ASYMPTOTIC INTEGRATION OF NAVIER–STOKES EQUATIONS WITH POTENTIAL FORCES. I

By

C. Foias

and

J.C. Saut

IMA Preprint Series # 722

November 1990

ASYMPTOTIC INTEGRATION OF NAVIER–STOKES EQUATIONS WITH POTENTIAL FORCES. I

By

C. Foias

and

J.C. Saut

IMA Preprint Series # 722

November 1990

Asymptotic Integration of Navier-Stokes Equations with Potential Forces. I

C. Foias(*)   J. C. Saut(**)

Introduction

In a previous work [4], we derived an asymptotic expansion and a normal form for the Navier-Stokes equations with potential forces. In the non resonant case, this normal form is the linear Navier-Stokes system.

More precisely let us consider the Navier-Stokes system

$$
(0.1) \qquad \begin{cases} \dfrac{\partial u}{\partial t} + (u \cdot \nabla)u - \nu \Delta u + \nabla p = 0 \\ \operatorname{div} u = 0 \\ u(x,0) = u_0(x) \end{cases}
$$

in $\Omega \times (0,\infty)$ where Ω is a smooth bounded domain in $\mathbb{R}^n$, $n = 2,3$ or the cube $(0,L)^n$. This system is supplemented with the Dirichlet boundary condition

(*) Department of Mathematics, Indiana University, Bloomington, Indiana, 47405, U.S.A.
(**) Mathématiques, Université Paris-Val de Marne and Laboratoire d'Analyse Numérique, CNRS et Université Paris-Sud, Bat. 425, 91405 ORSAY, France

(0.2) $$u_{|\partial\Omega} = 0$$

or with the spatial periodic condition

(0.3) $$u(x + Le_j) = u(x) \quad \forall x \in R^n,\ 1 \leq j \leq n$$

where (e_j), $1 \leq j \leq n$ is the canonical basis in $\mathbb{R}^n$.

The unknowns are the velocity field $u(x,t) = (u_1(x,t),\ldots,u_n(x,t)$ and the scalar pressure $p(x,t)$.

The normalization mapping W built in [4] associates to a function u_0 belonging to a natural space of initial data an element $W(u_0)$ of a Fréchet space $\mathcal{S}_A$ in such a way that for every solution $u(t)$ or (0.1), the function $v(t) = W(u(t))$ satisfies the equation in $\mathcal{S}_A$ (normal form):

(0.4) $$\frac{dv(t)}{dt} + \nu Av + \mathcal{B}(v) = 0$$

Here A is the Stokes operator (see §1 for a precise definition and other classical notations). $\mathcal{B}$ is a (generally) nonlinear operator which involves only terms corresponding to resonances in the spectrum of the Stokes operator. Moreover, (0.3) is equivalent to an infinite triangular system of non homogeneous linear ordinary differential equations, each in some $\mathbb{R}^k$ space. This system can be elementarily integrated. Moreover, the components of $\mathcal{B}$ can be expressed in terms of a sequence of polynomials $\{P_j\}_{j=1}^{\infty}$ and B .

The main purpose of the present work is to derive other properties of the normalizing mapping W and of the normal form (0.3). Namely we shall prove uniqueness of the polynomials P_j and give a way to compute them by induction. Hence, given Ω and the viscosity ν , the normal form (0.3) is canonical. Also, in the periodic case and under a technical assumption, we determine the inverse of the normalizing map W . Let us furthermore emphasize that our construction of the polynomials P_j could be used to compute asymptotically the flow associated to (0.1), although the algorithm which gives the P_j's is somewhat complicated.

Our methods and results are not only relevant to the Navier-Stokes equations. They can be used for other nonlinear parabolic equations (such as the Kuramoto-Sivashinski equation or reaction-diffusion equations) or for ordinary differential equations, but we shall not develop this here. The applications of the methods of [4] to other nonlinear parabolic equations has been made in [1].

The content of the paper is as follows:

1. Notations and some previous results.

2. The algebra of the asymptotic expansion.

3. Computation of the P_j's

4. The image and the inverse of the normalizing map.

1. Notations and previous results.

We firstly recall a classical functional setting for the Navier-Stokes system (0.1). This setting is developed e.g. in [2] and [5].

Let Ω be either a bounded regular open set in $\mathbb{R}^n$, or the cube $(0,L)^n$, where $n = 2,3$. In the first case, we set

$$\mathcal{V} = \{u \in C_0^\infty(\Omega)^n;\ \operatorname{div} u = 0\}$$

and in the second case

$$\mathcal{V} = \{u = \text{trigonometric polynomials with values in } \mathbb{R}^n;\ \operatorname{div} u = 0,\ \int_\Omega u\,dx = 0,\ u \text{ satisfy } (0.3)\}$$

In both cases we introduce the spaces

$$H = \text{closure of } \mathcal{V} \text{ in } L^2(\Omega)^n$$

$$V = \text{closure of } \mathcal{V} \text{ in } H^1(\Omega)^n$$

where $H^\ell(\Omega)$ $(\ell = 1,2,\ldots)$ denotes the Sobolev spaces of functions u in $L^2(\Omega)$ such that $D^\alpha u \in L^2(\Omega)$, $|\alpha| \le \ell$.

Also we set $E^m(\Omega) = H \cap H^m(\Omega)^n$, $E^\infty(\Omega) = \bigcap_{m\ge 0} E^m(\Omega)$, $m \ge 0$.

We shall use the notations:

$$|u|^2 = \int_\Omega |u(x)|^2 dx \quad \|v\|^2 = \int_\Omega |\nabla v(x)|^2 dx ,$$

$(u \in H,\ v \in V)$ for norms in H (resp. V). The corresponding scalar products will be denoted by $(.,.)$, resp. $((.,.))$. The norm in $H^m(\Omega)$ is indicated as $\|\cdot\|_m$.

