$$

then $\int_{-\infty}^{0} f(s)\tilde{u}(s)\, ds = \alpha$, so that $\tilde{u} \in K$. Notice that (2.6) is equivalent to

$$A_1 f(t_1) + A_2\, f(t_2) = \sigma_1(\delta_0) \tag{2.7}$$

for some $\sigma_1(\delta_0)$ such that $\sigma_1(\delta_0) \to 0$ if $\delta_0 \to 0$.

From the maximality of u_0 it follows that (2.6), or (2.7), implies $J(\tilde{u}) \leq J(u_0)$, that is

$$A_1 \int_{G_1} g(s)\, ds + A_2 \int_{G_2} g(s)\, ds \leq 0 \,, \tag{2.8}$$

i.e.,

$$A_1\, g(t_1) + A_2\, g(t_2) \leq \sigma_2(\delta_0) \tag{2.9}$$

for some $\sigma_2(\delta_0)$ such that $\sigma_2(\delta_0) \to 0$ if $\delta_0 \to 0$.

Choosing

$$A_1 = -A_2\, \frac{f(t_2)}{f(t_1)} + \frac{\sigma_1(\delta_0)}{f(t_1)} \tag{2.10}$$

so that (2.7) is satisfied, (2.9) must then hold and, upon letting $\delta_0 \to 0$, we get

$$A_2 \left[-\frac{g(t_1) f(t_2)}{f(t_1)} + g(t_2) \right] \leq 0 \,. \tag{2.11}$$

Since A_2 is arbitrary, it follows that the expression in brackets must vanish. Thus

$$\frac{g(t_1)}{f(t_1)} = \frac{g(t_2)}{f(t_2)}$$

for all t_1, t_2 in G. Since G has a positive measure, this is a contradiction to (2.2).

Denote by D the set of all points t such that $f(t) \neq 0$ and t is a Lebesgue point of u_0. Thus almost all t in $(-\infty, 0)$ belong to D. Take any t_1, t_2 in D with

$$\frac{g(t_1)}{f(t_1)} > \frac{g(t_2)}{f(t_2)} \,. \tag{2.12}$$

We shall prove that a.e.

$$u_0(t_2) = \operatorname{sgn} f(t_2) \quad \text{implies} \quad u_0(t_1) = \operatorname{sgn} f(t_1) \tag{2.13}$$

and

$$u_0(t_1) = -\operatorname{sgn} f(t_1) \quad \text{implies} \quad u_0(t_2) = -\operatorname{sgn} f(t_2) \,. \tag{2.14}$$

These two statements clearly imply the assertion (2.4).

To prove (2.13) suppose the assertion is not true. Then the set $\tilde{G}$ of pair (t_1, t_2) for which (2.13) is not true has positive measure. Choose t_1, t_2 at which $\tilde{G}$ has density 1. Then for any $\delta_0 > 0$ we can find sets G_1, G_2 such that G_i is contained in δ_0-neighborhood of t_i, $\operatorname{meas} G_1 = \operatorname{meas} G_2 = \delta > 0$, and

$$u_0(t) = \operatorname{sgn} f(t) \quad \text{for all} \quad t \in G_2,$$
$$u_0(t) = -\operatorname{sgn} f(t) \quad \text{for all} \quad t \in G_1 .$$

We again form the function (2.5). If

$$A_2 \operatorname{sgn} f(t_2) < 0 , \quad A_1 \operatorname{sgn} f(t_1) > 0 \tag{2.15}$$

then $|\tilde{u}| \leq 1$ if ϵ is sufficiently small.

If we can further choose A_1, A_2 such that (2.6) (or (2.7)) holds, then (2.8) (or 2.9)) must be satisfied. The condition (2.7) is satisfied by the choice (2.10) of A_1, and if $A_2 \operatorname{sgn} f(t_2) < 0$ then clearly also $A_1 \operatorname{sgn} f(t_1) > 0$ provided δ_0 is sufficiently small. We conclude, after letting $\delta_0 \to 0$, that (2.11) must hold provided $A_2 \operatorname{sgn} f(t_2) < 0$. Dividing (2.11) by $A_2 f(t_2)$ we arrive at the inequality

$$-\frac{g(t_1)}{f(t_1)} + \frac{g(t_2)}{f(t_2)} \geq 0$$

which is a contradiction to (2.12). This completes the proof of (2.13); the proof of (2.14) is similar.

