Volume of Moduli Space of Vortex Equations and Localization

Norisuke Sakai (Tokyo Woman's Christian University)

In collaboration with **Kazutoshi Ohta** (Meiji Gakuin University)

and **Akiko Miyake** (Kushiro National College of Technology)

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Contents

1 Moduli Space of Vortices

Vortex : relevant to understand confinement

Non-Abelian Vortices are especially useful

BPS solitons : no force between static solitons $\longrightarrow$ Rich **Moduli** Space

Moduli of Non-Abelian Vortices : Position and Orientation

Moduli = **Effective Fields** $\longrightarrow$ Dynamics of Solitons

Volume of **Moduli Space** is useful for

Thermodynamic Properties,

Nonperturbative Effects,

volume of the moduli space of the instantons

$=$ non-perturbative effective prepotential of $\boldsymbol{\mathcal{N} = 2}$ SUSY Yang-Mills

N. Nekrasov, Adv.Theor. Math.Phys.**7** (2004) 831; $\cdots$

Volume of moduli space obtained for Abelian local vortices ($\boldsymbol{N_c = N_f = 1}$)

metric defined by **Effective Lagrangian**

integrate over the moduli space with **topological** considerations

N.Manton and S.Nasir, Commun.Math.Phys.**199** (1999) 591; $\cdots$

Localization technique (Topological Field Theory)

Integral over Kähler metric + Deformation respecting BRST (SUSY)

$\longrightarrow$ Integral is **Localized at isometry fixed points**

G.Moore, N.Nekrasov and S.Shatashvili, Commun.Math.Phys.**209** (2000) 97; A.Gerasimov and S.Shatashvili, Commun.Math.Phys.**277** (2008) 323; $\cdots$

Our purpose: Apply the localization technique to moduli space of Vortices including semi-local, and/or non-Abelian vortices

2 BPS Vortices on Riemann Surface

Compact **Riemann surface** Σ_h, conformal factor $\sigma = g_{z\bar{z}}$, $(z = x+iy)$

$$ds^2 = -dt^2 + \sigma[(dx)^2 + (dy)^2] = g_{\mu\nu} dx^\mu dx^\nu$$

Area $\mathcal{A}$ is given by means of **Kähler 2-form** $\omega = \frac{i}{2} g_{z\bar{z}} dz \wedge d\bar{z}$

$$\mathcal{A} = \int dx dy \, \sigma = \int \omega$$

$U(N_c)$ gauge fields A_μ ($N_c \times N_c$ matrix)

N_f Higgs fields H in N_c representation ($N_c \times N_f$ matrix)

$$L = \int dxdy \sqrt{-\det(g_{\mu\nu})}\, \mathcal{L} = \int dtdxdy\, \sigma\, \mathcal{L}$$

$$\mathcal{L} = \mathrm{Tr}\Big[-\frac{1}{2g^2} F_{\mu\nu}F^{\mu\nu} + \mathcal{D}_\mu H (\mathcal{D}^\mu H)^\dagger - \frac{g^2}{4}\left(HH^\dagger - c\mathbf{1}_{N_c}\right)^2\Big]$$

BPS bound : Energy is bounded by **vortex number** k (vorticity)

$$E = \int dxdy\ \mathrm{Tr}\left[4D_{\bar z}H D_z H^\dagger + \frac{1}{g^2\sigma}\left(F_{12} - \frac{g^2\sigma}{2}(c - HH^\dagger)\right)^2 + cF_{12}\right]$$

$$\geq c \int dxdy\ \mathrm{Tr}(F_{12}) = 2\pi c k$$

BPS equations are given by moment map constraints: $\mu_r = \mu_{\bar z} = \mu_z = 0$

$$\mu_r \equiv F - \frac{g^2}{2}(c - HH^\dagger)\omega$$

$$\mu_{\bar z} \equiv D_{\bar z} H, \qquad \mu_z \equiv D_z H^\dagger$$

Moduli space is a Kähler quotient space

$$\mathcal{M}_k = \frac{\mu_r^{-1}(0) \cap \mu_z^{-1}(0) \cap \mu_{\bar z}^{-1}(0)}{U(1)}$$

