A DERIVATION THEOREM FOR CAPACITIES WITH RESPECT TO A RADON MEASURE

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#	Author(s) Title
78	**Abstracts** for the Workshop on Bayesian Analysis in Economics and Game Theory
79	**G. Chichilnisky, G.M.Heal,** Existence of a Competitive Equilibrium in L_p and Sobolev Spaces
80	**Thomas Seidman,** Time-dependent Solutions of a Nonlinear System in Semiconductory Theory, II: Boundedness and Periodicity
81	**Yakar Kannai,** Engaging in R&D and the Emergence of Expected Non-convex Technologies
82	**Herve Moulin,** Choice Functions over a Finite Set: A Summary
83	**Herve Moulin,** Choosing from a Tournament
84	**David Schmeidler,** Subjective Probability and Expected Utility Without Additivity
85	**I.G. Kevrekidis, R. Aris, L.D. Schmidt, and S. Pelikan,** The Numerical Computation of Invariant Circles of Maps
86	**F. William Lawvere,** State Categories, Closed Categories, and the Existence Semi-Continuous Entropy Functions
87	**F. William Lawvere,** Functional Remarks on the General Concept of Chaos
88	**Steven R. Williams,** Necessary and Sufficient Conditions for the Existence of a Locally Stable Message Process
89	**Steven R. Williams,** Implementing a Generic Smooth Function
90	**Dilip Abreu,** Infinitely Repeated Games with Discounting: A General Theory
91	**J.S. Jordan,** Instability in the Implementation of Walrasian Allocations
92	**Myrna Holtz Wooders, William R. Zame,** Large Games: Fair and Stable Outcomes
93	**J.L. Noakes,** Critical Sets and Negative Bundles
94	**Graciela Chichilnisky,** Von Neumann-Morgenstern Utilities and Cardinal Preferences
95	**J.L. Ericksen,** Twinning of Crystals
96	**Anna Nagurney,** On Some Market Equilibrium Theory Paradoxes
97	**Anna Nagurney,** Sensitivity Analysis for Market Equilibrium
98	**Abstracts** for the Workshop on Equilibrium and Stability Questions in Continuum Physics and Partial Differential Equations
99	**Millard Beatty,** A Lecture on Some Topics in Nonlinear Elasticity and Elastic Stability
100	**Filomena Pacella,** Central Configurations of the N-Body Problem via the Equivariant Morse Theory
101	**D. Carlson and A. Hoger,** The Derivative of a Tensor-valued Function of a Tensor
102	**Kenneth Mount,** Privacy Preserving Correspondence
103	**Millard Beatty,** Finite Amplitude Vibrations of a Neo-hookean Oscillator
104	**D. Emmons and N. Yannelis,** On Perfectly Competitive Economies: Loeb Economies
105	**E. Mascolo and R. Schianchi,** Existence Theorems in the Calculus of Variations
106	**D. Kinderlehrer,** Twinning of Crystals (II)
107	**R. Chen,** Solutions of Minimax Problems Using Equivalent Differentiable Equations
108	**D. Abreu, D. Pearce, and E. Stacchetti,** Optimal Cartel Equilibria with Imperfect Monitoring
109	**R. Lauterbach,** Hopf Bifurcation from a Turning Point
110	**C. Kahn,** An Equilibrium Model of Quits under Optimal Contracting
111	**M. Kaneko and M. Wooders,** The Core of a Game with a Continuum of Players and Finite Coalitions: The Model and Some Results
112	**Haim Brezis,** Remarks on Sublinear Equations
113	**D. Carlson and A. Hoger,** On the Derivatives of the Principal Invariants of a Second-order Tensor
114	**Raymond Deneckere and Steve Pelikan,** Competitive Chaos
115	**Abstacts** for the Workshop on Homogenization and Effective Moduli of Materials and Media
116	**Abstracts** for the Workshop on the Classifying Spaces of Groups
117	**Umberto Mosco,** Pointwise Potential Estimates for Elliptic Obstacle Problems
118	**J. Rodrigues,** An Evolutionary Continuous Casting Problem of Stefan Type
119	**C. Mueller and F. Weissler,** Single Point Blow-up for a General Semilinear Heat Equation
120	**D.R.J. Chillingworth,** Three Introductory Lectures on Differential Topology and its Applications
121	**Giorgio Vergara Caffarelli,** Green's Formulas for Linearized Problems with Live Loads
122	**F. Chiarenza and N. Garofalo,** Unique Continuation for Nonnegative Solutions of Schrodinger Operators
123	**J.L. Ericksen,** Constitutive Theory for some Constrained Elastic Crystals
124	**Minoru Murata,** Positive solutions of Schrodinger Equations
125	**John Maddocks and Gareth P. Parry,** A Model for Twinning
126	**M. Kaneko and M. Wooders,** The Core of a Game with a Continuum of Players and Finite Coalitions: Nonemptiness with Bounded Sizes of Coalitions
127	**William Zame,** Equilibria in Production Economies with an Infinite Dimensional Commodity Space
128	**Myrna Holtz Wooders,** A Tiebout Theorem
129	**Abstracts** for the Workshop on Theory and Applications of Liquid Crystals
130	**Yoshikazu Giga,** A Remark on A Priori Bounds for Global Solutions of Semilinear Heat Equations
131	**M. Chipot and G. Vergara-Caffarelli,** The N-Membranes Problem
132	**P.L. Lions and P.E. Souganidis,** Differential Games and Directional Derivatives of Viscosity Solutions of Isaacs' Equations II
133	**G. Capriz and P. Giovine,** On Virtual Effects During Diffusion of a Dispersed Medium in a Suspension
134	**Fall Quarter Seminar Abstracts**
135	**Umberto Mosco,** Wiener Criterion and Potential Estimates for the Obstacle Problem
136	**Chi-Sing Man,** Dynamic Admissible States, Negative Absolute Temperature, and the Entropy Maximum Principle
137	**Abstracts** for the Workshop on Oscillation Theory, Computation, and Methods of Compensated Compactness
138	**Arie Leizarowitz,** Tracking Nonperiodic Trajectories with the Overtaking Criterion
139	**Arie Leizarowitz,** Convex Sets in R^n as Affine Images of some Fixed Set in R
140	**Arie Leizarowitz,** Stochastic Tracking with the Overtaking Criterion
141	**Abstracts** from the Workshop on Amorphous Polymers and Non-Newtonian Fluids
142	**Winter Quarter Seminar Abstracts**
143	**D.G. Aronson and J.L. Vazquez,** The Porous Medium Equation as a Finite-spee Approximation to a Hamilton-Jacobi Equation
144	**E. Sanchez-Palencia and H. Weinberger,** On the Edge Singularities of a Composite Conducting Medium
145	**Jon C. Luke,** Soliton Solutions in a Class of Fully Discrete Nonlinear Wave Equations
146	**Chi-Sing Man and H. Cohen,** A Coordinate-Free Approach to the Kinematics of Membranes
147	**J.L. Lions,** Asymptotic Problems in Distributed Systems
148	**Reiner Lauterbach,** An Example of Symmetry Breaking with Submaximal Isotrop Subgroup
149	**Abstracts** from the Workshop on Metastability and Incompletely Posed Problem
150	**B. Bozar-Karakiewicz and Jerry Bona,** Wave-dominated Shelves: A Model of Sand-Ridge Formation by Progressive, Infragravity Waves
151	**Abstracts** from the Workshop on Dynamical Problems in Continuum Physics
152	**V.I. Oliker,** The problem of Embedding S^n into R^{n+1} with Prescribed Gauss
153	**R. Batra,** The force on a Lattice Defect in an Elastic Body
154	**J. Fleckinger and Michael Lapidus,** Eigenvalues of Elliptic Boundary Value Problems with and Indefinite Weight Function
155	**R. Kohn and M. Vogelius,** Thin Plates with Rapidly Varying Thickness, and Their relation to Structural Optimization
156	**M. Gurtin,** Some Results and Conjectures in the Gradient Theory of Phase Transitions
157	**A. Novick-Cohen,** Energy Methods for the Cahn-Hilliard Equation