Let P be the orthogonal projection in H in $L^2(\Omega)^n$, and let $Au = -P\Delta u$, $B(v,w) = P((v.\nabla)w)$, defined for $u,v,w \in E^2(\Omega) \cap V = D(A)$ and H-valued. We also set $b(u,v,w) = (B(u,v),w)$, and recall that $b(u,v,w) = -b(u,w,v)$.

The operators A and B can be extended by continuity to linear (resp. bilinear) operators from V(resp. $V \times V$) into the dual $V' \supset H \supset V$ of V .

The equations (0.1) completed with (0.2) or (0.3) are equivalent to

$$\text{(1.1)} \qquad \frac{du}{dt} + \nu Au + B(u,u) = 0 \quad u(0) = u_0 \in V$$

We will be interested mainly in <u>regular solutions</u> of (1.1); that is solutions u such that $u \in C(\mathbb{R}_+; V)$. Such solutions are smooth and unique (actually $u \in C^\infty(\tau_0 t_0, +\infty); E^\infty(\Omega) \cap V, \forall t_0 > 0)$. There exists an open set $\mathfrak{R}$ in V, such that if $u_0 \in \mathfrak{R}$, the corresponding solution $u(t) = S(t)u_0$ of (1.1) lies in $\mathfrak{R}$. If $n = 2$, $\mathfrak{R} = V$, and if $n = 3$, $\mathfrak{R}$ contains 0 .

Finally, let us recall that A is the Stokes operator: the equations

$$\begin{cases} -\nu\Delta u + \nabla p = f \ , \ f \in L^2(\Omega)^n \\ \nabla . u = 0 \quad \text{in} \quad \Omega \end{cases}$$

completed with the boundary conditions (0.2) or (0.3) are equivalent to

$$\nu Au = Pf \quad \text{in} \quad H \ .$$

We shall denote by $0 < \lambda_1 \leq \lambda_2 \leq \ldots$ the increasing sequence of the eigenvalues of A , and by $0 < \Lambda_1 < \Lambda_2 < \ldots$ the sequence of <u>distinct</u> eigenvalues of A, Λ_k having multiplicity $m_k \geq 1$.

Let also $(w_m)_{m=1}^{\infty}$ be the orthonormal (in H or V) basis of the associated eigenvectors

$$Aw_m = \lambda_m w_m, \quad m = 1,2,\dots .$$

The orthogonal projection on H on the linear span of $w_1,\dots,w_m$ will be denoted by P_m ; R_j will stand for the orthogonal projection on the eigenspace of Λ_j :

$$R_j w = \sum_{\lambda_k=\Lambda_j} (w,w_k)w_k .$$

Then, we have of course

$$R_j R_k = 0 \quad \text{if} \quad j \neq k \quad \text{and} \quad R_1 + R_2 + \dots = I .$$

We shall also consider the Fréchet space containing H

$$\mathcal{S}_A = R_1 H \oplus R_2 H \oplus \dots,$$

equipped with the topology of convergence of components. The Stokes operator A and the semi-group e^{-At} generated by A extend in the obvious way to $\mathcal{S}_A$.

A few more notions are also needed. We call resonance in the spectrum of A a relation of the type

(1.2) $a_1\Lambda_1 + a_2\Lambda_2 + \dots + a_k\Lambda_k = \Lambda_{k+1}$ where $a_i \in \mathbb{N}$, $1 \leq i \leq k$.

Finally we shall denote

$$0 < \mu_1 = \Lambda_1 < \mu_2 < \dots$$

the elements of the additive semi-group $\mathcal{S}$ generated by the Λ_k's and

$$k_j = \mathrm{Max}\{k,\ \Lambda_k \le \mu_j\}$$

The following Theorem summarize some of the main results of [4].

Theorem 1.1

(i) There exists a one-to-one, analytic mapping $W : \mathcal{R} \to \mathcal{S}_A$, satisfying $W'(0) = I\alpha$, such that for every regular solution $u(t)$ of (1.1), $v(t) = W(u(t))$ satisfies the equation

$$\frac{dv}{dt}(t) + \nu Av(t) + \mathcal{B}(v(t)) = 0 \tag{1.3}$$

where

$$\mathcal{B}_k(v) \underset{\text{def}}{=} R_k\mathcal{B}(v) = \sum_{\mu_\ell+\mu_j=\Lambda_k} R_k B(P_\ell(v_1,\dots,v_{k_\ell}),P_j(v_1,\dots,v_{k_j}), \tag{1.4}$$

and $v = v_1 \oplus v_2 \oplus \dots \in \mathcal{S}_A$.

(ii) In (1.4), for $j = 1,2,3,\dots$, P_j are $E^\infty(\Omega) \cap V$-valued polynomials defined on $R_1H \oplus\dots\oplus R_{k_j}H$ depending on the spectrum of A, B, ν. Moreover if $\mathcal{M}(x_1,\dots,x_{k_j})$ is a monomial in $P_j(x_1,\dots x_{k_j})$ of degree $m_1,\dots,m_{k_j}$ in $x_1,\dots,x_{k_j}$ respectively, then $m_1\Lambda_1+\dots+m_{k_j}\Lambda_{k_j} = \mu_j$. Furthermore, if μ_j is an eigenvalue Λ_k, i.e., $\mu_j = \Lambda_{k_j}$ then $P_j(x_1,\dots,x_k) = x_k +$ higher order terms in $x_1,\dots,x_{k-1}$. □

Actually the mapping W was constructed in [4] from a asymptotic expansion of every solution u of (1.1). We recall in the next theorem the properties of this expansion which will be used in the present paper.

