LARGE-SCALE AVERAGING ANALYSIS OF SINGLE PHASE FLOW IN FRACTURED RESERVOIRS

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IMA Preprint Series # 996

July 1992

LARGE-SCALE AVERAGING ANALYSIS OF SINGLE PHASE FLOW IN FRACTURED RESERVOIRS+

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Abstract. The method of large-scale averaging is introduced to derive and analyze the most general form of the dual-porosity model of single phase flow in naturally fractured reservoirs. The dual-porosity model contains the usual equations based on Darcy's law and the coupling terms representing the fluid transfer between the matrix and the fractures. A transient closure problem is developed in order to obtain and analyze the fracture and matrix permeability tensors and the fluid transfer terms. The spatial averaging theorem is presented from a mathematical point of view and proved rigorously by the distribution theory. The problem of well-posedness of the dual-porosity model is also considered. The techniques developed here are not restricted to either regular geometric fractures or spatially periodic reservoirs.

Key words. large-scale averaging, single phase flow, fractured reservoir, spatial averaging theorem

AMS(MOS) subject classifications. 76S05

1. Introduction.

It has been known that flow in naturally fractured reservoirs is not like that in unfractured reservoirs [2], [3], [5], [10], [19], [24], [30]; the flow acts as if the fractured reservoirs possessed two porous structures, one associated with the system of fractures and the other with the matrix. This dual-porosity concept has been used to model the flow of fluid within naturally fractured reservoirs since the 1960's [5], [19], [30], [33].

Recently, a general form of the dual-porosity model of single phase flow has been described [2], [3], [4], [12]. Some of these models, including the earlier ones, were derived on the basis of physical intuition under the main assumption that the fluid pressure (or density) is uniform at the surface of each matrix block. The rest were obtained from the point of view of homogenization theory [7], [27], which limited to reservoirs having spatially a locally periodic structure with geometrically regular fractures.

The critical process in any naturally fractured reservoir is the transfer of fluid between the matrix and the fractures. There exists an extensive literature on the modelling of the fluid transfer [2], [3], [4], [10], [12], [17], [19], [20], [29], [31]. Some of these papers consider models that define the matrix-fracture interaction by introducing various *ad hoc* parameters; the rest handle the interaction directly through boundary conditions imposed on the surface of the matrix blocks. However, the applicability of these models is restricted to fractured reservoirs having a fine and specific geometry of the fractures such as those mentioned above.

In this paper we shall derive and analyze the most general form of the dual-porosity model of single phase flow in naturally fractured reservoirs. We consider the fluid in the fracture system and the fluid in the matrix as separate continua. We make the continuum hypothesis so that the interfaces between the matrix blocks and the fractures are no longer recognizable; i.e., we consider transport on a scale that is

+This work was supported in part by the Army Research Office contract number DAAL03-89-C-0038 with the University of Minnesota Army High Performance Computing Research Center.

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much larger than the individual geometric heterogeneities of the rock. In this sense, the explicit geometric features of the fractures are not important.

The method of large-scale averaging [25] is introduced to derive the dual-porosity model. Traditionally, one uses this averaging process to analyze heterogeneous porous media with the object to capture the effects of heterogeneities [25]. The idea here is to treat a naturally fractured reservoir as a heterogeneous porous medium composed of the fractures and the matrix blocks at the reservoir scale. The interaction between these two very distinct porous structures, as pointed out earlier, has a strong influence on the flow of fluid in the reservoir. The influence will be analyzed by the large-scale averaging method; in particular, we shall derive an explicit expression for the transfer of fluid between the matrix and fractures, which is incorported into both the fracture and the matrix differential equations in a very general way.

A method of closure is also developed in order to obtain expressions for the fracture and matrix permeability tensors. In previous studies [8], [23], [25], [26], a considerable amount of effort has been put in favor of the use of quasi-steady closure schemes for transient processes. In this paper the complete transient closure problem will be solved directly.

The key mathematical tool used in the large-scale averaging method is the spatial averaging theorem [1], [28], [34]. Its derivation has been so far initiated with a temporal form known as the general transport theorem [35] rather than with a direct argument. We shall here present this theorem from a mathematical point of view and rigorously prove it by the distribution theory.

The rest of the paper is organized as follows. In the next section we shall establish the spatial averaging theorem in its most general form. After giving some basic background on the large-scale averaging method in §3, we shall derive the dual-porosity model of single phase flow in §4. The present model will also be compared with other dual-porosity models. Finally, in §5, we shall state a result on the existence, uniqueness, and continuous dependence on data of solution to the present model.

We close this section with a remark. It is relatively simple to model the flow of single phase within a naturally fractured reservoir; the two-phase flow is much more complicated and is of greater practical interest. The problem of two-phase flow in a naturally fractured reservoir is investigated in a forthcoming paper.

2. The spatial averaging theorem. The spatial averaging theorem was presented independently in 1967 by Anderson and Jackson [1], Slattery [28], and Whitaker [34]. The final result was obtained by three different methods and since then many other workers [11], [13], [15], [35] have presented their own versions of this important theorem. Recently, questions have been raised about the validity of the theorem since its derivations were initiated [32], [28], [35] with a temporal form known as the general transport theorem [36] rather than directly with a rigorous argument, as mentioned in the introduction. Howes and Whitaker [18] have re-examed the derivation of this theorem and confirmed its correctness. However, while their argument is formally correct, it is still a re-illustration of the approach above with great care and is not mathematically rigorous. Therefore, the spatial averaging theorem needs to be proven.

We shall restrict our analysis below to transport phenomena associated with a single fluid phase, which will be called the α-phase, in a porous medium $\Omega \in \mathbf{R}^3$. We begin by considering the most general form of the phase average, introduced in [1], which will make use of a weighting function $\widehat{m}(\xi)$, assumed integrable with finite

support such that

$$\int_{\mathbb{R}^3} \widehat{m}(\xi)d\xi = 1.$$

For some generic function ψ_α associated with the α-phase, the phase average is then defined by

$$\langle\psi_\alpha\rangle \equiv \langle\psi_\alpha\rangle(x) = \int_{\mathbb{R}^3} \psi_\alpha(x+\xi)\chi_\alpha(x+\xi)\widehat{m}(\xi)d\xi, \quad x \in \Omega, \tag{2.1}$$

where χ_α is the characteristic function of the α-phase (i.e., χ_α is unity in the α-pahse and zero elsewhere). This expression can be used to rigorously match theory and experiment since the weighting function can be chosen to correspond to the characteristics of measuring devices [6], [11].

If the transformation $x + \xi \to y$ is introduced in (2.1) along with the transposed weighting function $m(\xi) = \widehat{m}(-\xi)$, the resulting expression is

$$\begin{aligned}\langle\psi_\alpha\rangle &= \int_{\mathbb{R}^3} \psi_\alpha(y)\chi_\alpha(y)m(x-y)dy \\ &= \int_{\mathbb{R}^3} \psi_\alpha(x-y)\chi_\alpha(x-y)m(y)dy,\end{aligned} \tag{2.2}$$

which is, by definition, the convolution product of $\psi_\alpha\chi_\alpha$ and m, denoted by

$$\langle\psi_\alpha\rangle = m * (\psi_\alpha\chi_\alpha). \tag{2.3}$$

This formal definition, if understood in the sense of distribution, can be exploited to generalize (2.1) mathematically so that ψ_α needs not be a continuous or piecewise function but can be a generalized function. It is in the most general case that we shall establish the spatial averaging theorem.