A DERIVATION THEOREM FOR CAPACITIES WITH RESPECT TO A RADON MEASURE

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§ 1. INTRODUCTION

The problem dealt with in this paper is the identification of the Radon-Nikodym derivative

(1.1) $$f = \frac{d\mu}{d\nu}$$

of a Borel measure μ on R^n with respect to a Radon measure ν. The measure μ is supposed not to charge polar sets but may possibly take the value $+\infty$ on subsets of positive capacity.

A classical device is provided by the ***derivation theorems*** on special families of sets, e.g. the derivation theorems on ***balls***. The density f can be obtained as the limit

(1.2) $$f(x) = \liminf_{\rho\to 0} \frac{\mu(B_\rho(x))}{\nu(B_\rho(x))}$$

provided, for instance,

i) $f \in L^1_{loc}(R^n,\nu)$,

ii) $f(x) < +\infty$ for every $x \in R^n$.

This result, however, is of no help in some variational applications we are interested in (see for instance [4] and [7]), involving functionals of the type

$$\int_\Omega |Du|^2\, dx + \int_{\overline{\Omega}} u^2\, d\mu$$

and their limits.

In many of these problems, a measure μ is known to exist and we want to identify it by relying on the additional information available. This does not include the knowledge of the value of u on some family of sets, e.g. balls, as required in (1.2).

Instead, our information on μ is of variational nature, and it can be conveniently expressed in terms of a capacity associated with μ. The μ-capacity of a subset E of an open set $A \subset \mathbf{R}^n$ is defined by

$$\mathrm{cap}_\mu(E,A) = \min \{ \int_A |Du|^2\, dx + \int_E u^2\, d\mu \ :\ u-1 \in H_0^1(A) \}.$$

Our main result, Theorem 2.3, shows that the knowledge of the capacities

$$\mathrm{cap}_\mu(B_\rho(x), B_{2\rho}(x))$$

at a point $x \in \mathbf{R}^n$ gives the density (1.1) as the limit

$$f(x) = \liminf_{\rho \to 0} \frac{\mathrm{cap}_\mu(B_\rho(x), B_{2\rho}(x))}{\nu(B_\rho(x))} \tag{1.3}$$

under the assumptions i) and ii) above.

The role of the "masses" $\mu(B_\rho(x))$ of the classical derivation theorem is thus replaced by that of the "energies" $\mathrm{cap}_\mu(B_\rho(x), B_{2\rho}(x))$. The advantage is that the latter are preserved under broad classes of variational perturbations that may wildly modify the masses $\mu(B_\rho(x))$.

More precisely, a convergence of variational type, called γ-convergence, can be defined on the set $\mathfrak{M}_0$ of all Borel measures μ considered above, with respect to which ***both*** the (antagonistic) properties below hold (see [7]):

a) $\mathfrak{M}_0$ is compact;

b) the function $\mu \longmapsto \mathrm{cap}_\mu(B_\rho(x), B_{2\rho}(x))$ is continuous on $\mathfrak{M}_0$ for every $x \in \mathbf{R}^n$ and every $\rho > 0$, except possibly a countable set of values that may depend on x.

Given an arbitrary sequence (μ_h) in $\mathfrak{M}_0$, the property a) assures that there exists a measure μ in $\mathfrak{M}_0$ which is the γ-limit of some subsequence (μ_{h_k}) of (μ_h). The property b) then gives $\mathrm{cap}_\mu(B_\rho(x), B_{2\rho}(x))$ by just taking the limit of $\mathrm{cap}_{\mu_{h_k}}(B_\rho(x), B_{2\rho}(x))$, and this allows to evaluate the limit f in (1.3).