From Theorem 2.1 we immediately get

COROLLARY 2.2. *The constant λ in Theorem 2.1 is uniquely determined by*

$$\int_{\{\frac{g(s)}{f(s)} > \lambda\}} |f(s)|\, ds - \int_{\{\frac{g(s)}{f(s)} < \lambda\}} |f(s)| ds = \alpha \; ; \tag{2.16}$$

consequently the maximizer u_0 is also uniquely determined. As α decreases from $\int_{-\infty}^{0} |f(s)|\, ds$ to $-\int_{-\infty}^{0} |f(s)|\, ds$, $\lambda = \lambda(\alpha)$ increases monotonically from

$$\inf_{s<0} \{g(s)/f(s)\} \quad \textit{to} \quad \sup_{s<0} \{g(s)/f(s)\}.$$

§3. The structure of $u^{\pm}_{\tau,\alpha}$. Choose $h(t)$ as in §1, i.e.,

$$h \in L^1(0, \infty) \cap C^0[0, \infty) \tag{3.1}$$

and assume further that

$$h(t) \neq 0 \quad \text{a.e.} \tag{3.2}$$

$$\operatorname{meas}\{0 < t < \infty; \; \frac{h(t+\tau)}{h(t)} = \lambda\} = 0 \qquad \forall\, \tau > 0, \; \lambda \in \mathbf{R}. \tag{3.3}$$

Taking $f(t) = h(-t)$, $\quad g(t) = h(\tau - t)$ in Theorem 2.1 and Corollary 2.2, we get:

Theorem 3.1. *There exists a unique solution $u^+_{\tau,\alpha}$ of (1.9) and it is given by*

$$u^+_{\tau,\alpha}(s) = \begin{cases} \operatorname{sgn} h(-s) & \text{if} \quad \dfrac{h(\tau-s)}{h(-s)} > \lambda^+ \\ -\operatorname{sgn} h(-s) & \text{if} \quad \dfrac{h(\tau-s)}{h(-s)} < \lambda^+ \end{cases} \tag{3.4}$$

where λ^+ is determined by

$$\int\limits_{\{\frac{h(\tau-s)}{h(-s)} > \lambda^+\}} |h(-s)|\; ds - \int\limits_{\{\frac{h(t-s)}{h(-s)} < \lambda^+\}} |h(-s)|\; ds = \alpha\,. \tag{3.5}$$

Clearly also $u^+_{\tau,\alpha}(s) = \operatorname{sgn}\; h(\tau-s)$ if $0 < s < \tau$.

We now consider a special case.

Theorem 3.2. *If $h \in L^1(0,\infty)$, $h > 0, d^2(\log h)/dt^2 > 0$, then there is a unique solution of (1.9), given by*

$$u^+_{\tau,\alpha}(s) = \begin{cases} 1 & \text{if} \quad -\infty < s < \mu, \\ -1 & \text{if} \quad \mu < s < 0 \end{cases} \tag{3.6}$$

and $u^+_{\tau,\alpha}(s) = 1$ if $0 < s < \tau$, where μ is determined by

$$\int_{-\infty}^{\mu} h(-s)\; ds - \int_{\mu}^{0} h(-s)\; ds = \alpha\,. \tag{3.7}$$

Proof. By assumption

$$\frac{h'(s)}{h(s)} \quad \text{is strictly increasing,}$$

hence

$$\frac{h'(\tau+s)}{h(\tau+s)} > \frac{h'(s)}{h(s)}\,.$$

This means that

$$\frac{d}{ds}\quad \frac{h(\tau+s)}{h(s)} > 0$$

and thus

$$\frac{h(\tau-s)}{h(-s)} \quad \text{is strictly decreasing in } s.$$

Now apply Theorem 3.1 to complete the proof.

REMARK 3.1. If $\log h$ is convex (but not satisfying $d^2(\log h)/dt^2 > 0$) then we can approximate it by a smooth function h_n with $d^2(\log h_n)/dt^2 > 0$. Applying Theorem 3.2 to the corresponding maximizers $u^h_{\tau,\alpha}$ we deduce that there is a maximizer $u_{\tau,\alpha}$ (for h) having the form (3.6), (3.7). There may be other maximizers; for instance if $h(t) = e^{-t}$ then every $u \in K_{\tau,\alpha}$ is a maximizer. (Notice that (3.3) does not hold for $h(t) = e^{-t}$.)

THEOREM 3.3. *If $h \in L^1(0,\infty), h > 0$ and $d^2(\log h)/dt^2 < 0$ then there is a unique solution of (1.9) given by*

$$u^+_{\tau,\alpha}(s) = \begin{cases} -1 & \text{if} \quad -\infty < s < \tilde{\mu} \\ 1 & \text{if} \quad \tilde{\mu} < s < 0 \end{cases} \tag{3.8}$$

and $u^+_{\tau,\alpha}(s) = 1$ if $0 < s < \tau$, where $\tilde{\mu}$ is determined by

$$- \int_{-\infty}^{\tilde{\mu}} h(-s)\, ds + \int_{\tilde{\mu}}^{0} h(-s)\, ds = \alpha\ . \tag{3.9}$$

Notice that $d\mu/d\alpha > 0$, $d\tilde{\mu}/d\alpha < 0$ where $\mu = \mu(\alpha)$ and $\tilde{\mu} = \tilde{\mu}(\alpha)$ are defined by (3.7) and (3.9) respectively.