3 Abelian Vortices : $N_c = 1$

$U(1)$ gauge thoery, N_f charged Higgs fields H

BRST transformation

$$QA = i\lambda, \quad Q\lambda = -d\Phi$$
$$QH = i\psi, \quad Q\psi = \Phi H, \quad QH^\dagger = -i\psi^\dagger, \quad Q\psi^\dagger = H^\dagger\Phi$$
$$QY = \Phi * \chi, \quad Q\chi = Y, \quad Q\Phi = 0$$

1-form fields $A = A_z dz + A_{\bar{z}} d\bar{z}$, and λ, Y, χ

Fermionic fields λ, ψ, χ

Auxiliary fields Y, χ, Φ to implement BPS constraints

Action : **BRST invariant completion** of BPS constraint $\mu_r = 0$

with the Lagrange multiplier field Φ

$$S_0 = \int_{\Sigma_h} \left[i\Phi\mu_r + \frac{1}{2}\lambda \wedge \lambda + \frac{g^2}{2}\psi^\dagger\psi\omega \right] \to QS_0 = 0$$

BRST exact completion of remaining BPS constraints $\mu_z = \mu_{\bar{z}} = 0$

with the Lagrange mulitiplier fields $Y_z, Y_{\bar{z}}$ ($\mu_c = \mu_z dz + \mu_{\bar{z}} d\bar{z}$)

$$S_1 = t_1 Q \int_{\Sigma_h} i\chi \wedge *\mu_c = t_1 \int_{\Sigma_h} [-Y_z\mu_{\bar{z}} - Y_{\bar{z}}\mu_z + \cdots]$$

Moduli Space Volume: Path integral with the Total action S

$$S = S_0 + S_1$$

$$\mathcal{V}_k = \int \mathcal{D}\Phi \mathcal{D}^2 Y \mathcal{D}^2 \chi \mathcal{D}^2 A \mathcal{D}^2 \lambda \mathcal{D}^2 H \mathcal{D}^2 \psi \, e^{-S}$$

Deformation of S by **BRST exact** terms should not change $\mathcal{V}_k$

Another **BRST exact** term

$$S_2 = t_2 Q \int_{\Sigma_h} (-i)\chi \wedge *Y = t_2 \int_{\Sigma_h} [-iY \wedge *Y + i\Phi\chi \wedge \chi]$$

Choose the parameters to integrate most easily : $t_1 = 0, t_2 = 1$

$$S' = \int_{\Sigma_h} \left[i\Phi \left\{ F - \frac{g^2}{2}(c - HH^\dagger)\omega \right\} + \frac{1}{2}\lambda \wedge \lambda + \frac{g^2}{2}\psi^\dagger \psi \omega \right.$$
$$\left. -iY \wedge *Y + i\Phi\chi \wedge \chi \right]$$

$$\mathcal{V}_k = \int \mathcal{D}\Phi \mathcal{D}^2 Y \mathcal{D}^2 \chi \mathcal{D}^2 A \mathcal{D}^2 \lambda \mathcal{D}^2 H \mathcal{D}^2 \psi \, e^{-S'}$$

Gaussian integration of H, ψ (0-form), Y, χ (1-form) gives

$$\mathcal{V}_k = \int \mathcal{D}\Phi \mathcal{D}^2 A \mathcal{D}^2 \lambda \, (i\Phi)^{N_f(\dim \Omega^1 \otimes \mathcal{L}_k - \dim \Omega^0 \otimes \mathcal{L}_k)} e^{-\int_{\Sigma_h} \left[i\Phi(F - \frac{g^2 c}{2}\omega) + \frac{1}{2}\lambda \wedge \lambda \right]}$$

$\Omega^n \otimes \mathcal{L}_k$: holomorphic n-forms coupled with $U(1)$ gauge field of k-vorticity

Hirzebruch-Riemann-Roch theorem

$$\dim \Omega^0 \otimes \mathcal{L}_k - \dim \Omega^1 \otimes \mathcal{L}_k = 1 - h + \frac{1}{2\pi}\int_{\Sigma_h} F = 1 - h + k$$