<u>Theorem 1.2</u>

For each $N \in \mathbb{N}$ the solution $u(t)$ of (1.1) admits the expansion in H

$$(1.5)\quad u(t)=W_{\mu_1} e^{-\nu\mu_1 t}+W_{\mu_2}(t)e^{-\nu\mu_2 t}+\cdots+W_{\mu_N}(t)e^{-\nu\mu_N t}+v_N(t)\ .$$

where $W_{\mu_j}(t)$ $(1 \le j \le N)$ is a $E^{\infty}(\Omega) \cap V$-valued polynomial in t and $v_N \in C([0,\infty);V) \cap C^{\infty}([t_0,\infty);\ E^{\infty}(\Omega) \cap V)$, $t_0 > 0$. This expansion satisfies the following properties:

(i) $\quad \|v_N(t)\|_m = O(e^{-\nu(\mu_N+\epsilon_N)t})$, for some $\epsilon_N > 0$, and all $m = 0,1,2,\ldots$

(ii) $\quad d_j^0 \underset{\text{def}}{=} \deg W_{\mu_j} \le j-1$, $j = 1,\ldots,N$.

(iii) If $\Lambda_j \le \mu_N$ is a nonresonant eigenvalue, W_{Λ_j} is a constant in t and $R_j W_{\Lambda_j} = W_{\Lambda_j}$.

(iv) If $\mu_j \leq \mu_N$ is not a nonresonant eigenvalue, $W_{\mu_j}(t)$ satisfies the equation

$$(1.6) \qquad \frac{dW_{\mu_j}}{dt} + \nu(A-\mu_j)W_{\mu_j} + \sum_{\mu_\ell+\mu_k=\mu_j} B(W_{\mu_\ell}(t), W_{\mu_k}(t)) = 0$$

If Λ_j is a resonant eigenvalue, one has

$$(1.7) \qquad \deg W_{\Lambda_j} \leq \operatorname*{Max}_{\mu_\ell+\mu_k=\Lambda_j} (d_\ell^0 + d_k^0) + 1$$

(v) For any $\mu_j \in \mathcal{S}$,

$$W_{\mu_j}(0;u_0) = P_j(W_1(u_0), W_2(u_0), \ldots, W_{k_j}(u_0))$$

where the P_j's are the polynomials defined in Theorem 1.1 and

$$W_k(u_0) \underset{\text{def}}{=} R_k W_{\Lambda_k}(0;u_0)$$

are the components of the normalizing mapping W.

(vi) Let $\pi_N : \mathcal{S}_A \to R_1H \oplus \cdots \oplus R_NH$ be the canonical projection, $N = 1,2,\ldots$. Then the range of $\pi_N W$ contains a ball centered at 0.

<u>Remark 1.1</u>. We will henceforth use the notation

$$S(t,u_0) \sim \sum_1^\infty W_{\mu_j}(t,u_0)e^{-\nu\mu_j t}$$

for the asymptotic expansion (1.3).

2. The algebra of the asymptotic expansion

Proposition 2.1. For every $u \in \mathcal{R}$, the asymptotic expansion (1.3) can be rewritten as follows

$$(2.1) \qquad S(t)u \sim \sum_{j=1}^{\infty} P_j(W_1(S(t)u),\dots,W_{k_j}(S(t)u)) \ .$$

Proof. We recall that $k_j = \operatorname{Sup}\{k,\ \Lambda_k \leq \mu_j\}$. Let $u \in \mathcal{R}$. Then $S(t)u$ has the asymptotic expansion

$$S(t)u \sim \sum_{j=1}^{\infty} e^{-\nu\mu_j t} W_{\mu_j}(t;u)$$

where

$$W_{\mu_j}(0;u) = P_j(W_1(u),\dots,W_{k_j}(u))$$

(See Theorem 1.2). We also have for any $t,\ t_0 \geq 0$

$$S(t+t_0;u) \sim \sum_{j=1}^{\infty} e^{-\nu\mu_j(t+t_0)} W_{\mu_j}(t+t_0;u)$$

$$S(t+t_0;u) \sim \sum_{j=1}^{\infty} e^{-\nu\mu_j t} W_{\mu_j}(t;S(t_0)u)$$

By the uniqueness of the expansion (1.3), we obtain therefore

$$W_{\mu_j}(t;S(t_0)u) = e^{-\nu\mu_j t_0} W_{\mu_j}(t+t_0;u) \ ,$$

which, when evaluated at $t = 0$ leads to

$$W_{\mu_j}(0;S(t)u) = e^{-\nu\mu_j t} W_{\mu_j}(t;u) \quad \forall t \geq 0 \ .$$

So by Theorem 1.2,

$$(2.3)\qquad W_{\mu_j}(t;u) = e^{\nu\mu_j t} P_j(W_1(S(t)u),\ldots,W_{k_j}(S(t)u))$$
$$\forall t \geq 0, \quad \forall u \in \mathfrak{H}$$

which obviously implies formula (2.1). This formula, although a simple consequence of [4] was not explicitely stated there. □

The next Proposition gives further properties of the P_j's .

Proposition 2.2

(i) If $\mu_j \in \mathscr{S}$ is the eigenvalue $\bar{\Lambda}_{k_j}$, then

$$P_j(\xi_1,\xi_2,\ldots,\xi_{k_j}) = \xi_{k_j} + Q_{k_j}(\xi_1,\ldots,\xi_{k_k-1}) ,$$

where $d^O Q_{k_j} \geq 2$ and

$$R_{k_j} Q_{k_j}(\xi_1,\ldots,\xi_{k_j-1}) = 0 .$$

(ii) For $u \in \mathfrak{H}$ denote $\xi_k(t) = W_k(u(t))$. Then, for every j

$$(2.4)\qquad \sum_{m=2}^{k_j} (D_m P_j)\mathscr{B}_m = \nu(A - \mu_j)P_j + \sum_{\mu_i+\mu_\ell=\mu_j} B(P_i,P_\ell)$$

when D_m is the derivative with respect to the m^{th} component.

In (2.4) the polynomials are evaluated at $\xi_1(t),\ldots,\xi_{k_j}(t)$ but the formula is valid for arbitrary $\xi_1,\ldots,\xi_{k_j}$ since $W'(0) = I$.

Proof. Part (i) is a direct consequence of Lemma 7 and formula (4.29) in [4].

