TEMPORAL ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF THE BENJAMIN-ONO-BURGERS EQUATION

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TEMPORAL ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF THE BENJAMIN-ONO-BURGERS EQUATION*

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Abstract. We study certain similarity solutions of the Benjamin-Ono-Burgers (BOB) equation and their role in the asymptotic behavior of the general solution. For small initial data in $L^1(\mathbb{R})$ we prove that a solution of the BOB equation exists in $BC(\mathbb{R}^+, L^1(\mathbb{R}))$ and depends continuously on its initial data. Results about existence, uniqueness, regularity, and spacial asymptotics of solutions of a similarity reduction of the BOB equation are proved. Furthermore, the solutions of the BOB equation are proved to converge as $t \to \infty$ to appropriate similarity solutions faster than the typically sharp rate of decay of the solutions.

CONTENTS

§0 Introduction

The necessity of understanding the effects of dissipation on the propagation of nonlinear dispersive waves is widely recognized. In this paper we will consider a particular model equation describing the motion of such waves, the so-called Benjamin-Ono-Burgers equation

$$u_t + \bar{u}\bar{u}_x - (\nu + \rho\mathcal{H})u_{xx} = 0, \tag{0.1}$$

*This paper is based on a portion of the author's Ph.D. thesis, directed by Charles J. Amick, at the University of Chicago, 1988.

where $x \in \mathbb{R}$, $t \in (0, \infty)$, u is a complex-valued function of x and t, $\nu > 0$ and $\rho \in \mathbf{R}$ are constants, and $\mathcal{H}$ denotes the Hilbert transform, defined (for a smooth test function v) in terms of the Cauchy principal value integral

$$(0.2) \qquad (\mathcal{H}v)(x) = \frac{1}{\pi}\,\mathrm{p.\,v.} \int_{-\infty}^{\infty} \frac{v(y)}{x-y}\, dy.$$

We will henceforth refer to (0.1) as the BOB equation. Edwin and Roberts **[6]** have derived (0.1) (for u real-valued) by means of formal asymptotic expansions in order to describe wave motions supported by intense magnetic flux tubes in the solar atmosphere. The dissipative effects in that context are due to heat conduction. However the BOB equation is also a natural ad hoc candidate for an equation including nonlinear, dispersive, and dissipative effects, since it a combination of the Benjamin-Ono equation ((0.1) with $\nu = 0$) and the Burgers' equation ((0.1) with $\rho = 0$). Burgers' equation with a complex conjugate present in the nonlinear term is related to a system studied by Burgers **[4]** as a model in the statistical theory of turbulence. This type of nonlinear term does not contribute to the associated equation of balance of energy (see §3), and therefore seems to be the appropriate generalization to the context of complex-valued solutions. However, since this paper is concerned with small initial data in $L^1(\mathbb{R})$ all our results and proofs survive even if the complex conjugate is removed. The BOB equation is also interesting because it occupies a distinguished place in the following family of generalized nonlinear-dispersive-dissipative equations

$$(0.3) \qquad u_t + \alpha(\bar{u}^p)_x + \nu|D|^q u - \rho|D|^{r-1}u_x = 0.$$

Here $p \geq 2$ is an integer, and $q > 0$, $r > 1$, $\nu > 0$, α and ρ are real numbers. $|D|^q$ is the Fourier multiplier operator defined by $(|D|^q v)\hat{}\,(k) = |k|^q \hat{v}(k)$. The BOB equation can be obtained by taking $p = q = r = 2$, and is the simplest equation of this family for which the long-time form of the solution is determined by a precise three-way balance of nonlinear, dispersive, and dissipative effects. We will elaborate on this point presently.

But first we need some concepts useful for describing the temporal asymptotic behavior of solutions of evolution equations like (0.3). Let X and Y denote Banach spaces of complex-valued functions (or distributions) on $\mathbb{R}$, U a nonempty open subset of X, and $S: U \times (0,\infty) \to Y: (f,t) \mapsto S(t)f$ a continuous map where $u(t) = S(t)f$ is the solution with initial data f of the evolution equation in question. Suppose $\gamma: (0,\infty) \to (0,\infty)$ is a continuous function, and $\mathcal{O}$: range$(S) \to Y$ some operator, possibly unbounded and nonlinear.

DEFINITION 0.1.

(1) *Suppose for every $f \in U$ we have $\sup_{t>0} \gamma(t)^{-1}\|\mathcal{O}S(t)f\|_Y < \infty$. Then we say this estimate is* **typically sharp** *[for data in U] as $t \to \infty$ if the set of all $f \in U$ for which $\limsup_{t\to\infty} \gamma(t)^{-1}\|\mathcal{O}S(t)f\|_Y > 0$ is a dense open subset of U.*

(2) *Suppose for every $f \in U$ we have $\limsup_{t\to\infty} \gamma(t)^{-1}\|\mathcal{O}S(t)f\|_Y = 0$. Then we say this asymptotic result is* **optimal** *[for data in U] if whenever $\gamma_1: (0,\infty) \to (0,\infty)$ is a continuous function such that $\gamma_1(t)\gamma(t)^{-1} \to 0$ as $t \to \infty$ there must exist $f \in U$ such that $\limsup_{t\to\infty} \gamma_1(t)^{-1}\|\mathcal{O}S(t)f\|_Y = \infty$.*

The function γ depends on the choice of X and Y as well as the particular equation. For example if S is the flow of the linearized version of (0.3), i.e. $\alpha = 0$, it is not hard to show using Plancherel's theorem that $\|S(t)f\|_{L^2(\mathbb{R})} = O(t^{-1/(2q)})$ for $t > 0$ and data in $L^1(\mathbb{R})$, and that the estimate is sharp as $t \to \infty$, since the L^2-norm of the fundamental solution decays at precisely the desired rate. In fact in this case we can obtain typically sharp decay estimates on all the spacial derivatives of the solution as well, i.e. $\mathcal{O} = |D|^m$ for real numbers $m \geq 0$. In fact there exists a positive constant C_m such that for all $f \in L^1(\mathbb{R})$ we have $\lim_{t\to\infty} t^{(m+1/2)/q} \| \, |D|^m S(t)f\|_{L^2(\mathbb{R})} = C_m | \int_{-\infty}^{\infty} f(x)\, dx|$. So the set of initial data for which the estimate is sharp is the complement in $L^1(\mathbb{R})$ of the hyperplane where $\int_{-\infty}^{\infty} f(x)\, dx = 0$. On the other hand, using the methods in §2 (see Theorem 2.2.3, and the remarks thereafter) we have that $\|S(t)f\|_{L^2(\mathbb{R})} = o(1)$ as $t \to \infty$ for data in $L^2(\mathbb{R})$ and that this estimate is optimal for data in $L^2(\mathbb{R})$. For nonlinear equations the known methods of proof frequently break down for large data and thus we can only prove such decay estimates for initial data in some open set $U \subset X$ containing zero. The reader should realize that the term "optimal decay" is frequently used more loosely to mean that solutions of the nonlinear equation are known to decay at least as rapidly as solutions (arising from the same class of initial data) of its linearized equation. This usage assumes that nonlinearity can never enhance or decrease decay rates. Whenever we use this term in this sense we will enclose it in quotes to distinguish it from the technical usage introduced in Definition 0.1(2).

Beyond the question of the size of the solution as $t \to \infty$ it is also desirable to know the asymptotic shape of the solution. Thus we seek "intermediate asymptotics" for the solutions, i.e. continuous functions $S_1 : U \times (0, \infty) \to Y : (f, t) \mapsto S_1(t)f$ of simple dependence on x, t, and depending on f only through the values of a finite number of parameters, defined throughout U. We require $S_1(t)f$ to accurately describe the form of the solution $S(t)f$ for large t in the following sense.

DEFINITION 0.2. *Suppose $\|S_1(t)f\|_Y = O(\gamma(t))$ is typically sharp for data in U as $t \to \infty$, and $\gamma_1 : (0, \infty) \to (0, \infty)$ is a continuous function. We say S_1 is an* **intermediate asymptotic** *[with rate $O(\gamma_1)$ (resp. $o(\gamma_1)$ as $t \to \infty$)] for S [in the Y-norm, for data in U] if for every $f \in U$ we have $\|S(t)f - S_1(t)f\|_Y = O(\gamma_1(t))$ (resp. $o(\gamma_1(t))$ as $t \to \infty$). If the rate is not specified it is assumed to be $o(\gamma)$; if the rate is $O(\gamma_1)$ (resp. $o(\gamma_1)$ as $t \to \infty$) then we require that $\gamma_1(t) = o(\gamma(t))$ as $t \to \infty$ (resp. $\gamma_1(t) = O(\gamma(t))$) so that the definition has substance.*

This terminology is standard; see Sachdev [**15**]. For example in §2 we prove that if $s \geq 0$ and $U = X = L^1_s(\mathbb{R})$, i.e. the set of all f satisfying $\int_{-\infty}^{\infty} (1 + |x|)^s |f(x)|\, dx < \infty$, and if $Y = L^2(\mathbb{R})$, then an intermediate asymptotic with rate $o(t^{-(s+1/2)/2})$ as $t \to \infty$ for the solution $u(t) = S(t)f$ of the linearized BOB equation can be taken in the form of the special solution

$$u_1(t) = S_1(t)f = \sum_{j=0}^{n} \frac{(-1)^j \mu_j}{j!} \partial_x^j F(t), \tag{0.4}$$

where $n = [\![s]\!]$ is the greatest integer not exceeding s, $F(x, t) = t^{-1/2} G(xt^{-1/2})$ is the fundamental solution (the function G is defined in terms of its Fourier transform by the rule $\hat{G}(k) = e^{-(\nu - i\rho\,\mathrm{sgn}(k))k^2}$, and $\mu_j = \int_{-\infty}^{\infty} x^j f(x)\,dx$ for $j = 0, \ldots, n$. Notice that $u_1(t) = S_1(t)f$ is a sum of similarity solutions which depends on f only through the values of the parameters μ_j. Since $\|S_1(t)f\|_{L^2(\mathbb{R})} = \mathrm{O}(t^{-1/4})$ is typically sharp for data in $L^1_s(\mathbb{R})$ as $t \to \infty$ it follows that $\|S(t)f\|_{L^2(\mathbb{R})} = \mathrm{O}(t^{-1/4})$ is also typically sharp. Clearly then, one of the best ways to prove that a decay estimate is typically sharp is to find the corresponding intermediate asymptotics.

In order to see how the BOB equation is especially interesting we must get a general idea of the particular asymptotic balance between the effects of nonlinearity, dispersion, and dissipation inherent in each of the equations (0.3). One crude measure of this balance is the relative decay rates of the various terms in the equation. Some idea about the behavior of these rates can be obtained by making the assumption that solutions of the nonlinear equation with initial data in $L^1(\mathbb{R})$ decay at the same rates (at least) as solutions of the linearized equation. Use of the estimates mentioned above and the interpolation inequality $\|v\|_{L^\infty(\mathbb{R})} \leq 2^{1/2}\|v\|^{1/2}_{L^2(\mathbb{R})}\|v'\|^{1/2}_{L^2(\mathbb{R})}$ leads to the following (possibly nonsharp) estimates of the terms in equation (0.3):

$$
\begin{aligned}
\|\bar{u}(t)^{p-1}\bar{u}_x(t)\|_{L^2(\mathbb{R})} &= \mathrm{O}(t^{-(p+1/2)/q}), \\
\|\,|D|^q u(t)\|_{L^2(\mathbb{R})} &= \mathrm{O}(t^{-(q+1/2)/q}), \\
\|\,|D|^{r-1} u_x(t)\|_{L^2(\mathbb{R})} &= \mathrm{O}(t^{-(r+1/2)/q}).
\end{aligned} \tag{0.5}
$$

The u_t-term must decay at least as fast as the most slowly decaying term in the equation. If these estimates are typically sharp and it is not the case that $p = q = r$ then one or two of the three effects (nonlinearity, dissipation, dispersion) is relatively unimportant in the long time regime in comparison with the other(s).

The nature of this balance (or imbalance) appears in the form of the intermediate asymptotics. For example consider the Korteweg-deVries-Burgers (KdVB) equation:

$$u_t + \alpha u u_x - \nu u_{xx} + \rho u_{xxx} = 0. \tag{0.6}$$

This (for real-valued solutions) is a special case of (0.3) where $p = q = 2$, and $r = 3$. Amick, Bona, and Schonbek [**2**] have shown that the solutions of this equation satisfy a decay estimate $\|u(t)\|_{L^2(\mathbb{R})} = \mathrm{O}(t^{-1/4})$ which is typically sharp for initial data in $L^1(\mathbb{R})$. Using their methods the present author has shown [**5**] that if u_1 is the solution of Burgers' equation with the same initial data $f \in L^1(\mathbb{R})$ then $\|u(t) - u_1(t)\|_{L^2(\mathbb{R})} = \mathrm{O}(t^{-1/2})$. Since it is well-known [**7**] that the intermediate asymptotics for solutions of Burgers' equation are nonlinear diffusion waves with a single hump, the so-called triangular waves (see Whitham [**20**]), we see that these single hump waves serve as intermediate asymptotics with rate $\mathrm{o}(t^{-1/4})$ in the L^2-norm for solutions of KdVB arising from data in $L^1(\mathbb{R})$. Thus the multi-hump behavior one might expect of solutions of the KdVB equation is not present in the intermediate asymptotics appropriate for this rate. However the "decaying multi-soliton" behavior will undoubtedly be present in intermediate asymptotics with rate $\mathrm{O}(t^{-a})$ for solutions of KdVB if a is sufficiently large. In any case the estimates in (0.5) are typically sharp for the KdVB and thus the dispersive term decays more rapidly *than the other terms* of the equation. In fact it was precisely this extra decay which enabled Amick et al. to obtain their decay result.

The BOB equation is the simplest example of a model equation of the type (0.3) ($p = q = r = 2$) where all three effects are perfectly balanced so as to equally affect the nature of the intermediate asymptotics. The decay results proved in this paper will imply (at least for small initial data) the typical sharpness of the estimates (0.5) for the BOB equation. One manifestation of this balance is the existence of a similarity reduction of the BOB equation. Suppose one looks for solutions of the BOB equation in the similarity form $u_1(x,t) = (t+a)^{-1/2}w(x(t+a)^{-1/2})$, where a can be any nonnegative constant, usually either 0 or 1. If $w \in L^2(\mathbb{R})$ and u_1 satisfies the BOB equation then w must satisfy what we are calling the Reduced Benjamin-Ono-Burgers equation (RBOB)

$$\xi w(\xi) - \bar{w}(\xi)^2 + 2(\nu + \rho\mathcal{H})w'(\xi) = \frac{\eta}{\pi}. \tag{0.7}$$

η/π is a constant of integration. Being a first order equation we expect to be able to specify a single parameter and then get a unique solution. For reasons which will become clear later, the natural parameter to specify is $\mu = [\hat{w}(0^+) + \hat{w}(0^-)]/2$. In order to understand the significance of this similarity reduction and its solutions we introduce, following Hopf [**7**], new independent and dependent variables ξ, τ, v by the transformations

$$\begin{array}{lll} \xi = x(t+a)^{-1/2} & \tau = \ln(t+a) & v(\xi,\tau) = e^{\tau/2}u(\xi e^{\tau/2}, e^{\tau} - a) \\ x = \xi e^{\tau/2} & t = e^{\tau} - a & u(x,t) = (t+a)^{-1/2}v(x(t+a)^{-1/2}, \ln(t+a)). \end{array} \tag{0.8}$$

So if $u(x,t)$ satisfies the BOB equation then $v(\xi,\tau)$ satisfies

$$v_\tau - [\xi v/2 - \bar{v}^2/2 + (\nu + \rho\mathcal{H})v_\xi]_\xi = 0. \tag{0.9}$$

We will call this the BOB equation in similarity variables, or BOB' for short. The usual initial value problems are transformed into one another when $a = 1$. Clearly the τ-independent solutions of BOB' are exactly the solutions of RBOB for some value of η. Hopf showed in the case of Burgers' equation ($\rho = 0$) that the solution v of (0.9) arising from initial data $f \in L^1(\mathbb{R})$ tends in the norm of $BC^m(\mathbb{R})$ as $\tau \to \infty$ to a τ-independent solution w, i.e. to a solution of (0.7) (with $\rho = 0$ and $\eta = 0$). This solution turns out to be the spacial form of the triangular wave solution of Burgers' equation mentioned above. Thus the intermediate asymptotic for the solution of Burgers' equation is converted by the change of variables (0.8) into the rest state to which the transformed solution tends as $\tau \to \infty$. Because the BOB equation has the exact same scaling properties it is natural to suspect that this sort of behavior also holds for solutions of the BOB', i.e. that solutions of the RBOB are spacial forms for the intermediate asymptotics for solutions of the BOB equation. This is what we will prove.

Now we will describe the contents of this paper. In §2 we consider the linearized BOB equation. The results therein constitute the technical heart of §3 and §6. In §2.1 we examine the fundamental solution of the LBOB equation and its Hilbert transform. Explicit formulas are given in terms of well-known special functions and asymptotic expansions are recorded. Since these are similarity solutions, their decay rates are easily computed. In §2.2 we give a detailed account of the temporal asymptotic behavior of solutions of the homogeneous LBOB equation. As we have indicated above, in this case we can explicitly give the intermediate asymptotics corresponding to any desired algebraic decay rate. In §2.3 we examine the inhomogeneous form of the LBOB equation. Using the Duhamel representation of the solution we derive careful estimates which measure its regularity and decay properties.

In §3 we study narrowly the initial value problem for the BOB equation. References to other accounts of this problem are given. We confine our attention to extending these results to cover the kind of initial data for which we can also analyze the temporal asymptotic behavior, namely small elements of $L^1(\mathbb{R})$. We use a standard contraction mapping argument in specially weighted spaces based on the estimates of §2.3.

One corollary of our proof of global existence is that all the solutions so obtained satisfy the decay estimate $\|u(t)\|_{L^2(\mathbb{R})} = O(t^{-1/4})$, which we will eventually (see §6) show to be typically sharp for data in $L^1(\mathbb{R})$. In §4 we comment on the problem of extending this decay estimate to large data in $L^1(\mathbb{R})$. We will show there that this estimate holds for solutions with data in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ if and only if the L^1-norm of the solution is bounded independently of time. We will also briefly consider some non-"optimal" decay estimates for data in $L^2(\mathbb{R})$.

In §5 we analyze the RBOB equation. In §5.1 we derive some integral equations which are equivalent to the RBOB equation in the context of L^2 solutions. In §5.2 we show that any L^2 solution of the RBOB equation must be in $H^\infty(\mathbb{R})$ and we prove the a priori validity of a two term asymptotic expansion for $w(\xi)$ as $|\xi| \to \infty$ whose coefficients are expressed in terms of the parameters ν, ρ, μ, η. In §5.3 we prove that L^2 solutions of the RBOB equation exist provided μ and η are sufficiently small in relation to ν. We do this by a contraction mapping argument in an appropriate ball in $L^2(\mathbb{R})$. Thus we obtain uniqueness of solutions in this ball and continuous dependence on the parameters μ and η.

Finally in §6 we prove that if $0 \leq s < 1$, $f \in L^1_s(\mathbb{R})$, and $\|f\|_{L^1(\mathbb{R})}$ is sufficiently small, then the solution of the BOB equation whose existence was proven in §3 possesses as an intermediate asymptotic the similarity solutions ($a = 0$) with the same net mass. The rate is $o(t^{(s+1-1/p)/2})$ in the L^p-norm. Alternatively our results can be expressed in similarity variables, where they take on a dynamical systems flavor. The "flow" of the BOB' equation is a continuous mapping $S'\colon U_1 \to BC([0,\infty), L^1(\mathbb{R}))\colon f \mapsto S'f = (\tau \mapsto S'(\tau)f)$, where U_1 is the open set of all $f \in L^1(\mathbb{R})$ such that $\|\hat{f}\|_{L^\infty(\mathbb{R})} < c\nu$, $c \doteq 0.956$. Each of the closed hyperplanes $H_\mu = \{f \in L^1(\mathbb{R}) \mid \mu = \int_{-\infty}^{\infty} f(x)\,dx\}$ is invariant under the "flow" determined by S'. Suppose D is the open disc of all $\mu \in \mathbb{C}$ such that $|\mu| < 2^{1/2}\nu$. Then there is a continuous map $w\colon D \to L^1(\mathbb{R}) \cap L^2(\mathbb{R})\colon \mu \mapsto w_\mu$ such that $w_0 = 0$, and for all $\mu \in D$ we have $w_\mu \in H_\mu$ and w_μ is a solution of the RBOB equation ($\eta = 0$). Then the following results are true.

(1) There exists a constant $C > 0$ such that for every $f \in U_1 \cap H_\mu$ we have the estimate $\|S'f - w_\mu\|_{BC(\mathbb{R}^+, L^1(\mathbb{R}))} \leq C\|f - w_\mu\|_{L^1(\mathbb{R})}$.

(2) If $f \in U_1 \cap H_\mu$ then $\lim_{\tau\to\infty} \|S'(\tau)f - w_\mu\|_{L^1(\mathbb{R})} = 0$.

Combining these two we see that if $w_\mu \in U_1$ then w_μ is (Lyapunov) stable and in fact asymptotically stable with respect to the flow of the BOB' equation restricted to H_μ. If

$\eta \neq 0$ then solutions of RBOB are also asymptotically stable with respect to the flow of the BOB' equation restricted to hyperplanes $H_{\mu,\eta} \subset L^1(\mathbb{R}) + \mathcal{H}L^1(\mathbb{R})$ of constant values of μ and η, where $\mathcal{H}L^1(\mathbb{R})$ denotes the space of all Hilbert transforms of integrable functions.

Given the nature of the balance inherent in the BOB equation and the fact that solutions of the RBOB equation determine the intermediate asymptotics of solutions of the BOB equation arising from small initial data one is lead to make certain conjectures concerning the picture for large data.

(1) The BOB equation determines a *bounded* C_0 semigroup on *all* $L^1(\mathbb{R})$. In particular we expect global solutions to *exist* for large data in $L^1(\mathbb{R})$ and to be *bounded in dependently of time* in the L^1-norm. By the results of §4 this would imply that $\|S(t)f\|_{L^2(\mathbb{R})} = O(t^{-1/4})$ for all $f \in L^1(\mathbb{R})$.

(2) Solutions to the RBOB equation exist in $L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ for all $\mu \in \mathbb{R}$, and $\eta = 0$. It is likely that L^2 solutions may fail to exist for large values of η, as is the case when $\rho = 0$.