§4. Properties of the spread.

Theorem 3.1 implies that

$$\begin{aligned}
\sigma^+(\tau,\alpha) = & \int_{\left\{\frac{h(\tau-s)}{h(-s)} > \lambda^+\right\}} [\operatorname{sgn} h(-s)] h(\tau - s)\, ds \\
& - \int_{\left\{\frac{h(\tau-s)}{h(-s)} < \lambda^+\right\}} [\operatorname{sgn} h(-s)] h(\tau - s)\, ds + \int_0^\tau |h(\tau - s)|\, ds \\
= & \int_{\left\{\frac{h(\tau-s)}{h(-s)} > \lambda^+\right\}} \frac{h(\tau - s)}{h(-s)} |h(-s)|\, ds \\
& - \int_{\left\{\frac{h(\tau-s)}{h(-s)} < \lambda^+\right\}} \frac{h(\tau - s)}{h(-s)} |h(-s)|\, ds + \int_0^\tau |h(\tau - s)|\, ds\ ,
\end{aligned} \tag{4.1}$$

where λ^+ is determined by (3.5). Similarly, one can show that

$$\begin{aligned}
\sigma^-(\tau,\alpha) = & - \int_{\left\{\frac{h(\tau-s)}{h(-s)} > \lambda^-\right\}} \frac{h(\tau - s)}{h(-s)} |h(-s)|\, ds + \int_{\left\{\frac{h(\tau-s)}{h(-s)} < \lambda^-\right\}} |h(-s)|\, ds \\
& - \int_0^\tau |h(\tau - s)|\, ds\ ,
\end{aligned} \tag{4.2}$$

where λ^- is determined by

$$(4.3) \qquad -\int_{\left\{\frac{h(\tau-s)}{h(-s)}>\lambda^-\right\}} |h(-s)|\, ds + \int_{\left\{\frac{h(\tau-s)}{h(-s)}<\lambda^-\right\}} |h(-s)|\, ds = \alpha\ .$$

As α decreases from $\int_0^\infty |h(s)|\, ds$ to $-\int_0^\infty |h(s)|\, ds, \lambda^-(\alpha)$ decreases monotonically from $\sup_{s<0}\{h(\tau-s)/h(-s)\}$ to $\inf_{s<0}\{h(t-s)/h(-s)\}$. Also, $\lambda^-(0) = \lambda^+(0)$.

Combining (4.1) and (4.2) gives the spread

(4.4)

$$\begin{aligned} \frac{1}{2}\,\sigma(\tau,\alpha) &= \frac{1}{2}\,[\sigma^+(\tau,\alpha) - \sigma^-(\tau,\alpha)] \\ &= \int_{\left\{\frac{h(\tau-s)}{h(-s)}>\lambda_M\right\}} \frac{h(\tau-s)}{h(-s)}\,|h(-s)|\, ds - \int_{\left\{\frac{h(\tau-s)}{h(-s)}<\lambda_m\right\}} \frac{h(\tau-s)}{h(-s)}\,|h(-s)|ds \\ &\qquad + \int_0^\tau |h(\tau-s)|\, ds \end{aligned}$$

where $\lambda_M = \max(\lambda^-,\lambda^+)$ and $\lambda_m = \min(\lambda^-,\lambda^+)$,

THEOREM 4.1. *There holds:*

$$(4.5) \qquad \frac{\partial\sigma^\pm(\tau,\alpha)}{\partial\alpha} = \lambda^\pm\ .$$

Proof. Since $\lambda^+(\alpha)$ is a monotonically decreasing function of α, we can write

$$(4.6) \qquad \int_{\left\{\frac{h(\tau-s)}{h(-s)}>\lambda^++\Delta\lambda\right\}} |h(-s)|\, ds - \int_{\left\{\frac{h(\tau-s)}{h(-s)}<\lambda^++\Delta\lambda\right\}} |h(-s)|\, ds = \alpha - \Delta\alpha$$

where $\Delta\lambda, \Delta\alpha > 0$. Subtracting (4.6) from (4.1) gives

$$(4.7) \qquad 2\int_{\left\{\lambda^+<\frac{h(\tau-s)}{h(-s)}<\lambda^++\Delta\lambda\right\}} |h(-s)|\, ds = \Delta\alpha\ .$$

From (4.1), (4.6), and (4.7),

$$\begin{aligned} \sigma^+(\tau,\alpha) - \sigma^+(\tau,\alpha-\Delta\alpha) &= 2\int_{\left\{\lambda^+<\frac{h(\tau-s)}{h(-s)}<\lambda^++\Delta\lambda\right\}} \frac{h(\tau-s)}{h(-s)}\,|h(-s)|\, ds \\ &= [\lambda^+ + \epsilon(\Delta\lambda)]\,\Delta\alpha \end{aligned}$$

where $\epsilon(\Delta\lambda) \to 0$ as $\Delta\lambda \to 0$. Letting $\Delta\alpha \to 0$ while observing (4.7) gives $\partial\sigma^+/\partial\alpha = \lambda^+$. A similar argument shows that $\partial\sigma^-/\partial\alpha = \lambda^-$.

