A DIFFUSION EQUATION WITH LOCALIZED CHEMICAL REACTIONS

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IMA Preprint Series # 970

May 1992

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Abstract

In some chemical reaction-diffusion processes, the reaction takes place only at some local sites, due to the presence of the catalyst. In this paper we study the well-posedness of a model problem. Sufficient conditions are found to ensure global existence and finite time blowup. The blowup rate and the blowup set are also investigated for a special nonlinearity.

Key Words and Phrases: Localized Reaction-Diffusion, Global Solvability, Blowup.
AMS(MOS) Subject Classification: 35K57, 35B05.

[1]This work is supported by NSERC Canada

1. Introduction

Let Ω be a bounded domain in R^N with smooth boundary $S = \partial\Omega$. Let Γ be a C^2-hypersurface in Ω. Consider the following problem:

$$u_t - \Delta u = \mathcal{F}f(u),\ (x,t) \in Q_T = \Omega \times (0,T]; \tag{1.1}$$

$$u(x,t) = 0, \qquad (x,t) \in S_T = \partial\Omega \times [0,T]; \tag{1.2}$$

$$u(x,0) = u_0(x), \qquad x \in \Omega, \tag{1.3}$$

where $\mathcal{F}$ is a functional from $C(\bar{\Omega}) \to R^1$ defined by

$$\int_\Omega \mathcal{F}g(x)dx = \int_\Gamma g(x)ds,\ g(x) \in C(\bar{\Omega}), \tag{1.4}$$

where ds represents the surface area element in Γ.

The problem describes chemical reaction-diffusion processes in which, due to the effects of a catalyst, the reaction takes place only at some local sites. This causes the chemical concentration to be continuous, but the gradient of the concentration to have a jump at these local sites. The magnitude of the jump typically depends on the concentration. Similar phenomena are also frequently observed in biological systems, for instance on active membranes. The original model equation to describe the above phenomenon was given (cf. [4]) by

$$u_t = \Delta u + f(u)\delta(x),\ in\ Q_T = \Omega \times (0,T], \tag{1.5}$$

where Ω contains the origin and $\delta(x)$ is the Dirac-delta (generalized) function.

The steady-state structures of solutions in one-space dimension as well as the radial symmetrical region in higher space dimension was studied in [4] and [13]. However, the model equation (1.5) is not well-defined in a general domain of higher space dimension. To see this we let $G(x, y; t, \tau)$ be the Green's function with a homogeneous Dirichlet condition on the lateral boundary. Then we have the following representation:

$$u(x,t) = \int_\Omega G(x, y; t, 0) u_0(y) dy + \int_0^t G(x, 0; t, \tau) f(u(0,\tau)) d\tau. \tag{1.6}$$

It is known that

$$G(x, y; t, \tau) = Z(x, y; t, \tau) - g(x, y; t, \tau),$$

where $Z(x, y; t, \tau)$ is the fundamental solution of the heat equation and $g(x, y; t, \tau)$ is the solution of the heat equation with the boundary value $Z(x, y; t, \tau)$.

In a neighborhood of $x = 0$, $g(x, 0; t, \tau)$ is a smooth function but $G(x, 0; t, \tau)$ has the singularity $(t-\tau)^{-\frac{N}{2}}$. Hence, for any $t > 0$, by (1.6), $u(x,t)$ is not defined at $x = 0$ if $N > 1$. Consequently, the generalized function $\delta(x)$ can not act on the function $f(u(x,t))$. It follows that $u(x,t)$ can not satisfy the equation (1.1) in any sense unless the space dimension is equal to 1. In order to model the described phenomenon in several space dimensions, we recall that

$$\delta(x) = \frac{d}{dx} H(x),$$

where $H(x)$ is the Heaviside function. In several space dimensions if $\Gamma = \partial A$, where A is a subdomain of Ω, the natural generalization of the reaction term should be to replace $\delta(x)$ by

$$\nabla_{n(x)} \, \chi_A(x),$$

where $\nabla_{n(x)}$ denotes the inward normal derivative at the boundary $x \in \partial A$ and $\chi_A(x)$ is the characteristic function. Then for any continuous function $g(x)$ we have (assume $A \bigcap \partial\Omega = empty$):

$$\int_\Omega \nabla_{n(x)} \chi_A(x) g(x) dx = -\int_\Omega \chi_A(x) (g(x) n_i(x))_{x_i} dx$$

$$= \int_{\partial A} g(x)ds,$$

where we have extended the normal vector $\{n(x)\}$ to a vector field defined in the whole domain Ω. Thus the functional (1.4) is the natural extension for Γ to be an arbitrary C^2-hypersurface in Ω.