We now prove Part (ii). From (1.4) and (2.1) we derive for every $\mu_j \in \mathscr{S}$

$$\begin{aligned}
&\frac{d}{dt} e^{\nu\mu_j t} P_j(\xi_1(t),\dots,\xi_{k_j}(t)) \\
(2.5)\qquad &+ \nu(A-\mu_j) e^{\nu\mu_j t} P_j(\xi_1(t),\dots,\xi_{k_j}(t)) \\
&+ \sum_{\mu_i+\mu_\ell=\mu_j} B(e^{\nu\mu_i t} P_i(\xi_1(t),\dots\xi_{k_i}(t)), e^{\nu\mu_\ell t} P_\ell(\xi_1(t),\dots\xi_{k_\ell}(t)) = 0.
\end{aligned}$$

Using Theorem 1.1, (2.5) can be rewritten as

$$\begin{aligned}
&\frac{d}{dt} P_j(e^{\nu\Lambda_1 t}\xi_1(t),\dots,e^{\nu\Lambda_{k_j} t}\xi_{k_j}(t)) \\
&+ \nu(A-\mu_j) P_j(e^{\nu\Lambda_1 t}\xi_1(t),\dots,e^{\nu\Lambda_{k_j} t}\xi_{k_j}(t)) \\
&+ \sum_{\mu_i+\mu_\ell=\mu_j} B(P_i(e^{\nu\Lambda_1 t}\xi_1(t),\dots,e^{\nu\Lambda_{k_i} t}\xi_{k_i}(t)), P_\ell(e^{\nu\Lambda_1 t}\xi_1(t),\dots, \\
&\qquad e^{\nu\Lambda_{k_\ell} t}\xi_\ell(t))) = 0
\end{aligned}$$

Now we compute the time derivative of $e^{\nu\Lambda_k t}\xi_k(t)$, $k = 1,2,\dots$.

$$\frac{d}{dt}(e^{\nu\Lambda_k t}\xi_k(t)) = \nu\Lambda_k e^{\nu\Lambda_k t}\xi_k(t) + e^{\nu\Lambda_k t}\dot{\xi}_k(t)$$

and, by (2.4),

$$\frac{d\xi_k}{dt} = \dot{\xi}_k(t) = -\nu\Lambda_k - \mathscr{B}_k(\xi(t))$$

so that

$$\frac{d}{dt}(e^{\nu\Lambda_k t}\xi_k(t)) = -e^{\nu\Lambda_k t}\mathscr{B}_k .$$

Coming back to (2.6) we obtain the formula

$$\sum_{m=1}^{k_j} (D_m P_j)\mathfrak{z}_m = \nu(A-\mu_j)P_j + \sum_{\mu_i+\mu_\ell=\mu_j} B(P_i,P_\ell) .$$

The summation starts up at $m = 2$ since $\mathfrak{z}_1 = 0$. □

3. Construction of the P_j's

In what follows we will denote $R_{\mu_j} = R_k$ if $k = k_j$ and $R_{\mu_j} = 0$ otherwise, $Q_{\mu_j} = I - R_{\mu_j}$ and we are given

$$\xi_1^o \in R_1 H , \ \xi_2^o \in R_2 H , \ \ldots .$$

The basic result concerning the construction of the P_j's is the following

<u>Theorem 3.1</u>. We define a sequence of polynomials q_j as follows:

$$q_1(t) \equiv \xi_1^o$$

and then by induction

$$(3.1) \quad \begin{cases} q_j(t) = \displaystyle\sum_{n\geq 0} (-1)^n [\nu(A-\mu_j)]^{-n-1} (\tfrac{d}{dt})^n \sum_{\mu_i+\mu_\ell=\mu_j} B(q_i,q_\ell) \\ \text{if } \mu_j \notin \sigma(A) \end{cases}$$

or

$$(3.2)\begin{cases} q_j(t) = \xi^0_{k_j} + \int_0^t \sum_{\mu_i+\mu_\ell=\mu_j} R_{k_j} B(q_i(\tau),q_\ell(\tau))d\tau + \\ \sum_{n\geq 0} (-1)^n [\nu(A-\mu_j)(I-R_{k_j})]^{-n-1} (\frac{d}{dt})^n \sum_{\mu_i+\mu_\ell=\mu_j} (I-R_{k_j})B(q_i,q_\ell) \\ \text{if} \quad \mu_j = \Lambda_{k_j} \end{cases}$$

Then

$$(3.3) \qquad q_j(t) = P_{j0}(\xi^0_1,\dots,\xi^0_{k_j}) + tP_{j1}(\xi^0_1,\dots,\xi^0_{k_j}) + \cdots + t^{j-1}P_{j,j-1}(\xi^0_1,\dots,\xi^0_{k_j}) \ ,$$

where $P_{j,i}$ are polynomial maps in $\xi^0_1,\dots,\xi^0_{k_j}$ and

$$P_j(\xi^0_1,\dots,\xi^0_{k_j}) \equiv P_{j0}(\xi^0_1,\dots,\xi^0_{k_j}) \ .$$

__Proof__. We start with a simple algebraic lemma

__Lemma 3.2__. Let L be an invertible (not necessarily bounded) linear operator in some Hilbert space H . Then for any H-valued polynomial $b(t)$, the equation

$$(3.4) \qquad q' = -Lq + b$$

has a unique polynomial solution given by

$$(3.5)\qquad q(t) = \sum_{k=0}^{\infty} (-1)^k L^{-k} b^{(k)}(t) \ .$$

Proof of Lemma 3.2. An immediate computation shows that (3.5) solves (3.4). On the other hand, if q_0 and q_1 are two polynomial solutions of (3.4), the difference $q = q_0 - q_1$ solves the equation

$$(3.6)\qquad q' = -L_q \ .$$

Since q is a polynomial, $q^{(n)} = 0$ for $n > \deg.q$ which with (3.6) yields $0 = Lq^{(n-1)}$, and since L is invertible $q^{(n-1)} = 0$. Inductively one gets $q = 0$. □

Let us go back to the proof of Theorem 3.1. We assume first that $\mu_j \notin \sigma(A)$. Let us consider the equation, when q_j is a polynomial in t,

$$(3.7)\qquad q_j' + \nu(A-\mu_j)q_j + \sum_{\mu_i+\mu_\ell=\mu_j} B(q_i,q_\ell) = 0$$

$$(3.8)\qquad q_j(0) = P_j(\xi_1^0,\ldots,\xi_{k_j}^0) \ .$$

By Lemma 3.2 we get exactly the expression (3.1), and of course (3.3).