THEOREM 4.2. *(i)* $\sigma^+(\tau,\alpha)$ *is concave in* $\alpha, \sigma^-(\tau,\alpha)$ *is convex in* α, *and thus* $\sigma(\tau,\alpha)$ *is concave in* α;

(ii) $\sigma^\pm(\tau,\alpha) = -\sigma^\pm(\tau,-\alpha)$ *and therefore* $\sigma(\tau,\alpha)=\sigma(\tau,-\alpha)$;

(iii) $\dfrac{\partial\sigma(\tau,\alpha)}{\partial\alpha} \leq 0$ *if* $\alpha > 0$.

Proof. Assertions (i) and (ii) follow immediately from Theorem 4.1 and the fact that $\partial\lambda^+/\partial\alpha$ $(\partial\lambda^-/\partial\alpha)$ is negative (positive) for all α. The assertion (ii) is obvious from the definition of $\sigma^\pm$. Finally, since $\sigma(\tau,\alpha)$ is concave in α (by (i)) and $\partial\sigma(\tau,\alpha)/\partial\alpha = 0$ at $\alpha = 0$ (by (ii)), (iii) follows.

We now specialize to the case where either $\log h$ is convex, so that

$$\sigma^+(\tau,\alpha) = \int_{-\infty}^{\mu} h(\tau-s)\,ds - \int_{\mu}^{0} h(\tau-s)\,ds + \int_{0}^{\tau} h(s')\,ds' \tag{4.8}$$

where μ is determined by (3.7), or $\log h$ is concave so that

$$\sigma^+(\tau,\alpha) = -\int_{-\infty}^{\tilde\mu} h(\tau-s)\,ds + \int_{\tilde\mu}^{0} h(\tau-s)\,ds + \int_{0}^{\tau} h(s')\,ds' \tag{4.9}$$

where $\tilde\mu$ is determined by (3.9).

THEOREM 4.3. *If* $h' < 0$ *and* $\log h$ *is convex or concave then*

$$\frac{\partial\sigma^\pm(\tau,\alpha)}{\partial\tau} > 0\,. \tag{4.10}$$

Proof. If $\log h$ is convex then from (4.8) we get,

$$\begin{aligned}\frac{\partial\sigma^+(\tau,\alpha)}{\partial\tau} &= -\int_{-\infty}^{\mu} \frac{d}{ds}\,h(\tau-s)\,ds + \int_{\mu}^{0} \frac{d}{ds}\,h(\tau-s)\,ds + h(\tau)\\ &= 2h(\tau) - 2h(\tau-\mu) > 0.\end{aligned}$$

Similarly, if $\log h$ is concave then

$$\begin{aligned}\frac{\partial\sigma^+(\tau,\alpha)}{\partial\tau} &= \int_{-\infty}^{\tilde\mu} \frac{d}{ds}\,h(\tau-s)\,ds - \int_{\tilde\mu}^{0} \frac{d}{ds}\,h(\tau-s)\,ds + h(\tau)\\ &= 2h(\tau-\tilde\mu) > 0\,.\end{aligned}$$

Finally, the second inequality in (4.10) follows from the first inequality and Theorem 4.2 (ii).

§5. Examples.

If $h(t) = \exp\{-k(t)\}$ where $k(t) \to \infty, k$ convex (k concave) then $\log h$ is concave (convex). For $h(t) = (t+a)^b$ where $a > 0, \quad b > 0, \quad \log h$ is convex.

We shall now consider two functions $h(t)$ of special interest.

THEOREM 5.1. *Let*

$$h(t) = \sum_{i=1}^{N} a_i \, e^{-\beta_i t} \qquad (a_i > 0, \; \beta_i > 0) \; . \tag{5.1}$$

Then $d^2 \log h/dt^2 > 0$.