The solution of the problem (1.1)-(1.4) is defined as follows:

Definition: A continuous function $u(x,t)$ defined on $\bar{Q}_T$ is said a solution of (1.1)-(1.4), if $u(x,t)$ satisfies

$$\int\int_{Q_T} [u\psi_t + u\Delta\psi]dxdt - \int_0^t \int_\Gamma f(u(x,t))\psi(x,t)dsdt + \int_\Omega u_0(x)\psi(x,0)dx = 0$$

for any test function $\psi(x,t) \in C^{2,1}(\bar{Q}_T)$ with $\psi(x,t) = 0$ on S_T and $\psi(x,T) = 0$.

It is easy to see (cf. [10]), by choosing $\psi(x,t)$ properly, that (1.1)-(1.4) is equivalent to the following :

$$u_t - \Delta u = 0, \qquad in\ Q_T\backslash\Gamma; \tag{1.7}$$

$$u(x,t) = 0, \qquad (x,t) \in S_T; \tag{1.8}$$

$$u(x,t) = u_0(x), \qquad x \in \Omega; \tag{1.9}$$

and on the interface $\Gamma_T = \Gamma \times [0,T]$

$$u^-(x,t) = u^+(x,t), \tag{1.10}$$

$$\frac{\partial u^-(x,t)}{\partial n(x)} - \frac{\partial u^+(x,t)}{\partial n(x)} = f(u(x,t)), \tag{1.11}$$

where $n(x)$ is the normal direction at $x \in \Gamma$ and

$$\frac{\partial u^\pm(x,t)}{\partial n(x)} = \lim_{z\to 0\pm}[u(x+zn(x),t) - u(x,t)].$$

This is a reformulation of the problem (1.1)-(1.4) in a more traditional way. In the sequel we shall study the probelm in the form of (1.7)-(1.11).

In the present paper we shall study the global solvability and finite time blowup for the problem (1.1)-(1.4). Comparing the reaction-diffusion equation

$$u_t - \triangle u = u^p, \ (p > 1), \tag{1.12}$$

with ours, the equation (1.1) illustrates some interesting differences. For example, it is known that in one space dimension for a regular reaction-diffusion equation, the blowup set depends on the sign of the derivative of initial data. It may be a single or several points. However, for the solution of (1.1)-(1.4) we show that for $f(u) = u^p$ the blowup set consists of a single point without any restriction on the initial value. Moreover, we show that the blowup rate is

$$u(x,t) \sim C(T-t)^{-\frac{1}{2(p-1)}}$$

as t tends to the blowup time T. This growth rate is the same as that for the heat equation with a nonlinear boundary condition (cf. [6] for N=1):

$$\frac{\partial u(x,t)}{\partial n(x)} = u^p. \tag{1.14}$$

This is not surprising if one thinks of the equivalent problem (1.7)-(1.11). Therefore, our problem (1.1)-(1.4) is more or less similar to the heat equation with a nonlinear boundary condition. The global solvability is based on various *a priori* energy estimates. The blowup property in finite time is derived by using Levine's convexity method ([11]). By exploiting the maximum principle, we establish that the blowup occurs only at the interface for the case of one space dimension.

The plan of the paper is as follows. The following section deals with local and global existence. The finite time blowup is shown in Section 3. Section 4 is devoted to the study of the blowup rate and blowup set.

2. Local and Global Solvability

The following conditions on the data are always assumed throughout the paper:
H(A): $u_0(x) \in C^\alpha(\bar{\Omega}) \bigcap C^{2+\alpha}(\Omega \backslash \Gamma)$ with $u_0(x) = 0$ on $\partial\Omega$. The function $f(u)$ is differentiable in $[0, \infty)$. There is no intersection point in which the normal of the hypersurface Γ and the normal of the boundary of the domain Ω lies in the same direction.