Under the assumptions i) and ii) our derivation result will lead to identify μ via its density f. Moreover, whenever the limit

$$\lim_{h\to+\infty} \operatorname{cap}_{\mu_h}(B_\rho(x),B_{2\rho}(x))$$

exists, the whole sequence (μ_h) is seen to γ-converge to μ.

The stability of $\operatorname{cap}_\mu(B_\rho(x),B_{2\rho}(x))$ with respect to γ-convergence can also be exploited for "computing" the capacities $\operatorname{cap}_\mu(B_\rho(x),B_{2\rho}(x))$, and hence the density (1.1), by ***approximating*** μ with suitable measures $\mu_h \in \mathcal{M}_0$.

In this connection, we mention that ***density*** results are known which show that ***every*** measure $\mu \in \mathcal{M}_0$ can be approximated (in the sense of γ-convergence), for instance, by sequences (μ_h) of the form

$$\mu_h = \infty_{E_h} = \begin{cases} +\infty & \text{on } E_h \\ 0 & \text{elsewhere} \end{cases},$$

as well as by sequences (μ_h) of the form

$$d\mu_h = q_h(x)\ dx\ .$$

In a forthcoming paper our derivation result, and in particular its boundary formulation of Section 4, will be applied to a class of so-called ***reinforcement*** problems (see [4]).

§ 2. PRELIMINARIES AND STATEMENT OF THE RESULT.

In this section we give the notation we shall use in the following, and we state the main result of the paper.

Let Ω be an open subset of R^n, and let $\mathcal{B}(\Omega)$ be the σ-field of Borel subsets of Ω. By a ***Borel measure*** on Ω we mean a non-negative countably additive set function defined on $\mathcal{B}(\Omega)$ and with values in $[0,+\infty]$; by a ***Radon measure*** on Ω we mean a Borel measure which is finite on every compact subset of Ω.

If μ is a Borel measure on Ω and $f:\Omega\longrightarrow[0,+\infty]$ is a Borel function, we shall denote by $f\mu$ the Borel measure on Ω defined by

$$(f\mu)(B) = \int_B f\,d\mu \qquad \text{for every } B\in \mathcal{B}(\Omega).$$

For every $x\in R^n$ and every $r>0$ we set

$$B_r(x) = \{y\in R^n : |x-y|<r\} .$$

For Radon measures, the following ***derivation theorem*** holds (see [11] Section 2.9).

THEOREM 2.1. *Let μ,ν be two Radon measures on R^n. Assume that ν is absolutely continuous with respect to μ (i.e. $\mu(E)=0 \Longrightarrow \nu(E)=0$). Then*

i) *there exists a μ-measurable function $f:R^n \longrightarrow [0,+\infty[$ such that for μ-a.e. $x\in R^n$*

$$f(x) = \lim_{r\to 0} \frac{\nu(B_r(x))}{\mu(B_r(x))}$$

ii) $\nu=f\mu$.

If Ω is a bounded open subset of R^n, for every compact subset K of Ω the ***capacity*** of K with respect to Ω is defined by

$$\mathrm{cap}(K,\Omega) = \inf\left\{ \int_\Omega |Dv|^2\,dx \ :\ v\in C_0^\infty(\Omega),\ v\geq 1 \text{ on } K \right\} ;$$

the definition is extended to open sets $A\subset\Omega$ by

$$\mathrm{cap}(A,\Omega) = \sup\left\{ \mathrm{cap}(K,\Omega) : K\subset A,\ K \text{ compact} \right\} ,$$

and to arbitrary sets $E\subset\Omega$ by

$$\mathrm{cap}(E,\Omega) = \inf\left\{ \mathrm{cap}(A,\Omega) : A\supset E,\ A \text{ open} \right\} .$$

We say that a set $E\subset R^n$ has ***capacity zero*** if $\mathrm{cap}(E\cap\Omega)=0$ for every bounded open subset Ω of R^n. It is well known that a bounded set $E\subset R^n$ has capacity zero if and only if $\mathrm{cap}(E,\Omega)=0$ for some (hence for all) bounded open sets $\Omega\subset R^n$ with $\Omega\supset E$.

If a property P(x) holds for all $x\in E$ except for a set $E_0\subset E$ with capacity zero, then we say that P(x) holds ***quasi everywhere*** on E (q.e. on E).

If μ is a Borel measure on Ω and $f:R^n\longrightarrow [0,+\infty]$ is μ-measurable, then for every Borel set $E\subset\Omega$ with $0\leq\mu(E)<+\infty$ we set

$$⨍_E f\,d\mu = \frac{1}{\mu(E)}\int_E f\,d\mu \qquad \text{(with the convention } 0/0=1).$$

We denote by $H^1(\Omega)$ the usual Sobolev space of all functions in $L^2(\Omega)$ with first order distribution derivatives in $L^2(\Omega)$, by $H_0^1(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $H^1(\Omega)$, and by $H^{-1}(\Omega)$ the dual space of $H_0^1(\Omega)$, with dual pairing $\langle\cdot,\cdot\rangle$.

It is well known (see [12]) that for every $u\in H^1(\Omega)$ the limit

$$\lim_{r\to 0} \fint_{B_r(x)} u(y)\, dy$$

exists and is finite q.e. on Ω; thus the pointwise value $u(x)$ of a function $u\in H^1(\Omega)$ can be defined q.e. on Ω by the limit above, and in this way the function u is quasi-continuous on Ω (see [12]). Moreover, if Ω is bounded, it can be proved that for every $E\subset\Omega$

$$\mathrm{cap}(E,\Omega) = \min\left\{ \int_\Omega |Dv|^2\, dx \;:\; v\in H_0^1(\Omega),\ v\geq 1 \text{ q.e. on } E \right\} .$$

We denote by $\mathcal{M}_0$ the class of all Borel measures μ on R^n such that $\mu(E)=0$ for every Borel set $E\subset R^n$ with capacity zero. Note that we do not require the measures of the class $\mathcal{M}_0$ to be regular or σ-finite.