Proof. As in the proof of Theorem 3.2, the assertion is equivalent to showing that

$$\frac{d}{ds} \, \frac{h(\tau - s)}{h(-s)} = \frac{h(-s) \; \Sigma a_i \beta_i \; e^{-\beta_i(\tau - s)} \; - h(\tau - s) \Sigma \; a_i \beta_i e^{\beta_i s}}{h^2(-s)}$$

is negative for any $\tau > 0$. But the numerator is equal to

$$\begin{aligned}
&\Sigma\Sigma \; a_i \beta_i a_j (e^{\beta_i(\tau-s)+\beta_j s} \; - e^{-\beta_j(\tau - s)+\beta_i s}) \\
&\quad = \Sigma\Sigma \; a_i a_j \beta_i \; e^{s(\beta_i+\beta_j)} (e^{-\beta_i \tau} \; - e^{-\beta_j \tau}) \\
&\quad = \frac{1}{2} \; \Sigma\Sigma \; a_i a_j \; e^{s(\beta_i+\beta_j)} \left[\beta_i \left(e^{-\beta_i \tau} \; - e^{-\beta_j \tau} \right) + \beta_j \left(e^{-\beta_j \tau} \; - e^{-\beta_i \tau} \right) \right] \\
&\quad = \frac{1}{2} \; \Sigma\Sigma \; a_i a_j \; e^{s(\beta_i+\beta_j)} \; (\beta_i - \beta_j)(e^{-\beta_i \tau} - e^{-\beta_j \tau})
\end{aligned}$$

and each term in the last sum is negative if $\beta_i \neq \beta_j$.

For the function (5.1), the μ determined by (3.7) is given by

$$\sum_{i=1}^{N} \; \frac{a_i}{\beta_i} \, (2 \; e^{\beta_i \mu} - 1) = \alpha \; .$$

The next example is

$$h(t) = e^{-\beta t} \; \cos \omega t \qquad (\beta > 0, \; \omega > 0). \tag{5.2}$$

Since

$$\frac{h(\tau - s)}{h(-s)} = e^{-\beta \tau} (\cos \; \omega \tau + \sin \; \omega \tau \; \tan \; \omega s),$$

one can check that the optimal solution $u^+_{\tau,\alpha}$, which for simplicity we shall denote by u_0, satisfies:

$$u_0(s) = \begin{cases} \text{sgn} \; h(-s) & \text{if} \quad \gamma - n\pi < \omega s < -\dfrac{(2n+1)\pi}{2} \\[2ex] -\,\text{sgn} \; h(-s) & \text{if} \quad -\dfrac{(2n+3)\pi}{2} < \omega s < \gamma - n\pi \end{cases}$$

if $n = 0, 1, 2, \ldots$, and

$$u_0(s) = \begin{cases} -\operatorname{sgn}\ h(-s) & \text{if} \quad -\frac{\pi}{2} < \omega s < \min(\gamma+\pi, 0) \\ \operatorname{sgn}\ h(-s) & \text{if} \quad \min(\gamma+\pi,0) < \omega s \le 0 \end{cases}$$

where $\gamma \in [-\frac{3\pi}{2}, -\frac{\pi}{2}]$ is to be selected such that

$$\int_{-\infty}^{0} h(-s) u_0(s)\, ds = \alpha\ . \tag{5.3}$$

Recalling (5.2) one can check that

$$u_0(s) = \begin{cases} -1 & \text{if} \quad \gamma - 2n\pi < \omega s < \gamma - (2n-1)\pi \\ 1 & \text{if} \quad \gamma - (2n-1)\pi < \omega s < \gamma - 2(n-1)\pi \end{cases}$$

for $n = 1, 2, \ldots$, and

$$u_0(s) = \begin{cases} -1 & \text{if} \quad \gamma < \omega s < \min(\gamma+\pi, 0) \\ 1 & \text{if} \quad \min(\gamma+\pi,0) < \omega s < 0. \end{cases}$$

Setting $\gamma' = \min(\gamma + \pi, 0)$ and using the formula

$$\int_a^b h(-s)\, ds = -Re\, \{\frac{1}{\beta + i\omega}\, [e^{-(\beta+i\omega)b} - e^{-(\beta+i\omega)a}]\}\ ,$$

we can compute

$$\int_{-\infty}^{0} h(-s)u_0(s)\, ds = \sum_{n=1}^{\infty}\Big[- \int_{(\gamma-2n\pi)/\omega}^{(\gamma-(2n-1)\pi)/\omega} h(-s)\, ds + \int_{(\gamma-(2n-1)\pi)/\omega}^{(\gamma-2(n-1)\pi)/\omega} h(-s)\, ds \Big]$$
$$- \int_{\gamma/\omega}^{\gamma'/\omega} h(-s)\, ds + \int_{\gamma'/\omega}^{0} h(-s)\, ds\ .$$

After somewhat lengthy calculations we get the expression

$$Re\, \Big\{ \frac{\beta - i\omega}{\beta^2+\omega^2}\, e^{\beta+i\omega)\gamma/\omega}\ \frac{1+e^{-\beta\pi/\omega}}{1-e^{-\beta\pi/\omega}} + \frac{\beta - i\omega}{\beta^2+\omega^2}\Big[1 - 2\, e^{(\beta+i\omega)\gamma'/\omega} + e^{(\beta+i\omega)\gamma/\omega}\Big]\Big\}, \tag{5.4}$$

or

$$\frac{\beta}{\beta^2+\omega^2}+\frac{2e^{\beta\gamma/\omega}}{(\beta^2+\omega^2)(1-e^{-\beta\pi/\omega})}(\beta\cos\gamma+\omega\sin\gamma)$$
$$-\frac{2\,e^{\beta\gamma'/\omega}}{\beta^2+\omega}(\beta\cos\gamma'+\omega\sin\gamma')\,.$$