Genus h in terms of **curvature 2-form** $R^{(2)}$: $2(1-h) = \frac{1}{4\pi}\int_{\Sigma_h} R^{(2)}$

Vortex number k in terms of **field strength 2-form** F : $k = \frac{1}{2\pi}\int_{\Sigma_h} F$

$$\mathcal{V}_k = \int \mathcal{D}\Phi \mathcal{D}^2 A \mathcal{D}^2 \lambda \, e^{-S_{\rm eff}}, \qquad S_{\rm eff} = S_R + S_F + S_V$$

$$\begin{aligned} S_R &= \frac{1}{8\pi}\int_{\Sigma_h} \log(i\Phi) R^{(2)}, \\ S_F &= \int_{\Sigma_h}\left[i\left(\Phi + \frac{1}{2\pi i}\log i\Phi\right) F + \frac{1}{2}\lambda \wedge \lambda\right], \\ S_V &= -i\frac{g^2 c}{2}\int_{\Sigma_h} \Phi \omega \end{aligned}$$

S_F is not invariant under the BRST : **Anomaly** due to ψ, χ

BRST invariant completion of the effective action $S_F \longrightarrow S'_F$

$$S'_F = \int_{\Sigma_h} \left[i \frac{\partial \mathcal{W}_{\text{eff}}}{\partial \Phi} F + \frac{1}{2} \frac{\partial^2 \mathcal{W}_{\text{eff}}}{\partial \Phi^2} \lambda \wedge \lambda \right], \quad \mathcal{W}_{\text{eff}}(\Phi) = \frac{1}{2} \Phi^2 + \frac{N_f}{2\pi i} \Phi (\log i\Phi - 1)$$

We obtain the volume

$$\mathcal{V}_k = \int \mathcal{D}\Phi \mathcal{D}^2 A \mathcal{D}^2 \lambda \frac{1}{(i\Phi)^{N_f(1-h+k)}} e^{-\int_{\Sigma_h} \left[i\Phi \left(F - \frac{g^2 c}{2} \omega \right) + \frac{\mu(\Phi)}{2} \lambda \wedge \lambda \right]}$$

$$\mu(\Phi) = \frac{\partial^2 \mathcal{W}_{\text{eff}}}{\partial \Phi^2} = 1 + \frac{N_f}{2\pi i \Phi}$$

Integration over A and λ : $d\Phi = 0 \longrightarrow \Phi = \phi$ (**constant mode**)

$$\mathcal{V}_k = \int_{-\infty}^{\infty} \frac{d\phi}{2\pi} \frac{\mu(\phi)^h}{(i\phi)^{N_f(1-h+k)}} e^{i\phi \left(\frac{g^2 c}{2} \mathcal{A} - 2\pi k \right)}$$

Convergence of H path integral : $\phi \longrightarrow \phi - i\epsilon$

Volume is nonzero only for $\mathcal{A} \geq \frac{4\pi k}{g^2 c}$: **Bradlow inequality**

Each vortex occupies at least the intrinsic area $4\pi/(g^2 c)$

$$\mathcal{V}_k = \sum_{j=0}^{\min(h,d)} \frac{h!}{j!(h-j)!} \left(\frac{N_f}{2\pi} \right)^{h-j} \frac{1}{(d-j)!} \left(\frac{g^2 c}{2} \mathcal{A} - 2\pi k \right)^{d-j}$$

$d \equiv kN_f + (1-h)(N_f - 1)$, Volume vanishes for $d < 0$

ANO Vortices $N_c = N_f = 1$

Volume of the vacuum moduli space : $\mathcal{V}_0 = \frac{1}{(2\pi)^h}$

(Dimensionless) Area in unit of intrinsic area of vortex : $\tilde{\mathcal{A}} = \frac{g^2 c}{4\pi}\mathcal{A}$