LEMMA 2.1. *For some quantity ψ_α associated with the α-phase, we have*

$$\langle\nabla\psi_\alpha\rangle = \nabla\langle\psi_\alpha\rangle + \int_{\partial\Omega_\alpha} \psi_\alpha(y)n_{\alpha\sigma}(y)m(x-y)da(y), \tag{2.4}$$

where Ω_α denotes the volume occupied by the α-phase with the boundary $\partial\Omega_\alpha$ and $n_{\alpha\sigma}$ represents the normal unit-vector to the boundary $\partial\Omega_\alpha$ outwardly directed from the α-phase.

Proof. Equation (2.4) follows by the differentiation of the convolution:

$$\begin{aligned}\nabla\langle\psi_\alpha\rangle &= m * (\chi_\alpha\nabla\psi_\alpha) + m * (\psi_\alpha\nabla\chi_\alpha) \\ &= \langle\nabla\psi_\alpha\rangle + \int_{\mathbb{R}^3} \psi_\alpha(y)\nabla\chi_\alpha(y)m(x-y)dy \\ &= \langle\nabla\psi_\alpha\rangle - \int_{\partial\Omega_\alpha} \psi_\alpha(y)n_{\alpha\sigma}(y)m(x-y)da(y),\end{aligned}$$

by the definition of Ω_α. □

As an application of this lemma, we shall now consider the local spatial averaging theorem introduced in [28], [34]. For $x \in \Omega$, let V be the local averaging volume

centered at x (see Figure 1) and let the weighting function $m(\xi)$ be the characteristic function of $\widehat{V} \equiv x - V$:

$$m(\xi) = \frac{1}{|V|}\chi_{\widehat{V}}(\xi), \tag{2.5}$$

where $|V|$ is the measure of the set V. Then, as a particular case of Lemma 2.1, we have

COROLLARY 2.2. (The local spatial averaging theorem).

$$\langle \nabla \psi_\alpha \rangle = \nabla \langle \psi_\alpha \rangle + \frac{1}{|V|} \int_{A_{\alpha\sigma}} \psi_\alpha n_{\alpha\sigma} da, \tag{2.6}$$

where the local phase average is defined by

$$\langle \psi_\alpha \rangle = \frac{1}{|V|} \int_{V_\alpha} \psi_\alpha dy, \tag{2.7}$$

with V_α being the volume of the α-phase contained within the averaging volume V, and $A_{\alpha\sigma}$ indicates part of the interface $\partial\Omega_\alpha$ contained within V. □

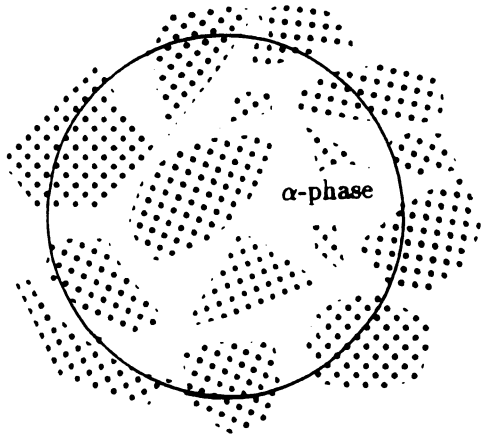

FIG. 1. The local averaging volume V.

Taking $\psi_\alpha = 1$ in (2.6), we obtain the useful relation

$$\frac{1}{|V|} \int_{A_{\alpha\sigma}} n_{\alpha\sigma} da = -\nabla \epsilon_\alpha, \tag{2.8}$$

where ϵ_α is the volume fraction of the α-phase given by

$$\epsilon_\alpha = \frac{|V_\alpha|}{|V|}.$$

3. The large-scale averaging. As pointed out in the introduction, the objective of the method of large-scale averaging is to capture the effects of heterogeneities in a formal manner that can be applied to all transport processes in poroue media. It is important to note that all large-scale averaging precesses incorporate the influence of heterogeneities into averaged equations [25], while local volume averaging procedures as originally put forth [1], [28], [34] incorporate boundary conditions into averaged equations. Here, we would expect that the large-scale averaging methods incorporate the effects of the interaction between the matrix blocks and the fractures into the averaged differential equations.

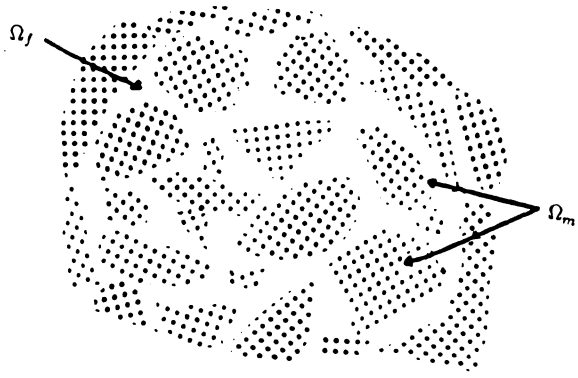

FIG. 2. The fractured reservoir Ω.

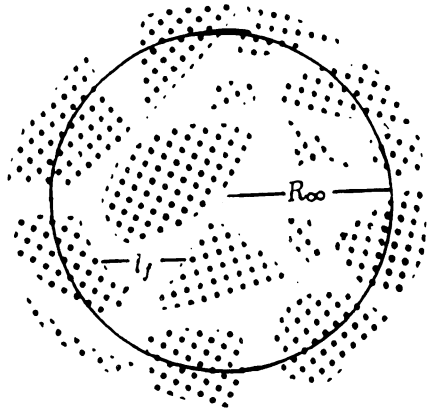

FIG. 3. The large-scale average V_∞.

The large-scale averaging method is applied to average the Darcy-scale equations over a volume V_∞, which will be called the large-scale averaging volume. Crucial to the validity of the large-scale averaging process is the assumption that the averaging volume size is independent of the location in the medium. Thus, we would expect that a sufficient number of fractures and matrix blocks exist in the averaging volume. If we denote by l_f and l_m the characteristic scales of the fractures and the matrix blocks, respectively, the radius R_∞ of the volume V_∞ must be large compared to l_f and l_m:

$$l_\alpha \ll R_\infty, \quad \alpha = f,\ m. \tag{3.1}$$

In addition, because the averaging volume acts as the smallest discernible dimensions that are indicative of the continuum scale, we shall also assume that the radius R_∞ is small relative to the macroscopic length scale L of the fractured reservoir:

$$R_\infty \ll L. \tag{3.2}$$

While these constraints are usually satisfied, in the analysis later we shall be careful to point out where they are applied.

A typical reservoir is shown in Figure 2 and a large-scale averaging volume is illustrated in Figure 3.