The local existence is quite standard. There are many ways to prove it. Here we sketch the proof via the Schauder fixed point theorem. Let

$$K = \{v(x,t) : v(x,t) \in C(\bar{\Omega}), ||v(x,t)||_0 \leq K_0\}.$$

If we replace the function $f(u(x,t))$ by $f(v(x,t))$ in (1.11) and solve (1.7)-(1.11), it is a linear diffraction problem and then there exists a unique solution (cf. [10])

$$u(x,t;v) \in C^\alpha(\bar{Q}_T) \bigcap C^{2+\alpha, 1+\frac{\alpha}{2}}(Q_T \backslash \Gamma_T).$$

Moreover, it is differentiable up to the interface Γ_T since Γ is C^2-hypersurface. Then we can define a mapping

$$M : K \to C^\alpha(\bar{\Omega}).$$

One can show that the mapping M is continuous and compact. Moreover, it can be shown that M maps K into itself, provided that T is small. The Schauder fixed point theorem gives a fixed point which is a solution of (1.7)-(1.11). The uniqneness is also easy to prove. Thus, we have

Theorem 2.1: Under the condition H(A), the problem (1.1)-(1.4) admits a unique solution for some $T > 0$.

To obtain a global solution, we need to derive *a priori* bound of $u(x,t)$. This can be achieved by energy estimates.

Theorem 2.2: Under the assumption H(A), if there exists a constant C such that

$$f(u)u \leq C[u^2 + 1],$$

then the problem (1.1)-(1.4) admits a global solution.

Proof: Let $m \geq 2$ be even. We multiply the equation (1.1) by u^{m+1} and integrate over Q_T, we have

$$
\begin{aligned}
&\frac{1}{m+2}\int_\Omega u^{m+2}(x,t)dx + (m+1)\int\int_{Q_t} u^m u_x^2 dxdt \\
&= \frac{1}{m+2}\int_\Omega u_0^{m+2}(x)dx + \int_0^t\int_\Gamma f(u)u^{m+1}dsdx \\
&\leq \frac{1}{m+2}\int_\Omega u_0^{m+2}(x)dx + C\int_0^t\int_\Gamma [u^{m+2}+u^m]dsdt \\
&\leq \frac{1}{m+2}\int_\Omega u_0^{m+2}(x)dx + (m+2)C + C\int_0^t\int_\Gamma u^{m+2}dsdt,
\end{aligned} \tag{2.1}
$$

where at the final step we have used Hölder's and Young's inequalities.

Now we need the following trace-type interpolation inequality (cf. [12], page 265): For $u \in H^1(\Omega)$,

$$\int_\Gamma u^2 dx \leq \varepsilon \int_\Omega u_x^2 dx + \frac{C}{\varepsilon}\int_\Omega u^2 dx.$$

Applying the above inequality for $v(x,t) = u^{\frac{m+2}{2}}(x,t)$ to the final term in (2.1), we have

$$
\begin{aligned}
&\frac{1}{m+2}\int_\Omega u^{m+2}(x,t)dx + (m+1)\int\int_{Q_t} u^m u_x^2 dxdt \\
&\leq \frac{1}{m+2}\int_\Omega u_0^{m+2}(x)dx + (m+2)C + C\varepsilon\int_0^t\int_\Omega [u^{\frac{m+2}{2}}]_x^2 dxdt + \frac{C}{\varepsilon}\int_0^t\int_\Omega u^{\frac{m+2}{2}}dxdt.
\end{aligned}
$$

It follows by taking $\varepsilon = \frac{m+1}{2(m+2)^2 C}$ that

$$
\begin{aligned}
&\frac{1}{m+2}\int_\Omega u^{m+2}(x,t)dx + \frac{(m+1)}{2}\int\int_{Q_t} u^m u_x^2 dxdt \\
&\leq \frac{1}{m+2}\int_\Omega u_0^{m+2}(x)dx + (m+2)C + C(m+2)\int_0^t\int_\Omega u^{\frac{m+2}{2}}dxdt,
\end{aligned}
$$

where the constant C is independent of m. By using Alikakos' iteration technique (cf. [1]), we can deduce

$$||u(\cdot,t)||_{L^\infty(\Omega)} \leq C.$$

Now we can apply $W_p^{2,1}$-estimates to obtain

$$||u||_{C^{1+\alpha,\frac{1+\alpha}{2}}(\bar{\Omega}\backslash\Gamma)} \leq C,$$

where $\alpha \in (0,1)$ is arbitrary and C depends only on the known data. Finally, by the standard boostrap argument we can establish existence on Q_T for any $T > 0$. This completes our proof.

The next two results are concerned with comparison principle and the continuous dependence upon the known data.