Volume of k vortices divided by the vacuum volume

$$\tilde{\mathcal{V}}_k \equiv \frac{\mathcal{V}_k}{\mathcal{V}_0} = (2\pi)^k \sum_{j=0}^{\min(h,k)} \frac{h!}{j!(k-j)!(h-j)!} \left(\tilde{\mathcal{A}} - k\right)^{k-j}$$

In **agreement** with the result of **effective Lagrangian**

for all genus h and vortex number k : $\tilde{\mathcal{V}}_k = \frac{\mathrm{Vol}^{(\mathrm{eff})}(\mathcal{M}_k)}{(2\pi)^k}$

(Moduli Space Metric has intrinsic scale ambiguity)

Abelian Semi-local Vortices : $N_f > N_c = 1$

Volume of k-Vortices on Sphere ($h = 0$)

$$\mathcal{V}_k(S^2) = \frac{(2\pi)^{kN_f + N_f - 1}}{(kN_f + N_f - 1)!} \left(\tilde{\mathcal{A}} - k\right)^{kN_f + N_f - 1}$$

Volume of $\boldsymbol{k}$-Vortices on Torus ($\boldsymbol{h=1}$)

$$\boldsymbol{\mathcal{V}_k(T^2) = \frac{N_f}{2\pi}\frac{(2\pi)^{kN_f}}{(kN_f)!}\tilde{\mathcal{A}}\left(\tilde{\mathcal{A}}-k\right)^{kN_f-1}}$$

Non-zero Volume requires $\boldsymbol{d = kN_f + (1-h)(N_f-1) \geq 0}$

vorticity has to be high enough for higher genus Riemann surfaces ($\boldsymbol{h \geq 2}$)

$$\boldsymbol{k \geq (h-1)\left(1-\frac{1}{N_f}\right)}$$

for BPS vortex solutions to exist, except $\boldsymbol{N_f = 1}$ case

Non-zero moduli space volume at the **Bradlow limit** $\boldsymbol{\tilde{\mathcal{A}} \to k}$

$$\boldsymbol{\mathcal{V}_k = \frac{h!}{d!(h-d)!}\left(\frac{N_f}{2\pi}\right)^{h-d} + \mathcal{O}\left(\tilde{\mathcal{A}}-k\right)}$$

provided $\boldsymbol{(h-1)\left(1-\frac{1}{N_f}\right) \leq k \leq h-1+\frac{1}{N_f}}$

For $\boldsymbol{N_f = N_c = 1}$, this agrees with the known result ($\boldsymbol{0 \leq k \leq h}$)

4 Non-Abelian Vortices : $N_c > 1$

Volume of moduli space from **BRST invariant action**

$$S_0 = \int_{\Sigma_h} \mathrm{Tr}\left[i\Phi\left\{ F - \frac{g^2}{2}(c - HH^\dagger)\omega \right\} + \frac{1}{2}\lambda\wedge\lambda + \frac{g^2}{2}\psi^\dagger\psi\omega \right]$$

$$S_2 = Q\int_{\Sigma_h} \mathrm{Tr}\left[\frac{1}{2}g^{z\bar{z}}(\chi_z Y_{\bar{z}} + Y_z\chi_{\bar{z}})\right] d^2z = \int_{\Sigma_h} \mathrm{Tr}\left[g^{z\bar{z}}(Y_z Y_{\bar{z}} + i\Phi\chi_z\chi_{\bar{z}})\right] d^2z$$

$$\mathcal{V}_k^{N_c,N_f}(\Sigma_h) = \int \mathcal{D}\Phi\mathcal{D}^2A\mathcal{D}^2\lambda\mathcal{D}^2H\mathcal{D}^2\psi\mathcal{D}^2Y\mathcal{D}^2\chi\, e^{-S_0-S_2}$$

Diagonal gauge choice : $\Phi = \mathrm{diag}(\phi_1, \phi_2, \ldots, \phi_{N_c})$