We say that a Radon measure μ on R^n belongs to $H^{-1}(R^n)$ if there exists $\lambda\in H^{-1}(R^n)$ such that

(2.1) $$\langle\lambda,v\rangle = \int v\, d\mu$$

for every $v\in C_0^\infty(R^n)$. It is well known that in this case $\mu\in\mathcal{M}_0$ and equality (2.1) also holds for $v\in H^1(R^n)$ because of the definition above of the pointwise values of the function v. In this paper we always identify λ and μ .

Other examples of measures in $\mathcal{M}_0$ are the following (see [7]):

i) the n-dimensional Lebesgue measure and the (n-1)-dimensional Hausdorff measure belong to the class $\mathcal{M}_0$;

ii) if $\mu\in\mathcal{M}_0$ and $f:R^n\longrightarrow[0,+\infty]$ is a Borel function, then the Borel measure $f\mu$ belongs to $\mathcal{M}_0$;

iii) if $\mu\in\mathcal{M}_0$ and $B\in\mathcal{B}(R^n)$, then the Borel measure μ_B defined by $\mu_B(E)=\mu(E\cap B)$ belongs to $\mathcal{M}_0$;

iv) for every Borel set $B\subset R^n$ the Borel measure ∞_B defined by

(2.2) $$\infty_B(E) = \begin{cases} 0 & \text{if } B\cap E \text{ has capacity zero} \\ +\infty & \text{otherwise} \end{cases}$$

belongs to $\mathcal{M}_0$.

The following property for measures of $\mathcal{M}_0$ holds (see [7] Lemma 4.15).

PROPOSITION 2.2. *For every* $\mu\in\mathcal{M}_0$ *there exists a Radon measure* $\lambda\in H^{-1}(\mathbb{R}^n)$ *and a Borel function* $g:\mathbb{R}^n\longrightarrow[0,+\infty]$ *such that*

$$\int_A u^2\,d\mu=\int_A u^2 g\,d\lambda$$

for every bounded open subset A *of* $\mathbb{R}^n$ *and for every* $u\in H^1(A)$.

Consider now an elliptic operator of the form

$$Lu=-\sum_{i,j=1}^{n} D_i(a_{ij}(x)\,D_j) \tag{2.3}$$

where $a_{ij}=a_{ji}\in L^\infty(\mathbb{R}^n)$ and

$$\alpha|z|^2\le\sum_{i,j=1}^{n} a_{ij}(x)\,z_i\,z_j\le\beta|z|^2 \qquad \text{for every } x\in\mathbb{R}^n,\ z\in\mathbb{R}^n$$

for suitable constants $0<\alpha\le\beta$.

Let $\mu\in\mathcal{M}_0$; for every bounded open set $A\subset\mathbb{R}^n$ and for every Borel set $E\subset A$ we define the μ-capacity relative to L by

$$\mathrm{cap}_\mu^L(E,A)=\inf\left\{\int_A\sum_{i,j=1}^{n} a_{ij}(x)D_ivD_jv\,dx+\int_E v^2\,d\mu\ :\ v-1\in H_0^1(A)\right\} \tag{2.4}$$

(see [8]). For example, if L is the Laplace operator $-\Delta$, μ is the measure ∞_B defined in (2.2), and $E\subset A$, then

$$\mathrm{cap}_\mu^L(E,A)=\mathrm{cap}(E\cap B,A)\ .$$

In any case, for every open set A and every Borel set $E\subset A$ we have

$$\mathrm{cap}_\mu^L(E,A)\le k\,\mathrm{cap}(E,A) \qquad (\text{with } k=\max\{1,\beta\}).$$

We are now in a position to state our main result.

THEOREM 2.3. *Let* μ *be a Borel measure of the class* $\mathcal{M}_0$, *let* ν *be a Radon measure of the class* $\mathcal{M}_0$, *and for every* $x\in\mathbb{R}^n$ *let*

$$f(x)=\liminf_{r\to0}\frac{\mathrm{cap}_\mu^L(B_r(x),B_{2r}(x))}{\nu(B_r(x))} \qquad (\textit{with the convention } 0/0=1)\ . \tag{2.5}$$

Assume that

i) $f\in L^1_{loc}(\mathbb{R}^n,\nu)$;

ii) $f(x)<+\infty$ q.e. *on* $\mathbb{R}^n$.

Then μ *is a Radon measure and we have* $\mu=f\nu$. *Moreover, the* liminf *in* (2.5) *is a li-*

mit for ν*-a.e.* $x \in \mathbf{R}^n$.

REMARK 2.4. The conclusion of Theorem 2.3 may fail if we drop hypothesis ii), as the following example shows.

Take $n=1$, μ the Dirac measure at the origin, and ν the Lebesgue measure. We have that μ and ν are in $\mathfrak{M}_0$, and ν is a Radon measure. It is easy to see that the function f defined in (2.5) is given by

$$f(x) = \begin{cases} +\infty & \text{if } x=0 \\ 0 & \text{otherwise} \end{cases} .$$

Then $f \in L^1(\mathbf{R},\nu)$, but the equality $\mu = f\nu$ does not hold.

§ 3. PROOF OF THE RESULT

For the proof of Theorem 2.3 we shall need some preliminary lemmas.

LEMMA 3.1. *Let* μ *be a Radon measure belonging to* $H^{-1}(\mathbf{R}^n)$. *Then*

$$\lim_{r \to 0} \frac{\mathrm{cap}^L_\mu(B_r(x), B_{2r}(x))}{\mu(B_r(x))} = 1 \tag{3.1}$$

for μ*-a.e.* $x \in \mathbf{R}^n$.