(ii) Assume now that $\mu_j = \Lambda_{k_j} \in \sigma(A)$. We write

$$q_j(t) = \eta_j(t) + Q_{\Lambda_{k_j}} q_j(t)$$

where $\eta_j(t) \in R_{\Lambda_{k_j}} H$ and $Q_{\Lambda_{k_j}} = I - R_{\Lambda_{k_j}}$.

The evolution of $\eta_j(t)$ is immediate and gives the integral contribution in (3.2). It suffices now to apply Lemma 3.2 to $Q_{\Lambda_{k_j}} q_j(t)$ to get (3.2) and of course (3.3) . □

Theorem 3.1 has the obvious consequence:

<u>Corollary 3.3</u>. The polynomials P_j's are unique.

<u>Remark 3.4</u>. Theorem 3.1 could be used to get the asymptotic integration of the Navier-Stokes equations (0.1).

4. <u>The image and the inverse of the normalizing map</u>

We start this section by giving a reinterpretation of the asymptotic expansion given in Proposition 2.1. First let us recall that the set $M_k = \{u \in \mathfrak{R},\ W_1(u) = \cdots = W_{k-1}(u) = 0\}$ is an analytic, invariant under $S(t)$, manifold in $\mathfrak{R}$, of codimension $m_1 + \ldots + m_k$ ([3]), characterized also by the condition

$$S(t)u = O(e^{-\nu\Lambda_{k+1}t}) .$$

Also, setting $M_0 = \mathcal{H}$, one has

$$u \in M_{k-1} \backslash M_k \Leftrightarrow \begin{cases} |S(t)u| = O(e^{-\nu\Lambda_k t}) \\ \quad \text{and} \\ e^{-\nu\Lambda_k t} = O(|S(t)u|) \end{cases}$$

Therefore, if $u \in M_{k-1}$, the expansion (2.1) takes the form

$$(4.1) \qquad S(t)u \sim \sum_{k_j \geq k} P_j(0,\ldots,W_k(S(t)u),\ldots W_{k_j}(S(t)u)) .$$

We want now to show that (4.1) can be viewed as an asymptotic expansion of $v = S(t)u$ in terms of its coordinates $W_k(v)$, $W_{k+1}(v)$, ... Indeed for N fixed, with $k_N \geq k$, let d_N denotes the degree of the polynomial

$$\sum_{\substack{k_j \geq k \\ j \leq N}} P_j(0,\ldots,W_k(v),\ldots,W_{k_j}(v)) .$$

Then it is easy to check using Theorem 1.1 (ii) that $d_N \leq \mu_N/\Lambda_k$. Therefore, if $v \in M_{k-1}\backslash M_k$, we can write

$$(4.2) \quad |v - \sum_{\substack{k_j \geq k \\ j \leq N}} P_j(0,\ldots,W_k(v),\ldots,W_{k_j}(v))| = O(|v|^{d_N+\epsilon_N/\Lambda_k})$$

where ϵ_N is as Theorem 1.2 (i). In particular, if $v \in \mathcal{H}\backslash M_1$, then

$$(4.3) \qquad \left|v - \sum_{j=1}^{N} P_j(W_1(v),\ldots,W_{k_j}(v))\right| = O(|v|^{d_N+\epsilon_N/\Lambda_1})$$

when d_N now is the degree of the polynomial

$$\sum_{j=1}^{N} P_j \quad \text{in} \quad W_1(v),\ldots,W_{k_N}(v) \ .$$

Notice that the asymptotic expansions (4.2), (4.3) hold on trajectories. With our present knowledge we don't know if they are global. Anyway, they strongly suggest that

$$(W_1(v), W_2(v),\ldots) \longmapsto \sum_{j=1}^{\infty} P_j(W_1(v),\ldots,W_{k_j}(v))$$

is a natural candidate for the inverse map of the normalizing map W . Here below we'll give a result which enforces this belief. Namely, we exhibit a subset in the range of W for which the asymptotic expansion is a convergent series and indeed provides the inverse of the normalizing map. However, there is a caveat to be made, namely, we were not able to prove that our subset of the range of W is not reduced to zero.

In what follows, we take $\Omega = (0,2\pi)^3$, hence $\Lambda_1 = 1$, $\mathcal{S} = \mathbb{N}$, and $\nu = 1$.

<u>Proposition 4.1</u>. Let $\xi = (\xi_1^0,\ldots,\xi_k^0,\ldots) \in \mathcal{S}_A$ satisfies

$$(4.4) \qquad \chi = \sum_{j=1}^{\infty} \sum_{\ell=0}^{j-1} \left| AP_{j,\ell}(\xi_1^0,\ldots,\xi_{k_j}^0) \right| e^{-\ell} \left(\frac{\ell}{j}\right)^{\ell} < +\infty .$$

Then, there exists a (unique) $u_0 \in \mathfrak{M}$ such that $W(v_0) = \xi$. The corresponding solution of the Navier-Stokes equation is given by

$$(4.5) \quad S(t)u_0 = \sum_{j=0}^{\infty} q_j(t) e^{-\mu_j t} = \sum_{j=0}^{\infty} P_j(W_1(u(t)),\ldots,W_{k_j}(u(t)))$$

where the q_j's are defined in Theorem 3.1.