There are two types of volume averages that will be used in the theoretical development of averaged equations. The first of these is the large-scale phase average defined by

$$\langle \psi_f \rangle = \frac{1}{|V_\infty|} \int_{V_f} \psi_f dx, \tag{3.3}$$

$$\langle \psi_m \rangle = \frac{1}{|V_\infty|} \int_{V_m} \psi_m dx, \tag{3.4}$$

where ψ_f and ψ_m represent some generic functions associated with the fracture and matrix systems and V_f and V_m indicate the volumes of the fractures and the matrix blocks contained within the averaging volume V_∞, respectively. To describe more closely measured values or values imposed on a boundary, one usually uses the second volume average; i.e., the intrinsic phase average defined by

$$\langle \psi_f \rangle^f = \frac{1}{|V_f|} \int_{V_f} \psi_f dx, \tag{3.5}$$

$$\langle \psi_m \rangle^m = \frac{1}{|V_m|} \int_{V_m} \psi_m dx. \tag{3.6}$$

With the definitions above, we shall derive some useful relations. First, it follows from Corollary 2.2 that

$$\langle \nabla \psi_f \rangle = \nabla \langle \psi_f \rangle + \frac{1}{|V_\infty|} \int_{A_{fm}} n_{fm} \psi_f da, \tag{3.7}$$

$$\langle \nabla \psi_m \rangle = \nabla \langle \psi_m \rangle + \frac{1}{|V_\infty|} \int_{A_{mf}} n_{mf} \psi_m da, \tag{3.8}$$

where $A_{fm} = A_{mf}$ is the interface between the fractures and the matrix blocks contained within the averaging volume V_∞ and n_{fm} and n_{mf} denote the unit outwardly directed normal vectors for the fracture system and the matrix, respectively. Secondly, the phase average is related to the intrinsic phase average by

$$\langle \psi_f \rangle = \epsilon_f \langle \psi_f \rangle^f, \tag{3.9}$$

$$\langle \psi_m \rangle = \epsilon_m \langle \psi_m \rangle^m, \tag{3.10}$$

where ϵ_f nad ϵ_m are the volume fractions of the fractures and the matrix given by

$$\epsilon_f = \frac{|V_f|}{|V_\infty|}, \tag{3.11}$$

$$\epsilon_m = \frac{|V_m|}{|V_\infty|}. \tag{3.12}$$

Consequently,

$$\epsilon_f + \epsilon_m = 1. \tag{3.13}$$

Finally, taking $\psi_f = 1$ and $\psi_m = 1$ in equations (3.7) nad (3.8), we see that

$$\frac{1}{|V_\infty|}\int_{A_{fm}} n_{fm}\,da = -\nabla\epsilon_f, \tag{3.14}$$

$$\frac{1}{|V_\infty|}\int_{A_{mf}} n_{mf}\,da = -\nabla\epsilon_m. \tag{3.15}$$

In the following, the large-scale averaging volume V_∞ will be taken to be a sphere of constant radius as shown in Figure 3, while V_f and V_m depend on the nature of the reservoir under consideration and will be functions of the space only. However, V_∞ is not necessarily chosen to be a sphere; in the analysis below it can be taken to be a parallelepiped constructed by the lattice vectors [35], for example.

4. Derivation of the dual-porosity model. In this section we shall derive the dual-porosity model of single phase flow in a naturally fractured reservoir Ω by using the large-scale averaging method introduced in the previous sections. The microscopic model, which for simplicity will be taken to have all physical parameters as constants, is given by a single porosity system with discontinuous porosity and permeability; it does so without requiring any concept of dual-porosity [3], [4]. The equations describing single phase flow in a single porosity system are thus posed over the whole domain Ω. However, for use of the averaging method, we shall write these equations on the fractures Ω_f and the matrix Ω_m separately.

Let ρ_f be the density of the fluid and p_f be the pressure in the fracture domain and let ϕ_f^* and k_f^* represent the porosity and permeability of an individual fracture. Denote the correponding matrix quantities by ρ_m, p_m, ϕ_m^*, and k_m^*. Then, the flow in the fracture system is controlled by

$$\phi_f^*\frac{\partial\rho_f}{\partial t} - \nabla\cdot\left(\frac{k_f^*}{\mu c}\nabla\rho_f\right) = q_{f,\text{ext}}, \quad x\in\Omega_f,\ t>0, \tag{4.1}$$

and the flow in the matrix domain by

$$\phi_m^*\frac{\partial\rho_m}{\partial t} - \nabla\cdot\left(\frac{k_m^*}{\mu c}\nabla\rho_m\right) = q_{m,\text{ext}}, \quad x\in\Omega_m,\ t>0, \tag{4.2}$$

where we have denoted by μ the viscosity of the fluid, assumed that the fluid is of a constant compressibility c, i. e., a fluid that satisfies the equation of state

$$d\,\rho_\alpha = c\rho_\alpha d\,p_\alpha, \quad \alpha = f, m, \tag{4.3}$$

indicated wells as external source terms of the form $q_{\alpha,\text{ext}}$, $\alpha = f, m$, and, for simplicity, ignored gravity. Note that equations (4.1) and (4.2) represent conservation of mass combined with Darcy's law and the equations of state given by (4.3).

On the interface $\partial\Omega_m$ between the two domains, we impose continuity of density and continuity of mass flux. Namely,

$$\rho_f = \rho_m, \qquad x\in\partial\Omega_m,\ t>0, \tag{4.4}$$

$$\left(\frac{k_f^*}{\mu c}\nabla\rho_f\right)\cdot n_{fm} = \left(\frac{k_m^*}{\mu c}\nabla\rho_m\right)\cdot n_{fm}, \quad x\in\partial\Omega_m,\ t>0. \tag{4.5}$$

The microscopic model is completed by specifying outer boundary conditions and the initial densities. However, we shall ignore the outer boundary conditions since they play no role in the analysis below. But, the initial densities must be given as we shall be concerned with transient closure problems later:

$$\rho_f(x,0) = \rho_f^0, \quad x \in \Omega_f, \tag{4.6}$$

$$\rho_m(x,0) = \rho_m^0, \quad x \in \Omega_m. \tag{4.7}$$

We now begin the large-scale averaging process with equation (4.1) to obtain

$$\phi_f^* \left\langle \frac{\partial \rho_f}{\partial t} \right\rangle - \left\langle \nabla \cdot \left(\frac{k_f^*}{\mu c} \nabla \rho_f \right) \right\rangle = \langle q_{f,\text{ext}} \rangle .$$

Since averaging volumes under consideration are independent of time, this together with relation (3.7) leads to

$$\phi_f^* \frac{\partial}{\partial t} \langle \rho_f \rangle - \nabla \cdot \left\langle \frac{k_f^*}{\mu c} \nabla \rho_f \right\rangle - \frac{1}{|V_\infty|} \int_{A_{fm}} n_{fm} \frac{k_f^*}{\mu c} \nabla \rho_f \, da = \langle q_{f,\text{ext}} \rangle .$$