Theorem 2.3: Let $u_0(x)$ and $u_0^*(x)$ be two initial data which satisfy the hypothesis H(A). Let $u(x,t)$ and $u^*(x,t)$ be the corresponding solutions in Q_T. Then, if $u_0(x) \geq u_0^*(x)$ on $\bar{\Omega}$,

$$u(x,t) \geq u^*(x,t)$$

for all $(x,t) \in \bar{Q}_T$.

Proof: This can be shown by the classical maximum principle. Indeed, if $w(x,t) = u(x,t) - u^*(x,t)$, then $w(x,t)$ satisfies the heat equation (1.7). Moreover, $w(x,t) \geq 0$ on the parabolic boundary $\partial_p Q_T$. We claim that $w(x,t)$ can not attain its negative minimum on the interface Γ_T. To see this we note that on the interface Γ_T, $w(x,t)$ satisfies

$$w^-(x,t) = w^+(x,t), \quad \frac{\partial w^-(x,t)}{\partial n(x)} - \frac{\partial w^+(x,t)}{\partial n(x)} = f'(\theta(x,t))w(x,t), \tag{2.2}$$

where $\theta(x,t)$ lies between $u(x,t)$ and $u^*(x,t)$.

We first assume that $u_0(x) > u_0^*(x)$ on Γ. Then the continuity of the solution implies that there exists a $T_0 > 0$ such that $w(x,t) > 0$ on $\Gamma \times [0, T_0]$. Let $T^* = sup\{t : w(x,t) > 0,\ 0 \leq \tau \leq t,\ x \in \Gamma\}$. If $T^* \geq T$, nothing needs to be proved. If $T^* < T$, then $w(x,t) > 0$ for all $(x,t) \in \Gamma \times [0, T^*)$ and there exists a $x_0 \in \Gamma$ such that $w(x_0, T^*) = 0$. The maximum principle implies $w(x,t) \geq 0$ in $\bar{Q}_{T^*}$. Since

$w(x_0, T^*) = 0$, it follows that on Q_{T^*}, $w(x,t)$ attains its minimum at (x_0, T^*). Hopf's lemma implies that

$$\frac{\partial w^-(x_0,T^*)}{\partial n(x_0)} < 0, \quad \frac{\partial w^+(x_0,T^*)}{\partial n(x_0)} > 0.$$

This contradicts the condition (2.2) at (x_0, T^*). To remove the strict inequality between the two initial values, we can use $u_0(x) + \varepsilon$ as the initial value which satisfies the strict inequaliy $u_0(x) + \varepsilon > u_0^*(x)$. Then we have

$$u_\varepsilon(x,t) \geq u^*(x,t).$$

By taking the limit, we can obtain the desired result.

Corollary 2.4: If $u_0(x) \geq 0$, then $u(x,t) \geq 0$ on $\bar{Q}_T$.

Theorem 2.5: Let $u_0(x)$ and $u_0^*(x)$ be two initial data. Let $u(x,t)$ and $u^*(x,t)$ be the corresponding solutions in Q_T. Then

$$||u - u^*||_0 \leq C||u_0 - u_0^*||_0.$$

Proof: By Green's representation, we have

$$u(x,t) = \int_\Omega G(x,y;t,0)u_0(y)dy + \int_0^t \int_\Gamma G(x,y;t,\tau)f(u(y,\tau))dsd\tau$$

and similarly for $u^*(x,t)$. Note that

$$|G(x,y;t,\tau)| \leq C[\frac{1}{\sqrt{t-\tau}}]^{-N} exp\{-c\frac{|x-y|^2}{t-\tau}\}$$

and

$$\int_\Gamma [\frac{1}{\sqrt{t-\tau}}]^{-N} exp\{-c\frac{|x-y|^2}{t-\tau}\}ds \leq \frac{C}{\sqrt{t-\tau}}.$$

It follows that

$$||u - u^*||_0 \leq C||u_0 - u_0^*||_0 + C\int_0^t \frac{||u-u^*||_0}{\sqrt{t-\tau}}d\tau.$$

Consequently, Gronwall's inequality yields the desired inequality.

3. Blowup Property

To prove the blowup property, in addition to the condition H(A), we need the following hypotheses:

H(B): With $F(u) = \int_0^u f(s)ds$,

(1) $-\frac{1}{2}\int_\Omega u_{0x}^2 dx + \int_\Gamma F(u_0(x))ds > 0$;

(2) $f(u) \geq 0$ for $u \geq 0$. There exists a constant $a > 0$ such that $f(u)u \geq 2(1+a)F(u)$ for $u \geq 0$,

Theorem 3.1: Under the assumptions H(A) and H(B), the solution of (1.1)-(1.4) blows up in finite time.