(3) These solutions of the RBOB equation for $\mu \in \mathbb{R}$ and $\eta = 0$ are globally asymptotically stable solutions of the BOB' equation restricted to H_μ. In other words, the similarity solutions of the BOB equation whose spacial forms are the solutions of the RBOB equation under consideration are intermediate asymptotics for general solutions of the BOB equation, even those arising from large initial data.

(4) Finally, we describe our expectation as to the appearance of the large data solutions of the RBOB equation ($\eta = 0$) whose existence was conjectured above. If $\mu = 4\pi n\rho$, where n is a positive integer, and ν is sufficiently small, then we expect that an integrable solution w exists with precisely n local maxima located at points $0 < \xi_1 < \cdots < \xi_n$ such that $0 < w(\xi_1) < \cdots < w(\xi_n)$. These solutions represent the spacial forms of pure "decaying n-solitons" (c.f. "explode-decay solitons" of Nakamura [**13**]). For all other values of μ the solutions should exhibit oscillations for negative values of ξ which represent "decaying dispersive tails". If $\mu < 0$ the solutions should be monotonically decreasing for positive ξ. The amount of oscillation in the tail region, and the distinctiveness of the humps should become more pronounced as $\nu \to 0^+$.

These conjectures, especially the last ones, may seem unwarranted. They represent the author's intuition, based on his investigations of the intermediate asymptotics for the linearized BOB equation, Burgers' equation, and the Benjamin-Ono equation. The critical values $\mu = 4\pi n\rho$, where n is a positive integer, coincide with the net masses of the pure n-soliton solutions of the Benjamin-Ono equation. One point, at least, deserves further elaboration. Why do we expect "decaying solitons" to behave like similarity solutions? Edwin and Roberts [**6**] and Matsuno [**12**] have computed by means of formal asymptotic expansions the effect of a small amount of Burgers-type dissipation on the propagation of a Benjamin-Ono soliton. Their approximate expression for the "decaying soliton" is

$$u(x,t) = \frac{4a}{\sqrt{1+2a^2\nu t}}\left\{1 + \left[\frac{a}{\sqrt{1+2a^2\nu t}}\left(x - \frac{2at}{1+\sqrt{1+2a^2\nu t}}\right)\right]^2\right\}^{-1}, \tag{0.10}$$

where a is a positive real constant. This function decays in amplitude, spreads out, and slows down at precisely the same rates as would a similarity solution whose spacial form

w has a single hump centered about some positive value of ξ. Since the expression (0.10) is supposed to become more accurate as $\nu \to 0^+$ (which takes us outside the realm of our smallness assumptions), this is an indication that there are "decaying soliton" solutions of the BOB equation which are of similarity form.

§1 Notation, Facts about the Hilbert Transform

$\mathbb{R}$ and $\mathbb{C}$ will denote the real and complex numbers respectively, and $\mathbb{R}^+ = (0, \infty)$. $[\![s]\!]$ will denote the greatest integer less than or equal to s. We will always use $\beta(k)$ to denote the function $\beta(k) = (1 + k^2)^{1/2}$ for $k \in \mathbb{R}$. Define $\alpha(k) = (\nu - i\rho\,\mathrm{sgn}(k))k^2$, where $\mathrm{sgn}(k) = 1$ if $k > 0$ and $\mathrm{sgn}(k) = -1$ when $k < 0$. We will use $\Re(z)$ and $\Im(z)$ for the real and imaginary parts of the complex number z. $B(r, s) = \int_0^1 (1-t)^{r-1} t^{s-1}\, dt$ denotes the Beta function, which is finite whenever $r, s > 0$.

If u is a function defined on $\mathbb{R} \times \mathbb{R}^+$ then we will use $u(\cdot, t)$ or simply $u(t)$ to denote the function $x \mapsto u(x,t)$. If X is a normed space of functions of x then we will use $u(x,t) \in X(x)$ and $u(t) \in X$ interchangeably and denote by $\|u(x,t)\|_{X(x)} = \|u(t)\|_X$ the norm in the space X. A class of mappings from one set X into another set Y will be denoted by something like $\mathcal{A}(X, Y)$ where $\mathcal{A}$ will indicate the type of mappings in the class. For example if X is a Banach space, then $L(X, X)$ denotes the class of bounded linear operators, and $C(\mathbb{R}^+, X)$ denotes the class of all continuous mappings. This class will contain unbounded mappings. The Banach space of bounded continuous mappings $U\colon \mathbb{R}^+ \to X$ will be denoted by $BC(\mathbb{R}^+, X)$, and will be assumed to be equipped with the sup norm. $C^n(\mathbb{R}^+, X)$ denotes the space of all n-times continuously Fréchet differentiable mappings.

For spaces of test functions and distributions we will use the notations: $\mathfrak{D}(\mathbb{R})$ for the space of C^∞ functions on $\mathbb{R}$ with compact support equipped with the usual inductive limit topology; $\mathfrak{D}'(\mathbb{R})$ is the space of all continuous linear functionals on $\mathfrak{D}(\mathbb{R})$ equipped with the topology of uniform convergence on bounded subsets of $\mathfrak{D}(\mathbb{R})$ (i.e. the strong topology); $\mathcal{S}(\mathbb{R})$ is the space of tempered test functions, i.e. C^∞ functions on $\mathbb{R}$ which are bounded together with all derivatives even after multiplying by polynomials, equipped with the obvious Fréchet space topology; $\mathcal{S}'(\mathbb{R})$ will denote its dual space again with the strong topology. $\mathfrak{D}'((0,T), X)$ where X is a locally convex topological vector space will denote the space of all continuous linear maps $\mathfrak{D}((0,T)) \to X$. If $X = \mathfrak{D}'(\mathbb{R})$ or $X = \mathcal{S}'(\mathbb{R})$ then it is essential that continuity be interpreted relative to the strong topologies on $\mathfrak{D}'(\mathbb{R})$ and $\mathcal{S}'(\mathbb{R})$. This allows us to use the Schwartz Kernel Theorem (see Treves [**19**]), $\mathfrak{D}'((0,T), \mathfrak{D}'(\mathbb{R})) \cong \mathfrak{D}'((0,T) \times \mathbb{R})$. However for convergence of sequences of for limits with respect to a real parameter we can equivalently use the weak star topologies on $\mathfrak{D}'(\mathbb{R})$ and $\mathcal{S}'(\mathbb{R})$. δ will denote the Dirac delta distribution and $\mathrm{p.v.}\,\frac{1}{x}$ will denote the distribution defined by $\langle \mathrm{p.v.}\,\frac{1}{x}, \varphi \rangle = \lim_{\epsilon \to 0^+} \int_{|x| \geq \epsilon} \frac{\varphi(x)}{x}\, dx$ for all $\varphi \in \mathcal{S}(\mathbb{R})$.

$L^p(\mathbb{R})$ will denote the space of equivalence classes of measurable complex-valued functions on $\mathbb{R}$ such that the pth power of their absolute value is Lebesgue integrable on $\mathbb{R}$ (the usual modification for $p = \infty$ is understood). We will let $L^p_s(\mathbb{R})$ denote the class of measurable functions f on $\mathbb{R}$ such that $(1 + |x|)^s f(x) \in L^p(x)$, the L^p-norm of this function being the norm in $L^p_s(\mathbb{R})$. If $f \in L^p(\mathbb{R})$ and $g \in L^q(\mathbb{R})$ then denote by $f * g$ the L^r function defined almost everywhere by the formula

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-y)g(y)\,dy,$$

where $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, $1 \leq p, q, r \leq \infty$.

Our Fourier transform is defined by

$$(\mathfrak{F}f)(k) = \hat{f}(k) = \int_{-\infty}^{\infty} e^{-ikx} f(x)\,dx,$$

for functions $f \in \mathcal{S}(\mathbb{R})$, and by transposition for all tempered distributions. Fourier multiplier operators will be denoted by $m(D)$, where $m\colon \mathbb{R} \to \mathbb{C}$ is a function and $D = -i\partial_x$. This operator is defined formally in terms of the Fourier transform by $[m(D)f]\hat{}(k) = m(k)\hat{f}(k)$.

Our Hilbert transform is defined to be the operator $-i\,\mathrm{sgn}(D)$. The usual pointwise definition is in terms of the limit

$$(\mathcal{H}f)(x) = \frac{1}{\pi} \lim_{\epsilon \to 0+} \int_{|y| \geq \epsilon} \frac{f(x-y)}{y}\,dy,$$

which exists for almost every $x \in \mathbb{R}$ for all $f \in L^p(\mathbb{R})$, $1 \leq p < \infty$. For $1 < p < \infty$ the resulting function $\mathcal{H}f \in L^p(\mathbb{R}) \subset \mathcal{S}'(\mathbb{R})$ and $\mathcal{H}f = -i\,\mathrm{sgn}(D)f$. When $f \in L^1(\mathbb{R})$ it may happen that $\mathcal{H}f$ is not locally integrable on any interval. Nevertheless the function $\mathcal{H}f$ can be made into a tempered distribution by means of the so-called Titschmarsh integral, i.e. if $\phi \in \mathcal{S}(\mathbb{R})$ then we define

$$\langle \mathcal{H}f, \phi \rangle = \lim_{a \to \infty} \int_{|(\mathcal{H}f)(x)| \leq a} (\mathcal{H}f)(x)\phi(x)\,dx.$$

The proof that this limit exists and defines a tempered distribution is in [**9**]. It then follows that this tempered distribution coincides with the tempered distribution $-i\,\mathrm{sgn}(D)f$. An important property of the Hilbert transform which enables the Hilbert transforms of certain functions to be explicit computed is the following. Suppose F is a holomorphic function, defined for $\Im(z) > 0$, and lying in the Hardy space $\mathcal{H}^2$, i.e. $\sup_{y>0} \|F(x+iy)\|_{L^2(x)} < \infty$. Then for almost every $x \in \mathbb{R}$ the $\lim_{y \to 0+} F(x+iy) = f(x) + ig(x)$ exists, $f, g \in L^2(\mathbb{R}, \mathbb{R})$, and $g = \mathcal{H}f$.

If $\mathcal{O}$ is an operator and $X \subset \mathcal{S}'(\mathbb{R})$ is a space, then $\mathcal{O}X$ will denote the image of the space X under the operator $\mathcal{O}$. If X is a normed space and $\mathcal{O}$ is injective on X, then we give $\mathcal{O}X$ the norm which makes $\mathcal{O}$ an isometry. For example $\mathfrak{F}L^p(\mathbb{R})$ denotes the space of all Fourier transforms of L^p functions, or equivalently, the space of tempered distributions whose Fourier transforms are L^p functions. For all $s \in \mathbb{R}$ denote by $H^s(\mathbb{R})$ the usual Hilbert-Sobolev space $\beta(D)^{-s}L^2(\mathbb{R})$ consisting of all tempered distributions f whose Fourier transforms are locally integrable functions $\hat{f}$ such that $(1+k^2)^{s/2}\hat{f}(k) \in L^2(k)$. The L^p based Sobolev space will be denoted by $W^{k,p}(\mathbb{R})$. It consists of all L^p *functions on*

$\mathbb{R}$ whose first k distributional derivatives come from L^p functions. If $X \subset H^s(\mathbb{R})$ then we will denote by $\mathcal{H}X \subset H^s(\mathbb{R})$ the space of all Hilbert transforms of distributions in X. Thus for example $\mathcal{H}L^1(\mathbb{R}) \subset H^{-1}(\mathbb{R})$. $L^1(\mathbb{R}) + \mathcal{H}L^1(\mathbb{R})$ will denote the linear span of $L^1(\mathbb{R})$ and $\mathcal{H}L^1(\mathbb{R})$ in $H^{-1}(\mathbb{R})$. The norm in this space is defined by $\|f\|_{L^1(\mathbb{R})+\mathcal{H}L^1(\mathbb{R})} = \inf \|g_0\|_{L^1(\mathbb{R})} + \|g_1\|_{L^1(\mathbb{R})}$ where the infimum is taken over all $g_0, g_1 \in L^1(\mathbb{R})$ such that $f = g_0 + \mathcal{H}g_1$. $L^1(\mathbb{R}) \cap \mathcal{H}L^1(\mathbb{R})$ will denote the space of functions in $L^1(\mathbb{R})$ whose Hilbert transform is in $L^1(\mathbb{R})$. It is equipped with the norm $\|f\|_{L^1(\mathbb{R})\cap\mathcal{H}L^1(\mathbb{R})} = \|f\|_{L^1(\mathbb{R})} + \|\mathcal{H}f\|_{L^1(\mathbb{R})}$. Notice that $L^1(\mathbb{R}) \cap \mathcal{H}L^1(\mathbb{R}) \hookrightarrow L^1(\mathbb{R}) \hookrightarrow L^1(\mathbb{R}) + \mathcal{H}L^1(\mathbb{R}) \hookrightarrow \mathcal{F}L^\infty(\mathbb{R})$ and that these inclusions are bounded.

§2. The Linearized BOB Equation

2.0 Introduction. In this section we collect together results about existence, uniqueness, regularity, spacial and temporal asymptotic behavior of solutions of the linearized Benjamin-Ono-Burgers' (LBOB) equation:

$$u_t - (\nu + \rho\mathcal{H})u_{xx} = -(gh)_x/2 \qquad \text{in } \mathfrak{D}'(\mathbb{R}^+, \mathcal{S}'(\mathbb{R})), \tag{2.0.1}$$

$$u(0) \to f \qquad \text{in } \mathcal{S}'(\mathbb{R}) \text{ as } t \to 0^+, \tag{2.0.2}$$

where $\nu > 0$ and ρ are real constants. If $\nu = 0$ then (2.0.1) is called the linearized Benjamin-Ono (LBO) equation. We will not consider this case. Our results will however contain the case $\rho = 0$, i.e. the heat equation. By scaling the x variable the constant ν can be replaced by 1, but we will not do this. The solution of (2.0.1) and (2.0.2) is given by the usual Duhamel formula

$$u(t) = e^{-\alpha(D)t}f - \frac{1}{2}\int_0^t e^{-\alpha(D)(t-\tau)}\partial_x[g(\tau)h(\tau)]\,d\tau. \tag{2.0.3}$$

In subsection 2.1 we study (2.0.1) and (2.0.2) in the case $g = h = 0$ and $f = \mu\delta + \eta\mathcal{H}\delta$. These solutions turn out to be of similarity form. Convolution of these special solutions with the initial data f yield the solution to the initial value problem for the homogeneous equation ($g = h = 0$). The temporal asymptotic behavior of this solution is studied in subsection 2.2. Finally, in subsection 2.3 we study the inhomogeneous equation (2.0.1) where g and h are in certain weighted spaces. and $f = 0$. The solution is analyzed in regard to its regularity and temporal asymptotic behavior in relation to that assumed of g and h. Since our discussion of the inhomogeneous equation is aimed toward the analysis of the nonlinear equation we will not consider more general right-hand-sides than that shown in (2.0.1). In the existence theory (Section 3) for the nonlinear equation we will have $g = h = \bar{u}$. In the analysis of the temporal asymptotic behavior (Section 6) of solutions u_1, u_2 of the nonlinear equation we will have $u = g = \bar{u}_1 - \bar{u}_2$ and $h = \bar{u}_1 + \bar{u}_2$.

2.1 Similarity and fundamental solutions. Suppose $s \geq 0$ is a real number and $w \in L^1_s(\mathbb{R}) + \mathcal{H}L^1_s(\mathbb{R})$ is a function such that $u_1(x,t)$ defined by $u_1(x,t) = t^{-(s+1)/2}w(xt^{-1/2})$ is a solution of (2.0.1). The solution u_1 is called a similarity solution and w will be called its similarity form. If $\xi = xt^{-1/2}$ then w satisfies the reduced equation

$$(s+1)w(\xi) + \xi w'(\xi) + 2\nu w''(\xi) + 2\rho(\mathcal{H}w'')(\xi) = 0. \tag{2.1.1}$$

We will concentrate on the case $s = 0$. The solutions in the case where s is a positive integer can be obtained from those with $s = 0$ by applying ∂_x^s. If $w = g_0 + \mathcal{H}g_1$ where $g_0, g_1 \in L^1(\mathbf{R})$ and μ and η are defined by $\mu = \int_{-\infty}^{\infty} g_0(x)\,dx$, $\eta = \int_{-\infty}^{\infty} g_1(x)\,dx$, then μ and η do not depend on the particular choice for g_0 and g_1. In fact $\mu = (\hat{w}(0^+) + \hat{w}(0^-))/2$ and $\eta = i(\hat{w}(0^+) - \hat{w}(0^-))/2$. In terms of these parameters we can easily see that the similarity solution u_1 satisfies the initial condition

$$u_1(t) \to \mu\delta + \frac{\eta}{\pi}\,\mathrm{p.v.}\,\frac{1}{x} \qquad \text{in } \mathcal{S}'(\mathbf{R}) \text{ as } t \to 0^+. \tag{2.1.2}$$

If $\mu_0 = 1$ and $\eta = 0$ then the unique solution in $L^\infty((0,\infty), L^1(\mathbf{R}))$ to the problem (2.0.1) and (2.1.2) is called the fundamental solution, and will be denoted by $F_{\nu,\rho}(x,t)$. Its similarity form will be denoted by $G_{\nu,\rho}(\xi)$. If the values of ν and η are understood then we will denote these functions simply by $F(x,t)$ and $G(\xi)$. Since $\frac{1}{\pi}\,\mathrm{p.v.}\,\frac{1}{x} = \mathcal{H}\delta$, the solution to the problem (2.0.1) and (2.1.2) with $\mu = 0$ and $\eta = 1$ is $(\mathcal{H}F(t))(x)$ with similarity form $(\mathcal{H}G)(\xi)$. If w is as above ($s = 0$) then (2.1.1) can be integrated once to obtain

$$\xi w(\xi) + 2\nu w'(\xi) + 2\rho(\mathcal{H}w')(\xi) = \frac{\eta}{\pi}. \tag{2.1.3}$$

Taking the Fourier transform of this equation one can easily show that its general solution in $L^1(\mathbf{R}) + \mathcal{H}L^1(\mathbf{R})$ is the inverse Fourier transform of $(\mu - i\eta\,\mathrm{sgn}(k))e^{-\alpha(k)t}$, where μ and η are arbitrary complex constants, and $\alpha(k) = \nu k^2 + i\rho k|k|$. Thus $\hat{G}(k) = e^{-\alpha(k)}$. G and $\mathcal{H}G$ can also be expressed in terms of well-known special functions by considering (2.1.3) as the real part of the ordinary differential equation

$$\xi\phi(\xi) + 2(\nu - \rho i)\phi'(\xi) = \frac{\eta + \mu i}{\pi}, \tag{2.1.4}$$

with solution $\phi = w + i\mathcal{H}w$. Without difficulty one can show that (2.1.4) has a solution of this form if and only if the initial datum is chosen to be

$$\phi(0) = \frac{\mu - \eta i}{2\sqrt{\pi(\nu - \rho i)}}, \tag{2.1.5}$$

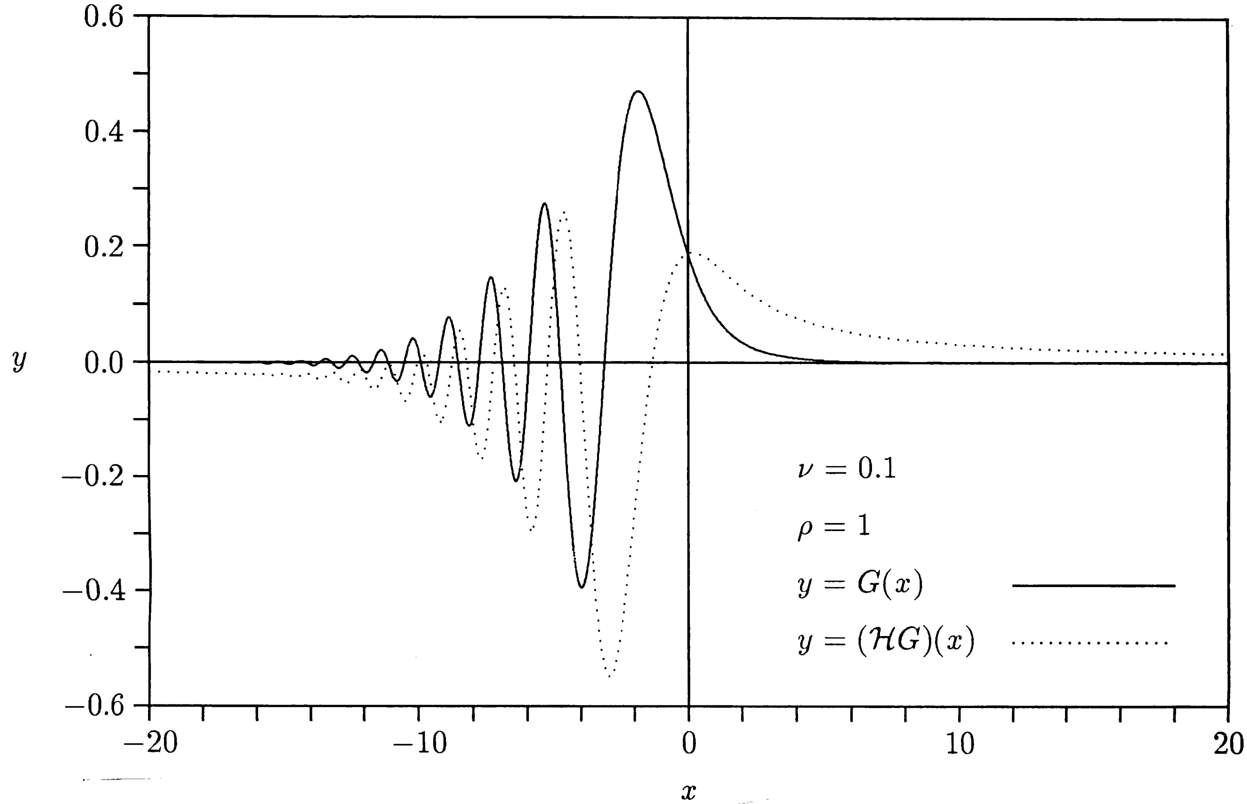
this taken to be in the first or fourth quadrant. With this datum (2.1.4) can be solved by the use of an integrating factor to obtain

$$\phi(x) = \frac{\mu - \eta i}{2\sqrt{\pi(\nu - \rho i)}}\, W\left(\frac{x}{2\sqrt{\nu - \rho i}}\right) \tag{2.1.6}$$

where the so called complementary error function of a complex variable $W(\zeta)$ is defined by (see [**1**])

$$W(\zeta) = e^{-\zeta^2}\left[1 + \frac{2i}{\sqrt{\pi}}\int_0^\zeta e^{s^2}\,ds\right]. \tag{2.1.7}$$

FIGURE 1. The Similarity Forms of the Fundamental Solution and its Hilbert Transform



Since $w = \Re\phi = \mu G + \eta \mathcal{H} G$ we have that

$$G_{\nu,\rho}(\xi) = \frac{1}{2\sqrt{\pi}} \Re\left[\frac{1}{\sqrt{\nu - \rho i}} W\left(\frac{\xi}{2\sqrt{\nu - \rho i}}\right)\right], \tag{2.1.8}$$

$$(\mathcal{H}G_{\nu,\rho})(\xi) = \frac{1}{2\sqrt{\pi}} \Im\left[\frac{1}{\sqrt{\nu - \rho i}} W\left(\frac{\xi}{2\sqrt{\nu - \rho i}}\right)\right]. \tag{2.1.9}$$

If $\rho = 0$ then a well-known asymptotic expansion of $W(\zeta)$ yields a correct asymptotic expansion for the similarity form of the fundamental solution of the heat equation (which decays exponentially) and its Hilbert transform, the Dawson's integral (see [**1**]). If on the other hand $\rho \neq 0$, then we have the formula

$$G_{\nu,\rho}(\xi) = \operatorname{sgn}(\rho)|\rho|^{-1/2} G_{\nu/|\rho|,1}(\xi \operatorname{sgn}(\rho)|\rho|^{-1/2}). \tag{2.1.10}$$

So without loss of generality we can set $\rho = 1$. When $\nu > 0$ the line $\zeta = \xi[4(\nu - i)]^{-1/2}$ lies inside the sector of validity of the above-mentioned asymptotic expansion of $W(\zeta)$. Using (2.1.8) and (2.1.9) one can derive asymptotic expansions for G and $\mathcal{H}G$ which are valid as

$|\xi| \to \infty$:

(2.1.11)
$$G_{\nu,1}(\xi) \sim \frac{1}{\pi}\left[\quad \frac{2}{\xi^3} + \frac{24\nu}{\xi^5} + \frac{120(3\nu^2-1)}{\xi^7} + \frac{6720\nu(\nu^2-1)}{\xi^9} + \dots \right],$$

(2.1.12)
$$\mathcal{H}G_{\nu,1}(\xi) \sim \frac{1}{\pi}\left[\frac{1}{\xi} + \frac{2\nu}{\xi^3} + \frac{12(\nu^2-1)}{\xi^5} + \frac{120\nu(\nu^2-3)}{\xi^7} + \frac{1680(\nu^4-6\nu^2+1)}{\xi^9} + \dots\right].$$

The derivatives of these expansions yield valid asymptotic expansions of the derivatives of G and $\mathcal{H}G$. Now we will state some consequences of the above calculations.