A second application of equation (3.7) to this expression implies that

$$\begin{aligned} \phi_f^* \frac{\partial}{\partial t} \langle \rho_f \rangle - \nabla \cdot & \left(\frac{k_f^*}{\mu c} \nabla \langle \rho_f \rangle + \frac{1}{|V_\infty|} \int_{A_{fm}} n_{fm} \frac{k_f^*}{\mu c} \rho_f \, da \right) \\ & - \frac{1}{|V_\infty|} \int_{A_{fm}} n_{fm} \frac{k_f^*}{\mu c} \nabla \rho_f \, da = \langle q_{f,\text{ext}} \rangle . \end{aligned} \tag{4.8}$$

We now repeat the argument above combined with use of relation (3.8) for equation (4.2) to obtain

$$\begin{aligned} \phi_m^* \frac{\partial}{\partial t} \langle \rho_m \rangle - \nabla \cdot & \left(\frac{k_m^*}{\mu c} \nabla \langle \rho_m \rangle + \frac{1}{|V_\infty|} \int_{A_{mf}} n_{mf} \frac{k_m^*}{\mu c} \rho_m \, da \right) \\ & - \frac{1}{|V_\infty|} \int_{A_{mf}} n_{mf} \frac{k_m^*}{\mu c} \nabla \rho_m \, da = \langle q_{m,\text{ext}} \rangle . \end{aligned} \tag{4.9}$$

As mentioned in the last section, the phase average is not the preferred one; indeed, one normally requires the intrinsic phase average since it more accurately corresponds to measured values. This needs to use relations (3.9)-(3.10) along with the following decompositions [14], [37]:

$$\rho_f = \langle \rho_f \rangle^f + \widehat{\rho}_f, \tag{4.10}$$

$$\rho_m = \langle \rho_m \rangle^f + \widehat{\rho}_m, \tag{4.11}$$

where $\widehat{\rho}_f$ and $\widehat{\rho}_m$ are spatial deviations of the density in the fracture and matrix domains, respectively, which we shall assume to satisfy the relationship on the interface $\partial\Omega_m$

$$\widehat{\rho}_f = \widehat{\rho}_m, \quad x \in \partial\Omega_m. \tag{4.12}$$

That is, we require continuity of the spatial deviations of the density on $\partial\Omega_m$.

In the performance below we shall remove the averaged quantities $\langle\rho_f\rangle^f$ and $\langle\rho_m\rangle^m$ from the integrals over A_{fm}. By doing this, we are committed to assuming these quantities to be constant with respect to integration over A_{fm}. The process of removing the large-scaled quantities from integrals over A_{fm} or over V_α, $\alpha = f, m$, will be repeated in the analysis of the closure problem later on and gives rise to a constraint of the form [9]

$$\left(\frac{R_\infty}{L}\right)^2 \ll 1. \tag{4.13}$$

This is satisfied by assumption (3.2).

We now turn to equations (4.8) and (4.9). Substitute expressions (4.10) and (4.11) into these two equations and use relations (3.9)-(3.10) and (3.14)-(3.15) to find that

(4.14)
$$\phi_f \frac{\partial}{\partial t}\langle\rho_f\rangle^f - \nabla\cdot\left(\frac{k_f^*}{\mu c}\epsilon_f\nabla\langle\rho_f\rangle^f + \frac{1}{|V_\infty|}\int_{A_{fm}} n_{fm}\frac{k_f^*}{\mu c}\widehat{\rho}_f da\right)$$
$$-\frac{1}{|V_\infty|}\int_{A_{fm}} n_{fm}\frac{k_f^*}{\mu c}\nabla\langle\rho_f\rangle^f da - \frac{1}{|V_\infty|}\int_{A_{fm}} n_{fm}\frac{k_f^*}{\mu c}\nabla\widehat{\rho}_f da = \langle q_{f,\text{ext}}\rangle,$$

and

(4.15)
$$\phi_m \frac{\partial}{\partial t}\langle\rho_m\rangle^m - \nabla\cdot\left(\frac{k_m^*}{\mu c}\epsilon_m\nabla\langle\rho_m\rangle^m + \frac{1}{|V_\infty|}\int_{A_{mf}} n_{mf}\frac{k_m^*}{\mu c}\widehat{\rho}_m da\right)$$
$$-\frac{1}{|V_\infty|}\int_{A_{mf}} n_{mf}\frac{k_m^*}{\mu c}\nabla\langle\rho_m\rangle^m da - \frac{1}{|V_\infty|}\int_{A_{mf}} n_{mf}\frac{k_m^*}{\mu c}\nabla\widehat{\rho}_m da = \langle q_{m,\text{ext}}\rangle,$$

where $\phi_f = \epsilon_f\phi_f^*$ and $\phi_m = \epsilon_m\phi_m^*$ are clearly the fracture and matrix porosities, respectively. For convenience below, we rewrite (4.14) and (4.15), using the fact that $\langle\widehat{\rho}_f\rangle^f = \langle\widehat{\rho}_m\rangle^m = 0$ and relations (3.7)-(3.8) and (3.14)-(3.15), as

$$\phi_f \frac{\partial}{\partial t}\langle\rho_f\rangle^f - \epsilon_f\nabla\cdot\left(\frac{k_f^*}{\mu c}\nabla\langle\rho_f\rangle^f\right) - \left\langle\nabla\cdot\left(\frac{k_f^*}{\mu c}\nabla\widehat{\rho}_f\right)\right\rangle = \langle q_{f,\text{ext}}\rangle, \tag{4.16}$$

(4.17)
$$\phi_m \frac{\partial}{\partial t}\langle\rho_m\rangle^m - \epsilon_m\nabla\cdot\left(\frac{k_m^*}{\mu c}\nabla\langle\rho_m\rangle^m\right) - \left\langle\nabla\cdot\left(\frac{k_m^*}{\mu c}\nabla\widehat{\rho}_m\right)\right\rangle = \langle q_{m,\text{ext}}\rangle.$$

From equations (4.14)-(4.15) or (4.16)-(4.17), it becomes clear that representations for $\widehat{\rho}_f$ and $\widehat{\rho}_m$ are required in order to develop a deterministic set of equations. This will be done by considering a so-called closure problem.

4.1. A transient closure problem. Substitute equations (4.10) and (4.11) into equations (4.1) and (4.2), respectively, and use expressions (4.16) and (4.17) to

see that

$$\phi_f^* \frac{\partial \widehat{\rho}_f}{\partial t} - \nabla \cdot \left(\frac{k_f^*}{\mu c} \nabla \widehat{\rho}_f \right) = q_{f,\text{ext}} - \langle q_{f,\text{ext}} \rangle^f - \left\langle \nabla \cdot \left(\frac{k_f^*}{\mu c} \nabla \widehat{\rho}_f \right) \right\rangle^f,$$

(4.18) $$x \in \Omega_f, \ t > 0,$$

$$\phi_m^* \frac{\partial \widehat{\rho}_m}{\partial t} - \nabla \cdot \left(\frac{k_m^*}{\mu c} \nabla \widehat{\rho}_m \right) = q_{m,\text{ext}} - \langle q_{m,\text{ext}} \rangle^m - \left\langle \nabla \cdot \left(\frac{k_m^*}{\mu c} \nabla \widehat{\rho}_m \right) \right\rangle^m,$$

(4.19) $$x \in \Omega_m, \ t > 0.$$

On the boundary $\partial\Omega_m$, in addition to continuity assumption (4.12), we also have

$$\left(\frac{k_f^*}{\mu c} \left(\langle \nabla \rho_f \rangle^f + \nabla \widehat{\rho}_f \right) \right) \cdot n_{fm} = \left(\frac{k_m^*}{\mu c} \left(\langle \nabla \rho_m \rangle^m + \nabla \widehat{\rho}_m \right) \right) \cdot n_{fm},$$

(4.20) $$x \in \partial\Omega_m, \ t > 0,$$

by (4.5).