Hence (5.3) determines γ by the following formulas:

$$\text{(5.5)}\qquad \begin{aligned}&\frac{2\,e^{\beta\gamma\omega}}{1-e^{-\beta\pi/\omega}}(\beta\cos\gamma+\omega\sin\gamma)=\alpha(\omega^2+\beta^2)+\beta \quad \text{if} \quad -\pi<\gamma<0,\\ &[\frac{2e^{\beta\gamma/\omega}}{1-e^{-\beta\pi/\omega}}+2e^{\beta(\gamma+\pi)/\omega}](\beta\cos\gamma+\omega\sin\gamma)\\ &\qquad=\alpha(\omega^2+\beta^2)-\beta \qquad \text{if} \qquad -\frac{3\pi}{2}<\gamma<-\pi.\end{aligned}$$

Since

$$\int_{-\infty}^{0} h(\tau-s)u_0(s)\,ds=Re\Big\{\int_{-\infty}^{0} e^{-(\beta+i\omega)(\tau-s)}u_0(s)\,ds\Big\}$$
$$=Re\Big\{e^{-(\beta+i\omega)\tau}\int_{-\infty}^{0} h_0(-s)u_0(s)\,ds\Big\}$$

and the last integral is equal to the expression in braces in (5.4), we find that

$$\sigma^+(\tau,\alpha)=\frac{2e^{\beta(\frac{\gamma}{\omega}-\tau)}}{(\beta^2+\omega^2)(1-e^{-\beta\pi/\omega})}[\beta\cos(\gamma-\omega\tau)+\omega\sin(\gamma-\omega\tau)]$$
$$-\frac{e^{-\beta\tau}}{\beta^2+\omega^2}(\beta\cos\omega\tau-\omega\sin\omega\tau)+\int_0^{\tau}|h(s)|\,ds \quad \text{if} \quad -\pi<\gamma<0,$$

$$\sigma^+(\tau,\alpha)=\frac{2e^{\beta(\frac{\gamma+\pi}{\omega}-\tau)}}{(\beta^2+\omega^2)(1-e^{-\beta\pi/\omega})}[\beta\cos(\gamma-\omega\tau)+\omega\sin(\gamma-\omega\tau)]$$
$$+\frac{e^{-\beta\tau}}{\beta^2+\omega^2}(\beta\cos\omega\tau-\omega\sin\omega\tau)+\int_0^{\tau}|h(s)|ds$$
$$\text{if} \quad -\frac{3\pi}{2}<\gamma<-\pi.$$

§6. **Several constraints.** The results of the previous sections can be extended to the case of several constraints. In fact it all hinges upon generalizing Theorem 2.1 to the problem:

$$\text{(6.1)}\qquad \max_{u\in K_\alpha}\int_{-\infty}^{0} g(s)u(s)\,ds$$

where K_α is the set of all measurable functions $u(s)$ satisfying:

$$-1 \le u(s) \le 1 \quad \text{for} \quad -\infty < s \le 0, \tag{6.2}$$

$$\int_{-\infty}^{0} f_i(s)u(s)\,ds = \alpha_i \qquad (i = 1, 2, \ldots, N). \tag{6.3}$$

Here g and f_i are given functions in $L^1(-\infty, 0) \cap C^0(-\infty, 0]$ and α_i are given real numbers.

THEOREM 6.1. *Assume that* $f_1 \ne 0$ a.e. *and that, for any real numbers* $\mu_1, \ldots, \mu_N$,

$$\text{measure}\{g = \sum_{i=1}^{N} \mu_i f_i\} = 0 \ .$$

Then there exist sequences $u_m, \lambda_{i,m}, \alpha_{i,m}$ *with* $u_m \to u_0$ *weakly in* L^1_{loc}, $\alpha_{1,m} = \alpha_1$, $\alpha_{i,m} \to \alpha_i$ *for* $2 \le i \le N$, *where* u_0 *is a maximizer of (6.1), and*

$$u_m(s) = \text{sgn}\ [g(s) - \sum_{i=1}^{N} \lambda_{i,m} f_i(s)]\ , \tag{6.4}$$

$$\int_{-\infty}^{0} f_i(s)u_m(s)\,ds = \alpha_{i,m} \qquad (i = 1, 2, \ldots, N)\ . \tag{6.5}$$

Thus, in order to evaluate (6.1) we need to analyze the u_m from (6.4), (6.5) and then compute $\int_{-\infty}^{0} g u_m$, noting that

$$\int_{-\infty}^{0} g(s)u_m(s)\,ds \to \int_{-\infty}^{0} g(s)u_0(s)\,ds = \max_{u \in K_\alpha} \int_{-\infty}^{0} g(s)u(s)\,ds.$$