Proof: The method follows from H. Levine's convexity method (cf. [11]). Multiplying the equation (1.1) by $u(x,t)$ and $u_t(x,t)$, respectively, we have, after integrating over Ω, that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2 dx + \int_\Omega u_x^2 dx = \int_\Gamma f(u)u ds. \tag{3.1}$$

$$\int_\Omega u_t^2 dx + \frac{1}{2}\frac{d}{dt}\int_\Omega u_x^2 dx = \int_\Gamma f(u)u_t ds. \tag{3.2}$$

We define

$$J(t) = -\frac{1}{2}\int_\Omega u_x^2 dx + \int_\Gamma F(u(x,t))ds,$$

From (3.2), we have

$$J(t) - J(0) = \int_0^t \int_\Omega u_t^2 dx dt. \tag{3.3}$$

Let

$$I(t) = \int_0^t \int_\Omega u^2 dx dt + A,$$

where A is a constant to be specified later.

Then we see

$$I'(t) = \int_\Omega u(x,t)^2 dx,\ I''(t) = \frac{d}{dt}\int_\Omega u^2 dx.$$

From (3.1) and the assumption H(B), one obtains

$$I''(t) \geq 4(1+a)J(t).$$

For $\alpha > 0$ sufficiently small, to be chosen later, we calculate

$$\begin{aligned}
&I''(t)I(t) - (1+\alpha)I'(t)^2 \\
&\geq 4(1+a)J(t)I(t) - (1+\alpha)I'(t)^2 \\
&= 4(1+a)[J(0) + \int_0^t \int_\Omega u_t^2 dxdt][\int_0^t \int_\Omega u^2 dxdt + A] - (1+\alpha)I'(t)^2.
\end{aligned}$$

Now

$$\begin{aligned}
I'(t) &= \int_\Omega u^2 dx \\
&= 2\int_0^t \int_\Omega uu_t dxdt + \int_\Omega u_0^2 dx \\
&\leq 2[\int_0^t \int_\Omega u^2 dxdt]^{\frac{1}{2}}[\int_0^t \int_\Omega u_t^2 dxdt]^{\frac{1}{2}} + \int_\Omega u_0^2 dx
\end{aligned}$$

by Hölder's inequality.

It follows by Cauchy's inequality that

$$I'(t)^2 \leq 4(1+\varepsilon)\int_0^t \int_\Omega u^2 dxdt \int_0^t \int_\Omega u_t^2 dxdt + (1+\frac{1}{\varepsilon})[\int_\Omega u_0^2(x)dx]^2.$$

Hence,

$$\begin{aligned}
&I''(t)I(t) - (1+\alpha)I'(t)^2 \\
&\geq 4(1+a)\{1 - \frac{(1+\alpha)(1+\varepsilon)}{1+a}\}[\int_0^t \int_\Omega u_t^2 dxdt][\int_0^t \int_\Omega u^2 dxdt] \\
&\quad +4(1+a)A\int_0^t \int_\Omega u_t^2 dxdt + 4(1+a)J(0)[\int_0^t \int_\Omega u^2 dxdt + A] \\
&\quad -(1+\alpha)(1+\frac{1}{\varepsilon})[\int_\Omega u_0^2 dx]^2.
\end{aligned}$$

Consequently, if we choose α and ε sufficiently small such that

$$(1+\alpha)(1+\varepsilon) < 1+a$$

while we take A to be large enough such that ($J(0) > 0$ by H(B)),

$$A \geq \frac{(1+\alpha)(1+\frac{1}{\varepsilon})[\int_\Omega u_0^2 dx]^2}{4(1+a)J(0)},$$

then we obtain

$$I''(t)I(t) - (1+\alpha)I'(t)^2 \geq 0.$$

This implies that $I(t)^{-\alpha}$ must become zero in finite time. That is $u(x,t)$ will become infinite in finite time. Moreover, the blowup time, say T, obeys the following estimate (cf. [11])

$$T \leq \frac{(1+\alpha)(1+\frac{1}{\varepsilon})\int_\Omega u_0^2 dx}{4(1+a)\alpha J(0)}.$$

Corollary 3.2: For $f(u) = u^p$ with $p > 1$, the solution will blowup in finite time, provided that the initial data satisfies H(B)-(1).

Proof: It is easy to verify that the assumption H(B)-(2) holds for $f(u) = u^p$. The result follows from Theorem 3.1.