Integrate H, ψ, Y, χ and off-daigonal A, λ and ghosts

Off-diagonal A and ghosts $\longrightarrow$ Vandermonde determinant $\prod_{a\neq b}(i\phi_a - i\phi_b)$

$$\mathcal{V}_k^{N_c,N_f} = \int \prod_{a=1}^{N_c}(\mathcal{D}\phi_a\mathcal{D}^2A_a\mathcal{D}^2\lambda_a)\, e^{-\sum_{a=1}^{N_c}\int_{\Sigma_h}\left[i\phi_a(F^{(a)} - \frac{g^2c}{2}\omega) + \frac{1}{2}\lambda_a\wedge\lambda_a\right]}$$

$$\times \frac{\prod_{a\neq b}(i\phi_a - i\phi_b)^{\dim\Omega^0\otimes\mathcal{L}_{k_a}\otimes\mathcal{L}_{k_b}^{-1} - \dim\Omega^1\otimes\mathcal{L}_{k_a}\otimes\mathcal{L}_{k_b}^{-1}}}{\prod_{a=1}^{N_c}(i\phi_a)^{N_f(\dim\Omega^0\otimes\mathcal{L}_{k_a} - \dim\Omega^1\otimes\mathcal{L}_{k_a})}}$$

diagonal $U(1)$ vortex numbers $\frac{1}{2\pi}\int F^{(a)} = k_a$

Genus and vortex number rewritten in terms of $R^{(2)}$ and $F^{(a)}$ gives

Path-integral with **Abelian effective action** : $S_{\text{eff}} = S_R + S_F + S_V$

BRST invariant completion : $S_F \to S'_F$ incorporates anomaly

Integrating over $A^{(a)}, \lambda^{(a)}$: $\phi_a(z,\bar{z}) \longrightarrow$ constant mode ϕ_a

$$\mathcal{V}_k^{N_c,N_f} = \sum_{\sum_a k_a = k} (-1)^\sigma \int \prod_{a=1}^{N_c} \frac{d\phi_a}{2\pi} \frac{\mu(\phi)^h \prod_{a<b}(i\phi_a - i\phi_b)^{2-2h}}{\prod_{a=1}^{N_c}(i\phi_a)^{N_f(1-h+k_a)}} e^{2\pi i \sum_{a=1}^{N_c} \phi_a(\tilde{\mathcal{A}} - k_a)}$$

$$\mu(\phi) = \prod_{a=1}^{N_c}\left(1 + \frac{1}{2\pi i}\frac{N_f}{\phi_a}\right)$$

$$\sigma = \tfrac{1}{2}N_c(N_c-1)(1-h) - \textstyle\sum_{a<b}(k_a - k_b)$$

On sphere

$N_c = 2$ case:

Asymptotic power of $\mathcal{A}$ for large area $\mathcal{A} \to \infty$

$$\mathcal{V}_k^{2,N_f}(S^2) \propto \tilde{\mathcal{A}}^{kN_f + 2(N_f - 2)}$$

Local Vortex : $N_c = N_f = 2$ case

Reduction of asymptotic power of $\mathcal{A}$ for $N_f = N_c$ (Local Vortex)

$$\mathcal{V}_k^{2,2}(S^2) = \frac{2(2\pi)^{2k}\tilde{\mathcal{A}}^k}{k!} + \mathcal{O}(\tilde{\mathcal{A}}^{k-1})$$

$$\begin{aligned}
\mathcal{V}_0^{2,2}(S^2) &= 2 \\
\mathcal{V}_1^{2,2}(S^2) &= 2(2\pi)^2(\tilde{\mathcal{A}} - 1) \\
\mathcal{V}_2^{2,2}(S^2) &= \frac{2(2\pi)^4}{2!}\left(\tilde{\mathcal{A}}^2 - \frac{20}{6}\tilde{\mathcal{A}} + \frac{17}{6}\right) \\
\mathcal{V}_3^{2,2}(S^2) &= \frac{2(2\pi)^6}{3!}\left(\tilde{\mathcal{A}}^3 - 7\tilde{\mathcal{A}}^2 + \frac{331}{20}\tilde{\mathcal{A}} - \frac{793}{60}\right) \\
\mathcal{V}_4^{2,2}(S^2) &= \frac{2(2\pi)^8}{4!}\left(\tilde{\mathcal{A}}^4 - 12\tilde{\mathcal{A}}^3 + \frac{818}{15}\tilde{\mathcal{A}}^2 - \frac{2336}{21}\tilde{\mathcal{A}} + \frac{18047}{210}\right)
\end{aligned}$$