Proof. Let A be a fixed open ball in $\mathbf{R}^n$; it will be enough to prove that (3.1) holds for μ-a.e. $x \in A$. For every $x \in A$ and every $r>0$ with $B_{2r}(x) \subset A$ we denote by u_r^x the solution in the distribution sense of the problem

$$\begin{cases} Lu + u\mu_{B_r(x)} = 0 & \text{on } A \\ u-1 \in H^1_0(A) \end{cases} \tag{3.2}$$

where $\mu_{B_r(x)}(E)=\mu(E\cap B_r(x))$. It is easy to see that the function u_r^x minimizes the integral

$$\int_A \sum_{i,j=1}^{n} a_{ij}(y) D_j v D_i v \, dy + \int_{B_r(x)} v^2 \, d\mu(y)$$

among the functions v with $v-1\in H_0^1(A)$. Multiplying (3.2) by u-1 and integrating by parts, we get

$$\mathrm{cap}_\mu^L(B_r(x),A) = \int_{B_r(x)} u_r^x(y) \, d\mu(y) \quad .$$

Since

$$\mathrm{cap}_\mu^L(B_r(x),A) \le \mathrm{cap}_\mu^L(B_r(x),B_{2r}(x)) \le \mu(B_r(x))$$

the proof will be achieved if we prove that

$$\text{(3.3)} \qquad \liminf_{r\to 0} ⨍_{B_r(x)} u_r^x(y) \, d\mu(y) \ge 1 \qquad \text{for } \mu\text{-a.e. } x\in A.$$

Define now a function v_r^x by setting for every $y\in A$

$$v_r^x(y) = \int_{B_r(x)} G(y,z) \, d\mu(z)$$

where G is the Green function for the Dirichlet problem in A relative to the operator L. Note that the values of v_r^x are defined everywhere on A, and not only q.e. on A. Then we have

$$\begin{cases} L\, v_r^x = \mu_{B_r(x)} \\ v_r^x \in H_0^1(A) \quad . \end{cases}$$

Moreover, it is easy to see by comparison, that $u_r^x \ge 1-v_r^x$ q.e. on A; therefore to prove (3.3) it will suffice to show that

$$\text{(3.4)} \qquad \lim_{r\to 0} ⨍_{B_r(x)} v_r^x(y) \, d\mu(y) = 0 \qquad \text{for } \mu\text{-a.e. } x\in A \, .$$

Again by comparison, we get

$$\text{(3.5)} \qquad B_r(x) \subset B_R(y) \Longrightarrow v_r^x \le v_R^y \quad .$$

We shall prove (3.4) in several steps.

<u>Step 1.</u> *There exists a Borel set* $N\subset A$ *with* $\mu(N)=0$, *such that*

$$\lim_{r\to 0} ⨍_{B_r(y)} v_R^x(z) \, d\mu(z) = v_R^x(y) < +\infty$$

for all $x\in A$, *for all* $R>0$ *with* $B_{2R}(x)\subset A$, *and for all* $y\in B_{R/2}(x)-N$.

In fact, for every $x\in A$ and $R>0$, the function v_R^x is μ-integrable. By a classical derivation theorem (see Theorem 2.1) there exists a Borel set $N\subset A$ depending on x and R such that $\mu(N)=0$ and

(3.6) $$\lim_{r\to 0} \fint_{B_r(y)} v_R^x(z)\, d\mu(z) = v_R^x(y) < +\infty \qquad \text{for every } y\in A-N.$$

If $X\subset\mathbb{R}^n$ and $S\subset]0,+\infty[$ are countable and dense, we can easily construct a Borel set $N\subset A$ with $\mu(N)=0$, such that (3.6) holds for every $x\in X$, $R\in S$.

We show now, by an approximating procedure, that this set N satisfies the requirement of Step 1.

Let x be any point of A and R any positive number; there exist two sequences $x_h\longrightarrow x$ and $R_h\longrightarrow R$ such that $x_h\in X$, $R_h\in S$ and

$$B_{3R/4}(x) \subset B_{R_h}(x_h) \subset B_R(x) \quad .$$

For every $y\in B_{R/2}(x)$ we have by (3.5)

(3.7) $$0 \le v_R^x(y) - v_{R_h}^{x_h}(y) = \int_{B_R(x)-B_{R_h}(x_h)} G(y,z)\, d\mu(z) \le c(x,R)\, \mu(B_R(x)-B_{R_h}(x_h))$$

where $c(x,R)$ is a constant depending only on x,R. By using (3.6),(3.7) we obtain for every $y\in B_{R/2}(x)-N$ and every $h\in\mathbb{N}$

$$\limsup_{r\to 0} \left| v_R^x(y) - \fint_{B_r(y)} v_R^x(z)\, d\mu(z) \right|$$

$$\le \limsup_{r\to 0} \left\{ |v_R^x(y) - v_{R_h}^{x_h}(y)| + \left| v_{R_h}^{x_h}(y) - \fint_{B_r(y)} v_{R_h}^{x_h}(z)\, d\mu(z)\right| + \fint_{B_r(y)} |v_{R_h}^{x_h}(z) - v_R^x(z)|\, d\mu(z) \right\}$$

$$\le 2\, c(x,R)\, \mu(B_R(x)-B_{R_h}(x_h)) \quad ,$$

and by letting $h\longrightarrow +\infty$ the proof of Step 1 is achieved.

<u>Step 2.</u> *There exists a Borel set* $N\subset A$ *with* $\mu(N)=0$ *such that*

$$\limsup_{r\to 0} \fint_{B_r(x)} v_r^x(z)\, d\mu(z) \le v_R^x(x)$$

for all $x\in A-N$ *and all* $R>0$ *with* $B_{2R}(x)\subset A$.