THEOREM 2.1.1. *Suppose $G_{\nu,\rho}$ is as defined above. Suppose also that $u_1(x,t) = t^{-1/2}w(xt^{-1/2})$ is a similarity solution of (2.0.1) where $w \in L^1(\mathbb{R}) + \mathcal{H}L^1(\mathbb{R})$. Define the differential operators $\mathcal{O}_s$ for $s \in \mathbb{R}$ by $\mathcal{O}_s = (1/2) - s - (1/2)\xi\partial_\xi$. Then for $x \in \mathbb{R}$, $t > 0$, $j \geq 0$, $k \geq 0$, and $1 \leq p \leq \infty$ we have*

1. $\partial_x^j \partial_t^k u_1(x,t) = t^{-(j+2k+1)/2}(\partial_\xi^j \mathcal{O}_k \dots \mathcal{O}_2\mathcal{O}_1 w)(xt^{-1/2})$;
2. $\|\partial_x^j \partial_t^k u_1(x,t)\|_{L^p(x)} = t^{-(j+2k+1-1/p)/2}\|\partial_\xi^j \mathcal{O}_k \dots \mathcal{O}_2\mathcal{O}_1 w\|_{L^p}$;
3. $\partial_\xi^j \mathcal{O}_k \dots \mathcal{O}_2\mathcal{O}_1 G_{\nu,\rho} \in L^p$; *also* $\partial_\xi^j \mathcal{O}_k \dots \mathcal{O}_2\mathcal{O}_1 \mathcal{H}G_{\nu,\rho} \in L^p$ *except for the case where* $j = 0$, $k = 0$ *and* $p = 1$ *simultaneously;*
4. $\|G^{(j)}_{\nu,\rho}\|_{L^2} = [1 \cdot 3 \cdot \dots \cdot (2j-1)]^{1/2} 2^{-j-3/4}\pi^{-1/4}\nu^{-(2j+1)/4}$.

PROOF: (1) and (2) follow from the self-similar form of u_1. (3) follows from (2.1.11), (2.1.12), and the fact that $\mathcal{O}_{(n+1)/2}[\xi^{-n}] = 0$ for $n = 1, 3, 5, \dots$. (4) follows from Parseval's theorem. ∎

2.2 Temporal asymptotic behavior of solutions of the homogeneous equation. Iório [8] has investigated the LBOB equation as a semigroup in $H^s(\mathbb{R})$ for $s \in \mathbb{R}$. As is well-known if $f \in L^p(\mathbb{R})$ for $1 \leq p < \infty$ then the solution $u(t) = e^{-\alpha(D)t}f$ of the homogeneous version of the Linearized BOB (LBOB) equation can be obtained as the convolution $u(t) = F_{\nu,\rho}(t) * f$. If $f \in L^1(\mathbb{R}) + \mathcal{H}L^1(\mathbb{R})$, i.e. $f = g_0 + \mathcal{H}g_1$ where $g_0, g_1 \in L^1(\mathbb{R})$, then $u(t) = e^{-\alpha(D)t}f = F_{\nu,\rho}(t) * g_0 + \mathcal{H}F_{\nu,\rho}(t) * g_1$. Because of the smoothness of the fundamental solution and its Hilbert transform the solution lies in $H^\infty(\mathbb{R})$ for every positive time. To supplement these comments we state the following theorem whose proof can easily be supplied by the reader.

THEOREM 2.2.1. *Suppose $1 \leq p < \infty$, $f \in L^p(\mathbb{R})$, and $u(t) = F_{\nu,\rho}(t) * f$ for all $t \in \mathbb{R}^+$. Then $u(t) \to f$ in $L^p(\mathbb{R})$ as $t \to 0^+$, and for all $p \leq q \leq \infty$ and integers $m \geq 0$ we have $u \in C^m(\mathbb{R}^+, W^{m,q}(\mathbb{R}))$. There exists a unique C^∞ representative of u that is a classical solution of (2.0.1), i.e. the principle value limit defining the Hilbert transform exists for every $(x,t) \in \mathbb{R} \times \mathbb{R}^+$, and the equation holds at every such (x,t) when the partial derivatives are interpreted classically. Furthermore if $1 \leq p \leq q \leq 2$ then u is the unique distributional solution of (2.0.1) and (2.0.2) in the class $L^1_{\text{loc}}(\mathbb{R}^+, \mathcal{F}L^{q'}(\mathbb{R}))$, where $q^{-1} + (q')^{-1} = 1$. If $f \in L^1(\mathbb{R}) + \mathcal{H}L^1(\mathbb{R})$ then the same assertions as above hold except* *that $u(t)$ will not in general lie in $L^1(\mathbb{R})$ (although its derivatives will). In this case however we do have $u \in BC([0,\infty), L^1(\mathbb{R}) + \mathcal{H}L^1(\mathbb{R}))$.*

The following result will be useful to our discussion in §3 of the convergence of the solution of the nonlinear equation to its initial data.

PROPOSITION 2.2.2. *Suppose $f \in L^1(\mathbb{R}) + \mathcal{H}L^1(\mathbb{R})$ then $\|e^{-\alpha(D)t}f\|_{L^2(\mathbb{R})} = o(t^{-1/4})$ as $t \to 0^+$.*

PROOF: Using Parseval's formula we have

$$t^{1/4}\|e^{-\alpha(D)t}f\|_{L^2(\mathbb{R})} = (2\pi)^{-1/2}\left(\int_{-\infty}^{\infty} e^{-2\nu\omega^2}|\hat{f}(\omega t^{-1/2})|^2\,d\omega\right)^{1/2}. \tag{2.2.1}$$

Clearly this tends to 0 by the dominated convergence theorem. ∎

Now we are ready to pursue the question of how solutions of the LBOB equation decay in time. The temporal decay rates depend on which spacial norm is used to measure the solutions as well as the behavior of the Fourier transform as $k \to 0$ of the initial data. Our first theorem describes the situation when the Fourier transform of the data is assumed to possess "singularities" of various strengths near $k = 0$, leading to more or less slow decay.

THEOREM 2.2.3. *Suppose $1 < q < \infty$, $q \le p \le \infty$, and $f \in L^q(\mathbb{R})$. Then*

$$\lim_{t\to\infty} t^{(1/q-1/p)/2}\|F_{\nu,\rho}(t) * f\|_{L^p(\mathbb{R})} = 0. \tag{2.2.2}$$

PROOF: First we will prove (2.2.1) in the case $p = \infty$. Let q' be the conjugate exponent to q, and suppose $\epsilon > 0$ is given. Choose M large enough so that

$$\|G\|_{L^{q'}(\mathbb{R})}\left(\int_{|y|\ge M} |f(y)|^q\,dy\right)^{1/q} < \frac{\epsilon}{2}. \tag{2.2.3}$$

Now choose $T > 0$ large enough so that

$$\|G\|_{L^\infty(\mathbb{R})}\left(\frac{2M}{\sqrt{T}}\right)^{1/q'}\|f\|_{L^q(\mathbb{R})} < \frac{\epsilon}{2}. \tag{2.2.4}$$

Then if $t \ge T$ we have that

(2.2.4)

$$\begin{aligned} t^{1/(2q)}|(F(t)*f)(\sqrt{t}x)| &= t^{-1/(2q')}\left|\int_{-\infty}^{\infty} G\left(x - \frac{y}{\sqrt{t}}\right) f(y)\,dy\right| \\ &\le \left(\int_{|y|\le Mt^{-1/2}} |G(x-y)|^{q'}\,dy\right)^{1/q'}\left(\int_{|y|\le M} |f(y)|^q\,dy\right)^{1/q} \end{aligned}$$

$$+\left(\int_{|y|\geq Mt^{-1/2}} |G(x-y)|^{q'}\,dy\right)^{1/q'}\left(\int_{|y|\geq M} |f(y)|^q\,dy\right)^{1/q}$$

$$\leq \|G\|_{L^\infty(\mathbb{R})}\left(\frac{2M}{\sqrt{t}}\right)^{1/q'}\|f\|_{L^q(\mathbb{R})}+\|G\|_{L^{q'}(\mathbb{R})}\left(\int_{|y|\geq M} |f(y)|^q\,dy\right)^{1/q}<\epsilon.$$

Now we will prove (2.2.2) in the case $p=q$. The case for values of p between q and ∞ will then follow by interpolation. Let $\epsilon>0$ be given. Choose a function $g\in C_c(\mathbb{R})$ such that $\|f-g\|_{L^q(\mathbb{R})}<\epsilon/(2\|G\|_{L^1(\mathbb{R})})$ and choose $T>0$ such that $\|G\|_{L^q(\mathbb{R})}\|g\|_{L^1(\mathbb{R})}T^{-(1-1/q)/2}<\epsilon/2$. Then by Young's convolution inequality we have for $t\geq T$ that

(2.2.5)

$$\begin{aligned}\|F(t)*f\|_{L^q(\mathbb{R})}&\leq \|F(t)*(f-g)\|_{L^q(\mathbb{R})}+\|F(t)*g\|_{L^q(\mathbb{R})}\\&\leq \|F(t)\|_{L^1(\mathbb{R})}\|f-g\|_{L^q(\mathbb{R})}+\|F(t)\|_{L^q(\mathbb{R})}\|g\|_{L^1(\mathbb{R})}\\&<\|G\|_{L^1(\mathbb{R})}\cdot\frac{\epsilon}{2\|G\|_{L^1(\mathbb{R})}}+\frac{\|G\|_{L^q(\mathbb{R})}}{t^{(1-1/q)/2}}\cdot\|g\|_{L^1(\mathbb{R})}<\epsilon. \blacksquare\end{aligned}$$

This result in the case $q=p=2$ is optimal in the sense of Definition 0.1(2). To see this suppose that $\gamma\colon(0,\infty)\to(0,\infty)$ is a continuous function such that $\gamma(t)=o(1)$ as $t\to\infty$ and for all $f\in L^2(\mathbb{R})$ we have $\limsup_{t\to\infty}\gamma(t)^{-1}\|e^{-\alpha(D)t}f\|_{L^2(\mathbb{R})}<\infty$. Define for $t>0$ the function $\gamma_1(t)=\sup_{s\geq t}\gamma(s)$. For every $f\in L^2(\mathbb{R})$ let $T(f)>0$ be such that $\sup_{t\geq T(f)}\gamma(t)^{-1}\|e^{-\alpha(D)t}f\|_{L^2(\mathbb{R})}<\infty$. Since $\gamma_1(t)\geq\gamma(t)$ we have that $\sup_{t\geq T(f)}\gamma_1(t)^{-1}\|e^{-\alpha(D)t}f\|_{L^2(\mathbb{R})}<\infty$. But if $0<t<T(f)$ then

$$\gamma_1(t)^{-1}\|e^{-\alpha(D)t}f\|_{L^2(\mathbb{R})}\leq\gamma_1(T(f))^{-1}\|f\|_{L^2(\mathbb{R})}.$$

Thus for all $f\in L^2(\mathbb{R})$ we have $\sup_{t>0}\gamma_1(t)^{-1}\|e^{-\alpha(D)t}f\|_{L^2(\mathbb{R})}<\infty$. Then by the Uniform Boundedness Principle we have $\sup_{t>0}\gamma_1(t)^{-1}\|e^{-\alpha(D)t}\|_{L(L^2,L^2)}<\infty$. But

$$\|e^{-\alpha(D)t}\|_{L(L^2,L^2)}=\|e^{-\alpha(k)t}\|_{L^\infty(k)}=1$$

for all $t>0$, which yields a contradiction since $\gamma_1(t)=o(1)$ as $t\to\infty$. This argument is adapted from Littman and Markus [**11**]. The result for general p and q is undoubtedly optimal as well, but we will not pursue this question here.

An examination of Theorem 2.2.3 reveals that more rapidly (spacially) decaying initial data (less singular Fourier transform near $k=0$) generate more rapidly (temporally) decaying solutions as measured in a fixed norm. However if one considers data with even more spacial decay, say $f\in L^1_s(\mathbb{R})$ for $s\geq 0$ (smoothness of the Fourier transform near $k=0$), then the decay rate of the spacial norm of the solution depends also on which moments of the data vanish or do not vanish. This dependence can be greatly clarified by the consideration of the form of the solution as it decays. In the following theorem we show that appropriate linear combinations of the fundamental solution and its derivatives serve as intermediate asymptotics to the solution as $t \to \infty$. Since the temporal decay rates of spacial norms of the intermediate asymptotic are known exactly (since it is a sum of similarity solutions) the exact conditions under which we can expect a particular decay rate of the solution will be clear (see Corollary 2.2.7). Before we state our theorem we will state as a lemma an easy generalization of Taylor's theorem.

LEMMA 2.2.4. *Suppose $n \geq 0$ is an integer, and $K \in C^n(\mathbb{R})$. Define the nth Taylor remainder of K by*

$$(R_nK)(y)(\xi) = K(\xi + y) - \sum_{j=0}^{n} \frac{K^{(j)}(\xi)}{j!} y^j, \tag{2.2.6}$$

where $\xi, y \in \mathbb{R}$. If the distributional derivative $K^{(n+1)}$ is locally integrable then the usual integral representation of this remainder holds for all $\xi, y \in \mathbb{R}$:

$$(R_nK)(y)(\xi) = \frac{y^{n+1}}{n!} \int_0^1 (1-z)^n K^{(n+1)}(\xi + yz)\, dz. \tag{2.2.6}$$

Furthermore if $1 \leq p \leq \infty$ and for $j = 1, \ldots, n+1$ the distributional derivatives $K^{(j)}$ are in $L^p(\mathbb{R})$ then the following estimates hold for all $y \in \mathbb{R}$:

(2.2.7)
$$\|(R_nK)(y)\|_{L^p(\mathbb{R})} \leq \frac{|y|^{n+1}}{(n+1)!} \|K^{(n+1)}\|_{L^p(\mathbb{R})},$$

(2.2.8)
$$\|(R_nK)(y)\|_{L^p(\mathbb{R})} \leq \|K(\xi + y) - K(\xi)\|_{L^p(\xi)} + \sum_{j=1}^{n} \frac{|y|^j}{j!} \|K^{(j)}\|_{L^p(\mathbb{R})}.$$

Let $f = g_0 + \mathcal{H}g_1$ where $g_0, g_1 \in L^1_s(\mathbb{R})$ and $s \geq 0$. If the numbers μ_j and η_j are defined by $\mu_j = \int_{-\infty}^{\infty} x^j g_0(x)\, dx$ and $\eta_j = \int_{-\infty}^{\infty} x^j g_1(x)\, dx$ for $j = 0, \ldots, n = [\![s]\!]$, then it is easy to show that they depend only on f and not on the particular choice of g_0 and g_1. In fact for $j = 0, \ldots, n$ we have

$$\mu_j = i^j[\hat{f}^{(j)}(0^+) + \hat{f}^{(j)}(0^-)]/2 \qquad \eta_j = i^{j+1}[\hat{f}^{(j)}(0^+) - \hat{f}^{(j)}(0^-)]/2. \tag{2.2.9}$$

THEOREM 2.2.5. *Suppose $s \geq 0$, $1 \leq p \leq \infty$, $n = [\![s]\!]$, $m \geq 0$ an integer,and $f \in L^1_s(\mathbb{R}) + \mathcal{H}L^1_s(\mathbb{R})$, say $f = g_0 + \mathcal{H}g_1$. Define μ_j, η_j for $j = 0, 1, \ldots, n$ as in (2.2.9), and the asymptotic approximation $u_1(t)$ by*

$$u_1(t) = \sum_{j=0}^{n} \frac{(-1)^j}{j!} \partial_x^j \left[\mu_j F(t) + \eta_j \mathcal{H}F(t)\right]. \tag{2.2.10}$$

Then the asymptotic result

$$\lim_{t \to \infty} t^{(s+m+1-1/p)/2} \|\partial_x^m [e^{-\alpha(D)t} f - u_1(t)]\|_{L^p(\mathbb{R})} = 0 \tag{2.2.11}$$

holds in the following cases:

(1) *if* $p > 1$;
(2) *if* $p = 1$, *and* $m > 0$;
(3) *if* $p = 1$, $m = 0$, *and* $s > 0$;
(4) *if* $p = 1$, $m = 0$, $s = 0$, *and* $\int_{-\infty}^{\infty}[\ln(1+|x|)]^{\epsilon}|g_1(x)|\,dx < \infty$ *for some* $\epsilon > 0$.

If $p = 1$, $m = 0$, $s = 0$, *and* $f \in L^1(\mathbb{R}) + \mathcal{H}L^1(\mathbb{R})$ *is arbitrary, then*

$$\lim_{t\to\infty} \|e^{-\alpha(D)t}f - u_1(t)\|_{L^1(\mathbb{R})+\mathcal{H}L^1(\mathbb{R})} = 0.$$

PROOF: By Theorem 2.1.1(3) $K = G^{(m)}$ and $K = \mathcal{H}G^{(m)}$ are in $C^n(\mathbb{R})$, so we can define $R_n(G^{(m)})$ and $R_n(\mathcal{H}G^{(m)})$ as in (2.2.6). Then it is easy to check that

(2.2.12)
$$\begin{aligned}
&\partial_x^m[F(t) * g_0 + \mathcal{H}F(t) * g_1 - u_1(t)](x) \\
&\qquad = \sum_{l=0}^{1}\int_{-\infty}^{\infty} t^{-(m+1)/2}[R_n(\mathcal{H}^l G^{(m)})](-yt^{-1/2})(xt^{-1/2})g_l(y)\,dy,
\end{aligned}$$

where $F = F_{\nu,\rho}$ and $G = G_{\nu,\rho}$. Since

$$\|(R_nK)(y)(xt^{-1/2})\|_{L^p(x)} = t^{1/(2p)}\|(R_nK)(y)\|_{L^p(\mathbb{R})}$$

for any L^p function $(R_nK)(y)$, we have by Minkowski's integral inequality that

(2.2.13)
$$\begin{aligned}
&t^{(s+m+1-1/p)/2}\|e^{-\alpha(D)t}f - u_1(t)\|_{L^p(\mathbb{R})} \\
&\qquad \le t^{s/2}\sum_{l=0}^{1}\int_{-\infty}^{\infty}\|(R_n\mathcal{H}^l G^{(m)})(-yt^{-1/2})\|_{L^p(\mathbb{R})}|g_l(y)|\,dy \\
&\qquad = \sum_{l=0}^{1}\int_{-\infty}^{\infty}\frac{\|(R_n\mathcal{H}^l G^{(m)})(-yt^{-1/2})\|_{L^p(\mathbb{R})}}{|yt^{-1/2}|^s}|y|^s|g_l(y)|\,dy.
\end{aligned}$$

Now we apply from Lemma 2.2.4 the estimate (2.2.7) when $|yt^{-1/2}| \le 1$ and (2.2.8) when $|yt^{-1/2}| > 1$. If $p > 1$ or $m > 0$ we see by Theorem 2.1.1(3) that the quotients in the last integrals of (2.2.13) are bounded independently of y and t. If $p = 1$ and $m = 0$ but $s > 0$ then we must apply the estimate

$$\int_{-\infty}^{\infty}|\mathcal{H}G(\xi+y) - \mathcal{H}G(\xi)|\,d\xi \le C\ln(1+|y|), \tag{2.2.14}$$

where $y \in \mathbb{R}$ and $C > 0$, in (2.2.8). (2.2.14) follows from (2.1.12). This will again imply the second quotient ($l = 1$) is bounded. Thus under the assumptions of cases (1), (2), and (3) the quotients are bounded and by (2.2.7) tend to 0 as $t \to \infty$ for each fixed y. Thus (2.2.11) follows by the dominated convergence theorem. In case (4) the first ($l = 0$) quotient (not really a quotient since $s = 0$) is bounded and we may use dominated convergence as before. In the second ($l = 1$) integral of (2.2.13) the integrand can be estimated using (2.2.14) to obtain

(2.2.15)

$$\|\mathcal{H}G(\xi - yt^{-1/2}) - \mathcal{H}G(\xi)\|_{L^1(\xi)}|g_1(y)| \le C\ln(1 + |yt^{-1/2}|)|g_1(y)| \le [\ln(1+|y|)]^{\epsilon}|g_1(y)|,$$

when $t \ge \max\{1, \epsilon^{-2}\}$. Thus again we can apply the dominated convergence theorem. The last assertion is a consequence of the case (4) ($g_1 = 0$) and the definition of the norm in the space $L^1(\mathbb{R}) + \mathcal{H}L^1(\mathbb{R})$. ∎

Several remarks about this result are in order. Firstly, we only needed the fact that F is a solution of LBOB in order to show that G is in $W^{m,p}(\mathbb{R})$. Therefore Theorem 2.2.3 is really a result about large parameter asymptotic approximations to convolutions by a smooth self-similar kernel. Secondly, if the condition on g_1 in (4) does not hold then the difference $\mathcal{H}F(t) * g_1 - \eta_0\mathcal{H}F(t)$ will not in general lie in $L^1(\mathbb{R})$ and hence the result (2.2.11) fails for $p = 1$, $m = 0$, and $s = 0$. An example of an $g_1 \in L^1(\mathbb{R})$ for which this happens is

$$g_1(x) = \begin{cases} (1+x)^{-1}[1+\ln(1+x)]^{-1}\{1+\ln[1+\ln(1+x)]\}^{-2}, & \text{if } x \ge 0 \\ 0, & \text{otherwise.} \end{cases} \tag{2.2.16}$$

Thirdly, if we rephrase the statement of Theorem 2.2.5 in similarity variables (0.8) (with $a = 0$) we obtain an asymptotic expansion of the solution v of the LBOB' equation:

$$v(\tau) \sim v_1(\tau) = \sum_{j=0}^{n} \frac{(-1)^j}{j!} e^{-\tau j/2}[\mu_j G^{(j)} + \eta_j \mathcal{H}G^{(j)}]$$

as $\tau \to \infty$ uniformly in ξ in the sense that $\|v(\tau) - v_1(\tau)\|_{W^{m,p}(\mathbb{R})} = o(e^{\tau s/2})$ as $\tau \to \infty$. For technical reasons, we will need in §6 control of this error term in the H^m-norm for every real $m \ge 0$.