In order to obtain a result for exponential nonlinearities, we need the condition:

H(B): (3) Let $dist\{\Gamma, \partial\Omega\} > 0$. Assume that $\Delta u_0 \geq 0$ on $\Omega \backslash \Gamma$ and $u_0(x) \geq 2(1+a)$ in a neighborhood of Γ.

Corollary 3.3: For $f(u) = e^u$, in addition to the assumption H(B), if $u_0(x)$ satisfies H(B)-(3), then $u(x,t)$ blows up in finite time.

Proof: By the condition H(B)-(3) and the maximum principle, we can obtain $u_t \geq 0$. Therefore, there exists a neighborhood, say $K(\Gamma_T)$, of Γ_T such that $u(x,t) \geq 2(1+a)$. It follows that for $(x,t) \in K(\Gamma_T)$

$$ue^u \geq 2(1+a)(e^u - 1).$$

Thus, the assumption H(B)-(2) holds in $K(\Gamma_T)$, which is sufficient from the proof of Theorem 3.1. Hence, Theorem 3.1 implies the desired result.

4. Blowup Rate Estimates and Blowup Set

Throughout this section, T will be denoted the blowup time.

Theorem 4.1: For $f(u) = u^p$ with $p > 1$, there exists a constant $C > 0$ such that

$$\sup_{\bar{Q}_T} u(x,t) \geq C(T-t)^{-\frac{1}{2(p-1)}}.$$

Proof: The proof is based on the argument of [15]. For any $z \in (0,T)$, we have

$$u(x,t) = \int_\Omega G(x,y;t,z)u(y,z)dy + \int_z^t \int_\Gamma Gu^p ds d\tau.$$

Let

$$A(t) = \sup_{z \leq \tau \leq t} ||u(\cdot,\tau)||_{L^\infty(\Omega)}.$$

Then

$$\begin{aligned} A(t) \quad & \leq A(z) + C\int_z^t \frac{1}{\sqrt{t-\tau}} d\tau A(t)^p \\ & = A(z) + 2C\sqrt{t-z}A(t)^p. \end{aligned} \tag{4.1}$$

It follows that for all $t \in (z,T)$

$$(t-z)^{\frac{1}{2}} \geq \frac{\beta - A(t)}{C\beta^p}, \tag{4.2}$$

provided that $\beta > A(z)$ and $0 < z < T$. Otherwise, from (4.1) we would have $A(t) \leq \beta$ for all $t \in (z,T)$, which is a contradiction since $u(x,t)$ blows up at $t = T$. In particular, if we take $\beta = 2A(z)$ in (4.2) and let $t \to T$ to obtain

$$A(z) \geq C(T-z)^{-\frac{1}{2(p-1)}}.$$

Theorem 4.2: For $f(u) = u^p$ with $p > 1$, if $dist\{\Gamma, \partial\Omega\} > 0$, then

$$u(x,t) \sim \frac{C}{(T-t)^{\frac{1}{2(p-1)}}}, \quad \text{as } t \to T.$$

Proof: By Green's representation,

$$u(x,t) = \int_\Omega G(x,y;t,0)u_0(y)dy + \int_0^t \int_\Gamma G u^p ds d\tau.$$

Assume

$$u(x,t) \sim C(T-t)^{-\beta}$$

as t tends to T. Then

$$C(T-t)^{-\beta} \sim \int_0^t \int_\Gamma G C^p (T-\tau)^{-\beta p} ds d\tau. \tag{4.3}$$

It is known that

$$G(x,y;t,\tau) = Z(x,y;t,\tau) - g(x,y;t,\tau),$$

where $Z(x,y;t,\tau)$ and $g(x,y;t,\tau)$ are the same as in Section 1.

By the results of [2], there exist constants $c_i, i = 1,2,3,4$, such that

$$c_1(t-\tau)^{-\frac{N}{2}} exp\{-c_2 \frac{|x-y|^2}{t-\tau)}\} \leq Z(x,y;t,\tau) \leq c_3(t-\tau)^{-\frac{N}{2}} exp\{-c_4 \frac{|x-y|^2}{t-\tau}\}.$$

Moreover, $g(x,y;t,\tau)$ is bounded on Γ since Γ lies inside of Ω (cf. [10], page 411).