In fact, if N is the set constructed in Step 1, for every $x\in A-N$ and every $R>0$ we have $x\in B_{R/2}(x)-N$, and so by (3.5) and by Step 1 we get

$$\limsup_{r\to 0} \fint_{B_r(x)} v_r^x(z)\, d\mu(z) \le \limsup_{r\to 0} \fint_{B_r(x)} v_R^x(z)\, d\mu(z) = v_R^x(x) \quad .$$

<u>Step 3.</u> *For* μ*-a.e.* $x\in A$ *we have* $\lim_{R\to 0} v_R^x(x) = 0$.

In fact we have

(3.8) $$0 \le v_R^x(x) = \int_{B_R(x)} G(x,z)\, d\mu(z) \quad .$$

The function $w(x)=\int_A G(x,z)\, d\mu(z)$ is in $H^1_0(A)$; moreover it is quasi-continuous, so that $w(x)<+\infty$ q.e. on A (see [13]). Therefore for q.e. $x\in A$ the function $G(x,\cdot)$ is μ-integrable, and so, passing to the limit in (3.8) as $R\to 0$, the proof of Step 3 is achieved, and this, together with Step 2, proves (3.4). □

Proof of Theorem 2.3. By Proposition 2.2 there exist a Radon measure $\lambda\in H^{-1}(\mathbf{R}^n)$ and a Borel function $g:\mathbf{R}^n\to[0,+\infty]$ such that

$$\int_A u^2\, d\mu = \int_A u^2 g\, d\lambda \tag{3.9}$$

for every open subset A of $\mathbf{R}^n$ and for every $u\in H^1(A)$. Therefore

$$\mathrm{cap}^L_\mu(B_r(x),B_{2r}(x)) = \mathrm{cap}^L_{g\lambda}(B_r(x),B_{2r}(x)) \tag{3.10}$$

For every $k\in\mathbf{N}$ set $g_k=g\wedge k$; by Lemma 3.1 there exists a Borel set $E_1\subset\mathbf{R}^n$ with $\int_{E_1} g_k\, d\lambda = 0$ such that

$$\lim_{r\to 0} \frac{\mathrm{cap}^L_{g_k\lambda}(B_r(x),B_{2r}(x))}{(g_k\lambda)(B_r(x))} = 1 \tag{3.11}$$

for every $x\in\mathbf{R}^n-E_1$ and for every $k\in\mathbf{N}$. Since $\lambda+\nu$ is a Radon measure, by the classical derivation theorem (Theorem 2.1), there exists a Borel set $E_2\subset\mathbf{R}^n$ such that $(\lambda+\nu)(E_2)=0$ and

$$\lim_{r\to 0} \frac{(g_k\lambda)(B_r(x))}{(\lambda+\nu)(B_r(x))} = g_k(x)\frac{d\lambda}{d(\lambda+\nu)}(x) < +\infty \tag{3.12}$$

$$\lim_{r\to 0} \frac{\nu(B_r(x))}{(\lambda+\nu)(B_r(x))} = \frac{d\nu}{d(\lambda+\nu)}(x) \le 1 \tag{3.13}$$

for every $x\in\mathbf{R}^n-E_2$ and for every $k\in\mathbf{N}$. Finally, by hypothesis ii) of Theorem 2.3, there exists a Borel set $E_3\subset\mathbf{R}^n$ such that $(\lambda+\nu)(E_3)=0$ and

$$f(x) < +\infty \qquad \text{for every } x\in\mathbf{R}^n-E_3 . \tag{3.14}$$

Set $E=E_1\cup E_2\cup E_3$; if $x\in\mathbf{R}^n-E$ we have by (3.10),(3.11),(3.12),(3.13)

$$g_k(x)\frac{d\lambda}{d(\lambda+\nu)}(x) = \lim_{r\to 0}\frac{(g_k\lambda)(B_r(x))}{(\lambda+\nu)(B_r(x))}\ \lim_{r\to 0}\frac{\mathrm{cap}^L_{g_k\lambda}(B_r(x),B_{2r}(x))}{(g_k\lambda)(B_r(x))} =$$

$$= \lim_{r\to 0}\frac{\mathrm{cap}^L_{g_k\lambda}(B_r(x),B_{2r}(x))}{(\lambda+\nu)(B_r(x))} \le \liminf_{r\to 0}\frac{\mathrm{cap}^L_{g\lambda}(B_r(x),B_{2r}(x))}{(\lambda+\nu)(B_r(x))} =$$

$$= \liminf_{r\to 0} \frac{\mathrm{cap}^L_\mu(B_r(x),B_{2r}(x))}{\nu(B_r(x))} \cdot \lim_{r\to 0} \frac{\nu(B_r(x))}{(\lambda+\nu)(B_r(x))} = f(x) \frac{d\nu}{d(\lambda+\nu)}(x) \;.$$

Note that these inequalities hold even if $\frac{d\nu}{d(\lambda+\nu)}(x)=0$, because we have assumed $f(x)<+\infty$. This is a delicate point in the proof; indeed this is the only point where hypotesis ii) is used.

Therefore, for every Borel set $B\subset R^n-E_1$

$$\int_B \left[g_k \frac{d\lambda}{d(\lambda+\nu)} \right] d(\lambda+\nu) \le \int_B \left[f \frac{d\nu}{d(\lambda+\nu)} \right] d(\lambda+\nu) \quad ,$$

so that, since $f\in L^1_{loc}(R^n,\nu)$

$$\int_B g_k \, d\lambda \le \int_B f \, d\nu$$

for every Borel set $B\subset R^n$. Passing to the limit as $k\longrightarrow+\infty$, by the monotone convergence theorem, we get

$$\int_B g \, d\lambda \le \int_B f \, d\nu \qquad \text{for every Borel set } B\subset R^n .$$

Thus, by (3.9)

$$\mu(A) \le \int_A f \, d\nu \qquad \text{for every open set } A\subset R^n ,$$

which yields that μ is a Radon measure absolutely continuous with respect to ν. Then

$$(3.15) \qquad \mu(B) \le \int_B f \, d\nu \qquad \text{for every Borel set } B\subset R^n .$$