THEOREM 2.2.6. *Suppose that s, n, f, μ_j, η_j, and u_1 are as in Theorem 2.2.5, and $m \ge 0$ is any real number. Then*

$$\lim_{t\to\infty} t^{(s+1/2)/2}\|\beta(Dt^{1/2})^m[e^{-\alpha(D)t}f - u_1(t)]\|_{L^2(\mathbb{R})} = 0. \tag{2.2.17}$$

PROOF: To shorten writing we set $\mu_j^0 = \mu_j$ and $\mu_j^1 = \eta_j$. Define

$$\phi(k) = |k|^{-s}\left|e^{-ik} - \sum_{j=0}^{n} \frac{(-ik)^j}{j!}\right|. \tag{2.2.18}$$

Notice that ϕ is a bounded function which tends to 0 as $k \to 0$. Then

(2.2.19)
$$
\begin{aligned}
&t^{(s+1/2)/2}\|\beta(Dt^{1/2})^m[e^{-\alpha(D)t}f - u_1(t)]\|_{L^2(\mathbb{R})} \\
&\leq \frac{t^{(s+1/2)/2}}{(2\pi)^{1/2}}\left\|\beta(kt^{1/2})^m e^{-\alpha(k)t}\sum_{l=0}^{1}(-i\,\mathrm{sgn}(k))^l\left[\hat{g}_l(k) - \sum_{j=0}^{n}\frac{(-1)^j}{j!}(ik)^j\mu_j^l\right]\right\|_{L^2(k)} \\
&= \frac{t^{(s+1/2)/2}}{(2\pi)^{1/2}}\left\|\beta(kt^{1/2})^m e^{-\alpha(k)t}\sum_{l=0}^{1}(-i\,\mathrm{sgn}(k))^l\int_{-\infty}^{\infty}\left[e^{-ikx} - \sum_{j=0}^{n}\frac{(-ikx)^j}{j!}\right]g_l(x)\,dx\right\|_{L^2(k)} \\
&\leq \frac{1}{(2\pi)^{1/2}}\left\|\,|\omega|^s\beta(\omega)^m e^{-\nu\omega^2}\sum_{l=0}^{1}\int_{-\infty}^{\infty}\phi(\omega x t^{-1/2})|x|^s|g_l(x)|\,dx\right\|_{L^2(\omega)}.
\end{aligned}
$$

Now use dominated convergence twice. ∎

COROLLARY 2.2.7. *Suppose $s \geq 0$ and $n = [\![s]\!]$. Define the space X_s to consist of functions $f \in L_s^1(\mathbb{R}) + \mathcal{H}L_s^1(\mathbb{R})$ such that $\mu_j = \eta_j = 0$ for $j = 0, \ldots, n-1$, and equip it with the subspace topology inherited from $L_s^1(\mathbb{R}) + \mathcal{H}L_s^1(\mathbb{R})$. Then for every $1 \leq p \leq \infty$, $m \geq 0$ an integer, and $f \in X_s$ such that one of the 4 cases from Theorem 2.2.5 holds we have the estimate*

$$
\sup_{t>0} t^{(n+m+1-1/p)/2}\|\partial_x^m e^{-\alpha(D)t}f\|_{L^p(\mathbb{R})} < \infty \tag{2.2.20}
$$

and it is typically sharp for $f \in X_s$ as $t \to \infty$. In fact we have

$$
\lim_{t\to\infty} t^{(n+m+1-1/p)/2}\|\partial_x^m e^{-\alpha(D)t}f\|_{L^p(\mathbb{R})} = \frac{\|\mu_n G^{(m+n)} + \eta_n\mathcal{H}G^{(m+n)}\|_{L^p(\mathbb{R})}}{n!}. \tag{2.2.21}
$$

PROOF: (2.2.20) and (2.2.21) follow immediately from the proof of Theorem 2.2.5. Since $G^{(m+n)}$ and $\mathcal{H}G^{(m+n)}$ are linearly independent, the limit in (2.2.21) will vanish if and only if $\mu_n = \eta_n = 0$. This determines a closed subspace of X_s of codimension 2. Hence (2.2.20) is typically sharp for $f \in X_s$ as $t \to \infty$. ∎

2.3 Estimates on the Solution of the Inhomogeneous Linear Problem. Now we consider the inhomogeneous LBOB equation (2.0.1) and (2.0.2) where $f = 0$ and g, h are in weighted spaces which are appropriate for solutions of the nonlinear problem. A consideration of the behavior of solution $e^{-\alpha(D)t}f$ of the LBOB equation (2.0.1), where $f \in L_s^1(\mathbb{R}) + \mathcal{H}L_s^1(\mathbb{R})$ and $0 \leq s < 1$ will be our guide in constructing the appropriate spaces in which to work. The prototypical estimate is the following:

(2.3.1)
$$
\begin{aligned}
(\nu t)^{1/4}\|\beta(D\sqrt{\nu t})^r e^{-\alpha(D)t}f\|_{L^2(\mathbb{R})} &= \frac{(\nu t)^{1/4}}{\sqrt{2\pi}}\|\beta(k\sqrt{\nu t})^r e^{-\alpha(k)t}\hat{f}(k)\|_{L^2(k)} \\
&\leq \frac{(\nu t)^{1/4}}{\sqrt{2\pi}}\|\beta(k\sqrt{\nu t})^r e^{-\nu t k^2}\|_{L^2(k)}\|\hat{f}\|_{L^\infty(\mathbb{R})} \\
&= (2\pi)^{-1/2}\|\beta(\omega)^r e^{-\omega^2}\|_{L^2(\omega)}\|\hat{f}\|_{L^\infty(\mathbb{R})}.
\end{aligned}
$$

So define the weighted space $L^\infty_w(\mathbb{R}^+, H^r(\mathbb{R}))$ to be the class of all strongly measurable mappings $g\colon \mathbb{R}^+ \to H^r(\mathbb{R})$ satisfying

$$\|g\|_{L^\infty_w(\mathbb{R}^+,H^r(\mathbb{R}))} \stackrel{\text{def}}{=} \operatorname*{ess\,sup}_{t>0}(\nu t)^{1/4}\|\beta(D\sqrt{\nu t})^r g(t)\|_{L^2(\mathbb{R})} \tag{2.3.2}$$

Define $BC_w(\mathbb{R}^+, H^r(\mathbb{R}))$ as the intersection of $C(\mathbb{R}^+, H^r(\mathbb{R}))$ with $L^\infty_w(\mathbb{R}^+, H^r(\mathbb{R}))$. It is a Banach space with norm given in (2.3.2).

Define $K(f, gh)\colon \mathbb{R}^+ \to L^2(\mathbb{R})$ by

$$K(f,gh)(t) = e^{-\alpha(D)t}f - \frac{1}{2}\int_0^t e^{-\alpha(D)(t-\tau)}\partial_x[g(\tau)h(\tau)]\,d\tau. \tag{2.3.3}$$

for all $t \in \mathbb{R}^+$. We will show that this is the solution of the problem (2.0.1) and (2.0.2) for $f \in L^1(\mathbb{R}) + \mathcal{H}L^1(\mathbb{R})$ and $g, h \in L^\infty_w(\mathbb{R}^+, L^2(\mathbb{R}))$, and examine the uniqueness, regularity, and decay properties of this solution.

First we will need a pair of lemmas, the first of which will enable us to estimate the product of g and h.

LEMMA 2.3.1. *Suppose $a > 0$, $r \geq 0$ are real numbers, and $g, h \in L^2(\mathbb{R})$. Then*

$$\|\beta(ka)^r[gh]\hat{}\,(k)\|_{L^\infty(k)} \leq 2^{r/4}\|\beta(Da)^r g\|_{L^2(\mathbb{R})}\|\beta(Da)^r h\|_{L^2(\mathbb{R})}. \tag{2.3.4}$$

PROOF: Since g and h are in $L^2(\mathbb{R})$ we have by Petersen [, Corollary 4.9 page 81] the exchange formula:

$$[gh]\hat{}\,(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{g}(k-\xi)\hat{h}(\xi)\,d\xi. \tag{2.3.5}$$

To estimate this we will need Peetre's inequality (Petersen [, page 86]) which says that $(1+k^2)^\rho \leq 2^{|\rho|/2}(1+(k-\xi)^2)^{|\rho|}(1+\xi^2)^\rho$ for all $k, \xi, \rho \in \mathbb{R}$. So

$$\beta(ka)^r \leq 2^{r/4}\beta((k-\xi)a)^r\beta(\xi a)^r. \tag{2.3.6}$$

Applying (2.3.6) in (2.3.4) we have by Young's convolution inequality that

$$\begin{aligned}
&\|\beta(ka)^r[gh]\hat{}\,(k)\|_{L^\infty(k)} \\
&\leq \frac{2^{r/4}}{2\pi}\left\|\int_{-\infty}^{\infty}\beta((k-\xi)a)^r\hat{g}(k-\xi)\beta(\xi a)^r\hat{h}(\xi)\,d\xi\right\|_{L^\infty(k)} \\
&\leq \frac{2^{r/4}}{2\pi}\|\beta(ka)^r\hat{g}(k)\|_{L^2(k)}\|\beta(ka)^r\hat{h}(k)\|_{L^2(k)} \\
&\leq 2^{r/4}\|\beta(Da)^r g\|_{L^2(\mathbb{R})}\|\beta(Da)^r h\|_{L^2(\mathbb{R})}. \ \blacksquare
\end{aligned} \tag{2.3.7}$$

Define the constants

$$\begin{aligned}
A_{r,q,\delta} &= \frac{2^{r/4-3/2}}{\pi^{1/2}}\int_0^1 \frac{\|\omega\beta(\omega\sigma^{-1/2})^\delta\beta(\omega)^r e^{-\omega^2}\|_{L^2(\omega)}}{\sigma^{3/4}(1-\sigma)^q}\,d\sigma, \\
B_p &= (2\pi)^{-1/2}\|\beta(\omega)^p e^{-\omega^2}\|_{L^2(\omega)};
\end{aligned} \tag{2.3.8}$$

LEMMA 2.3.2. *$A_{r,q,\delta}$ is finite if $r \geq 0$, $q < 1$, and $\delta < 1/2$.*

PROOF: To see this note first that $(1+\sigma^{-1}\omega^2)^{\delta/2} \leq 1 + \sigma^{-\delta/2}|\omega|^\delta$, and therefore

$$\tag{2.3.9} \begin{aligned} &\int_0^1 \frac{\|\omega\beta(\omega\sigma^{-1/2})^\delta\beta(\omega)^r e^{-\omega^2}\|_{L^2(\omega)}}{\sigma^{3/4}(1-\sigma)^q}\,d\sigma \\ &\leq \int_0^1 \left(\frac{\|\omega\beta(\omega)^r e^{-\omega^2}\|_{L^2(\omega)}}{\sigma^{3/4}(1-\sigma)^q} + \frac{\|\,|\omega|^{1+\delta}\beta(\omega)^r e^{-\omega^2}\|_{L^2(\omega)}}{\sigma^{3/4+\delta/2}(1-\sigma)^q}\right) d\sigma \\ &= \|\omega\beta(\omega)^r e^{-\omega^2}\|_{L^2(\omega)} B(1/4, 1-q) \\ &\quad + \|\,|\omega|^{1+\delta}\beta(\omega)^r e^{-\omega^2}\|_{L^2(\omega)} B(1/4-\delta/2, 1-q). \end{aligned}$$

Since the Beta function $B(\alpha,\beta)$ is finite for $\alpha, \beta > 0$ we are done. ∎

Now we are ready to state the first main theorem of this section.

THEOREM 2.3.3. *Suppose $r \geq 0$ is a real number. Denote $L^\infty_w(\mathbb{R}^+, H^s(\mathbb{R}))$ by X_s for any $s \geq 0$ and suppose $g, h \in X_r$. Then*

(1) *for every $0 \leq \delta < \frac{1}{2}$ we have $K(0,gh) \in BC_w(\mathbb{R}^+, H^{r+\delta}(\mathbb{R}))$ and*

$$\|K(0,gh)\|_{X_{r+\delta}} \leq A_{r,1/2,\delta}\nu^{-1}\|g\|_{X_r}\|h\|_{X_r};$$

(2) *$K(0,gh) \in BC(\mathbb{R}^+, L^1(\mathbb{R}) \cap \mathcal{H}L^1(\mathbb{R}))$ and*

$$\|K(0,gh)\|_{L^\infty(\mathbb{R}^+, L^1(\mathbb{R})\cap\mathcal{H}L^1(\mathbb{R}))} \leq \frac{\pi}{2\nu^{1/2}}\|G'\|_{L^1(\mathbb{R})\cap\mathcal{H}L^1(\mathbb{R})}\|g\|_{X_0}\|h\|_{X_0};$$

(3) *$u = K(0,gh)$ satisfies (2.0.1) on $\mathbb{R}^+$;*

(4) *if $s < -1/2$ then $\|K(0,gh)(t)\|_{H^s(\mathbb{R})} \to 0$ as $t \to 0^+$, and in particular $u = K(0,gh)$ satisfies (2.0.2) with $f = 0$;*

(5) *if $\lim_{t\to 0^+} t^{1/4}\|g(t)\|_{L^2(\mathbb{R})} = 0$ then we have $\lim_{t\to 0^+} t^{1/4}\|K(0,gh)(t)\|_{L^2(\mathbb{R})} = 0$ and $\lim_{t\to 0^+}\|K(0,gh)(t)\|_{L^1(\mathbb{R})\cap\mathcal{H}L^1(\mathbb{R})} = 0$;*

(6) *if $\tilde{u} \in L^1_{\mathrm{loc}}(\mathbb{R}^+, L^2(\mathbb{R}))$ is any solution of the inhomogeneous problem (2.0.1) and (2.0.2) with $f = 0$ then $\tilde{u}(t) = K(0,gh)(t)$ in $L^2(\mathbb{R})$ for almost every $t \in \mathbb{R}^+$.*

PROOF: First we will establish the estimate in (1). Fix $t \in \mathbb{R}^+$. Clearly $g(\tau)h(\tau) \in L^1(\mathbb{R})$ for almost every $\tau \in (0,t)$. So by Young's convolution inequality, Theorem 2.2.1(2), and Theorem 2.2.1(4) we have the estimate

$$\tag{2.3.10} \begin{aligned} \|\partial_x F(t-\tau) * [g(\tau)h(\tau)]\|_{L^2(\mathbb{R})} &\leq \|\partial_x F(t-\tau)\|_{L^2(\mathbb{R})}\|g(\tau)h(\tau)\|_{L^1(\mathbb{R})} \\ &\leq \frac{\|g\|_{X_0}\|h\|_{X_0}}{2^{7/4}\pi^{1/4}[\nu(t-\tau)]^{3/4}(\nu\tau)^{1/2}}, \end{aligned}$$

holding for almost every $\tau \in (0,t)$. So $\tau \mapsto \partial_x F(t-\tau) * [g(\tau)h(\tau)]$ defines an element of $L^1((0,t), L^2(\mathbb{R}))$. So the pointwise integral of this map represents the Bochner integral of this map. Since the integral can be interpreted in a Bochner sense we have that the Fourier transform and the integral commute. So for almost every $k \in \mathbb{R}$ we have

$$(2.3.11) \qquad [K(f,gh)(t)]\hat{}(k) = e^{-\alpha(k)t}\hat{f}(k) - \frac{1}{2}\int_0^t ike^{-\alpha(k)(t-\tau)}[g(\tau)h(\tau)]\hat{}(k)\, d\tau.$$

Since $\nu^2(t-\tau)\tau k^4 \geq 0$ we have the inequality $(1+\nu t k^2) \leq (1+\nu(t-\tau)k^2)(1+\nu\tau k^2)$ holding for all $k \in \mathbb{R}$ and $0 \leq \tau \leq t$. So

$$(2.3.12) \qquad \beta(k\sqrt{\nu t}) \leq \beta(k\sqrt{\nu(t-\tau)})\beta(k\sqrt{\nu\tau}).$$

So by (2.3.11), (2.3.12), and Lemma 2.3.1 we have that

(2.3.13)

$$\begin{aligned}
&(\nu t)^{1/4}\left\|\beta(D\sqrt{\nu t})^{r+\delta}\left[\frac{1}{2}\int_0^t \partial_x F(t-\tau) * [g(\tau)h(\tau)]\, d\tau\right]\right\|_{L^2(\mathbb{R})} \\
&= \frac{(\nu t)^{1/4}}{2\sqrt{2\pi}}\left\|\int_0^t \beta(k\sqrt{\nu t})^{r+\delta} ike^{-\alpha(k)(t-\tau)}[g(\tau)h(\tau)]\hat{}(k)\, d\tau\right\|_{L^2(k)} \\
&\leq \frac{(\nu t)^{1/4}}{2\sqrt{2\pi}}\int_0^t \|ik\beta(k\sqrt{\nu t})^{\delta}\beta(k\sqrt{\nu(t-\tau)})^r e^{-\alpha(k)(t-\tau)}\|_{L^2(k)} \\
&\qquad\qquad \cdot \|\beta(k\sqrt{\nu\tau})^r[g(\tau)h(\tau)]\hat{}(k)\|_{L^\infty(k)}\, d\tau \\
&\leq \frac{2^{r/4}(\nu t)^{1/4}}{2\sqrt{2\pi}}\|g\|_{X_r}\|h\|_{X_r}\int_0^t \frac{\|ik\beta(k\sqrt{\nu t})^{\delta}\beta(k\sqrt{\nu\tau})^r e^{-\nu\tau k^2}\|_{L^2(k)}}{[\nu(t-\tau)]^{1/2}}\, d\tau \\
&\leq \frac{2^{r/4-3/2}}{\pi^{1/2}\nu}\|g\|_{X_r}\|h\|_{X_r}\int_0^1 \frac{\|\omega\beta(\omega\sigma^{-1/2})^{\delta}\beta(\omega)^r e^{-\omega^2}\|_{L^2(\omega)}}{\sigma^{3/4}(1-\sigma)^{1/2}}\, d\sigma.
\end{aligned}$$

This proves the estimate in (1).