We need to solve equations (4.12) and (4.18)-(4.20) for $\widehat{\rho}_f$ and $\widehat{\rho}_m$. In general, time- and length-scale constraints have been imposed to use quasi-steady closure schemes for transient processes. In analyses of diffusion [27], dispersion [8], heat conduction [23], and local heterogeneity in porous media [25], for example, one can evoke some time- and length-scale constraints in order to pass from time-dependent closure problems to the corresponding stationary problems.

In this paper the closure problem given by equations (4.12) and (4.18)-(4.20) will be solved. Toward that end, we shall now develop initial conditions for $\widehat{\rho}_f$ and $\widehat{\rho}_m$. For this, we make the two assumptions: The initial data ρ_f^0 and ρ_m^0 are constant and $\langle q_{\alpha,\text{ext}} \rangle^\alpha = q_{\alpha,\text{ext}}$, $\alpha = f, m$. The second condition implies that the external source terms $q_{f,\text{ext}}$ and $q_{m,\text{ext}}$ are constant with respect to integration over V_f and V_m, respectively. This is satisfied when the source terms are constant or when they are uniformly distributed over a spatially periodic reservoir (see Figure 4 in §4.2), for example.

With the first assumption, it follows from equations (4.6)-(4.7) and (4.10)-(4.11) that

(4.21) $$\widehat{\rho}_f(x, 0) = 0, \quad x \in \Omega_f,$$

(4.22) $$\widehat{\rho}_m(x, 0) = 0, \quad x \in \Omega_m.$$

We shall now seek solutions of the form

(4.23) $$\widehat{\rho}_f = b_f^1(x) \cdot \nabla \langle \rho_f \rangle^f + b_f^2(x) \cdot \nabla \langle \rho_m \rangle^m + \xi_f(x, t),$$

(4.24) $$\widehat{\rho}_m = b_m^1(x) \cdot \nabla \langle \rho_f \rangle^f + b_m^2(x) \cdot \nabla \langle \rho_m \rangle^m + \xi_m(x, t),$$

where ξ_f and ξ_m are completely arbitrary functions. The arbitrariness of ξ_f and ξ_m allows us to specify the coefficients in representations (4.23) and (4.24) in any way we want; we choose to specify them according to the following boundary-value problems:

(4.25a) $$\nabla \cdot (k_f^* \nabla b_f^1) = \langle \nabla \cdot (k_f^* \nabla b_f^1) \rangle^f, \quad x \in \Omega_f,$$

(4.25b) $$b_f^1 = b_m^1, \quad x \in \partial\Omega_m,$$

(4.25c) $$n_{fm} \cdot (k_f^* I + k_f^* \nabla b_f^1) = n_{fm} \cdot (k_m^* \nabla b_m^1), \quad x \in \partial\Omega_m,$$

(4.25d) $$\nabla \cdot (k_m^* \nabla b_m^1) = \langle \nabla \cdot (k_m^* \nabla b_m^1) \rangle^m, \quad x \in \Omega_m,$$

(4.25e) $$\langle b_f^1 \rangle^f = \langle b_m^1 \rangle^m = 0,$$

and

$$\nabla \cdot \left(k_f^* \nabla b_f^2\right) = \left\langle \nabla \cdot \left(k_f^* \nabla b_f^2\right)\right\rangle^f, \qquad x \in \Omega_f, \tag{4.26a}$$

$$b_f^2 = b_m^2, \qquad x \in \partial\Omega_m, \tag{4.26b}$$

$$n_{fm} \cdot \left(k_f^* \nabla b_f^2\right) = n_{fm} \cdot \left(k_m^* I + k_m^* \nabla b_m^2\right), \quad x \in \partial\Omega_m, \tag{4.26c}$$

$$\nabla \cdot \left(k_m^* \nabla b_m^2\right) = \left\langle \nabla \cdot \left(k_m^* \nabla b_m^2\right)\right\rangle^m, \qquad x \in \Omega_m, \tag{4.26d}$$

$$\left\langle b_f^2 \right\rangle^f = \left\langle b_m^2 \right\rangle^m = 0, \tag{4.26e}$$

where I is the identity tensor.

The initial-boundary value problems for ξ_f and ξ_m are obtained by substitution of equations (4.23)-(4.24) into (4.12) and (4.18)-(4.22) and use of equations (4.25)-(4.26) and the facts that $\langle q_{\alpha,\text{ext}}\rangle^\alpha = q_{\alpha,\text{ext}}$ and $\nabla \langle \rho_\alpha \rangle^\alpha (x,0) = 0$, $\alpha = f, m$:

$$\phi_f^* \frac{\partial \xi_f}{\partial t} - \nabla \cdot \left(\frac{k_f^*}{\mu c} \nabla \xi_f\right) = -\left\langle \nabla \cdot \left(\frac{k_f^*}{\mu c} \nabla \xi_f\right)\right\rangle^f, \quad x \in \Omega_f,\ t > 0, \tag{4.27a}$$

$$\xi_f = \xi_m, \qquad x \in \partial\Omega_m,\ t > 0, \tag{2.27b}$$

$$\xi_f(x,0) = 0, \qquad x \in \Omega_f, \tag{4.27c}$$

$$\langle \xi_f \rangle^f = 0, \qquad t > 0, \tag{4.27d}$$

and

$$\phi_m^* \frac{\partial \xi_m}{\partial t} - \nabla \cdot \left(\frac{k_m^*}{\mu c} \nabla \xi_m\right) = -\left\langle \nabla \cdot \left(\frac{k_m^*}{\mu c} \nabla \xi_m\right)\right\rangle^m, \quad x \in \Omega_m,\ t > 0, \tag{4.28a}$$

$$\left(k_f^* \nabla \xi_f\right) \cdot n_{fm} = \left(k_m^* \nabla \xi_m\right) \cdot n_{fm}, \qquad x \in \partial\Omega_m,\ t > 0, \tag{2.28b}$$

$$\xi_m(x,0) = 0, \qquad x \in \Omega_m, \tag{4.28c}$$

$$\langle \xi_m \rangle^m = 0, \qquad t > 0. \tag{4.28d}$$

In deriving equations (4.27a) and (4.28a-b), we have treated the large-scale averaged quantities, as mentioned before, as constants, which can be justified as usual [9], [37] by requiring the length-scale constraint

$$l_\alpha \ll L, \quad \alpha = f, m, \tag{4.29}$$

except terms of the form

$$\phi_\alpha^* \frac{\partial}{\partial t} \nabla \langle \rho_\alpha \rangle^\alpha, \quad \alpha = f, m.$$

Applying equations (4.16)-(4.17) and the fact that $\nabla \langle q_{\alpha,\text{ext}} \rangle^\alpha = 0$, the quantities above are equal to

$$\nabla\nabla \cdot \left(\frac{k_\alpha^*}{\mu c} \nabla \langle \rho_\alpha \rangle^\alpha\right) - \nabla \left\langle \nabla \cdot \left(\frac{k_\alpha^*}{\mu c} \nabla \widehat{\rho}_\alpha\right)\right\rangle^\alpha, \quad \alpha = f, m.$$

These high order terms can be analyzed by standard magnitude arguments [9], [37] and be neglected due to constraint (3.1) and (3.2).