It is easy to see that there exists two constants C_1 and C_2 such that

$$\frac{C_1}{\sqrt{t-\tau}} \leq \int_\Gamma (t-\tau)^{-\frac{N}{2}} exp\{-C \frac{|x-y|^2}{t-\tau)}\} ds \leq \frac{C_2}{\sqrt{t-\tau}}.$$

It follows that

$$\int_0^t \int_\Gamma G C^p (T-\tau)^{-\beta p} ds d\tau \sim C \int_0^t \sqrt{t-\tau}(T-\tau)^{-\beta p} d\tau.$$

Introduce a new variable

$$s = \frac{\sqrt{t-\tau}}{\sqrt{T-t}},$$

then

$$\int_0^t \sqrt{t-\tau}(T-\tau)^{-\beta p} d\tau$$
$$= (T-t)^{\frac{1}{2}-\beta p} \int_0^{\sqrt{\frac{t}{T-t}}} \frac{1}{(1+s^2)^{\beta p}} ds.$$

Comparing the exponents of the factor $T-t$ in (4.3), we have

$$\beta = \frac{1}{2(p-1)},$$

since the integral

$$\int_0^\infty \frac{1}{(1+s^2)^{\frac{p}{2(p-1)}}} ds$$

is convergent.

We now study the blowup set. We are only able to deal with the problem in one space dimension. In this case the model equation (1.1) deduces into (1.5) if one assumes that the interface $\Gamma = \{x = 0\}$. In the following, we take the domain $Q_T = (-1,1) \times (0,T)$.

Recall that a point x_0 is said to be a blowup point if there exists a sequence (x_n, t_n) with $t_n < T$ such that $(x_n, t_n) \to (x_0, T)$ as $n \to \infty$ and

$$\lim_{n\to\infty} u(x_n, t_n) = \infty.$$

Theorem 4.3: Assume $G(z) = \int_z^\infty \frac{ds}{f(s)}$ is defined on $[z_0, \infty)$ with some $z_0 > 0$ and $f'(s) \geq 0,\ f''(s) > 0$ for $s > 0$. Moreover, for $s \geq s_0 > 0$ assume there exists a constant $K(s_0) > 0$ such that

$$|\frac{f'(s)}{f''(s)}| \leq K(s_0),$$

then the blowup occurs only at $x = 0$.

Proof: First of all, it is clear that $x = 0$ is a blowup point.

Let $Q_T^* = [0,1] \times [0,T)$. On Q_T^*, we define a new function

$$w(x,t) = u(x,t) + u(-x,t).$$

It is easy to see that $w(x,t)$ satisfies

$$w_t - w_{xx} = 0, \qquad (x,t) \in Q_T^*; \tag{4.4}$$

$$w_x(0,t) = -f(\frac{w(0,t)}{2}), \qquad 0 \le t < T; \tag{4.5}$$

$$w(1,t) = 0, \qquad 0 \le t < T; \tag{4.6}$$

$$w(x,0) = w_0(x) \equiv u_0(x) + u_0(-x),\ 0 \le x \le 1. \tag{4.7}$$

For a fixed $\alpha > 0$, we can choose t_0 close T such that

$$w(0,t) \ge \alpha,\ t_0 \le t < T.$$

The monotonicity of $f(s)$ implies that $w_x(0,t) \le -f(\frac{\alpha}{2})$ in $t_0 \le t < T$. Consequently, there exists a neighborhood $N_a(x,t) = \{0 \le x \le a,\ t_0 \le t < T\}$ such that

$$w_x(x,t) \le -\frac{1}{2} f(\frac{\alpha}{2})$$

for all $(x,t) \in N_a(x,t)$.

On $N_a(x,t)$, we introduce an auxiliary function

$$J(x,t) = w_x(x,t) + v(x,t)f(w),$$

where $v(x,t)$ will be specified later. We would like to show that there exists a bounded function $v(x,t)$ such that $J(x,t) \le 0$. To this end we first choose $v(x,t)$ satisfying

$$v(0,t) = v(a,t) = 0,\ t_0 \le t < T.$$

Moreover, we choose $v_0(x) = v(x, t_0)$ to be small such that

$$||v_0(x)||_0 \leq \frac{f(\frac{\alpha}{2})}{2||f(w(x,t_0))||_0}.$$

With the above $v(x,t)$, we see that on the parabolic boundary $\partial_p N_a(x,t)$

$$J(x,t) \leq 0.$$

By a direct calculation, one finds

$$J_t - J_{xx} = [v_t - v_{xx}]f(w) - 2v_x f'(w)w_x - vf''(w)w_x^2.$$

Since $f''(w) > 0$, the Cauchy inequality implies

$$|2v_x f'(w)w_x| \leq vf''(w)w_x^2 + \frac{f'(w)}{f''(w)}\frac{v_x^2}{v}.$$

It follows from the assumptions that

$$|2v_x f'(w)w_x| \leq vf''(w)w_x^2 + \frac{K(s_0)}{f(\frac{\alpha}{2})}\frac{v_x^2}{v}f(w).$$