Since $\mathrm{cap}^L_\mu(B_r(x),B_{2r}(x)) \le \mu(B_r(x))$, we obtain for ν-a.e. $x\in R^n$

$$f(x) \le \liminf_{r\to 0} \frac{\mu(B_r(x))}{\nu(B_r(x))} = \frac{d\mu}{d\nu}(x) \;.$$

Therefore, by (3.15), for every Borel set $B\subset R^n$

$$\mu(B) \le \int_B f \, d\nu \le \int_B \left(\frac{d\mu}{d\nu}\right) d\nu = \mu(B) \quad ,$$

which implies that $\mu=f\nu$. Moreover, by (3.15), for ν-a.e. $x\in R^n$

$$f(x) = \liminf_{r\to 0} \frac{\mathrm{cap}^L_\mu(B_r(x),B_{2r}(x))}{\nu(B_r(x))} \le \limsup_{r\to 0} \frac{\mathrm{cap}^L_\mu(B_r(x),B_{2r}(x))}{\nu(B_r(x))} \le$$

$$\le \limsup_{r\to 0} \frac{\mu(B_r(x))}{\nu(B_r(x))} \le \limsup_{r\to 0} \frac{(f\nu)(B_r(x))}{\nu(B_r(x))} = f(x)$$

which achieves the proof. □

§ 4. THE BOUNDARY CASE

In order to apply the results of this paper to boundary reinforcement problems (see [1],[3],[4],[5],[6]) we are led to extend the theorems of the previous sections to a very particular class of non-uniformly elliptic operators.

Let Ω be a bounded open set of $\mathbf{R}^n$ with a Lipschitz boundary. We shall consider an elliptic operator L of the form (2.3) such that $a_{ij}(x)=0$ for every $x\in\mathbf{R}^n-\Omega$ and

$$\alpha|z|^2 \le \sum_{i,j=1}^{n} a_{ij}(x)z_jz_i \le \beta|z|^2 \qquad \text{for every } x\in\Omega \text{ and } z\in\mathbf{R}^n$$

for suitable constants $0<\alpha\le\beta$.

Let μ be a Borel measure in $\mathcal{M}_0$ with $\mathrm{spt}\mu\subset\overline{\Omega}$. Then, since $a_{ij}=0$ on $\mathbf{R}^n-\Omega$, the corresponding capacity cap_μ^L satisfies

$$(4.1)\qquad \mathrm{cap}_\mu^L(E,A) = \inf\left\{\int_{A\cap\Omega} \sum_{i,j=1}^{n} a_{ij}(x)D_juD_iu\,dx + \int_{E\cap\overline{\Omega}} u^2\,d\mu \;:\; u-1\in H_0^1(A)\right\}$$

for every open set $A\subset\mathbf{R}^n$ and every Borel set $E\subset A$.

Of course, we may apply Theorem 2.3 to the interior points $x\in\Omega$. We now give an extension of the derivation theorem to the boundary points $x\in\partial\Omega$.

THEOREM 4.1. *If* $\mathrm{spt}\,\mu\subset\overline{\Omega}$, *then Theorem 2.3 holds also for the degenerate operator* L *considered above.*

Proof. The only change is in the proof of Lemma 3.1 for $x\in\partial\Omega$. Since Ω has a Lipschitz boundary, for every $x_0\in\partial\Omega$ there exist a neighbourhood U of x_0 and a Lipschitz map ϕ from U into a ball $B_R(0)$ with a Lipschitz inverse ψ, such that $\phi(x_0)=0$ and

$$\phi(U\cap\Omega) = B_R^+ = \{y\in B_R(0) : y_n>0\} \quad .$$

Let $\tilde{\mu}$ be the measure on $B_R(0)$ defined by

$$\tilde{\mu}(E) = \mu(\psi(E)) \quad .$$

Note that $\tilde{\mu}$ is supported by $S_R^+=\{y\in B_R(0) : y_n\ge 0\}$.

Let $\tilde{L} = -\sum_{i,j=1}^{n} D_i(\tilde{a}_{ij}(y)D_j)$ be the elliptic operator on B_R^+ with

$$\tilde{a}_{ij}(y) = \sum_{h,k=1}^{n} (a_{hk}D_h\phi_iD_k\phi_j)(\psi(y))\,|\det D\psi(y)| \;,$$

and let $\mathrm{cap}_{\tilde{\mu}}^{\tilde{L}}$ be the $\tilde{\mu}$-capacity defined as in (2.4) with μ,L,A,E replaced by $\tilde{\mu}$,$\tilde{L}$, $A\cap B_R^+$, $E\cap S_R^+$ respectively. It is easy to see that for every $x\in U\cap\partial\Omega$ the left hand side of (3.1) becomes

$$(4.2) \qquad \lim_{r\to 0} \frac{\mathrm{cap}_{\tilde{\mu}}^{\tilde{L}}(\tilde{B}_r(y),\tilde{B}_{2r}(y))}{\tilde{\mu}(\tilde{B}_r(y))}$$

where $y=\phi(x)$ and for every $r>0$ $\tilde{B}_r(y)=\phi(B_r(x))$.

In order to reduce the problem to the case of an interior point, already considered in Lemma 3.1, we use a symmetrization argument.

Let $\tau:R^n\longrightarrow R^n$ be the map defined by

$$\tau(y_1,\cdots,y_{n-1},y_n) = (y_1,\cdots,y_{n-1},-y_n) .$$

For every Borel set $E\subset S_R^+$ we put

$$\hat{\mu}(E) = \tilde{\mu}(E\cap S_R^+) + \tilde{\mu}(\tau(E)\cap S_R^+) \;;$$

let $\hat{L} = -\sum_{i,j=1}^{n} D_i(\hat{a}_{ij}(y)D_j)$ be the elliptic operator on B_R^+ with

$$\hat{a}_{ij}(y) = \begin{cases} \tilde{a}_{ij}(y) & \text{if } y_n>0 \\ c_{ij}\tilde{a}_{ij}(\tau(y)) & \text{if } y_n<0 , \end{cases}$$

where $c_{ij}=1$ if $i,j<n$ or $i=j=n$, and $c_{ij}=-1$ otherwise.