4.2. A spatially periodic reservoir. It is appealing to note that the solution to the system given by equations (4.27)-(4.28) has the null solution: $\xi_f = \xi_m = 0$. While we have no general proof, this can be proved for a spatially periodic reservoir as shown in Figure 4. For the spatially periodic reservoir, we impose, in addition to conditions (4.27b) and (4.28b), the periodicity condition

$$\xi_\alpha(x + l_j) = \xi_\alpha(x), \quad j = 1,2,3, \ \alpha = f, m, \tag{4.30}$$

where l_j denote the three lattice vectors that are needed to characterize a unit cell (see Figure 4).

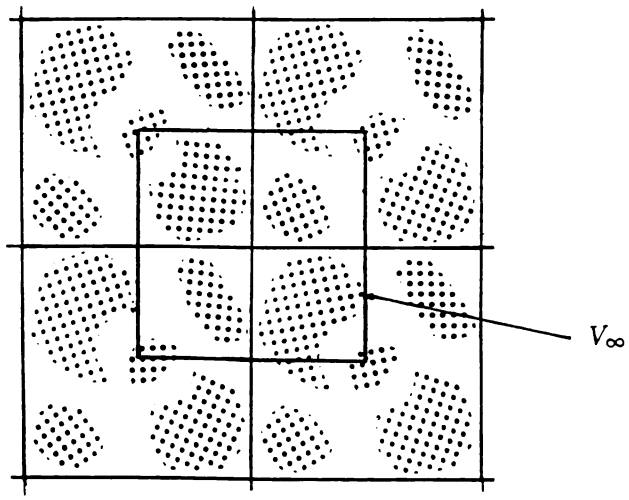

FIG. 4. The large-scale average in a spatially periodic reservoir.

It can be seen from the development above that the large-scale averaging volume is not necessarily a sphere. In fact, V_∞ can be taken to be a parallelepiped constructed by the lattice vectors l_j, $j = 1,2,3$, as in [37]. Note that this does not compromise the length-scale constraint (3.1), since one can always choose a unit cell having a characteristic length that is large relative to l_α, $\alpha = f, m$. When V_∞ is constructed in this manner, it can be easily seen that integration over V_f and V_m always takes place over the fractures and the matrix contained within an entire unit, respectively, while not necesarily the same unit cell. Hence, the periodicity condition (4.30) implies that the integrals on the right-hand side of equations (4.27a) and (4.28a) are constant.

Taking the scalar product of equation (4.27a) with ξ_f, integrating the resulting equation over V_f^*, and using (4.27d), we observe that

$$\int_{V_f^*} \phi_f^* \frac{\partial \xi_f}{\partial t} \xi_f \, dx - \int_{V_f^*} \nabla \cdot \left(\frac{k_f^*}{\mu c} \nabla \xi_f \right) \xi_f \, dx = 0, \tag{4.31}$$

where V_f^* represents the volume of the fractures contained in a single unit cell. It should be noted that V_f^* and V_f coincide only in the special case where the centroid

of V_∞ has the same coordinates with that of the unit cell. Use of Green's formula for the second term on the left-hand side of equation (4.31) and condition (4.30) and integration of the resulting equation with respect to t shows that

$$\frac{1}{2}\phi_f^* \int_{V_f^*} \xi_f^2(t)dx + \int_0^t \int_{V_f^*} \left(\frac{k_f^*}{\mu c}\nabla\xi_f\right)\cdot\nabla\xi_f dx d\tau$$

$$- \int_0^t \int_{A_{fm}^*} \left(\frac{k_f^*}{\mu c}\nabla\xi_f\right)\cdot n_{fm}\xi_f da d\tau = 0, \quad t > 0, \tag{4.32}$$

by (4.27c), where A_{fm}^* is the interface between the fractures and the matrix contained within the unit cell.

The same argument also applies to equation (4.28) to obtain

$$\frac{1}{2}\phi_m^* \int_{V_m^*} \xi_m^2(t)dx + \int_0^t \int_{V_m^*} \left(\frac{k_m^*}{\mu c}\nabla\xi_m\right)\cdot\nabla\xi_m dx d\tau$$

$$- \int_0^t \int_{A_{mf}^*} \left(\frac{k_m^*}{\mu c}\nabla\xi_m\right)\cdot n_{mf}\xi_m da d\tau = 0, \quad t > 0, \tag{4.33}$$

where V_m^* and A_{mf}^* have similar meanings to V_f^* and A_{fm}^*. Now, add (4.32) and (4.33) and use equations (4.27b) and (4.28b) and the fact that $A_{fm}^* = A_{mf}^*$ and $n_{fm} = -n_{mf}$ to find that

$$\frac{1}{2}\left(\phi_f^* \int_{V_f^*} \xi_f^2(t)dx + \phi_m^* \int_{V_m^*} \xi_m^2(t)dx\right) + \int_0^t \int_{V_f^*} \left(\frac{k_f^*}{\mu c}\nabla\xi_f\right)\cdot\nabla\xi_f dx d\tau$$

$$+ \int_0^t \int_{V_m^*} \left(\frac{k_m^*}{\mu c}\nabla\xi_m\right)\cdot\nabla\xi_m dx d\tau = 0.$$

If k_f^* and k_m^* are assumed to be nonnegative (nonnegative definite in the case of tensors), the expression above shows that $\xi_f(t) = \xi_m(t) = 0$, $t > 0$.

The results above are summarized in the next lemma.