To prove the continuity asserted in (1) let $0 < \epsilon < T' < T$ be given and let $\epsilon \leq t_1 < t_2 \leq T'$. By (2.3.11) we see that we must control the $L^2_{r+\delta}(\mathbb{R})$ norm of the following:

(2.3.14)

$$\begin{aligned}
&\int_0^{t_2} ike^{-\alpha(k)(t_2-\tau)}[g(\tau)h(\tau)]\hat{}(k)\, d\tau - \int_0^{t_1} ike^{-\alpha(k)(t_1-\tau)}[g(\tau)h(\tau)]\hat{}(k)\, d\tau \\
&\qquad = \int_{t_1}^{t_2} ike^{-\alpha(k)(t_2-\tau)}[g(\tau)h(\tau)]\hat{}(k)\, d\tau \\
&\qquad\quad + \int_0^{t_1} ike^{-\alpha(k)(t_1-\tau)}[e^{-\alpha(k)(t_2-t_1)} - 1][g(\tau)h(\tau)]\hat{}(k)\, d\tau.
\end{aligned}$$

We must show that the L^2 norm of the product of (2.3.14) and $\beta(k)^{r+\delta}$ tends to 0 as $t_2 - t_1 \to 0^+$. If $t \geq \epsilon$ then it is easy to see that $(1+k^2) \leq C^2(1+\nu t k^2)$ where $C = \max\{1, (\nu\epsilon)^{-1}\}^{1/2}$. Using this and (2.3.12) we get

$$(2.3.15) \qquad \beta(k)^{r+\delta} \leq C^r\beta(k)^{\delta}\beta(k\sqrt{\nu(t-\tau)})^r\beta(k\sqrt{\nu\tau})^r,$$

which holds for all $k \in \mathbf{R}$, $t \geq \epsilon$, and $0 \leq \tau \leq t$. Using (2.3.15) we can estimate the first term in (2.3.14) as follows.

$$(2.3.16)\quad \left\| \beta(k)^{r+\delta} \int_{t_1}^{t_2} ike^{-\alpha(k)(t_2-\tau)}[g(\tau)h(\tau)]\hat{}(k)\, d\tau \right\|_{L^2(k)}$$
$$\leq C^r \int_{t_1}^{t_2} \| ik\beta(k)^\delta \beta(k\sqrt{\nu(t_2-\tau)})^r e^{-\alpha(k)(t_2-\tau)} \|_{L^2(k)}$$
$$\cdot \| \beta(k\sqrt{\nu\tau})^r [g(\tau)h(\tau)]\hat{}(k) \|_{L^\infty(k)}\, d\tau$$

Now $\beta(k)^\delta = (1+k^2)^{\delta/2} \leq 1 + |k|^\delta$ so

(2.3.17)
$$\| ik\beta(k)^\delta \beta(k\sqrt{\nu(t_2-\tau)})^r e^{-\alpha(k)(t_2-\tau)} \|_{L^2(k)}$$
$$\leq \| k\beta(k\sqrt{\nu(t_2-\tau)})^r e^{-\nu(t_2-\tau)k^2} \|_{L^2(k)}$$
$$+ \| \, |k|^{1+\delta} \beta(k\sqrt{\nu(t_2-\tau)})^r e^{-\nu(t_2-\tau)k^2} \|_{L^2(k)}$$
$$\leq \frac{\| \omega\beta(\omega)^r e^{-\omega^2} \|_{L^2(\omega)}}{[\nu(t_2-\tau)]^{3/4}} + \frac{\| \, |\omega|^{1+\delta}\beta(\omega)^r e^{-\omega^2} \|_{L^2(\omega)}}{[\nu(t_2-\tau)]^{3/4+\delta/2}}.$$

Using (2.3.17) and Lemma 2.3.1 in (2.3.16) we obtain

$$(2.3.18)\quad (2.3.16) \leq 2^{r/4} C^r \|g\|_{X_r} \|h\|_{X_r}$$
$$\cdot \int_{t_1}^{t_2} \left[\frac{\| \omega\beta(\omega)^r e^{-\omega^2} \|_{L^2(\omega)}}{[\nu(t_2-\tau)]^{3/4}(\nu\tau)^{1/2}} + \frac{\| \, |\omega|^{1+\delta}\beta(\omega)^r e^{-\omega^2} \|_{L^2(\omega)}}{[\nu(t_2-\tau)]^{3/4+\delta/2}(\nu\tau)^{1/2}} \right] d\tau,$$

which clearly tends to 0 as $t_2 - t_1 \to 0^+$ since $3/4 + \delta/2 < 1$. The second term in (2.3.14) can be estimated in a similar manner:

(2.3.19)
$$\left\| \beta(k)^{r+\delta} \int_0^{t_1} ike^{-\alpha(k)(t_1-\tau)}[e^{-\alpha(k)(t_2-t_1)} - 1][g(\tau)h(\tau)]\hat{}(k)\, d\tau \right\|_{L^2(k)}$$
$$\leq C^r \int_0^{t_1} \| ik\beta(k)^\delta \beta(k\sqrt{\nu(t_1-\tau)})^r e^{-\alpha(k)(t_1-\tau)}[e^{-\alpha(k)(t_2-t_1)} - 1] \|_{L^2(k)}$$
$$\cdot \| \beta(k\sqrt{\nu\tau})^r [g(\tau)h(\tau)]\hat{}(k) \|_{L^\infty(k)}\, d\tau$$
$$\leq 2^{r/4} C^r \|g\|_{X_r} \|h\|_{X_r}$$
$$\cdot \int_0^{t_1} \| ik\beta(k)^\delta \beta(k\sqrt{\nu\tau})^r e^{-\alpha(k)\tau}[e^{-\alpha(k)(t_2-t_1)} - 1] \|_{L^2(k)} [\nu(t_1-\tau)]^{-1/2}\, d\tau.$$

Applying $\beta(k)^\delta \leq 1 + |k|^\delta$ in the L^2-norm in (2.3.19) and introducing the new variable

$\omega = k\sqrt{\nu\tau}$ we have

(2.3.20)
$$\|ik\beta(k)^\delta\beta(k\sqrt{\nu\tau})^r e^{-\alpha(k)\tau}[e^{-\alpha(k)(t_2-t_1)}-1]\|_{L^2(k)} \le \frac{\|\omega\beta(\omega)^r e^{-\omega^2}|e^{-\alpha(\omega/\sqrt{\nu\tau})(t_2-t_1)}-1|\,\|_{L^2(\omega)}}{(\nu\tau)^{3/4}} + \frac{\|\,|\omega|^{1+\delta}\beta(\omega)^r e^{-\omega^2}|e^{-\alpha(\omega/\sqrt{\nu\tau})(t_2-t_1)}-1|\,\|_{L^2(\omega)}}{(\nu\tau)^{3/4+\delta/2}}.$$

If we insert (2.3.20) into (2.3.19), and introduce the new varible $\sigma = \tau/t_1$, then we see that the integral in (2.3.19) is majorized by

(2.3.21)
$$\frac{1}{\nu(\nu t_1)^{1/4}}\left[\int_0^1 \frac{\|\omega\beta(\omega)^r e^{-\omega^2}|e^{-\alpha(\omega/\sqrt{\nu t_1\sigma})(t_2-t_1)}-1|\,\|_{L^2(\omega)}}{\sigma^{3/4}(1-\sigma)^{1/2}}\,d\sigma + \frac{1}{(\nu t_1)^{\delta/2}}\int_0^1 \frac{\|\,|\omega|^{1+\delta}\beta(\omega)^r e^{-\omega^2}|e^{-\alpha(\omega/\sqrt{\nu t_1\sigma})(t_2-t_1)}-1|\,\|_{L^2(\omega)}}{\sigma^{3/4+\delta/2}(1-\sigma)^{1/2}}\,d\sigma\right].$$

Since $\epsilon \le t_1 \le T' < \infty$ we can use the dominated convergence theorem to show that (2.3.21) converges to 0 as $t_2 - t_1 \to 0^+$. This completes the proof of (1).

To prove the estimate in (2) let $j = 0,1$, and $\mathcal{H}^0 = I$ and $\mathcal{H}^1 = \mathcal{H}$. Then

(2.3.22)
$$\|\mathcal{H}^j K(0,gh)(t)\|_{L^1(\mathbb{R})} \le \frac{\|\mathcal{H}^j G'\|_{L^1(\mathbb{R})}}{2}\int_0^t \frac{(\nu\tau)^{1/4}\|g(\tau)\|_{L^2(\mathbb{R})}(\nu\tau)^{1/4}\|h(\tau)\|_{L^2(\mathbb{R})}\,d\tau}{(t-\tau)^{1/2}(\nu\tau)^{1/2}} = \frac{\|\mathcal{H}^j G'\|_{L^1(\mathbb{R})}}{2\nu^{1/2}}\|g\|_{X_0}\|h\|_{X_0}\int_0^1 \frac{d\sigma}{(1-\sigma)^{1/2}\sigma^{1/2}}.$$

If we add the two estimates obtained by taking $j = 0,1$ we obtain the estimate of (2).

To prove the continuity asserted in (2) let $j = 0,1$, $0 < t_1 < t_2 < \infty$ and $1 < \lambda = t_2/t_1 \le 2$. We must show that the L^1-norm of

$$\int_{t_1}^{t_2} \partial_x\mathcal{H}^j F(t_2-\tau)*[g(\tau)h(\tau)]\,d\tau + \int_0^{t_1} \partial_x\mathcal{H}^j\big[F(t_2-\tau)-F(t_1-\tau)\big]*[g(\tau)h(\tau)]\,d\tau \tag{2.3.23}$$

tends to 0 as $\lambda \to 1^+$. The first term is trivial to deal with. But if $\sigma = \tau/t_1$ and $a = \sigma/(\lambda - 1 + \sigma)$ we have that the L^1-norm of the second term is bounded by

$$C\int_0^1 \frac{\|a(\mathcal{H}^j G')(\xi\sqrt{a}) - (\mathcal{H}^j G')(\xi)\|_{L^1(\xi)}}{\sigma^{1/2}(1-\sigma)^{1/2}}\,d\sigma \tag{2.3.24}$$

But for each fixed $\sigma \in (0,1]$ we have for all $1 < \lambda \le 2$ that $1 > a \ge \sigma/(1+\sigma)$. Thus by the asymptotic expansion (2.2.11) we have the estimate

$$|a(\mathcal{H}^j G')(\xi\sqrt{a}) - (\mathcal{H}^j G')(\xi)| \le C(1 + |\xi\sqrt{\sigma/(1+\sigma)}|)^{2j-4},$$

uniformly in λ. Thus by the dominated convergence theorem we have

$$\|a(\mathcal{H}^j G')(\xi\sqrt{a}) - (\mathcal{H}^j G')(\xi)\|_{L^1(\xi)} \to 0$$

as $\lambda \to 1^+$. But since

$$\|a(\mathcal{H}^j G')(\xi\sqrt{a}) - (\mathcal{H}^j G')(\xi)\|_{L^1(\xi)} \leq 2\|\mathcal{H}^j G'\|_{L^1(\mathbb{R})},$$

we may apply the dominated convergence theorem once more to obtain the desired result.

Now we will show (3), i.e. that u satisfies (2.0.1). Let $v(t) = K(0, gh)(t)$. From (2.3.14) we see that $v(t)\hat{}(k) = -ike^{-\alpha(k)t}\int_0^t e^{\alpha(k)\tau}[g(\tau)h(\tau)]\hat{}(k)\,d\tau/2$. This holds for every $t \in \mathbb{R}^+$ and almost every $k \in \mathbb{R}$, and since both sides are measurable functions of t and k the equation holds almost everywhere in $\mathbb{R}^+ \times \mathbb{R}$. Since

$$\tau^{1/2}\|[g(\tau)h(\tau)]\hat{}(k)\|_{L^\infty(k)} \leq \tau^{1/2}\|g(\tau)\|_{L^2(\mathbb{R})}\|h(\tau)\|_{L^2(\mathbb{R})} \leq \|g\|_X\|h\|_X$$

for almost all $\tau \in \mathbb{R}^+$, we see that $|\tau^{1/2}[g(\tau)h(\tau)]\hat{}(k)| \leq \|g\|_X\|h\|_X$ for almost all $(\tau, k) \in \mathbb{R}^+ \times \mathbb{R}$ and thus $\tau^{1/2}[g(\tau)h(\tau)]\hat{}(k) \in L^\infty(\tau)$ for almost every $k \in \mathbb{R}$, say for all k in the conull set $S \subset \mathbb{R}$. Thus for all $0 < T < \infty$ and $k \in S$ we have $[g(\tau)h(\tau)]\hat{}(k) \in L^1(\tau)$ for $\tau \in (0, T)$. So for $k \in S$ the map $t \mapsto \int_0^t e^{\alpha(k)\tau}[g(\tau)h(\tau)]\hat{}(k)\,d\tau$ is absolutely continuous on $(0, T)$ with t-partial derivative equal to $e^{\alpha(k)t}[g(t)h(t)]\hat{}(k)$ for almost every $t \in (0, T)$. Since the product of two absolutely continuous functions is absolutely continuous we see that the map $t \mapsto -ike^{-\alpha(k)t}\int_0^t e^{\alpha(k)\tau}[g(\tau)h(\tau)]\hat{}(k)\,d\tau/2$ is absolutely continuous with t-partial derivative equal to

$$(2.3.25)\quad \frac{\partial}{\partial t}\left[-ike^{-\alpha(k)t}\int_0^t e^{\alpha(k)\tau}[g(\tau)h(\tau)]\hat{}(k)\,d\tau/2\right]$$
$$= ik\alpha(k)e^{-\alpha(k)t}\int_0^t e^{\alpha(k)\tau}[g(\tau)h(\tau)]\hat{}(k)\,d\tau/2 - ik[g(t)h(t)]\hat{}(k)/2.$$

The right hand side of (2.3.25) is in $L^1_{\text{loc}}((0,T)\times\mathbb{R})$ so by Petersen [**14**, Theorem 9.4, page 24] $\{v(t)\hat{}(k)\}_t = \{-\alpha(k)v(t)\hat{}(k) - ik[g(t)h(t)]\hat{}(k)/2\}$ in $\mathfrak{D}'((0,T)\times\mathbb{R})$, where $\{v\}$ denotes the distribution induced by the mapping v. By definition $v(t)\hat{}(k)$ is a measurable representative of $\mathfrak{F}v\colon (0,T) \to \mathfrak{D}'(\mathbb{R})$. By the Schwartz Kernel Theorem, $\mathfrak{D}'((0,T)\times\mathbb{R}) \cong \mathfrak{D}'((0,T), \mathfrak{D}'(\mathbb{R}))$, and it is easy to see that $\{v(t)\hat{}(k)\}_t \mapsto \{\mathfrak{F}v(t)\}'$ under this isomorhism. Also $-\alpha(k)v(t)\hat{}(k) - ik[g(t)h(t)]\hat{}(k)/2$ is a measurable representative of $\nu\mathfrak{F}\partial_x^2 v + \rho\mathfrak{F}\mathcal{H}\partial_x^2 v - \mathfrak{F}\partial_x(gh)/2$ so again we have

$$\{-\alpha(k)v(t)\hat{}(k) - ik[g(t)h(t)]\hat{}(k)/2\} \mapsto \{\nu\mathfrak{F}\partial_x^2 v + \rho\mathfrak{F}\mathcal{H}\partial_x^2 v - \mathfrak{F}\partial_x(gh)/2\}.$$

So $\{\mathfrak{F}v(t)\}' - \nu\{\mathfrak{F}\partial_x^2 v\} - \rho\{\mathfrak{F}\mathcal{H}\partial_x^2 v\} = -\{\mathfrak{F}\partial_x(gh)\}/2$ in $\mathfrak{D}'((0,T), L^2_{-2}(\mathbb{R}))$. It is easy to see that the chain rule $\{\mathfrak{F}v(t)\}' = \mathfrak{F}(\{v\}')$ holds, so we have that $\mathfrak{F}(\{v\}') - \nu\mathfrak{F}\{\partial_x^2 v\} - \rho\mathfrak{F}\{\mathcal{H}\partial_x^2 v\} = -\mathfrak{F}\{\partial_x(gh)\}/2$. Taking the inverse Fourier transform of both side we see that v satisfies (2.0.1) on $(0,T)$. But $T > 0$ was arbitrary.

To prove (4) we proceed by using formula (2.3.11) and estimating as before:

(2.3.26)

$$\begin{aligned}
&\left\| \beta(D)^s \int_0^t \partial_x F(t-\tau) * [g(\tau)h(\tau)]\, d\tau \right\|_{L^2(\mathbb{R})} \\
&= (2\pi)^{-1/2} \left\| \beta(k)^s \int_0^t ike^{-\alpha(k)(t-\tau)}[g(\tau)h(\tau)]\hat{\ }(k)\, d\tau \right\|_{L^2(k)} \\
&\le (2\pi)^{-1/2} \int_0^t \|\, |k|\beta(k)^s e^{-\alpha(k)(t-\tau)}\|_{L^2(k)} \cdot \|[g(\tau)h(\tau)]\hat{\ }(k)\|_{L^\infty(k)}\, d\tau \\
&\le (2\pi)^{-1/2} \|g\|_{X_0} \|h\|_{X_0} \int_0^t \frac{\|\, |k|\beta(k)^s e^{-\nu k^2(t-\tau)}\|_{L^2(k)}}{(\nu\tau)^{1/2}}\, d\tau.
\end{aligned}$$

It will be convenient to assume $s > -3/2$. Once proved for such values of s it also follows for smaller values. Since $s < -1/2$ we have $\beta(k)^s \le |k|^s$. So we see that the integral in (2.3.26) is bounded by

$$\nu^{-1}(\nu t)^{-(s+1/2)/2} \|\, |\omega|^{1+s} e^{-\omega^2}\|_{L^2(\omega)} B((1/2-s)/2, 1/2). \tag{2.3.27}$$

This tends to 0 as $t \to 0^+$ under our assumptions. So (4) is true.

To prove (5) define $\phi(t) = (\nu t)^{1/4}\|g(t)\|_{L^2(\mathbb{R})}$. Estimating as in the proof of (1) and (2) we obtain for $j = 0, 1$

(2.3.28)

$$\begin{aligned}
(\nu t)^{1/4}\|K(0,gh)(t)\|_{L^2(\mathbb{R})} &\le \frac{\|h\|_{X_0}}{2^{3/2}\pi^{1/2}\nu} \int_0^1 \frac{\|\omega e^{-\omega^2}\|_{L^2(\omega)}\phi(t\sigma)}{\sigma^{3/4}(1-\sigma)^{1/2}}\, d\sigma, \\
\|\mathcal{H}^j K(0,gh)(t)\|_{L^1(\mathbb{R})} &\le \frac{\|\mathcal{H}^j G'\|_{L^1(\mathbb{R})}}{2\nu^{1/2}} \|h\|_X \int_0^1 \frac{\phi(t\sigma)\, d\sigma}{(1-\sigma)^{1/2}\sigma^{1/2}}.
\end{aligned}$$

The result then follows by the dominated convergence theorem.

Finally, to prove (6) we note that $v = u - \tilde{u} \in L^1_{\mathrm{loc}}(\mathbb{R}^+, L^2(\mathbb{R}))$ solves the homogeneous linear problem (2.0.1) with initial data 0. Thus by Theorem 2.1.1 $v = 0$ in $\mathfrak{D}'(\mathbb{R}^+, L^2(\mathbb{R}))$. This implies (6). ∎

Now we will present a Theorem which asserts that the solution of the inhomogeneous linear problem will decay more rapidly than usual provided one of the forcing functions also decays more rapidly. An important feature of this result is that a certain fractional number of extra derivatives of the solution can be shown to decay over the number of derivatives of the forcing function that are assumed to decay. This result will be useful in Section 6.

THEOREM 2.3.4. *Suppose $0 \le \delta < 1/2$, $0 \le s < 1$, and $r \ge 0$ are real numbers. Suppose $g, h \in L^\infty_w(\mathbb{R}^+, H^r(\mathbb{R}))$ where in addition we have*

$$\lim_{t\to\infty} (\nu t)^{(s+1/2)/2} \|\beta(D(\nu t)^{1/2})^r g(t)\|_{L^2(\mathbb{R})} = 0. \tag{2.3.29}$$

Then for $K(0,gh)$ as defined in (2.3.3) we have

$$\lim_{t\to\infty}(\nu t)^{(s+1/2)/2}\|\beta(D(\nu t)^{1/2})^{r+\delta}K(0,gh)(t)\|_{L^2(\mathbb{R})}=0. \tag{2.3.30}$$

Furthermore if $g,h\in\bigcap_{r\geq 0}L^\infty_w(\mathbb{R}^+,H^r(\mathbb{R}))$ and (2.3.29) is satisfied for all $r\geq 0$, then for all $1\leq p\leq\infty$ and for every integer $m\geq 0$ we have

$$\lim_{t\to\infty}t^{(s+m+1-1/p)/2}\|\partial_x^m K(0,gh)(t)\|_{L^p(\mathbb{R})}=0. \tag{2.3.31}$$

PROOF: Define $X_r=L^\infty_w(\mathbb{R}^+,H^r(\mathbb{R}))$ and

$$\phi(t)=(\nu t)^{(s+1/2)/2}\|\beta(D(\nu t)^{1/2})^r g(t)\|_{L^2(\mathbb{R})}, \tag{2.3.32}$$

Then by our usual methods we obtain the following estimate

(2.3.33)
$$\begin{aligned}
&(\nu t)^{(s+1/2)/2}\|\beta(D(\nu t)^{1/2})^{r+\delta}K(0,gh)(t)\|_{L^2(\mathbb{R})}\\
&\qquad\leq C(\nu t)^{(s+1/2)/2}\|h\|_{X_r}\int_0^t\frac{\|k\beta(k\sqrt{\nu t})^\delta\beta(k\sqrt{\nu\tau})^r e^{-\alpha(k)\tau}\|_{L^2(k)}\phi(t-\tau)}{[\nu(t-\tau)]^{(s+1)/2}}\,d\tau\\
&\qquad=C'\int_0^1\frac{\|\omega\beta(\omega\sigma^{-1/2})^\delta\beta(\omega)^r e^{-\omega^2}\|_{L^2(\omega)}\phi(t(1-\sigma))}{\sigma^{3/4}(1-\sigma)^{(s+1)/2}}\,d\sigma.
\end{aligned}$$

Since $g\in X_r$ and satisfies (2.3.29) we have that ϕ is bounded and $\lim_{t\to\infty}\phi(t)=0$. By Lemma 2.3.2 the integral in (2.3.33) is finite. Hence by the dominated convergence theorem (2.3.30) follows.

We will prove (2.3.31) for $p=\infty$ and $p=1$. The result for intermediate values of p will then follow by interpolation.

($p=\infty$) Define $u=K(0,gh)$. Then by Hölder's inequality and (2.3.30) we have

(2.3.34)
$$\begin{aligned}
\|\partial_x^m u(t)\|_{L^\infty(\mathbb{R})}&\leq 2^{1/2}\|\partial_x^m u(t)\|_{L^2(\mathbb{R})}^{1/2}\|\partial_x^{m+1}u(t)\|_{L^2(\mathbb{R})}^{1/2}\\
&\leq\pi^{-1/2}\big((\nu t)^{-m/2}\|\,|k\sqrt{\nu t}|^m u(t)\hat{\ }(k)\|_{L^2(k)}\big)^{1/2}\\
&\qquad\cdot\big((\nu t)^{-(m+1)/2}\|\,|k\sqrt{\nu t}|^{m+1}u(t)\hat{\ }(k)\|_{L^2(k)}\big)^{1/2}\\
&=o(t^{-(s+m+1/2)/2})^{1/2}o(t^{-(s+m+3/2)/2})^{1/2}=o(t^{-(s+m+1)/2})
\end{aligned}$$

as $t\to\infty$. This is the desired result.