Proof. We prove that the series in (4.5) converges in $D(A)$ and define a regular solution of the Navier-Stokes equations. We have by Theorem 3.1,

$$q_j(t) = \sum_{\ell=0}^{j-1} t^{\ell} P_{j,\ell} .$$

We set

$$\varphi_j(t) = e^{-\mu_j t} q_j(t)$$

and compute

$$\frac{d}{dt}\varphi_j = -\mu_j \varphi_j + e^{-\mu_j t} \frac{d}{dt} q_j = \text{(See proof of Theorem 3.1)}$$

$$= -A\varphi_j - \sum_{\mu_i+\mu_\ell=\mu_j} e^{-\mu_j t} B(q_i,q_\ell) =$$

$$= -A\varphi_j - \sum_{\mu_i+\mu_\ell=\mu_j} B(e^{-\mu_i t} q_i, e^{-\mu_\ell t} q_\ell) =$$

$$= -A\varphi_j - \sum_{\mu_i+\mu_\ell=\mu_j} B(q_i,q_\ell) \ .$$

We therefore obtain the estimate

$$\left|\frac{d\varphi_j}{dt}\right| \leq |A\varphi_j| + c \sum_{\mu_i+\mu_\ell=\mu_j} |A\varphi_i| \ |A\varphi_\ell| \ .$$

But (see (3.3)),

$$|A\varphi_j(t)| = \left|\sum_{\ell=0}^{j-1} e^{-\mu_j t} t^\ell AP_{j,\ell}\right|$$

(4.6)

$$\leq \sum_{\ell=0}^{j-1} e^{-\mu_j t} t^\ell |AP_{j,\ell}| \ .$$

At this stage, we are going to use the fact that $\Omega = (0,2\pi)^3$, so that $\mu_j = j$. In other situations it seems difficult to get precise enough information on the semi group $\mathcal{S}$ in order to carry on the following estimates.

By an elementary property of the function $e^{-jt}t^\ell$ we can therefore deduce from (4.6) that

$$(4.7) \qquad |A\varphi_j(t)| \leq \sum_{\ell=0}^{j-1} e^{-\ell} \left(\frac{\ell}{j}\right)^\ell |AP_{j,\ell}| \ ,$$

and we obtain

$$(4.8) \qquad \sum_{j=0}^{\infty} |A\varphi_j(t)| \leq \chi$$

$$(4.9) \qquad \sum_{j=0}^{\infty} \left|\frac{d\varphi_j}{dt}(t)\right| \leq \chi + c\chi^2$$

where χ was defined in (4.4). In particular the series in (4.5) converges and we set $u_0 = \sum_{j=0}^{\infty} q_j(0) = \sum_{j=0}^{\infty} P_j(\xi_1^0,\ldots,\xi_{k_j}^0)$. The fact that $S(t)u_0$ actually solves the Navier-Stokes equation is now an immediate consequence of (3.6) and Theorem 1.1 (ii).

It remains to show that (4.5) is the asymptotic expansion, i.e. to prove the correct decay estimate for the remainder.

Let us evaluate

$$\sum_{j=n+1}^{\infty} e^{-jt}|Aq_j(t)| \ .$$

One has

$$\phi(t) \underset{\text{def}}{=} \sum_{j=n+1}^{\infty} e^{-jt}|Aq_j(t)| = \sum_{j=n+1}^{\infty} e^{-jt} \sum_{\ell=0}^{j-1} |AP_{j,\ell}|t^{\ell} =$$

$$= \sum_{j=n+1}^{\infty} \sum_{\ell=0}^{j-1} \frac{e^{-j\ell}t^{\ell}}{(\frac{\ell}{j})^{\ell}e^{-\ell}} (\frac{\ell}{j})^{\ell} e^{-\ell}|AP_{j,\ell}| =$$

$$= \sum_{j=n+1}^{\infty} \sum_{\ell=0}^{j-1} \alpha_{j\ell} \frac{e^{-jt}t^{\ell}}{e^{-\ell}(\frac{\ell}{j})^{\ell}} = \sum_{j=n+1}^{\infty} \sum_{\ell=0}^{j-1} \alpha_{j\ell}e^{-jt}(\frac{etj}{\ell})^{\ell} \ .$$

Where we have set

$$\alpha_{j\ell} = e^{-\ell}\left(\frac{\ell}{j}\right)^{\ell} |AP_{j,\ell}| \ .$$

Note that by (4.2),

$$\sum_{j=0}^{\infty} \sum_{\ell=0}^{j-1} \alpha_{j\ell} < +\infty \ .$$

Since the function $\left(\frac{etj}{\ell}\right)^{\ell}$ is increasing in ℓ for $t \geq 1$ (its derivative has the sign of $\operatorname{Log}\frac{j}{\ell} + \operatorname{Log} t$), we have

$$\phi(t) \leq \sum_{j=n+1}^{\infty} e^{-jt}\left(\frac{j}{j-1}\, et\right)^{j-1} \sum_{\ell=0}^{j-1} \alpha_{j\ell} =$$

$$= \sum_{j=n+1}^{\infty} (e^{-t}et)^{j}\left(\frac{j}{j-1}\right)^{j}\left(\frac{j-1}{j}\right) \frac{1}{et} \sum_{\ell=0}^{j-1} \alpha_{j\ell} \ ,$$

$$\phi(t) \leq c_1 \sum_{j=n+1}^{\infty} \frac{1}{t}(ete^{-t})^{j} \sum_{\ell=0}^{j-1} \alpha_{j\ell} \ .$$

Since

$$ete^{-t} = ete^{\frac{-1/2}{n+1}t}\, e^{-\frac{n+1/2}{n+1}t} \leq c_2 e^{-\frac{n+1/2}{n+1}t} \quad \forall t \geq 0,$$

we have

$$\phi(t) \leq \frac{c_1}{t} \sum_{j=n+1}^{\infty} c_2^{j}\, e^{-\frac{n+1/2}{n+1}jt} \sum_{\ell=0}^{j-1} \alpha_{j\ell} =$$

$$= \frac{c_1}{t}\, e^{-(n+1/2)t} \sum_{j=n+1}^{\infty} c_2^{j}\, e^{-(n+1/2)\left(\frac{j-1}{n+1}\right)t} \sum_{\ell=0}^{j-1} \alpha_{j\ell} \ .$$