$N_c = 3$ case

Asymptotic power of $\mathcal{A}$ for large area $\mathcal{A} \to \infty$

$$\mathcal{V}_k^{N_c=3,N_f}(S^2) \propto \tilde{\mathcal{A}}^{kN_f+3(N_f-3)}$$

$N_c = 2, 3$ results suggest the asymptotic power of $\mathcal{A}$ for general $N_f > N_c$

$$\mathcal{V}_k^{N_c,N_f}(S^2) \propto \tilde{\mathcal{A}}^{kN_f+N_c(N_f-N_c)}$$

Local Vortex $N_c = N_f = 3$ case

$$
\begin{aligned}
\mathcal{V}_0^{3,3}(S^2) &= 3! \\
\mathcal{V}_1^{3,3}(S^2) &= 3!\frac{(2\pi)^3}{2}(\tilde{\mathcal{A}} - 1) \\
\mathcal{V}_2^{3,3}(S^2) &= \frac{3!}{2!}\frac{(2\pi)^6}{2^2}\left(\tilde{\mathcal{A}}^2 - \frac{46}{15}\tilde{\mathcal{A}} + \frac{36}{15}\right) \\
\mathcal{V}_3^{3,3}(S^2) &= \frac{3!}{3!}\frac{(2\pi)^9}{2^3}\left(\tilde{\mathcal{A}}^3 - \frac{31}{5}\tilde{\mathcal{A}}^2 + \frac{3641}{280}\tilde{\mathcal{A}} - \frac{23249}{2520}\right)
\end{aligned}
$$

Reduction of asymptotic power of $\mathcal{A}$ for $N_f = N_c$ (Local Vortex)

$$
\mathcal{V}_k^{3,3}(S^2) = \frac{3!}{k!}\left(\frac{(2\pi)^3\tilde{\mathcal{A}}}{2}\right)^k + \mathcal{O}(\tilde{\mathcal{A}}^{k-1})
$$

Expectation for general $N_c = N_f$ case

$$
\mathcal{V}_k^{N,N}(S^2) = \frac{N!}{k!}\left(\frac{(2\pi)^N\tilde{\mathcal{A}}}{N-1}\right)^k + \mathcal{O}(\tilde{\mathcal{A}}^{k-1})
$$

Volume of Moduli Space of Vortices on Torus can also be computed

<u>**Effective Lagrangian of Vortices**</u> on sphere : confirms our results

strong coupling $g^2 \to \infty \Rightarrow$ **Asymptotic power** for large $\mathcal{A}$

reduction of asymptotic power of $\mathcal{A}$ at $N_f = N_c$ (Local Vortex) :

internal modes other than position do not extend to the whole area $\mathcal{A}$

5 Conclusion

1. The volume of moduli space of vortices are computed for $U(N_c)$ gauge theory with N_f Higgs fields in the fundamental representation, using the **localization** technique of topological field theory.

2. Volume of moduli space of **ANO vortices** ($N_f = N_c = 1$) for any **vortex number** k and for any **genus** h of Riemann surfaces is obtained and **agrees** with the previous direct calculation using the **effective Lagrangian**.

3. Volume of moduli space is obtained both for Abelian **semi-local vortices** ($N_f > N_c = 1$) and **non-Abelian vortices** ($N_c > 1$).

4. We find that the **asymptotic power** of area $\mathcal{A}$ for $\mathcal{A} \to \infty$ is $\mathcal{A}^{kN_f + N_c(N_f - N_c)}$ for semi-local vortices ($N_f > N_c$), and that the power reduces to $\mathcal{A}^k$ for local vortices ($N_f = N_c$).

5. Reduction of asymptotic power is understood by noticing that **internal modes** other than position do not extend over the whole area $\mathcal{A}$ for local vortices.

6. **Localization** technique should be powerful to obtain the volume of moduli space of other solitons.