($p=1$) We will prove this for even values of $m=2n$ by induction on $n\geq 0$. The proof for odd values of m will then follow from the inequality $\|w'\|_{L^1(\mathbb{R})}\leq 2^{3/2}\|w\|_{L^1(\mathbb{R})}^{1/2}\|w''\|_{L^1(\mathbb{R})}^{1/2}$ in a way similar to (2.3.34). Define

$$\begin{aligned}
\phi_l(t)&=t^{(s+1/2)/2}\|[\sqrt{t}\partial_x]^l g(t)\|_{L^2(\mathbb{R})},\\
\psi_l(t)&=t^{1/4}\|[\sqrt{t}\partial_x]^l h(t)\|_{L^2(\mathbb{R})}
\end{aligned}$$

for all $t > 0$, and all $l = 0, \ldots, m$. Our approach is based on the formula

$$1 - t\partial_x^2 = [1 - (t-\tau)\partial_x^2][1 - \tau\partial_x^2] - [(t-\tau)\partial_x^2][\tau\partial_x^2] \tag{2.3.35}$$

which will accomplish for us what the estimate (2.3.12) did for us in the L^2 theory. Applying the binomial theorem yields

$$[1 - t\partial_x^2]^n = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} [1 - (t-\tau)\partial_x^2]^j [(t-\tau)\partial_x^2]^{n-j} [1 - \tau\partial_x^2]^j [\tau\partial_x^2]^{n-j}. \tag{2.3.36}$$

We intend to apply this operator on both sides of the integral equation

$$u(t) = -\frac{1}{2} \int_0^t \partial_x F(t-\tau) * [g(\tau)h(\tau)]\, d\tau, \tag{2.3.37}$$

and then apply the L^1-norm and estimate using the triangle inequality, Minkowski's integral inequality, and Young's convolution inequality. Multiplying both sides of the result by $t^{s/2}$ yields

(2.3.38)

$$\begin{aligned} t^{s/2} &\| [1 - t\partial_x^2]^n u(t) \|_{L^1(\mathbb{R})} \\ &\le \frac{t^{s/2}}{2} \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} \int_0^t \| p_j((t-\tau)\partial_x^2) \partial_x F(t-\tau) \|_{L^1(\mathbb{R})} \| p_j(\tau\partial_x^2)[g(\tau)h(\tau)] \|_{L^1(\mathbb{R})}\, d\tau, \end{aligned}$$

where $p_j(x) = x^{n-j}(1-x)^j$. An easy calculation shows that

$$\| [(t-\tau)\partial_x^2]^l \partial_x F(t-\tau) \|_{L^1(\mathbb{R})} = (t-\tau)^{-1/2} \| G^{(2l+1)} \|_{L^1(\mathbb{R})}. \tag{2.3.39}$$

Therefore $\| p_j((t-\tau)\partial_x^2) \partial_x F(t-\tau) \|_{L^1(\mathbb{R})} = \mathrm{O}((t-\tau)^{-1/2})$. Also Leibnitz's rule shows that

$$\| [\tau\partial_x^2]^j [g(\tau)h(\tau)] \|_{L^1(\mathbb{R})} \le \sum_{l=0}^{2j} \binom{2j}{l} \frac{\phi_l(\tau)\psi_{2j-l}(\tau)}{\tau^{(s+1/2)/2}\tau^{1/4}} = \mathrm{o}(\tau^{-(s+1)/2}) \tag{2.3.40}$$

and therefore $\| p_j(\tau\partial_x^2)[g(\tau)h(\tau)] \|_{L^1(\mathbb{R})} = \mathrm{o}(\tau^{-(s+1)/2})$ as $\tau \to \infty$. Using these estimates in (2.3.38) we see that

$$t^{s/2} \| [1 - t\partial_x^2]^n u(t) \|_{L^1(\mathbb{R})} \le C \int_0^1 \frac{\Phi(t\sigma)\, d\sigma}{(1-\sigma)^{1/2}\sigma^{(s+1)/2}}, \tag{2.3.41}$$

where $\Phi(t) = \mathrm{o}(1)$ as $t \to \infty$. By the dominated convergence theorem we have that $t^{s/2} \| [1 - t\partial_x^2]^n u(t) \|_{L^1(\mathbb{R})} = \mathrm{o}(1)$ as $t \to \infty$. By the induction hypothesis (null if $n = 0$) we are done. ∎

§3 The IVP for the BOB Equation in $L^1(\mathbb{R})$

A discussion of existence and uniqueness of a global solution of the BOB equation

$$(3.1)\qquad u_t + (\bar{u}^2/2)_x - (\nu + \rho\mathcal{H})u_{xx} = 0 \qquad \text{in } \mathfrak{D}'(\mathbb{R}^+, \mathcal{S}'(\mathbb{R})),$$

$$(3.2)\qquad u(0) \to f \qquad \text{in } \mathcal{S}'(\mathbb{R}) \text{ as } t \to 0^+,$$

with initial data in $f \in L^2(\mathbb{R}, \mathbb{R})$ could be based on the general results of Saut [**16**]. Iório [**8**] also has proved results about the specific initial value problem (3.1) and (3.2) as well as results on the spacial asymptotic behavior of solutions. These accounts deal only with real-valued solutions and do not address the questions of interior regularity (hypoellipticity), existence for more singular initial data (such as L^1), and continuous dependence on initial data. We will examine these problems when $f \in L^1(\mathbb{R}) + \mathcal{H}L^1(\mathbb{R})$ is small, since this is the context in which our asymptotic results hold. The consideration of complex-valued solutions and the proofs of interior regularity, decay properties, and continuous dependence on the initial data are easy corollaries of our method. Our results easily translate into results for the BOB' equation, and allow for the simple "ODE-like" description of the asymptotic behavior of its solutions. The local existence of solutions of the BOB equation corresponding to large initial data in $L^1(\mathbb{R})$ is apparently an open problem.

THEOREM 3.1. *Suppose for $f \in L^1(\mathbb{R}) + \mathcal{H}L^1(\mathbb{R})$ the number $\lambda(f)$ is defined by*

$$(3.3)\qquad \lambda(f) = \frac{B(1/4, 1/2)\|\hat{f}\|_{L^\infty(\mathbb{R})}}{2^{3/2}\pi^{1/2}\nu}.$$

Denote by U (resp. U_1) the open subset of $L^1(\mathbb{R}) + \mathcal{H}L^1(\mathbb{R})$ (resp. $L^1(\mathbb{R})$) consisting of all f such that $\lambda(f) < 1$. Define the integral operator K by (2.3.3).

(1) **Existence:** *For every $f \in U$ there exists a map $u = Sf \in L^\infty_w(\mathbb{R}^+, L^2(\mathbb{R}))$ satisfying $u(t) = K(f, \bar{u}^2)(t)$ in $L^2(\mathbb{R})$ for almost every $t > 0$, and*

$$(3.4)\qquad \operatorname*{ess\,lim}_{t \to 0^+} t^{1/4}\|u(t)\|_{L^2(\mathbb{R})} = 0,$$

$$(3.5)\qquad \|u\|_{L^\infty_w(\mathbb{R}^+, L^2(\mathbb{R}))} \le \frac{2^{7/4}\pi^{1/4}\nu}{B(1/4, 1/2)}\left[1 - \sqrt{1 - \lambda(f)}\right].$$

(2) **Regularity:** *For every $f \in U$ the map $u = Sf$ satisfies $u \in BC_w(\mathbb{R}^+, H^r(\mathbb{R}))$ for every real number $r \ge 0$, $u \in C^\infty(\mathbb{R} \times \mathbb{R}^+, \mathbb{C})$, and $u \in BC([0,\infty), L^1(\mathbb{R}) + \mathcal{H}L^1(\mathbb{R}))$; if $f \in U_1$ then $u \in BC([0,\infty), L^1(\mathbb{R}))$.*

(3) **Satisfies equation:** *For every $f \in U$ the map $u = Sf$ satisfies (3.1); it is also a classical solution of the BOB equation in the sense that the principal value integral defining the Hilbert transform of u_{xx} converges at every point, all the partial derivatives are classical, and the equation holds at every $(x,t) \in \mathbb{R} \times \mathbb{R}^+$.*

(4) **Convergence to data:** *For every $f \in U$ and $s < -1/2$ we have $\|S(t)f - f\|_{H^s(\mathbb{R})} \to 0$ as $t \to 0^+$; also $\|S(t)f - f\|_{L^1(\mathbb{R}) + \mathcal{H}L^1(\mathbb{R})} \to 0$ as $t \to 0^+$; in particular (3.2) is satisfied; if $f \in U_1$ then $\|S(t)f - f\|_{L^1(\mathbb{R})} \to 0$ as $t \to 0^+$.*

(5) **Uniqueness:** *If $f \in U$ and $u \in L^\infty_w(\mathbb{R}^+, L^2(\mathbb{R}))$ satisfies (3.1), (3.2), and*

$$\|u\|_{L^\infty_w(\mathbb{R}^+,L^2(\mathbb{R}))} < \frac{2^{7/4}\pi^{1/4}\nu}{B(1/4,1/2)} \tag{3.6}$$

then $u(t) = S(t)f$ in $L^2(\mathbb{R})$ for almost every $t > 0$.

(6) **Continuous dependence on data:** *The solution map $f \mapsto Sf$ is continuous from U into $BC_w(\mathbb{R}^+, H^r(\mathbb{R}))$ for every $r \geq 0$, from U into $BC([0,\infty), L^1(\mathbb{R}) + \mathcal{H}L^1(\mathbb{R}))$, and from U_1 into $BC([0,\infty), L^1(\mathbb{R}))$.*

(7) **Conserved Quantities:** *For every $f \in U$ the map $u = Sf$ has the property that the quantites $\mu = [u(t)\hat{}(0^+) + u(t)\hat{}(0^-)]/2$ and $\eta = i[u(t)\hat{}(0^+) - u(t)\hat{}(0^-)]/2$ are defined for all $t > 0$ and are independent of t.*

PROOF: (1) Define X to be the Banach space of all equivalence classes of mappings in $L^\infty_w(\mathbb{R}^+, L^2(\mathbb{R}))$ which satisfy (3.4). If we use (2.3.1) with $r = 0$ and set $\delta_1 = 0$ in Theorem 2.3.3(1) then we obtain the following estimates for all $u, v \in X$

$$\|K(f,\bar{u}^2)\|_X \leq B_0\|\hat{f}\|_{L^\infty(\mathbb{R})} + A_{0,1/2,0}\nu^{-1}\|u\|^2_X, \tag{3.7}$$

$$\|K(f,\bar{u}^2) - K(f,\bar{v}^2)\|_X \leq A_{0,1/2,0}\nu^{-1}\|u+v\|_X\|u-v\|_X. \tag{3.8}$$

Note that by Proposition 2.2.2 and Theorem 2.3.3(5) the map $u \mapsto K(f,\bar{u}^2)$ takes X into X. Thus $u \mapsto K(f,\bar{u}^2)$ will map the closed ball $B \subset X$ of radius R centered at 0 into itself and will be a contraction there provided

$$B_0\|\hat{f}\|_{L^\infty(\mathbb{R})} + A_{0,1/2,0}\nu^{-1}R^2 \leq R, \tag{3.9}$$

$$\kappa = 2A_{0,1/2,0}\nu^{-1}R < 1. \tag{3.10}$$

(3.9) says that a certain quadratic polynomial function evaluated at R is nonpositive. Since the coefficient of R^2 is positive this is only possible if this polynomial function has real roots. (3.10) rules out the case of two coincident roots (since then equality would hold in (3.10)), so the discriminant of the quadratic must be positive. Since $B_0 = 2^{-3/4}\pi^{-1/4}$ and $A_{0,1/2,0} = 2^{-11/4}\pi^{-1/4}B(1/4,1/2)$ this works out to be the condition $\lambda(f) < 1$. Therefore there exists an R satisfying (3.9) and (3.10) if and only if $f \in U$. For each fixed $f \in U$ we can take R to be the smaller of the two roots, namely the right-hand-side of (3.5). So there is a unique fixed point $u = K(f,\bar{u}^2)$ in the ball B of radius R. Define $S(t)f = K(f,\bar{u}^2)(t)$ for all $t > 0$. Then (1) is true .

(2) Suppose $f \in U$, $u = Sf \in L^\infty_w(\mathbb{R}^+, H^r(\mathbb{R}))$ for some $r \geq 0$, and $0 < \delta < 1/2$. Then by (2.3.1) and Theorem 2.3.3(1) we have that $u \in BC_w(\mathbb{R}^+, H^{r+\delta}(\mathbb{R}))$. Thus for every $r \geq 0$ we have $u \in BC_w(\mathbb{R}^+, H^r(\mathbb{R}))$. By Theorem 2.3.3(3) we have that u satisfies (3.1). So if $m \geq 0$ is an integer and $u \in C^m(\mathbb{R}^+, H^r(\mathbb{R}))$ for all $r \geq 0$ then the distributional (and Fréchet) time derivative of u also lies in $C^m(\mathbb{R}^+, H^r(\mathbb{R}))$ for all $r \geq 0$, which implies that $u \in C^{m+1}(\mathbb{R}^+, H^r(\mathbb{R}))$ for all $r \geq 0$. Therefore for all integers $m \geq 0$ we have $u \in C^m(\mathbb{R}^+, H^r(\mathbb{R}))$ for all $r \geq 0$. Thus the mapping u has a representative in $C^\infty(\mathbb{R} \times \mathbb{R}^+, \mathbb{C})$. The fact that $u \in BC([0,\infty), L^1(\mathbb{R}) + \mathcal{H}L^1(\mathbb{R}))$ for $f \in U$ and $u \in BC([0,\infty), L^1(\mathbb{R}))$ for $f \in U_1$ follows from Theorem 2.2.1 and Theorem 2.3.1(2).

(3) Since u_t, u_x, u_{xx} are all represented by continuous functions we have by Petersen [**14**, Theorem 12.2, page 30] that the distributional derivatives are represented by the classical derivatives. Also since $u_{xx}(t)$ is Hölder continuous in x we have by Titchmarsh [**18**,Theorem 106] that $\mathcal{H}u_{xx}(t)$ is Hölder continuous in x and the principal value integral converges everywhere. So since the continuous function $u_t + uu_x - \nu u_{xx} - \rho\mathcal{H}u_{xx}$ is equal almost everywhere to the continuous function 0 they must in fact be equal everywhere. So the C^∞ representative of u is a classical solution of the BOB equation.

(4) These assertions follow immediately from Theorem 2.2.1 and Theorem 2.3.3(4)(5).

(5) If $f \in U$ and u satisfies (3.1) and (3.2) then by Theorem 2.3.3(6) $v(t) = u(t) - e^{-\alpha(D)t}$ satisfies $v = K(0, \bar{u}^2)$. Thus $u = K(f, \bar{u}^2)$. But since (3.6) is satisfied, we can choose a radius R satisfying (3.9) and (3.10) so that the ball in X of radius R centered at 0 will contain u (see the proof of (1)). Since fixed points of contraction mappings are unique we have $u = Sf$ in $L^\infty_w(\mathbb{R}^+, L^2(\mathbb{R}))$.

(6) Let $\lambda < 1$ be given. Define $U^{(\lambda)}$ to be the subset of U consisting of all f such that $\lambda(f) \le \lambda$. Define $R = \nu(1 - \sqrt{1-\lambda})/(2A_{0,1/2,0})$ and $\kappa = 1 - \sqrt{1-\lambda}$. Then for every $f \in U^{(\lambda)}$ the integral mapping $u \mapsto K(f, \bar{u}^2)$ is a contraction with constant κ on the ball $B \subset X$ of radius R centered at 0. So if $f, g \in U$, $u = Sf$, and $v = Sg$ then by (2.3.1) we have

(3.11)

$$\|u - v\|_X = \|K(f, \bar{u}^2) - K(f, \bar{v}^2) + K(f, \bar{v}^2) - K(g, \bar{v}^2)\|_X \le \kappa\|u - v\|_X + B_0\|\hat{f} - \hat{g}\|_{L^\infty(\mathbb{R})}.$$

Since $\kappa < 1$ we have $\|u - v\|_X \le (1-\kappa)^{-1}B_0\|\hat{f} - \hat{g}\|_{L^\infty(\mathbb{R})}$. So the solution map is continuous from $U^{(\lambda)}$ to X, and since λ was arbitrary it is continuous from U to X.

Now let $X_r = BC_w(\mathbb{R}^+, \hat{r})$ for any $r \ge 0$. Using (2.3.1) and iterative applications of Theorem 2.3.3(1) we see that for every $r \ge 0$ there exists $C_r > 0$ such that for all $f \in U$ we have $\|Sf\|_{X_r} \le C_r$. Suppose $r \ge 0$ is such that the solution map is continuous from U to X_r, and let $0 < \delta < 1/2$. Then by (2.3.1) and Theorem 2.3.3(1) we have

$$\|u - v\|_{X_{r+\delta}} \le B_{r+\delta}\|\hat{f} - \hat{g}\|_{L^\infty(\mathbb{R})} + 2A_{r,1/2,\delta}C_r\|u - v\|_{X_r}, \tag{3.12}$$

where as before $f, g \in U$, $u = Sf$, and $v = Sg$. Therefore the solution map is continuous from U into $X_{r+\delta}$. Therefore by induction the solution map is continuous from U to X_r for all $r \ge 0$. The continuity into $BC([0,\infty), L^1(\mathbb{R}) + \mathcal{H}L^1(\mathbb{R}))$, and into $BC([0,\infty), L^1(\mathbb{R}))$ follows from the continuity into X_0, Theorem 2.3.3(2), and trivial estimates on the solution of the homogeneous LBOB equation.

(7) By Theorem 2.3.3(2) $u(t) = e^{-\alpha(D)t}f + K(0, \bar{u}^2)(t)$ where $K(0, \bar{u}^2)(t) \in L^1(\mathbb{R}) \cap \mathcal{H}L^1(\mathbb{R})$ for all $t > 0$. Thus $K(0, \bar{u}^2)(t)\hat{}(0) = 0$ for all $t > 0$. Since μ and η are conserved quantities for the solution $e^{-\alpha(D)t}f$ of the LBOB equation we are done. ∎

Notice that the same proof will work if we replace the fixed point equation $u = K(f, \bar{u}^2)$ with $u = K(f, u^2)$, or even $u = K(f, |u|^2)$. So the complex conjugate can be removed from the nonlinear term in (3.1) without affecting any of our results. The virtue of retaining the complex conjugate is that it enables us to obtain an a priori bound on the L^2 norm of the solution. For if we multiply (3.1) by $\bar{u}$, integrate over $\mathbb{R} \times (\epsilon, t)$, take the real part, assume that u, u_x decay to 0 as $|x| \to \infty$, use the fact that $\mathcal{H}^* = -\mathcal{H}$, and integrate by parts, then we find that the contribution from the nonlinear and dispersive terms vanishes and we obtain

$$(3.13)\qquad \int_{-\infty}^{\infty} |u(x,t)|^2\,dx + 2\nu \int_{\epsilon}^{t}\int_{-\infty}^{\infty} |u_x(x,\tau)|^2\,dx\,d\tau = \int_{-\infty}^{\infty} |u(x,\epsilon)|^2\,dx.$$

Using contraction mapping arguments in the weighted space of all $u \in C((0,T), H^r(\mathbb{R}))$ satisfying

$$(3.14)\qquad \sup_{0<t<T} (\nu t)^{|s|/2} \|\beta(D\sqrt{\nu t})^r u(t)\|_{L^2(\mathbb{R})} < \infty,$$

where $-1/2 < s \leq 0$ and $r \geq 0$, we can obtain local existence (T is sufficiently small) of a solution to (3.1) and (3.2) where $f \in H^s(\mathbb{R})$ can be of arbitrary size. The nature of the weighted space forces the solution to lie in $L^2(\mathbb{R})$ for every positive time. The number r can be chosen large enough so that solutions of (3.1) satisfying (3.14) obey the a priori bound (3.13). Local existence can then be proven in the space of u satisfying (3.14) with $s = 0$ starting with $u(T/2)$ as initial data. The time interval of existence will depend only on the L^2-norm of the initial data. Because the solution always obeys (3.13) this process can be repeated indefinitely to obtain a global solution. Thus global existence of solutions is true for (3.1) for data in $H^s(\mathbb{R})$, $s > -1/2$, whereas this may not be true if the nonlinear term is modified by the removal of complex conjugates, since in that case the a priori bound (3.13) is lost.

§4 Decay Estimates for the BOB Equation

In §2 we gave an account of the decay properties of solutions of the LBOB equation. The general rule was that greater smoothness of the Fourier transform of the initial data near $k = 0$ translates into a more rapidly decaying solution provided certain "obstructions" (e.g. nonvanishing moments) are not present. This rule is far from being verified for the nonlinear equation. In this section we will survey what has been done, critique various methods, and identify some open problems. To get started we will consider two types of assumptions on the initial data, namely either $f \in L^2(\mathbb{R})$ or $f \in L^1(\mathbb{R})$.

As we have seen, if $f \in H^s(\mathbb{R})$, $s > -1/2$, then the solution $u = Sf$ of the BOB equation lies in $H^\infty(\mathbb{R})$ for all positive time. The Fourier transform of the solution near $k = 0$ however is qualitatively the same for all time, namely like the Fourier transform of an L^2 function. The "optimal" decay results one should try to prove in this case are, for $2 \leq p \leq \infty$, that $\|u(t)\|_{L^p(\mathbb{R})} = o(t^{-(1/2-1/p)/2})$ as $t \to \infty$. However, the only known L^∞ decay result for solutions of the BOB equation, proved by Biler [**3**], is not "optimal". His result is the following.

THEOREM 4.1. *Suppose $f \in H^2(\mathbb{R},\mathbb{R})$ and $u \in BC(\mathbb{R}^+, L^2(\mathbb{R}))$ is the solution of the BOB equation with initial data f. Then*

$$(4.1)\qquad \lim_{t\to\infty} \|\partial_x u(t)\|_{L^2(\mathbb{R})} = \lim_{t\to\infty} \|u(t)\|_{L^\infty(\mathbb{R})} = 0.$$

He uses an "energy" method. It should be remarked that Amick, Bona, and Schonbek [**2**] have obtained, by more complicated a priori estimates akin to "energy" estimates, the "optimal" L^4 decay estimate listed above, namely $\|u(t)\|_{L^4(\mathbb{R})} = o(t^{-1/8})$ as $t \to \infty$, for solutions u of the Korteweg-deVries-Burgers equation (see Proposition 6.2 of that reference). This estimate is "optimal" in the sense that it is equally strong with what we can prove about solutions of the linearized Korteweg-deVries-Burgers equation, $u_t + u_{xxx} - \nu u_{xx} = 0$, with initial data in $L^2(\mathbb{R})$ (the analog of Theorem 2.2.3 holds, with a slightly modified proof). They also obtain, under the same assumptions on f, the "non-optimal" estimates $\|u_x(t)\|_{L^2(\mathbb{R})} = o(t^{-1/4})$ and $\|u(t)\|_{L^\infty(\mathbb{R})} = o(t^{-1/6})$. The author has been unsuccessful however in his attempts to apply their methods to the BOB equation.