LEMMA 4.1. *Assume that k_f^* and k_m^* are nonnegative. Then, the initial-boundary value problems* (4.27), (4.28), *and* (4.30) *have the null solutions only.* □

4.3. The dual-porosity model. We now turn to equations (4.14) and (4.15). Substitute expressions (4.23) and (4.24) with $\xi_f = \xi_m = 0$ into these two equations to have

$$\phi_f \frac{\partial}{\partial t}\langle\rho_f\rangle^f - \nabla\cdot\left(\frac{k_f}{\mu c}\nabla\langle\rho_f\rangle^f\right) + q_f = \langle q_{f,\text{ext}}\rangle, \tag{4.34}$$

$$\phi_m \frac{\partial}{\partial t}\langle\rho_m\rangle^m - \nabla\cdot\left(\frac{k_m}{\mu c}\nabla\langle\rho_m\rangle^m\right) + q_m = \langle q_{m,\text{ext}}\rangle, \tag{4.35}$$

where $k_f = \left(k_f^{ij}\right)$ and $k_m = \left(k_m^{ij}\right)$ define the fracture and matrix permeability tensors, respectively, by

$$k_f^{ij} = k_f^*\left(\epsilon_f\delta_{ij} + \frac{1}{|V_\infty|}\int_{A_{fm}} n_{fm}^i b_f^{1,j} da\right), \tag{4.36}$$

$$k_m^{ij} = k_m^*\left(\epsilon_m\delta_{ij} + \frac{1}{|V_\infty|}\int_{A_{mf}} n_{mf}^i b_m^{2,j} da\right), \tag{4.37}$$

with δ_{ij} denoting the Kronecker symbol, $n_{fm} = \left(n^i_{fm}\right)$, $b^1_f = \left(b^{1,i}_f\right)$, and $b^2_m = \left(b^{2,i}_m\right)$, and q_f and q_m are the coupling terms representing the fluid transfer between the fractures and the matrix:

(4.38)
$$q_f = -\frac{k^*_f}{\mu c}\left(\frac{1}{|V_\infty|}\int_{A_{fm}} n_{fm}(I + \nabla b^1_f)da \cdot \nabla \langle \rho_f \rangle^f + \langle \Delta b^2_f \rangle \cdot \nabla \langle \rho_m \rangle^m\right),$$

(4.39)
$$q_m = -\frac{k^*_m}{\mu c}\left(\frac{1}{|V_\infty|}\int_{A_{mf}} n_{mf}(I + \nabla b^2_m)da \cdot \nabla \langle \rho_m \rangle^m + \langle \Delta b^1_m \rangle \cdot \nabla \langle \rho_f \rangle^f\right),$$

by (4.25e) and (4.26e).

Note that k_f and k_m reflect the geometry of the fractures and the matrix through the interaction functions b^1_f and b^2_m, respectively.

In practice, one would never solve the boundary value problems (4.25) and (4.26) in their present forms to determine k_α and q_α, $\alpha = f, m$. Instead, one would determine these variables in some representative region of a reservoir. This region can be naturally treated as a unit cell in a spatially periodic reservoir such as the one illustrated in Figure 4. Then, as in the previous subsection, we consider the periodicity condition

$$b^i_\alpha(x + l_j) = b^i_\alpha(x), \quad i = 1, 2,\ j = 1, 2, 3,\ \alpha = f, m, \tag{4.40}$$

and use the same argument as in Lemma 4.1 to have

LEMMA 4.2. *If k^*_f and k^*_m are positive and condition* (4.40) *is satisfied, then the boundary value problems* (4.25) *and* (4.26) *have a unique solution.* □

By condition (4.40) and the divergence theorem, we see that

$$\langle \Delta b^2_f \rangle = \frac{1}{|V_\infty|}\int_{A^*_{fm}} n_{fm}\nabla b^2_f da.$$

Thus, by (4.38), q_f takes the form

(4.41)
$$q_f = -\frac{k^*_f}{\mu c|V_\infty|}\left(\int_{A^*_{fm}} n_{fm}(I + \nabla b^1_f)da \cdot \nabla \langle \rho_f \rangle^f + \int_{A^*_{fm}} n_{fm}\nabla b^2_f da \cdot \nabla \langle \rho_m \rangle^m\right).$$

Similarly, we get

(4.42)
$$q_m = -\frac{k^*_m}{\mu c|V_\infty|}\left(\int_{A^*_{mf}} n_{mf}(I + \nabla b^2_m)da \cdot \nabla \langle \rho_m \rangle^m + \int_{A^*_{mf}} n_{mf}\nabla b^1_m da \cdot \nabla \langle \rho_f \rangle^f\right).$$

It now becomes obvious from (4.25c) and (4.26c) that

$$q_{fm} \equiv -q_f = q_m, \tag{4.43}$$

since $A^*_{fm} = A^*_{mf}$ and $n_{fm} = -n_{mf}$. The term q_{fm} thus depends on the interaction functions b^i_α, $i = 1, 2$, $\alpha = f, m$, and is a dynamic function.

Finally, our dual-porosity model can be written as

$$\phi_f \frac{\partial}{\partial t} \langle \rho_f \rangle^f - \nabla \cdot \left(\frac{k_f}{\mu c} \nabla \langle \rho_f \rangle^f \right) - q_{fm} = \langle q_{f,\text{ext}} \rangle , \tag{4.44}$$

$$\phi_m \frac{\partial}{\partial t} \langle \rho_m \rangle^m - \nabla \cdot \left(\frac{k_m}{\mu c} \nabla \langle \rho_m \rangle^m \right) + q_{fm} = \langle q_{m,\text{ext}} \rangle . \tag{4.45}$$

As a final remark in this section, we shall compare the present model with other dual-porosity models. First, the so-called Warren and Root model [5], [33] assumes the following relationship for the fluid transfer term q_{fm}, usually called a quasi-steady approximation:

$$q_{fm} = \beta \left(\langle \rho_m \rangle^m - \langle \rho_f \rangle^f \right) , \tag{4.46}$$

wher β is a coefficient proportional to the product of the matrix permeability and the specific surface area of the fractures. That is, q_{fm} is assumed to essentially depend on the density drop between the matrix and the fracture system. However, from our rigorous analysis, the coupling term q_{fm} is much more complicated and in fact depends upon the density gradient drop even in the special case of

$$n_{mf} \cdot \nabla b_m^1 = -n_{mf} \cdot \left(I + \nabla b_m^2 \right) , \quad x \in \partial\Omega_m ,$$

by (4.42).

The other physical dual-porosity model, which has received somewhat greater attention in recent years, takes q_{fm} as [2], [3], [12]

$$q_{fm} = -\frac{k_m^*}{\mu c |V_\infty|} \int_{A_{mf}} n_{mf} \cdot \nabla \langle \rho_m \rangle^m \, da. \tag{4.47}$$

This model implies that the matrix blocks act as sources in the fracture system. A glance of (4.42) show that assumption (4.47) amounts to a special case of our model where $\nabla b_m^1 = \nabla b_m^2 = 0$ on the interface $\partial\Omega_M$.

5. Well-posedness of the model. We shall briefly consider the problem of existence, uniqueness, and continuous dependence on data of solution to the model derived in the previous section. Of course, our macroscopic coefficients should have appropriate properties. The necessary requirement for physical relevance that k_f be symmetric and positive definite is stated in the next theorem. Again, we shall here consider a periodic system as did in §4.