Finally, let $\mathrm{cap}_{\hat{\mu}}^{\hat{L}}$ be the $\hat{\mu}$-capacity defined as in (2.4). It is easy to see that for every $r>0$

$$\hat{\mu}(\hat{B}_r(y)) = 2\,\tilde{\mu}(\tilde{B}_r(y))$$

$$\mathrm{cap}_{\hat{\mu}}^{\hat{L}}(\hat{B}_r(y),\hat{B}_{2r}(y)) = 2\,\mathrm{cap}_{\tilde{\mu}}^{\tilde{L}}(\tilde{B}_r(y),\tilde{B}_{2r}(y))$$

where $\hat{B}_r(y) = [\tilde{B}_r(y)\cap S_R^+]\cup\tau(\tilde{B}_r(y)\cap S_R^+)$. Therefore, (4.2) becomes

$$(4.3) \qquad \lim_{r\to 0} \frac{\mathrm{cap}_{\hat{\mu}}^{\hat{L}}(\hat{B}_r(y),\hat{B}_{2r}(y))}{\hat{\mu}(\hat{B}_r(y))} .$$

Now, by repeating the proof of Lemma 3.1, and by using a more refined version of Theorem 2.1 (see for instance [11], Section 2.9), we obtain that the limit in (4.3) equals 1 for $\hat{\mu}$-a.e. $y\in B_R(0)\cap\{y_n=0\}$, therefore the limit in (4.1) equals 1 for μ-a.e. $x\in U\cap\partial\Omega$. □

§ 5. APPLICATIONS TO γ-CONVERGENCE

In this section we apply the derivation theorem proved in Section 3 to the characterization of the γ-limit of a sequence (μ_h) of measures of the class $\mathfrak{M}_0$.

Let L be a uniformly elliptic operator as in Section 2 (see(2.3)). With every $\mu\in\mathfrak{M}_0$ we associate the family of quadratic functionals

$$F^L_\mu(u,\Omega) = \begin{cases} \int_\Omega \left[\sum_{i,j=1}^n a_{ij}(x) D_j u D_i u\right] dx + \int_\Omega u^2 \, d\mu & \text{if } u\in H^1_0(\Omega) \\ +\infty & \text{elsewhere in } L^2(\Omega) \end{cases}$$

where Ω is an arbitrary bounded open subset of R^n.

The γ^L-convergence of a sequence (μ_h) is defined in terms of the Γ^-convergence of the corresponding functionals $(F^L_{\mu_h})$. For the definition and properties of Γ^-convergence we refer to [9],[10] and to [2], where it is called epi-convergence.

DEFINITION 5.1. *We say that a sequence* (μ_h) *in* $\mathfrak{M}_0$ γ^L*-converges to a measure* $\mu\in\mathfrak{M}_0$ *if, for every bounded open subset* Ω *of* R^n, *the functionals* $F^L_{\mu_h}(\cdot,\Omega)$ Γ^-*converge to* F^L_μ *in the space* $L^2(\Omega)$.

When L is the Laplace operator $-\Delta$, then the γ^L-convergence is the γ-convergence introduced in [7] (Definition 4.8). In the general case of an elliptic operator L, the properties of γ^L-convergence can be easily obtained from the corresponding properties of γ-convergence studied in Sections 4 and 5 of [7].

THEOREM 5.2. *Let* (μ_h) *be a sequence of Borel measures of the class* $\mathfrak{M}_0$, *let* ν *be a Radon measure of the class* $\mathfrak{M}_0$, *and let* $f:R^n \to R^n$ *be a nonnegative function. Assume that*

i) *for* q.e. $x\in R^n$

$$\liminf_{r\to 0} \liminf_{h\to\infty} \frac{\mathrm{cap}^L_{\mu_h}(B_r(x),B_{2r}(x))}{\nu(B_r(x))} = \liminf_{r\to 0} \limsup_{h\to\infty} \frac{\mathrm{cap}^L_{\mu_h}(B_r(x),B_{2r}(x))}{\nu(B_r(x))} = f(x) ;$$

ii) $f\in L^1_{loc}(R^n,\nu)$;

iii) $f(x) < +\infty$ q.e. *on* R^n .

Then (μ_h) γ^L*-converges to* $\mu=f\nu$.

Proof. Since the γ^L-convergence is metrizable and compact (see [7], Proposition 4.9 and Theorem 4.14) we may assume that (μ_h) γ^L-converges to a measure $\mu\in\mathcal{M}_0$ and we have only to prove that $\mu=f\nu$. By Theorem 5.15 of [7] we obtain that for every $x\in R^n$

$$\lim_{h\to\infty} \mathrm{cap}^L_{\mu_h}(B_r(x),B_{2r}(x)) = \mathrm{cap}^L_{\mu}(B_r(x),B_{2r}(x)) \tag{5.1}$$

for all $r>0$ in a complement of a countable set. From (5.1) and hypothesis i) we get

$$f(x) = \liminf_{r\to 0} \frac{\mathrm{cap}^L_{\mu}(B_r(x),B_{2r}(x))}{\nu(B_r(x))} \qquad \text{q.e. on } R^n .$$

Recalling that $\mu+\nu$ vanishes on the sets of capacity zero, we can apply Theorem 2.3 and we conclude that $\mu=f\nu$. □

REMARK 5.3. A similar result can be obtained for the operators L considered in Section 4 and for the corresponding capacities (see [4]).

REMARK 5.4. In the case when ν is the n-dimensional Lebesgue measure or the n-1 dimensional measure on a smooth manifold, Theorem 5.2 has been obtained by Marchenko and Hruslov (see [14],[15]).

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