Schonbek [**17**] has invented a method for proving decay estimates of the L^2-norm of solutions, which she and others (see e.g. Kajikiya and Miyakawa [**10**]) have used successfully on solutions of the Navier-Stokes equations in two or more spacial dimensions. Her method when applied in our one dimensional context is based on the following estimate.

THEOREM 4.2. *Suppose $\nu > 0$ and $u \in C((0,\infty), H^1(\mathbb{R})) \cap C([0,\infty), L^2(\mathbb{R}))$ are such that*

$$\frac{d}{dt}\int_{-\infty}^{\infty} |u(x,t)|^2\,dx + 2\nu \int_{-\infty}^{\infty} |u_x(x,t)|^2\,dx = 0, \tag{4.2}$$

for all $t > 0$. Then for all $t > 0$

$$\int_{-\infty}^{\infty} |u(x,t)|^2\,dx \le \frac{1}{2\pi t}\int_0^t \int_{|k|\le(2\nu\tau)^{-1/2}} |u(\tau)\hat{\ }(k)|^2\,dk\,d\tau. \tag{4.3}$$

The easy proof of this theorem is given in [**3**]. By (3.13) we see that the solution u of the BOB equation satisfies (4.2). The method then proceeds by obtaining some control of $|u(\tau)\hat{\ }(k)|$. The methods used in [**17**] and [**10**] to estimate this quantity apparently fail to yield, in our one dimensional case, estimates which are good enough to imply any sort of decay of $\|u(t)\|_{L^2(\mathbb{R})}$ as $t \to \infty$, regardless of one's assumptions on the initial data.

If we assume that $f \in L^1(\mathbb{R})$, then the "optimal" estimates to try to prove are given by: $\|\partial_x^m u(t)\|_{L^p(\mathbb{R})} = O(t^{-(m+1-1/p)/2})$ as $t \to \infty$. One immediate corollary of Theorem 3.1(2) is that this result is true if $p = 2$ and the initial data is sufficiently small. Since Amick, Bona, and Schonbek [**2**] prove the same thing for real-valued solutions of the Korteweg-deVries-Burgers' (KdVB) equation and the Benjamin-Bona-Mahoney-Burgers' (BBMB) equation except that they are able to avoid the small data assumption, a natural (open) problem is to extend our small data result to the large data case. The method of Amick, et al fails in the BOB case since it essentially exploits the asymptotic imbalance inherent in the KdVB equation, which as we noted in the introduction, is not present in the BOB case. An approach based on Theorem 4.2 also apparently fails as we mentioned above. But Theorem 4.2 does allow us to give equivalent formulations of the problem. If a solution exists satisfying $u \in L^\infty(\mathbb{R}^+, L^1(\mathbb{R}))$, then by Theorem 4.2 the desired L^2 decay estimate is satisfied. Conversely, if $f \in L^1(\mathbb{R})$ and a solution exists satisfying $u \in L^\infty_w(\mathbb{R}^+, L^2(\mathbb{R}))$ then by Corollary 2.2.7 and Theorem 2.3.3(2) we have that $u \in L^\infty(\mathbb{R}^+, L^1(\mathbb{R}))$.

We can separate this problem from the local existence problem mentioned in §3 by trying to prove the estimate for data in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, where global existence is assured. Also, it is not clear to the author whether or not this decay estimate should be expected to hold for complex-valued solutions. Thus a reasonable open problem is to prove that the L^2 decay estimate, $\|u(t)\|_{L^2(\mathbb{R})} = O(t^{-1/4})$ for $t > 0$, holds for all solutions u of the BOB equation arising from initial data $f \in L^1(\mathbb{R},\mathbb{R}) \cap L^2(\mathbb{R},\mathbb{R})$ of arbitrary size, and for all positive values of ν. Equivalently, one should try to prove that solutions starting in $L^1(\mathbb{R},\mathbb{R}) \cap L^2(\mathbb{R},\mathbb{R})$ lie in $L^\infty(\mathbb{R}^+, L^1(\mathbb{R}))$.

§5 The Reduced BOB (RBOB) Equation

5.0 Introduction. In this section we will examine the existence, uniqueness, regularity, and asymptotic behavior as $|\xi| \to \infty$ of the solutions of the Reduced BOB (RBOB) equation

$$\xi w(\xi) - \bar{w}(\xi)^2 + 2(\nu + \rho\mathcal{H})w'(\xi) = \frac{\eta}{\pi}, \tag{5.0.1}$$

where η is a complex constant. In analogy with first order ordinary differential equations we expect to be able to specify a free parameter and then have the solution be uniquely determined. In subsection 5.1 we derive integral equations which are equivalent to the integro-differential equation (5.0.1) in the context of L^2 solutions. The derivations show that the natural parameter that must be specified is $\mu = [\hat{w}(0^+) + \hat{w}(0^-)]/2$. This parameter is not in general defined for all $w \in L^2(\mathbb{R})$ but it will be for the solutions of (5.0.1) as we will show. The integral equations we derive will make sense for any $w \in L^2(\mathbb{R})$ and the numbers μ and η will appear explicitly. In subsection 5.2 we prove that if we have an L^2 solution of (5.0.1), then it must be in $H^\infty(\mathbb{R})$ and it must have a specific asymptotic behavior as $|\xi| \to \infty$. More precisely we give a two term asymptotic expansion of the form $a\xi^{-1} + b\xi^{-3}$ for the solution $w(\xi)$ of (5.0.1), where the coefficients a and b are explicitly given functions of ν, ρ, μ, and η. Using the special form of the equation we show that the difference between the solution and the asymptotic expansion is $o(\xi^{-3})$ as $|\xi| \to \infty$. Finally in subsection 5.3 we employ a contraction mapping argument based on the integral equations derived in subsection 5.1 to prove that an L^2 solution to (5.0.1) exists provided μ and η are sufficiently small relative to ν. The solution is uniquely determined in the ball in $L^2(\mathbb{R})$ in which a contraction is obtained. The existence theorem is more than adequate to provide intermediate asymptotics for all the solutions of the BOB equation which were shown to exist in section 3. We also prove that the map taking (μ, η) into the corresponding solution $w_{\mu,\eta}$ is continuous from an open set $D \subset \mathbb{C}^2$ into $L^2(\mathbb{R})$, $L^1(\mathbb{R}) + \mathcal{H}L^1(\mathbb{R})$, and in the case where $\eta = 0$ into $L^1(\mathbb{R})$.

5.1 Integral equations equivalent to RBOB. Now we will derive a pair of integral equations, one for $\hat{w}$ and one for w, which are equivalent to the RBOB equation (5.0.1). Suppose $w \in L^2(\mathbb{R})$ and w satisfies the reduced equation (5.0.1) in $\mathcal{S}'(\mathbb{R})$. Then taking the Fourier transform of (5.0.1) we obtain

$$\frac{d\hat{w}}{dk}(k) + \alpha'(k)\hat{w}(k) = -2i\eta\delta(k) - i(\bar{w}^2)\hat{\ }(k), \tag{5.1.1}$$

where as usual $\alpha(k) = \nu k^2 - i\rho k|k|$, and therefore $\alpha'(k) = 2\nu k - 2i\rho|k|$. Now define $\hat{w}_2(k) = -i\eta\,\mathrm{sgn}(k)e^{-\alpha(k)}$. Then we have that $\hat{w}_2(k)$ satisfies

$$\frac{d\hat{w}_2}{dk}(k) + \alpha'(k)\hat{w}_2(k) = -2i\eta\delta(k). \tag{5.1.2}$$

Therefore $\hat{w}_1$ defined by $\hat{w}_1 = \hat{w} - \hat{w}_2$ satisfies the equation

$$\frac{d\hat{w}_1}{d\omega}(\omega) + \alpha'(\omega)\hat{w}_1(\omega) = -i(\bar{w}^2)\hat{\ }(\omega). \tag{5.1.3}$$

By changing the representative of $\hat{w}$ we can assume $\hat{w}_1$ is absolutely continuous. Multiplying by an integrating factor we obtain

$$\frac{d}{d\omega}\left(e^{\alpha(\omega)}\hat{w}_1(\omega)\right) = -ie^{\alpha(\omega)}(\bar{w}^2)\hat{\ }(\omega). \tag{5.1.4}$$

Integrating both sides in ω over the interval $(0,k)$ and denoting $\hat{w}_1(0)$ by μ we get

$$\hat{w}_1(k) = \mu e^{-\alpha(k)} - ie^{-\alpha(k)}\int_0^k e^{\alpha(\omega)}(\bar{w}^2)\hat{\ }(\omega)\,d\omega. \tag{5.1.5}$$

Combining this equation with the definition of $\hat{w}_2$ we obtain our integral equation for $\hat{w}$, holding for almost every $k \in \mathbb{R}$:

$$\hat{w}(k) = (\mu - i\eta\,\mathrm{sgn}(k))e^{-\alpha(k)} - ie^{-\alpha(k)}\int_0^k e^{\alpha(\omega)}(\bar{w}^2)\hat{\ }(\omega)\,d\omega. \tag{5.1.6}$$

So the assumption that $w \in L^2(\mathbb{R})$ satisfies (5.0.1) implies that $\hat{w}$ satisfies (5.1.6) for some constant μ. The converse is also clearly true, i.e. $w \in L^2(\mathbb{R})$ and $\hat{w}$ satisfies (5.1.6) for some constant μ implies that w satisfies (5.0.1) in $\mathcal{S}'(\mathbb{R})$. So $w \in L^2(\mathbb{R})$ and (5.1.6) imply that for every representative $\hat{w}$ of the Fourier transform of w we have

$$\hat{w}(0^+) = \operatorname*{ess\,lim}_{k\to 0^+}\hat{w}(k) = \mu - i\eta \qquad \text{and} \qquad \hat{w}(0^-) = \operatorname*{ess\,lim}_{k\to 0^-}\hat{w}(k) = \mu + i\eta.$$

Thus the constant μ is determined from w by the rule $\mu = [\hat{w}(0^+) + \hat{w}(0^-)]/2$.

Now we will show that the associated similarity solution $v(x,t) = t^{-1/2}w(xt^{-1/2})$ of the BOB equation satisfies the integral equation $v = K(\mu\delta + \frac{\eta}{\pi}\mathrm{p.v.}\frac{1}{x}, \bar{v}^2)$. Then we will use that to derive an integral equation for w. Define the dilation operator Y_λ on functions ϕ by the rule $(Y_\lambda\phi)(x) = \phi(\lambda x)$. It is not hard to verify that $(Y_\lambda\phi)\hat{\ }(k) = \lambda^{-1}\hat{\phi}(\lambda^{-1}k)$. Since $v(t) = t^{-1/2}Y_{t^{-1/2}}w$ and $\bar{v}(\tau)^2 = \tau^{-1}Y_{\tau^{-1/2}}(\bar{w}^2)$ we have that $v(t)\hat{\ }(k) = \hat{w}(kt^{1/2})$ and $[\bar{v}(\tau)^2]\hat{\ }(k) = \tau^{-1/2}(\bar{w}^2)\hat{\ }(k\tau^{1/2})$. So substituting $kt^{1/2}$ for k in (5.1.6) we obtain

(5.1.7)

$$\begin{aligned}
v(t)\hat{\ }(k) &= (\mu - i\eta\,\mathrm{sgn}(kt^{1/2}))e^{-\alpha(kt^{1/2})} - ie^{-\alpha(kt^{1/2})}\int_0^{kt^{1/2}} e^{\alpha(\omega)}(\bar{w}^2)\hat{\ }(\omega)\,d\omega \\
&= (\mu - i\eta\,\mathrm{sgn}(k))e^{-\alpha(k)t} - ie^{-\alpha(k)t}\int_0^t e^{-\alpha(k)\tau}\tau^{1/2}[\bar{v}(\tau)^2]\hat{\ }(k)\frac{k\,d\tau}{2\tau^{1/2}} \\
&= (\mu - i\eta\,\mathrm{sgn}(k))e^{-\alpha(k)t} - \frac{1}{2}\int_0^t ike^{-\alpha(k)(t-\tau)}[\bar{v}(\tau)^2]\hat{\ }(k)\,d\tau.
\end{aligned}$$

So applying the Fourier transform to both sides we obtain the familiar integral equation

$$v(t) = \mu F(t) + \eta \mathcal{H} F(t) - \frac{1}{2}\int_0^t \partial_x F(t-\tau) * \bar{v}(\tau)^2 \, d\tau. \tag{5.1.8}$$

Now we claim that the similarity form of v and (5.1.8) together imply that w satisfies the integral equation

$$w = \mu G + \eta \mathcal{H} G - \frac{1}{2}\int_0^1 \left\{ Y_{\frac{1}{\sqrt{1-s}}}[G'] \right\} * \left\{ Y_{\frac{1}{\sqrt{s}}} \bar{w}^2 \right\} \frac{ds}{(1-s)s}. \tag{5.1.9}$$

In the above G is the normalized L^1 solution of the linearized reduced equation ($\eta = 0$) discussed in detail in section 2. To derive (5.1.9) first change variables $\tau = ts$ in the τ integral. Then use the relation $F(t)(x) = t^{-1/2}G(xt^{-1/2})$ and change variables appropriately in the convolution. The factors of $t^{-1/2}$ can then be cleared from both sides to yield (5.1.9). Clearly these steps are reversible so that (5.1.9), (5.1.6), and (5.0.1) are all equivalent for $w \in L^2(\mathbb{R})$.

5.2 Regularity and spacial asymptotics of solutions of RBOB. Now we will show that a generalized solution of the reduced equation must in fact be in $H^\infty(\mathbb{R})$ and a classical solution of the reduced equation. Once having obtained the regularity of solutions of the reduced equation we will then be able to exploit the form of the equation to obtain precise information about the asymptotic behavior of solutions. The associated similarity solutions v to the BOB equation will therefore be shown to have spatial asymptotic behavior which is entirely consistent with the results of Iório [8]. One should also note that the leading asymptotic behavior of solutions of the reduced equation is similar to that of solutions of the linearized reduced equation (2.1.3) (see (2.1.11) and (2.1.12)). This information about the asymptotic behavior applies to any solution of the reduced equation, not just to those solutions whose existence we will prove subsequently. First we will require a pair of simple lemmas.

LEMMA 5.2.1. *Suppose $w \in H^\infty(\mathbb{R})$ and θ is a locally bounded measurable function of at most polynomial growth as $|k| \to \infty$. Then $\theta\hat{w} \in L^1(\mathbb{R})$.*

PROOF: If $|\theta(k)| \le C\beta(k)^s$ for some positive constants C and s then by Hölder's inequality we have $\|\theta\hat{w}\|_{L^1(\mathbb{R})} \le C\|\beta^{-1}\|_{L^2(\mathbb{R})}\|\beta^{s+1}\hat{w}\|_{L^2(\mathbb{R})} < \infty$. ∎

LEMMA 5.2.2. *Suppose $v_1, v_2 \colon \mathbb{R} \to \mathbb{C}$ are both of the form $v_j(k) = f_j(k) + \operatorname{sgn}(k) g_j(k)$ where f_j and g_j, $j = 1, 2$, are absolutely continuous functions on every bounded interval and of at most polynomial growth as $|k| \to \infty$. Let $\dot{v}$ denote the classical derivative of v which exists almost everywhere. Let v' denote the distributional derivative of v. Then*

$$(v_1 v_2)' = [v_1(0^+)v_2(0^+) - v_1(0^-)v_2(0^-)]\delta + \dot{v}_1 v_2 + v_1 \dot{v}_2. \tag{5.2.1}$$

PROOF: The distribution $(v_1v_2)'$ evaluated at a test function $\varphi \in \mathcal{S}(\mathbb{R})$ is equal to v_1v_2 integrated against $-\varphi'$. Split the integral over $\mathbb{R}$ into one over $(-\infty, 0)$ and one over $(0, \infty)$ and then integrate by parts and use the product rule for absolutely continuous functions. The boundary terms at $\pm\infty$ vanish because of the polynomial growth of v_1 and v_2 together with the rapid decay of φ'. The result follows. ∎

THEOREM 5.2.3. *Suppose $\nu > 0$, $\rho \in \mathbb{R}$, $\mu, \eta \in \mathbb{C}$, and $w \in L^2(\mathbb{R})$ satisfies*

(1) $\xi w(\xi) - \bar{w}(\xi)^2 + 2[(\nu + \rho\mathcal{H})w'](\xi) = \eta/\pi$ *in* $\mathcal{S}'(\mathbb{R})$; *and*

(2) $\mu = [\hat{w}(0^+) + \hat{w}(0^-)]/2$, *where* $\hat{w}(0^\pm) = \operatorname{ess\,lim}_{k\to 0^\pm} \hat{w}(k)$;

Then the following statements are true

(1) $w \in H^\infty(\mathbb{R})$ *and w is a classical solution of the reduced equation.*

(2) $w(\xi) = (\eta/\pi)\xi^{-1} + [2(\rho\mu + \nu\eta)/\pi + (\bar{\eta}/\pi)^2]\xi^{-3} + o(\xi^{-3})$ *as* $|\xi| \to \infty$.

PROOF: (1) We have already seen that the above assumptions on w imply that the associated similarity solution v of the BOB equation satisfies the integral equation (5.1.8). Repeated applications of Theorem 2.3.3(1) show that $w \in H^\infty(\mathbb{R})$ and is a classical solution of the RBOB equation.

To prove (2) we need to show that

$$\xi^2[\xi w(\xi) - \eta/\pi] - 2(\rho\mu + \nu\eta)/\pi - (\bar{\eta}/\pi)^2 \tag{5.2.2}$$

is a bounded function of ξ which tends to 0 as $|\xi| \to \infty$. We will do this by showing that the Fourier transform of this function is in $L^1(\mathbb{R})$. From the reduced equation (5.0.1) we have

$$\xi w(\xi) - \eta/\pi = \bar{w}(\xi)^2 - 2\nu w'(\xi) - 2\rho(\mathcal{H}w')(\xi). \tag{5.2.3}$$

Substituting (5.2.3) into (5.2.2) and taking the Fourier transform of the result and using the fact that $\hat{1} = 2\pi\delta$ we obtain

$$[(\xi\bar{w})^2 - (\bar{\eta}/\pi)^2]\hat{} + 2[\nu w' + \rho\mathcal{H}w']\hat{}'' - 4(\rho\mu + \nu\eta)\delta. \tag{5.2.4}$$

First we claim that $(\xi\bar{w})^2 - (\bar{\eta}/\pi)^2 \in H^\infty(\mathbb{R})$. But since $w \in H^\infty(\mathbb{R})$ we have that the right-hand-side of (5.2.3) is in $H^\infty(\mathbb{R})$, and therfore so is the complex conjugate of the left-hand-side. Furthermore we have

$$(\xi\bar{w})^2 - (\bar{\eta}/\pi)^2 = \left[\xi\bar{w} - \bar{\eta}/\pi\right]\left[(\xi\bar{w} - \bar{\eta}/\pi) + 2\bar{\eta}/\pi\right].$$

Thus the claim follows from the fact that H^∞ is an algebra. Therefore by Lemma 5.2.1 the first term of (5.2.4) lies in $L^1(\mathbb{R})$. It remains to show that $i[\alpha'\hat{w}]'' - 4(\rho\mu + \nu\eta)\delta \in L^1(\mathbb{R})$. From (5.1.1) it is clear that

$$\hat{w}' = -2i\eta\delta - \alpha'\hat{w} - i(\bar{w}^2)\hat{} = -2i\eta\delta + \dot{\hat{w}}, \tag{5.2.5}$$

where $\hat{w}$ is defined by the right-hand-side of (5.1.6) and $\dot{\hat{w}} = -\alpha'\hat{w} - i(\bar{w}^2)\hat{}$ is the almost everywhere defined classical derivative of $\hat{w}$. Also by (5.1.6) we have that $v_1 = \alpha'$ and $v_2 = \hat{w}$ satisfy the hypotheses of Lemma 5.2.2. Since α' is absolutely continuous we can apply Lemma 5.2.2 to obtain

$$[\alpha'\hat{w}]' = \alpha''\hat{w} + \alpha'\dot{\hat{w}} = [\alpha'' - (\alpha')^2]\hat{w} - i\alpha'(\bar{w}^2)\hat{}. \tag{5.2.6}$$

Since $\xi\bar{w}^2 = \bar{w}(\xi\bar{w}-\bar{\eta}/\pi)+\bar{w}\bar{\eta}/\pi$ we see that it is an H^∞-function. Thus $(\bar{w}^2)^{\frown\prime} = [-i\xi\bar{w}^2]^\frown$ is an L^1-function. Thus $(\bar{w}^2)^\frown$ can be taken to be absolutely continuous. So applying Lemma 5.2.2 to each of the two terms in (5.2.6) and using $\dot{\hat{w}} = -\alpha'\hat{w} - i(\bar{w}^2)^\frown$ we obtain

(5.2.7)
$$\begin{aligned}[\alpha'\hat{w}]'' &= [\alpha''(0^+)\hat{w}(0^+) - \alpha''(0^-)\hat{w}(0^-)]\delta - 2\alpha'\alpha''\hat{w} + [\alpha'' - (\alpha')^2]\dot{\hat{w}} \\ &\quad - i\alpha''(\bar{w}^2)^\frown - i\alpha'(\bar{w}^2)^{\frown\prime} \\ &= -4i(\rho\mu+\nu\eta)\delta + [(\alpha')^3 - 3\alpha'\alpha'']\hat{w} - i[2\alpha'' - (\alpha')^2](\bar{w}^2)^\frown - \alpha'[\xi\bar{w}^2]^\frown.\end{aligned}$$

Therefore

$$(5.2.8)\qquad i[\alpha'\hat{w}]'' - 4(\rho\mu+\nu\eta)\delta = i[(\alpha')^3 - 3\alpha'\alpha'']\hat{w} + [2\alpha'' - (\alpha')^2](\bar{w}^2)^\frown - i\alpha'[\xi\bar{w}^2]^\frown.$$

By Lemma 5.2.1 this lies in $L^1(\mathbb{R})$. ∎

Notice that if the complex conjugate is removed from the nonlinear term in (5.0.1) then the same asymptotic expansion for w holds except that the term $(\bar{\eta}/\pi)^2$ is replaced by $(\eta/\pi)^2$.

5.3 Existence and Uniqueness of solutions of RBOB. Now we will present our existence and uniqueness result for solutions of the reduced equation. In light of our regularity result Theorem 5.2.3 it will be sufficient to prove existence of a solution in $L^2(\mathbb{R})$.