THEOREM 5.1. *The macroscopic fracture permeability tensor $k_f(x)$ is symmetric and positive definite.*

Proof. By (3.7) and (4.25e), we write k_f^{ij} in (4.36) as (with $\partial_i = \partial/\partial x_i$)

$$k_f^{ij} = k_f^* \left(\epsilon_f \delta_{ij} + \left\langle \partial_i b_f^{1,j} \right\rangle \right) , \tag{5.1}$$

and from equations (4.25) and (4.40), we see (exactly as for (4.32) and (4.33)) that

$$\left\langle \left(e_j + \nabla b_f^{1,j} \right) \cdot \nabla b_f^{1,i} \right\rangle - \frac{1}{|V_\infty|} \int_{A_{fm}^*} \left(e_j + \nabla b_f^{1,j} \right) \cdot n_{fm} b_f^{1,i} \, da = 0, \tag{5.2}$$

$$\left\langle \nabla b_m^{1,j} \cdot \nabla b_m^{1,i} \right\rangle - \frac{1}{|V_\infty|} \int_{A_{mf}^*} \nabla b_m^{1,j} \cdot n_{mf} b_m^{1,i} \, da = 0, \tag{5.3}$$

where e_j is the unit vector in the jth direction. Add equations (5.2) and (5.3) and use relations (4.25b-c) to obtain

$$\left\langle \partial_j b_f^{1,i} \right\rangle = -\left\langle \nabla b_f^{1,j} \cdot \nabla b_f^{1,i} \right\rangle - \left\langle \nabla b_m^{1,j} \cdot \nabla b_m^{1,i} \right\rangle,$$

which shows that k_f is symmetric by (5.1). In order to show positive-definiteness, we rewrite equation (5.1) as

$$\begin{aligned}
(k_f^*)^{-1} k_f^{ij} &= \epsilon_f \delta_{ij} + \left\langle \partial_j b_f^{1,j} \right\rangle \\
&= \langle \nabla x_j \cdot \nabla x_i \rangle + \left\langle \nabla b_f^{1,j} \cdot \nabla x^i \right\rangle \\
&\quad + \left\langle \nabla b_f^{1,i} \cdot \nabla x^j \right\rangle - \left\langle \nabla b_f^{1,i} \cdot \nabla x^j \right\rangle \\
&= \left\langle \nabla \left(x_j + b_f^{1,j} \right) \cdot \nabla \left(x_i + b_f^{1,i} \right) \right\rangle + \left\langle \nabla b_m^{1,j} \cdot \nabla b_m^{1,i} \right\rangle.
\end{aligned}$$

This implies that k_f is positive-semidefinite. Definiteness follows from the connectedness of V_f^* and the periodicity of $b_f^{1,i}$. Let z be any constant with components z_i. Then,

$$\begin{aligned}
0 &= \sum_{i,j} z_j \left(\left\langle \nabla \left(x_j + b_f^{1,j} \right) \cdot \nabla \left(x_i + b_f^{1,i} \right) \right\rangle + \left\langle \nabla b_m^{1,j} \cdot \nabla b_m^{1,i} \right\rangle \right) z_i \\
&= \sum_i \left(\left\langle \left(\partial_i \left(z \cdot (x + b_f^1) \right) \right)^2 \right\rangle + \left\langle \left(\partial_i (z \cdot b_m^1) \right)^2 \right\rangle \right).
\end{aligned}$$

Since V_f^* is connected, $z \cdot b_f^1(x) = c - z \cdot x$ for some constant c. Thus, by periodicity, z must be zero. The proof is complete. □

We remark that the tensor k_f may not be strictly positive definite for some disconnected geometries. For example, in one space dimension V_f^* consists of disjoint intervals, so $k_f = 0$, as noted in [2].

As for k_m, we must make the unphysical assumption that V_m^* is connected, so that k_m is symmetric and positive definite. But, for the situation under consideration, obviously ϕ_f and ϕ_m are uniformly positive.

We now introduce some notation. Denote the Sobolev space of functions with derivatives of order n in $L^2(\Omega)$ by $H^n(\Omega)$. Let $H_0^1(\Omega)$ denote the closure in $H^1(\Omega)$ of $C_0^\infty(\Omega)$, the infinitely differentiable functions with compact support. Set $J = (0, T]$, $T > 0$, the time interval of interest. For a Banach space X, let $L^2(J; X)$ denote the space of X-valued functions in $L^2(J)$ and $H^1(J; X)$ the space of those in $H^1(J)$.

We shall write the dual-porosity model in a slightly general form:

$$\phi_f \frac{\partial \sigma_f}{\partial t} - \nabla \cdot (K_f \nabla \sigma_f) + a_f \cdot \nabla \sigma_f - a_m \cdot \nabla \sigma_m = g_f, \tag{5.4}$$

$$\phi_m \frac{\partial \sigma_m}{\partial t} - \nabla \cdot (K_m \nabla \sigma_m) - a_f \cdot \nabla \sigma_f + a_m \cdot \nabla \sigma_m = g_m, \tag{5.5}$$

for $(x, t) \in \Omega \times J$, where $\sigma_\alpha = \langle \rho_\alpha \rangle^\alpha$, $K_\alpha = k_\alpha/(\mu c)$, g_α denotes the external source terms, and

$$a_\alpha = -\frac{k_\alpha^*}{\mu c |V_\infty|} \int_{A_{\alpha\beta}^*} n_{\alpha\beta} \left(I + \nabla b_\alpha^{\delta_\alpha} \right) da, \quad \alpha, \beta = f, m, \ \alpha \neq \beta,$$

with $\delta_\alpha = 1$ for $\alpha = f$ and $\delta_\alpha = 2$ for $\alpha = m$. Hence, we have a system of linear equations for σ_f and σ_m. If the initial and boundary conditions are given, the unique solvability of the system will be an ordinary matter provided that the coefficients in the system are smooth. We consider the Dirichlet boundary condition

$$\sigma_\alpha = \sigma_\alpha^D, \quad (x,t) \in \partial\Omega \times J, \tag{5.6}$$

and the initial condition

$$\sigma_\alpha(x,0) = \sigma_\alpha^0(x), \quad x \in \Omega. \tag{5.7}$$

Then (see, e.g., [21]),

THEOREM 5.2. *If $\partial\Omega$ is smooth and the following conditions are satisfied:*

$$\frac{\partial K_\alpha^{ij}}{\partial x_l},\ a_\alpha^i \in L^4(\Omega), \quad l = 1,2,3,\ \alpha = f,m,$$

where $K_\alpha = \left(K_\alpha^{ij}\right)$ and $a_\alpha = \left(a_\alpha^i\right)$, then the system given by equations (5.4)-(5.7) *has a unique solution $\sigma_\alpha \in H^1(J;L^2(\Omega)) \cap L^2(J;H^2(\Omega))$. Moreover, the solution varies continuously with the data $g_\alpha \in L^2(J;L^2(\Omega))$, $\sigma_\alpha^D \in L^2(J;H^1(\Omega))$, and $\sigma_\alpha^0 \in H^1(\Omega)$, $\alpha = f,m$.* □

ACKNOWLEDGEMENT. Partial results in this paper were completed while the author was at Purdue University. The author wishes to thank Professor Jim Douglas, Jr. for his many valuable suggestions.

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