Proof. For any small $\eta > 0$ introduce the "penalized" functional

$$J_\eta(u) = \int_{-\infty}^{0} g(s)u(s)\,ds - \frac{1}{\eta} \sum_{i=2}^{N} [\int_{-\infty}^{0} f_i(s)u(s)\,ds - \alpha_i]^2 \tag{6.6}$$

and consider the problem:

$$\text{maximize} \quad J_\eta(u) \quad \text{for} \quad u \in K \tag{6.7}$$

where K consists of all functions u satisfying:

$$-1 \leq u(s) \leq 1 \, , \quad \int_{-\infty}^{0} f_1(s) \, ds = \alpha_1 \, .$$

Proceeding as in the proof of Theorem 2.1 we deduce that if $|\tilde{u}| \leq 1$ where $\tilde{u}$ is defined by (2.6) with $u_0 = u_\eta$, then (2.6) implies

$$A_1 g(t_1) + A_2 g(t_2) - \frac{2}{\eta} \sum_{i=2}^{N} (\int_{-\infty}^{0} f_i u_\eta ds - \alpha_i)(A_1 f_i(t_1) + A_2 f_i(t_2)) \leq \sigma_2(\delta_0)$$

where u_η is a solution of (6.7) and $\sigma_2(\delta_0) \to 0$ if $\delta_0 \to 0$. Taking $\delta_0 \to 0$ we get the inequality

$$\begin{aligned} A_1[g(t_1) - \sum_{i=2}^{N} \lambda_{i,\eta} f_i(t_1)] \\ + A_2[g(t_2) - \sum_{i=2}^{N} \lambda_{i,\eta} f_i(t_2)] \leq 0 \end{aligned}$$

for some scalars $\lambda_{i,\eta}$. We can now proceed as in §2 to deduce that $\text{meas}\{|u_\eta| < 1\} = 0$; furthermore,

$$u_\eta(s) = \text{sgn} \, [g(s) - \sum_{i=1}^{N} \lambda_{i,\eta} f_i(s)] \, . \tag{6.8}$$

We note that

$$J_\eta(u_\eta) \geq J_\eta(\hat{u}) \qquad \forall \hat{u} \in K \, ;$$

From the inequality $J_\eta(u_\eta) \geq J_\eta(\hat{u})$ for some $\hat{u} \in K_\eta$ it follows that

$$\frac{1}{\eta} \sum_{i=2}^{N} \left[\int_{-\infty}^{0} f_i(s) u_\eta(s) ds - \alpha_i \right]^2 \leq C, \quad C \quad \text{independent of} \quad \eta \, .$$

Hence, as $\eta \to 0$,

$$\alpha_{i,\eta} \equiv \int_{-\infty}^{0} f_i(s) u_\eta(s) \, ds \to \alpha_i \quad (2 \leq i \leq N). \tag{6.9}$$

It is also easy to verify that for any convergent subsequence u_{η_m} (weakly in L^1_{loc}), the limit u_0 is a solution to problem (6.1). Indeed

$$J_\eta(u_\eta) \le \int_{-\infty}^{0} g u_\eta \, ds \le \max_{u \in K_{\alpha_\eta}} \int_{-\infty}^{0} g u ds = \int_{-\infty}^{0} g \hat{u}_\eta \, ds \tag{6.10}$$

where K_{α_η} is defined as K_α but with

$$\alpha_2 = \alpha_{2,\eta}, \ldots, \alpha_N = \alpha_{N,\eta} \ .$$

Since $\alpha_{j,\eta} \to \alpha_j$, if we take η to vary in a subsequence of η_m such that $\hat{u}_\eta \to \hat{u}$ weakly in L^1_{loc}, then $\int_{-\infty}^{0} g\hat{u}_\eta \to \int_{-\infty}^{0} g\hat{u}$ and $\hat{u} \in K_\alpha$ (i.e., $\hat{u}$ satisfies (6.2), (6.3)). Denoting by u_1 any solution of (6.1) – (6.3) we then have

$$\int_{-\infty}^{0} g\hat{u} \, ds \le \int_{-\infty}^{0} g u_1 ds;$$

also, by maximality of u_η (see (6.7))

$$\int_{-\infty}^{0} g u_1 ds = J_\eta(u_1) \le J_\eta(u_\eta) \ .$$

Using these relations in (6.10) and noting that

$$\int_{-\infty}^{0} g \ u_\eta \ ds \to \int_{-\infty}^{0} g \ u_0 \ ds \ ,$$

we conclude that

$$\int_{-\infty}^{0} g \ u_0 \ ds = \int_{-\infty}^{0} g \ u_1 \ ds = \max_{u \in K_\alpha} \int_{-\infty}^{0} g u \ ds \ .$$

Thus u_0 is a solution to (6.1)–(6.3). Recalling also (6.8), (6.9), the proof of Theorem 6.1 is complete.