Consequently,

$$J_t - J_{xx} \geq [v_t - v_{xx} - \frac{K_0 v_x^2}{v}]f(w),$$

where $K_0 = \frac{K(s_0)}{f(\frac{\alpha}{2})}$. Now we consider the following problem

$$v_t - v_{xx} - \frac{K_0 v_x^2}{v} = 0, (x,t) \in N_a(x,t), \tag{4.8}$$

$$v(0,t) = v(a,t) = 0, \quad t_0 \leq t \leq T, \tag{4.9}$$

$$v(x,t_0) = v_0(x), \qquad 0 \leq x \leq a. \tag{4.10}$$

Note that the equation (4.8) is singular at the boundary. However we claim that the problem (4.8)-(4.10) admits a bounded solution on $[t_0, T]$. For this purpose we consider the following regularized problem:

$$v_t - v_{xx} - \frac{K_0 v_x^2}{v} = 0,\ (x,t) \in N_a(x,t), \tag{4.11}$$

$$v(0,t) = v(a,t) = \varepsilon, \quad t_0 \le t \le T, \tag{4.12}$$

$$v(x,t_0) = v_0(x) + \varepsilon, \quad 0 \le x \le a. \tag{4.13}$$

The maximum principle implies

$$\varepsilon \le v(x,t) \le M = ||v_0||_0.$$

Since the consistency conditions hold at the corner points $(0,0)$ and $(a,0)$, the classical theory implies that the problem (4.11)-(4.13) possesses a unique classical solution $v(x,t) \in C(\bar{N}_a(x,t)) \bigcap C^{2+\beta,1+\frac{\beta}{2}}(N_a)$. Moreover, again by the maximum principle, $\{v_\varepsilon(x,t)\}$ is a monotone decreasing sequence. It follows that $\{v_\varepsilon(x,t)\}$ converges to a function, denoted by $v(x,t)$, uniformly on $N_a(x,t)$. Furthermore, the Schauder theory yields

$$||v_\varepsilon||_{C^{2+\beta,1+\frac{\beta}{2}}(N')} \le C,$$

where N' is any compact subdomain of $N_a(x,t)$ and C depends only on M, a and K_0, but not on ε.

The standard compactness argument implies that the limit function $v(x,t)$ solves the problem (4.8)-(4.10). Finally, we choose $v(x,t)$ to be the solution of (4.8)-(4.10), then on $N_a(x,t)$

$$J_t - J_{xx} \ge 0.$$

Combining the initial and boundary conditions, we obtain by the maximum principle that

$$J(x,t) \le 0,\ (x,t) \in N_a(x,t).$$

Now we integrate the above inequality over $[0, a]$ to obtain

$$\int_0^a \frac{w_x}{f(w)} dx \leq -\int_0^a v(x,t)dx = -C(t).$$

It follows that

$$\int_{w(0,t)}^{w(a,t)} \frac{1}{f(s)} ds \leq -C(t).$$

That is

$$G(w(a,t)) - G(w(0,t)) \geq C(t).$$

Since $x = 0$ is a blowup point, we have $G(w(0,t)) \to 0$ as $t \to T$. As $v(x,t) \geq 0$ and $v(x,t) \neq 0$, we have for any $t \in [t_0, T]$

$$C(t) = \int_0^a v(x,t)dx > 0,$$

It follows that

$$\lim_{t \to T} \inf_{t_0 \leq \tau \leq t} G(w(a,\tau)) \geq C(T) > 0$$

which implies

$$\lim_{t \to T} \sup_{t_0 \leq \tau \leq t} w(a,\tau) < \infty.$$

For any $x_0 \in (0, a)$, we can repeat the above procedure to conclude that x_0 is not a blowup point. On the interval $[a, 1]$, we can use the maximum principle to get

$$\lim_{t \to T} \sup_{t_0 \leq \tau \leq t} w(x,t) < \infty.$$

This completes our proof.

Corollary 4.4: For $f(u) = u^p$ or $f(u) = e^u$, the blowup set consists of the single point $x = 0$.

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