But

$$c_2^{j} e^{-(n+1/2)\left(\frac{j}{n+1}-1\right)t} \leq c_2^{n+1}\left(c_2 e^{-\frac{n+1/2}{n+1}t}\right)^{j-(n+1)}$$

$$\leq c_2^{n+1} \text{ for } t \text{ sufficiently large.}$$

We have therefore proven that

$$\phi(t) = O(e^{-(n+1/2)t}) \text{ as } t \to +\infty$$

which shows that

$$S(t) \sum_{n=1}^{\infty} q_n(0) \sim \sum_{n=1}^{\infty} e^{-\mu_n t} q_n(t)$$

and completes the proof.

Acknowledgements. Part of this work was done when the authors were participating in the 1989/90 Program on Dynamical Systems and their Applications at the Institute for Mathematics and its Applications, University of Minnesota. This work was also supported by the DOE Grant DE-FG02-86ER25020, NSF Grant DMS-8802596, and the Research Fund of Indiana University.

References

[1] M. Aissaoui, Thesis, Université Paris-Sud, Orsay, France (1988).

[2] P. Constantin, C. Foias, Navier-Stokes equations, University of Chicago Press, Chicago, 1988.

[3] C. Foias, J.C. Saut, On the smoothness of the nonlinear spectral manifolds of Navier-Stokes equations, Indiana Univ. Math. J., 33,6,1984, 911-926.

[4] C. Foias, J.C. Saut, Linearization and normal form of the Navier-Stokes equations with potential forces, Ann. Inst. Henri Poincare, Analyse non lineaire, 4,1,1987, 1-47.

[5] R. Temam, Navier-Stokes equations. Theory and numerical analysis, North-Holland, Amsterdam, 1979.

Recent IMA Preprints

#	Author/s / Title
637	**Xinfu Chen**, Generation and propagation of the interface for reaction–diffusion equations
638	**Philip Korman**, Dynamics of the Lotka–Volterra systems with diffusion
639	**Harlan W. Stech**, Generic Hopf bifurcation in a class of integro-differential equations
640	**Stephane Laederich**, Periodic solutions of non linear differential difference equations
641	**Peter J. Olver**, Canonical Forms and Integrability of BiHamiltonian Systems
642	**S.A. van Gils, M.P. Krupa and W.F. Langford**, Hopf bifurcation with nonsemisimple 1:1 Resonance
643	**R.D. James and D. Kinderlehrer**, Frustration in ferromagnetic materials
644	**Carlos Rocha**, Properties of the attractor of a scalar parabolic P.D.E.
645	**Debra Lewis**, Lagrangian block diagonalization
646	**Richard C. Churchill and David L. Rod**, On the determination of Ziglin monodromy groups
647	**Xinfu Chen and Avner Friedman**, A nonlocal diffusion equation arising in terminally attached polymer chains
648	**Peter Gritzmann and Victor Klee**, Inner and outer j- Radii of convex bodies in finite-dimensional normed spaces
649	**P. Szmolyan**, Analysis of a singularly perturbed traveling wave problem
650	**Stanley Reiter and Carl P. Simon**, Decentralized dynamic processes for finding equilibrium
651	**Fernando Reitich**, Singular solutions of a transmission problem in plane linear elasticity for wedge-shaped regions
652	**Russell A. Johnson**, Cantor spectrum for the quasi-periodic Schrödinger equation
653	**Wenxiong Liu**, Singular solutions for a convection diffusion equation with absorption
654	**Deborah Brandon and William J. Hrusa**, Global existence of smooth shearing motions of a nonlinear viscoelastic fluid
655	**James F. Reineck**, The connection matrix in Morse–Smale flows II
656	**Claude Baesens, John Guckenheimer, Seunghwan Kim and Robert Mackay**, Simple resonance regions of torus diffeomorphisms
657	**Willard Miller, Jr.**, Lecture notes in radar/sonar: Topics in Harmonic analysis with applications to radar and sonar
658	**Calvin H. Wilcox**, Lecture notes in radar/sonar: Sonar and Radar Echo Structure
659	**Richard E. Blahut**, Lecture notes in radar/sonar: Theory of remote surveillance algorithms
660	**D.V. Anosov**, Hilbert's 21st problem (according to Bolibruch)
661	**Stephane Laederich**, Ray–Singer torsion for complex manifolds and the adiabatic limit
662	**Geneviève Raugel and George R. Sell**, Navier-Stokes equations in thin 3d domains: Global regularity of solutions I
663	**Emanuel Parzen**, Time series, statistics, and information
664	**Andrew Majda and Kevin Lamb**, Simplified equations for low Mach number combustion with strong heat release
665	**Ju. S. Il'yashenko**, Global analysis of the phase portrait for the Kuramoto–Sivashinsky equation
666	**James F. Reineck**, Continuation to gradient flows
667	**Mohamed Sami Elbialy**, Simultaneous binary collisions in the collinear N–body problem
668	**John A. Jacquez and Carl P. Simon**, Aids: The epidemiological significance of two different mean rates of partner-change
669	**Carl P. Simon and John A. Jacquez**, Reproduction numbers and the stability of equilibria of SI models for heterogeneous populations
670	**Matthew Stafford**, Markov partitions for expanding maps of the circle
671	**Ciprian Foias and Edriss S. Titi**, Determining nodes, finite difference schemes and inertial manifolds
672	**M.W. Smiley**, Global attractors and approximate inertial manifolds for abstract dissipative equations
673	**M.W. Smiley**, On the existence of smooth breathers for nonlinear wave equations
674	**Hitay Özbay and Janos Turi**, Robust stabilization of systems governed by singular integro-differential equations
675	**Mary Silber and Edgar Knobloch**, Hopf bifurcation on a square lattice
676	**Christophe Golé**, Ghost circles for twist maps
677	**Christophe Golé**, Ghost tori for monotone maps
678	**Christophe Golé**, Monotone maps of $T^n \times R^n$ and their periodic orbits
679	**E.G. Kalnins and W. Miller, Jr.**, Hypergeometric expansions of Heun polynomials
680	**Victor A. Pliss and George R. Sell**, Perturbations of attractors of differential equations
681	**Avner Friedman and Peter Knabner**, A transport model with micro- and macro-structure
682	**E.G. Kalnins and W. Miller, Jr.**, A note on group contractions and radar ambiguity functions
683	**George R. Sell**, References on dynamical systems
684	**Shui-Nee Chow, Kening Lu and George R. Sell**, Smoothness of inertial manifolds
685	**Shui-Nee Chow, Xiao-Biao Lin and Kening Lu**, Smooth invariant foliations in infinite dimensional spaces