REMARK 6.1. The $\lambda_{i,m}$ satisfy

$$\int_{\{g > \Sigma \lambda_{j,m} f_j\}} f_i(s) \, ds - \int_{\{g < \Sigma \lambda_{j,m} f_j\}} f_i(s) \, ds = \alpha_i$$

$$(1 \le i \le N) \ .$$

From these equations one should be able to determine the $\lambda_{j,m}$, at least in some relatively simple examples, and show that $\lambda_{j,m} \to \lambda_j \quad (\lambda_j$ finite) as $m \to \infty$; consequently

$$u_0 = \operatorname{sgn}[g(s) - \sum_{i=1}^{N} \lambda_i f_i(s)] \ .$$

and $$\int_{-\infty}^{0} f_j(s) u_0(s)\, ds = \alpha_j \quad \text{for} \quad 1 \le j \le N.$$

REMARK 6.2. Since uniqueness has not been established for (6.1), it is worth noting that any solution u_0 of (6.1) can be recovered as a weak limit of maximizers u_η provided we modify the definitions of $J_\eta(u)$ in (6.6), adding the term

$$-\int_{-\infty}^{0} \frac{(u-u_0)^2}{1+s^2}\, ds$$

to the right-hand side. One can then show that

$$\int_{-\infty}^{0} \frac{(u_\eta - u_0)^2}{1+s^2}\, ds \to 0 \qquad \text{if} \quad \eta \to 0.$$

The only differences occurs in (6.8): In the first inequality we have to replace $\lambda_{1,\eta} f_1(s)$ by $\lambda_{1,\eta} f_1(s) + \gamma(\eta)$ and in the second inequality we have to replace $\lambda_{1,\eta} f_1(s)$ by $\lambda_{1,\eta} f_1(s) - \gamma(\eta)$, for some function $\gamma(\eta) \to 0$ if $\eta \to 0$; furthermore, (6.8) holds a.e. except for set of measure $\le \tilde{\gamma}(\eta)$ where $\tilde{\gamma}(\eta) \to 0$ if $\eta \to 0$.

REMARK 6.3. Theorem 2.1 can actually also be proved using the penalized functional

$$\int_{-\infty}^{0} guds - \frac{1}{\eta}\Big(\int_{-\infty}^{0} fuds - \alpha\Big)^2 - \int_{-\infty}^{0} \frac{(u-u_0)^2}{1+s^2}\, ds \ .$$

REMARK 6.4. Consider the problem

$$\max_{u \in K} \quad J_\tau(u) \tag{6.11}$$

where K is the set of all inputs u that satisfy

$$\int_{-\infty}^{t_j} h(t_j - s) u(s)\, ds = \alpha_j, \qquad j = 1, \ldots, N, \tag{6.12a}$$

and where $J_\tau(u)$ is defined by (1.2) and $0 = t_1 < t_2 < \cdots < t_{N-1} < t_N \equiv \tau$.

Set

$$\sigma^+(\tau; t_1, \alpha_1, \ldots, t_N, \alpha_N) = \max_{u \in K} J_\tau(u), \tag{6.13a}$$

$$\sigma^-(\tau; t_1, \alpha_1, \ldots, t_N, \alpha_N) = \min_{u \in K} J_\tau(u). \tag{6.13b}$$

Then there exists a solution to (6.11) if and only if

$$\begin{cases} |\alpha_1| \le \int_0^\infty |h(s)|\, ds, \\ \sigma^-(t_j; t_1, \alpha_1, \ldots, t_{j-1}, \alpha_{j-1}) \le \alpha_j \le \sigma^+(t_j; t_1, \alpha_1, \ldots, t_{j-1}, \alpha_{j-1}), \\ \qquad\qquad\qquad\qquad\qquad\qquad 1 < j \le N\,. \end{cases} \tag{6.14}$$

Assuming $h(s) = 0$ for $s < 0$ (6.12a) can be written as

$$\int_{-\infty}^{\tau} h(t_j - s) u(s)\, ds = \alpha_j, \qquad j = 1, \ldots, N, \tag{6.15}$$

so that Theorem 6.1 implies that

$$u(s) = \begin{cases} \operatorname{sgn}\,[h(\tau - s) - \sum_{i=1}^{j} \lambda_i h(t_i - s)], & t_j < s < t_{j+1} \\ \operatorname{sgn}\,[h(\tau - s) - \sum_{i=1}^{N} \lambda_i h(t_i - s)], & s < t_1, \end{cases} \tag{6.16}$$

where $\lambda_1, \ldots, \lambda_N$ are chosen to satisfy (6.12a). In general, it is necessary to resort to numerical techniques to compute $\lambda_1, \ldots, \lambda_N$.

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