THEOREM 5.3.1. *Suppose $\mu,\eta \in \mathbb{C}$ the number $\lambda(\mu,\eta)$ is defined by*

$$(5.3.1)\qquad \lambda(\mu,\eta) = \frac{\max\{|\mu - i\eta|, |\mu + i\eta|\}}{\nu\sqrt{2}}.$$

Let D (resp. D_1) denote the open subset of $\mathbb{C}^2$ (resp. $\mathbb{C}$) such that $\lambda(\mu,\eta) < 1$ (resp. $\lambda(\mu,0) < 1$). Then for every $(\mu,\eta) \in D$ there exists $w \in L^2(\mathbb{R})$ such that the following three conditions hold:

(1) $\xi w(\xi) - \bar{w}(\xi)^2 + 2[(\nu+\rho\mathcal{H})w'](\xi) = \eta/\pi$ *in* $\mathcal{S}'(\mathbb{R})$;
(2) $\mu = [\hat{w}(0^+) + \hat{w}(0^-)]/2$, *where* $\hat{w}(0^\pm) = \operatorname{ess\,lim}_{k\to 0^\pm} \hat{w}(k)$;
(3) $\|w\|_{L^2(\mathbb{R})} \le 2^{3/4}\nu^{3/4}\pi^{-1/4}[1 - \sqrt{1-\lambda(\mu,\eta)}]$.

The solution is uniquely determined in the open ball in $L^2(\mathbb{R})$ centered at 0 of radius $2^{3/4}\nu^{3/4}\pi^{-1/4}$. Furthermore the map $(\mu,\eta) \mapsto w$ is continuous from D into $L^2(\mathbb{R})$, from D into $L^1(\mathbb{R}) + \mathcal{H}L^1(\mathbb{R})$, and when $\eta = 0$ from D_1 into $L^1(\mathbb{R})$.

PROOF: Define the integral operator $L = L(\mu,\eta)\colon L^2(\mathbb{R}) \to L^2(\mathbb{R})$ by the rule Lw is equal to the right-hand side of (4.2.9). Then $(Lw)^\frown(k)$ is given by the right-hand side of (4.2.6). We will show that if $(\mu,\eta) \in D$ then L will be a contraction on a closed ball B centered at 0 in $L^2(\mathbb{R})$ of radius $R = 2^{3/4}\nu^{3/4}\pi^{-1/4}[1 - \sqrt{1-\lambda(\mu,\eta)}]$. Note that $\|\mu - i\eta\,\mathrm{sgn}(k)\|_{L^\infty(k)} = \max\{|\mu - i\eta|, |\mu + i\eta|\} = \lambda(\mu,\eta)\nu\sqrt{2}$. Since $|(\bar{w}^2)^\frown(\omega)| \le \|w\|^2_{L^2(\mathbb{R})}$

we have by Plancherel's theorem the following easy estimate on Lw.

(5.3.2)
$$\begin{aligned}
&\|Lw\|_{L^2(\mathbb{R})} \\
&\le \frac{\|\mu - i\,\mathrm{sgn}(k)\eta\|_{L^\infty(k)}}{\sqrt{2\pi}}\|e^{-\nu k^2}\|_{L^2(k)} + \frac{1}{\sqrt{2\pi}}\left\| e^{-\nu k^2}\int_0^{|k|} e^{\nu\omega^2}|(\bar{w}^2)\hat{}(\omega)|\,d\omega\right\|_{L^2(\omega)} \\
&\le \frac{\lambda(\mu,\eta)\nu^{3/4}}{2^{1/4}\pi^{1/4}} + \frac{\|w\|^2_{L^2(\mathbb{R})}}{2^{1/2}\pi^{1/2}\nu^{3/4}}\|F_1\|_{L^2(\mathbb{R})},
\end{aligned}$$

where $F_1(x) = e^{-x^2}\int_0^x e^{s^2}\,ds$ is a special function called Dawson's integral. Let W denote the complementary error function of a complex variable defined in (2.2.7). Recall that $W(\zeta)$ is in the Hardy space $\mathcal{H}^2$ of the upper half plane, $\Im(\zeta) > 0$, and $W(x) = e^{-x^2} + i\frac{2}{\sqrt{\pi}}F_1(x)$. Therefore we have that $\frac{2}{\sqrt{\pi}}F_1(x)$ is the Hilbert transform of e^{-x^2}. Since the Hilbert transform is an isometry on L^2 we can compute the L^2 norm of F_1 exactly, i.e. $\|F_1\|_{L^2(\mathbb{R})} = \frac{\sqrt{\pi}}{2}\|e^{-x^2}\|_{L^2(x)} = 2^{-5/4}\pi^{3/4}$. Using this in (5.3.2) we obtain

$$\|Lw\|_{L^2(\mathbb{R})} \le \frac{\lambda\nu^{3/4}}{2^{1/4}\pi^{1/4}} + \frac{\pi^{1/4}\|w\|^2_{L^2(\mathbb{R})}}{2^{7/4}\nu^{3/4}}. \tag{5.3.3}$$

Estimating in a very similar way leads to the estimate

$$\|Lw_1 - Lw_2\|_{L^2(\mathbb{R})} \le \frac{\pi^{1/4}\|w_1 + w_2\|_{L^2(\mathbb{R})}\|w_1 - w_2\|_{L^2(\mathbb{R})}}{2^{7/4}\nu^{3/4}}. \tag{5.3.4}$$

The estimates (5.3.3) and (5.3.4) imply that L will map the closed ball B into itself and be a contraction on that ball if R satisfies

$$\frac{\lambda(\mu,\eta)\nu^{3/4}}{2^{1/4}\pi^{1/4}} + \frac{\pi^{1/4}R^2}{2^{7/4}\nu^{3/4}} \le R; \tag{5.3.5}$$

$$\kappa = \frac{\pi^{1/4}R}{2^{3/4}\nu^{3/4}} < 1. \tag{5.3.6}$$

Such an R will exist satisfying these two conditions provided the discriminant of the quadratic (in R) determined by (5.3.5) is positive,

$$1 - 4\cdot\frac{\pi^{1/4}}{2^{7/4}\nu^{3/4}}\cdot\frac{\lambda(\mu,\eta)\nu^{3/4}}{2^{1/4}\pi^{1/4}} > 0. \tag{5.3.7}$$

This is the same as $\lambda(\mu,\eta) < 1$, which we are assuming. So by Banach's contraction mapping principle there exists a unique fixed point $w \in B$ of the operator L. By the arguments given in paragraph 4.2 this fixed point w satisfies conditions (1) and (2) in the statement of the theorem. Choosing R as small as possible so that (5.3.5) holds we obtain (3). Uniqueness follows since any $w_1 \in L^2(\mathbb{R})$ satisfying (1), (2), and (3) must be a fixed point of the operator $L(\mu,\eta)$ in the ball of radius R for some R satisfying (5.3.6), and this fixed point must be unique.

To prove the asserted continuity into $L^2(\mathbb{R})$ fix $\lambda < 1$. It will suffice to prove that this map is continuous on the set $D^{(\lambda)}$ of $(\mu,\eta) \in \mathbb{C}^2$ satisfying $\lambda(\mu,\eta) \leq \lambda$. Choose $R = 2^{3/4}\nu^{3/4}\pi^{-1/4}[1-\sqrt{1-\lambda}]$. Therefore L is a contraction on the ball in $L^2(\mathbb{R})$ of radius R with contraction constant $\kappa = 1-\sqrt{1-\lambda}$ uniformly in $(\mu,\eta) \in D^{(\lambda)}$. This implies the asserted continuity in the L^2-topology. The continuity into $L^1(\mathbb{R}) + \mathcal{H}L^1(\mathbb{R})$ follows from the estimate

(5.3.9)

$$\begin{aligned}
&\|w_1 - w_2\|_{L^1(\mathbb{R})+\mathcal{H}L^1(\mathbb{R})} = \|L(\mu_1,\eta_1)w_1 - L(\mu_2,\eta_2)w_2\|_{L^1(\mathbb{R})+\mathcal{H}L^1(\mathbb{R})} \\
&\quad \leq |\mu_1-\mu_2|\cdot\|G\|_{L^1(\mathbb{R})} + |\eta_1-\eta_2|\cdot\|G\|_{L^1(\mathbb{R})} \\
&\qquad + \frac{1}{2}\int_0^1 \left\|Y_{\frac{1}{\sqrt{1-s}}}[G']\right\|_{L^1(\mathbb{R})} \left\|Y_{\frac{1}{\sqrt{s}}}[\bar{w}_1^2 - \bar{w}_2^2]\right\|_{L^1(\mathbb{R})} \frac{ds}{(1-s)s} \\
&\quad \leq (|\mu_1-\mu_2| + |\eta_1-\eta_2|)\|G\|_{L^1(\mathbb{R})} + R\|w_1-w_2\|_{L^2(\mathbb{R})}\|G'\|_{L^1(\mathbb{R})}\int_0^1 \frac{ds}{(1-s)^{1/2}s^{1/2}}.
\end{aligned}$$

When $\eta = 0$ then a very similar estimate shows that the map $\mu \mapsto w$ is continuous from D_1 into $L^1(\mathbb{R})$. ∎

This Theorem is not expected to hold for large values of η since in the case $\rho = 0$ (similarity solutions of Burgers' equation) solutions are generally singular and do not lie in $L^2(\mathbb{R})$. However for $\eta = 0$ and μ real we do expect solutions to the RBOB equation to exist.

§6 Intermediate Asymptotics for Solutions of BOB

Now we will show that the similarity solutions of the BOB equation are the intermediate asymptotics for the general solutions. First we will need the following lemma.

LEMMA 6.1. *Suppose $\alpha,\beta > 0$, $0 \leq \delta < \beta$ are real numbers. Denote by X_δ the Banach space of all $\rho \in C(\mathbb{R}^+,\mathbb{R})$ such that $\|\rho\|_{X_\delta} = \sup_{t>0}(1+t)^\delta|\rho(t)| < \infty$. Let $Y_\delta \subset X_\delta$ be the closed subspace consisting of those ρ satisfying $\rho(t) = o(t^{-\delta})$ as $t \to \infty$. For all $\rho \in X_\delta$ define for $C \geq 0$ the integral operator J by*

$$(6.1) \qquad (J\rho)(t) = C\int_0^1 \rho(t\sigma)(1-\sigma)^{\alpha-1}\sigma^{\beta-1}\,d\sigma.$$

Then J is a bounded linear order-preserving operator $J\colon X_\delta \to X_\delta$, mapping Y_δ into itself. If $CB(\alpha,\beta-\delta) < 1$ (where B denotes the Beta function) then $(I-J)^{-1}$ is also a bounded linear order-preserving operator $(I-J)^{-1}\colon X_\delta \to X_\delta$ which maps Y_δ into itself.

PROOF: Use the inequality $(1+t)/(1+t\sigma) \leq \sigma^{-1}$ to show that J maps X_δ into itself, with operator norm $\|J\|_{L(X_\delta,X_\delta)} \leq CB(\alpha,\beta-\delta)$. To see that J maps Y_δ into itself use the dominated convergence theorem. The rest follows from the Neumann series for $(I-J)^{-1}$. ∎

THEOREM 6.2. *Suppose* $0 \le s < 1$, $f \in L^1_s(\mathbb{R}) + \mathcal{H}L^1_s(\mathbb{R})$, $\mu = [\hat{f}(0^+) + \hat{f}(0^-)]/2$, *and* $\eta = i[\hat{f}(0^+) - \hat{f}(0^-)]/2$. *Let* $\lambda(f)$ *and* $\lambda(\mu,\eta)$ *be defined by (3.3) and (5.3.1) respectively. Suppose* $\lambda(f) < 1$ *and (if* $s > 0$*)*

$$\frac{1-\sqrt{1-\lambda(f)}}{2} + \frac{B(1/4,1/2)}{2\pi^{1/2}}\left(\frac{1-\sqrt{1-\lambda(\mu,\eta)}}{2}\right) < \frac{B(1/4,1/2)}{B(1/4,(1-s)/2)}. \tag{6.2}$$

Suppose $u = Sf$ *is the solution of the BOB equation with initial data* f. *Let* w *be the solution of the RBOB equation with the parameters* μ *and* η. *Let* u_1 *be the solution of the BOB equation defined by* $u_1(x,t) = t^{-1/2}w(xt^{-1/2})$, *and* $u_2(t) = u_1(t+1) = S(t)w$ *for all* $t > 0$. *Then there is a constant* $C > 0$ *independent of* f *such that*

$$\|u - u_2\|_{BC_w(\mathbb{R}^+,L^2(\mathbb{R}))} \le C\|\hat{f} - \hat{w}\|_{L^\infty(\mathbb{R})}. \tag{6.3}$$

Furthermore if $1 \le p \le \infty$ *and* $m \ge 0$ *is an integer so that one of the four conditions on* p, s, m, f *given in Theorem 2.2.5 hold, then*

$$\lim_{t\to\infty} t^{(s+m+1-1/p)/2}\|\partial_x^m[u(t) - u_1(t)]\|_{L^p(\mathbb{R})} = 0, \tag{6.4}$$

Even if $p = 1$, $s = 0$, $m = 0$, *and* $f \in L^1(\mathbb{R}) + \mathcal{H}L^1(\mathbb{R})$ *is arbitrary we have*

$$\lim_{t\to\infty} \|u(t) - u_1(t)\|_{L^1(\mathbb{R})+\mathcal{H}L^1(\mathbb{R})} = 0. \tag{6.5}$$

PROOF: Define $Z = BC_w(\mathbb{R}^+, L^2(\mathbb{R}))$. By the definitions of $\lambda(f)$ and $\lambda(\mu,\eta)$ we see that $\lambda(\mu,\eta) \le 2\pi^{1/2}B(1/4,1/2)^{-1}\lambda(f)$. Since $2\pi^{1/2}B(1/4,1/2)^{-1} \doteq 0.676 < 1$ we see that $\lambda(\mu,\eta) < 1$. Thus by Theorem 5.3.1 a solution w of the RBOB equation exists corresponding to the constants μ and η. Define $f_1 = \mu\delta + \eta\mathcal{H}\delta$ and $f_2 = w$. For $j = 1,2$ define

$$\phi_j(t) = (\nu t)^{1/4}\|u(t) - u_j(t)\|_{L^2(\mathbb{R})}; \tag{6.6}$$

$$\psi_j(t) = (\nu t)^{1/4}\|e^{-\alpha(D)t}(f - f_j)\|_{L^2(\mathbb{R})}. \tag{6.7}$$

Clearly u_j satisfies the integral equation $u_j = K(f_j, \bar{u}_j^2)$. So subtracting the two integral equations that u and u_j satisfy yields

$$u(t) - u_j(t) = e^{-\alpha(D)t}(f - f_j) - \frac{1}{2}\int_0^t \partial_x F(t-\tau) * [\bar{u}(\tau)^2 - \bar{u}_j(\tau)^2]\, d\tau. \tag{6.8}$$

Taking the L^2 norm of both sides of (6.8) and using the above definitions (6.6) and (6.7) we obtain

(6.9)

$$\begin{aligned}
\phi_j(t) &\le \psi_j(t) + \frac{(\nu t)^{1/4}}{2}\int_0^t \|F_x(\nu, t-\tau)\|_{L^2(\mathbb{R})}\|u(\tau) + u_j(\tau)\|_{L^2(\mathbb{R})}\|u(\tau) - u_j(\tau)\|_{L^2(\mathbb{R})}\, d\tau \\
&\le \psi_j(t) + \frac{(\|u\|_Z + \|u_j\|_Z)(\nu t)^{1/4}}{2^{11/4}\pi^{1/4}}\int_0^t \frac{\phi_j(\tau)}{[\nu(t-\tau)]^{3/4}(\nu\tau)^{1/2}}\, d\tau, \\
&\le \psi_j(t) + \frac{\|u\|_Z + \|u_j\|_Z}{2^{11/4}\pi^{1/4}\nu}\int_0^1 \frac{\phi_j(t\sigma)}{(1-\sigma)^{3/4}\sigma^{1/2}}\, d\sigma = \psi_j(t) + (J\phi_j)(t).
\end{aligned}$$

We adopt the notation of Lemma 6.1, where $\alpha = 1/4$, $\beta = 1/2$, and $\delta = s/2$. To apply the Lemma we need to show that

$$\frac{\|u\|_Z + \|u_1\|_Z}{2^{11/4}\pi^{1/4}\nu} B(1/4,(1-s)/2) < 1. \tag{6.10}$$

Estimate $\|u\|_Z$ by (3.5). Estimate $\|u_j\|_Z$ as follows

$$\|u_j\|_Z = \nu^{1/4}\|w\|_{L^2(\mathbb{R})} \leq \frac{2^{3/4}\nu}{\pi^{1/4}}[1 - \sqrt{1-\lambda(\mu,\eta)}]. \tag{6.11}$$

In the above we used Theorem 5.3.1(3). Thus (6.2) implies (6.10).

To prove (6.3) we need to show that the right-hand-side of (6.10) stays away from 1 for $s = 0$ and all f such that $\lambda(f) < 1$. But using the estimates at the beginning of this proof we see that the right-hand-side of (6.10) is always less than 0.819. Applying (2.3.1) with f replaced by $f - f_2$ and applying the Lemma we get (6.3).

(6.4) with $p = 2$ and $m = 0$ follows from the Lemma and Theorem 2.2.5. By (6.10) we have $u(t) - u_1(t) = e^{-\alpha(D)t}f - \mu F(t) - \eta\mathcal{H}F(t) + K(0,(\bar{u}-\bar{u}_1)(\bar{u}+\bar{u}_1))$. Now we apply Theorem 2.2.6 and Theorem 2.3.4 inductively to prove that for all $m \geq 0$ we have

$$\lim_{t\to\infty} t^{(s+1/2)/2}\|\beta(Dt^{1/2})[u(t) - u_1(t)]\|_{L^2(\mathbb{R})} = 0. \tag{6.12}$$

Now Theorem 2.2.5 and Theorem 2.3.4 imply (6.4) for all p and m satisfying the assumptions, as well as (6.5). ∎

One should notice that as $s \to 1^-$ in the above theorem the subset of all f satisfying (6.2) shrinks to $\emptyset$. Thus more accurate asymptotic approximations to the solution, such as are known for the linearized equation (see Theorem 2.2.5), seem to be beyond the reach of our methods.

As corollaries to the above theorem we see that the decay estimates $\|\partial_x^m u(t)\|_{L^p(\mathbb{R})} = \mathrm{O}(t^{(m+1-1/p)/2})$ are typically sharp as $t \to \infty$ for small data in $L^1(\mathbb{R}) + \mathcal{H}L^1(\mathbb{R})$ if $1 < p \leq \infty$ and $m \geq 0$ is an integer. We also obtain this result for $p = 1$ if $m > 0$. If $p = 1$, $m = 0$, and $f \in L^1(\mathbb{R}) + \mathcal{H}L^1(\mathbb{R})$ is arbitrary then the estimate $\|u(t)\|_{L^1(\mathbb{R})+\mathcal{H}L^1(\mathbb{R})} = \mathrm{O}(1)$ is typically sharp as $t \to \infty$. Another corollary is that the decay estimates (0.5) on the various terms in the BOB equation ($p = q = r = 2$) are typically sharp for small data in $L^1(\mathbb{R})$. The first estimate, being the only one requiring any argument, follows since

$$\begin{aligned}
&\|\bar{u}(t)\bar{u}_x(t) - \bar{u}_1(t)\bar{u}_{1x}(t)\|_{L^2(\mathbb{R})} \\
&\qquad \leq \|u(t) - u_1(t)\|_{L^\infty(\mathbb{R})}\|u_x(t)\|_{L^2(\mathbb{R})} + \|u_1(t)\|_{L^\infty(\mathbb{R})}\|u_x(t) - u_{1x}(t)\|_{L^2(\mathbb{R})} \\
&\qquad = \mathrm{o}(t^{-1/2})\mathrm{O}(t^{-3/4}) + \mathrm{O}(t^{-1/2})\mathrm{o}(t^{-3/4}) = \mathrm{o}(t^{-5/4})
\end{aligned} \tag{6.13}$$

as $t \to \infty$. Thus $\lim_{t\to\infty} t^{5/4}\|u(t)u_x(t)\|_{L^2(\mathbb{R})} = \|ww'\|_{L^2(\mathbb{R})}$, which is positive unless both $\mu = 0$ and $\eta = 0$. A final corollary is the Lyapunov asymptotic stability of the solutions of the RBOB equation thought of as a stationary points of the restricted flow of the BOB' equation.

COROLLARY 6.3. *Suppose $s = 0$, and $f, \mu, \eta, \lambda(f), \lambda(\mu, \eta)$, and w are as in Theorem 6.2. Let $v = S'f$ be the solution of the BOB' equation with initial data f. Then there is a constant $C > 0$, independent of f, such that*

$$\|v - w\|_{BC([0,\infty), L^1(\mathbb{R}) + \mathcal{H}L^1(\mathbb{R}))} \leq C\|f - w\|_{L^1(\mathbb{R}) + \mathcal{H}L^1(\mathbb{R})}, \tag{6.14}$$

$$\|v - w\|_{BC([0,\infty), L^1(\mathbb{R}))} \leq C\|f - w\|_{L^1(\mathbb{R})}. \tag{6.15}$$

Furthermore we have $\lim_{\tau \to \infty} \|v(\tau) - w\|_{L^1(\mathbb{R}) + \mathcal{H}L^1(\mathbb{R})} = 0$; and if $f \in L^1(\mathbb{R})$ we have $\lim_{\tau \to \infty} \|v(\tau) - w\|_{L^1(\mathbb{R})} = 0$. Thus if $\lambda(w) < 1$ then w is (Lyapunov) asymptotically stable with respect to the flow of the BOB' equation restricted to the hyperplanes

$$H_{\mu,\eta} = \{f \in L^1(\mathbb{R}) + \mathcal{H}L^1(\mathbb{R}) \mid \mu = [\hat{f}(0^+) + \hat{f}(0^-)]/2, \eta = i[\hat{f}(0^+) - \hat{f}(0^-)]/2\}$$

and $H_\mu = \{f \in L^1(\mathbb{R}) \mid \hat{f}(0) = \mu\}$.

PROOF: (6.14) and (6.15) follow from Theorem 2.3.3(2) and trivial estimates on the solution of the LBOB equation. The last two assertions follow by changing variables in (6.4) and (6.5) respectively, and using the fact that $\|g(x) - \gamma g(x\gamma)\|_{L^1(x)} \to 0$ as $\gamma \to 1$ for any $g \in L^1(\mathbb{R})$. ∎

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