686 **Kening Lu**, A Hartman–Grobman theorem for scalar reaction-diffusion equations

687 **Christophe Golé and Glen R. Hall**, Poincaré's proof of Poincaré's last geometric theorem

688 **Mario Taboada**, Approximate inertial manifolds for parabolic evolutionary equations via Yosida approximations

689 **Peter Rejto and Mario Taboada**, Weighted resolvent estimates for Volterra operators on unbounded intervals

690 **Joel D. Avrin**, Some examples of temperature bounds and concentration decay for a model of solid fuel combustion

691 **Susan Friedlander and Misha M. Vishik**, Lax pair formulation for the Euler equation

692 **H. Scott Dumas**, Ergodization rates for linear flow on the torus

693 **A. Eden, A.J. Milani and B. Nicolaenko**, Finite dimensional exponential attractors for semilinear wave equations with damping

694 **A. Eden, C. Foias, B. Nicolaenko & R. Temam**, Inertial sets for dissipative evolution equations

695 **A. Eden, C. Foias, B. Nicolaenko & R. Temam**, Hölder continuity for the inverse of Mañé's projection

696 **Michel Chipot and Charles Collins**, Numerical approximations in variational problems with potential wells

697 **Huanan Yang**, Nonlinear wave analysis and convergence of MUSCL schemes

698 **László Gerencsér and Zsuzsanna Vágó**, A strong approximation theorem for estimator processes in continuous time

699 **László Gerencsér**, Multiple integrals with respect to L-mixing processes

700 **David Kinderlehrer and Pablo Pedregal**, Weak convergence of integrands and the Young measure representation

701 **Bo Deng**, Symbolic dynamics for chaotic systems

702 **P. Galdi, D.D. Joseph, L. Preziosi, S. Rionero**, Mathematical problems for miscible, incompressible fluids with Korteweg stresses

703 **Charles Collins and Mitchell Luskin**, Optimal order error estimates for the finite element approximation of the solution of a nonconvex variational problem

704 **Peter Gritzmann and Victor Klee**, Computational complexity of inner and outer j-radii of polytopes in finite-dimensional normed spaces

705 **A. Ronald Gallant and George Tauchen**, A nonparametric approach to nonlinear time series analysis: estimation and simulation

706 **H.S. Dumas, J.A. Ellison and A.W. Sáenz**, Axial channeling in perfect crystals, the continuum model and the method of averaging

707 **M.A. Kaashoek and S.M. Verduyn Lunel**, Characteristic matrices and spectral properties of evolutionary systems

708 **Xinfu Chen**, Generation and Propagation of interfaces in reaction diffusion systems

709 **Avner Friedman and Bei Hu**, Homogenization approach to light scattering from polymer-dispersed liquid crystal films

710 **Yoshihisa Morita and Shuichi Jimbo**, ODEs on inertial manifolds for reaction-diffusion systems in a singularly perturbed domain with several thin channels

711 **Wenxiong Liu**, Blow-up behavior for semilinear heat equations:multi-dimensional case

712 **Hi Jun Choe**, Hölder continuity for solutions of certain degenerate parabolic systems

713 **Hi Jun Choe**, Regularity for certain degenerate elliptic double obstacle problems

714 **Fernando Reitich**, On the slow motion of the interface of layered solutions to the scalar Ginzburg–Landau equation

715 **Xinfu Chen and Fernando Reitich**, Local existence and uniqueness of solutions of the Stefan problem with surface tension and kinetic undercooling

716 **C.C. Lim, J.M. Pimbley, C. Schmeiser and D.W. Schwendeman**, Rotating waves for semiconductor inverter r

717 **W. Balser, B.L.J. Braaksma, J.-P. Ramis and Y. Sibuya**, Multisummability of formal power series solutions of linear ordinary differential equations

718 **Peter J. Olver and Chehrzad Shakiban**, Dissipative decomposition of partial differential equations

719 **Clark Robinson**, Homoclinic bifurcation to a transitive attractor of Lorenz type, II

720 **Michelle Schatzman**, A simple proof of convergence of the QR algorithm for normal matrices without *shifts*

721 **Ian M. Anderson, Niky Kamran and Peter J. Olver**, Internal, external and generalized symmetries

722 **C. Foias and J.C. Saut**, Asymptotic integration of Navier–Stokes equations with potential forces. I

723 **Ling Ma**, The convergence of semidiscrete methods for a system of reaction-diffusion equations

724 **Adelina Georgescu**, Models of asymptotic approximation

725 **A. Makagon and H.Salehi**, On bounded and harmonizable solutions on infinite order arma systems

726 ***San-Yih Lin and Yan-Shin Chin***, *An upwind finite-volume scheme with a triangular mesh for conservation laws*

727 ***J.M. Ball, P.J. Holmes, R.D. James, R.L. Pego & P.J. Swart***, *On the dynamics of fine structure*

728 ***KangPing Chen and Daniel D. Joseph***, *Lubrication theory and long waves*