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and
Computable Economies

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Abstract: This paper studies economic equilibrium theory with a "uniformity principle" constraining the magnitudes (prices, quantities, etc.) and the operations (to perceive, evaluate, choose, communicate, etc.) that agents can use.

For the special case of computability constraints, all prices, quantities, preference relations, utility functions, demand functions, etc. are required to be computable by finite algorithms. Then we obtain sharper versions of several traditional assertions on utility representation, existence of consumer demand functions, the fundamental welfare theorems, characterizations of market excess demands, and others. These positive results hold despite the fact that commodity and price spaces are no longer topologically complete.

On the other hand, we give "computable counterexamples" to several traditional assertions, including the existence of a competitive equilibrium.

The results can be interpreted as possibility and impossibility results in both computability-bounded rationality and in computational economics.

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Waitress. How much pizza would you like?

Homo Œconomicus, Jr. □.

Waitress. How much?

Homo Œconomicus, Jr. I don't have a word for the number.

Waitress. Would you like a drink with that?

Homo Œconomicus, Jr. □.

Waitress. Well?

Homo Œconomicus, Jr. I can't figure it out.

Junior has neither the communication abilities nor the decision making abilities of his famous father. But then who does?

1. INTRODUCTION

How relevant to economics are all the real numbers and functions of mathematics? It is hard to take seriously the idea that economic agents can perceive, work with, and communicate arbitrary real numbers. There are continuumly many real numbers (and even more real functions and relations), but a human being can reason with and communicate with only the countably many words and sentences that his finite alphabet can form. In particular, it is unrealistic to assume agents can perform arbitrarily complex operations with these arbitrarily complex magnitudes.

Of course, we knew all along that the fantastic communicating and calculating abilities in our models were only approximations to some real world abilities.[1] But to understand what becomes of our theories when more realistic human capabilities are assumed, we must incorporate such limitations in our models from the start. If the results don't change, we should be able to prove that. If they do change, we should know it.

In this paper, we propose a framework within which many models of bounded rationality can be developed. We also examine a particular model based on computability theory: agents can work only with "computable real numbers," "computable relations," and "computable functions."

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(1) Or did we? Why did we spend so much effort proving existence of equilibria, based on arbitrary numbers and functions, if approximate equilibria are the only realistic ones?

One way to view our framework and our specific models[2] is to link them to theories of bounded rationality. While we do not propose specific decision making rules of thumb, we nevertheless take seriously the idea that agents are limited in what they can perceive, work with, and communicate. Our approach limits both magnitudes and operations in a parallel fashion. We mention two distinct applications, one to definable economies and the other to computable economies. In each of these applications, the limitations we impose on agents' abilities can be considered minimal restrictions; they are not as strong, for example, as those found in Simon (1959) and others, for specific situations.

For the second application, in computable economies we examine the validity of several traditional assertions, from utility representation to existence of general equilibrium. For some of them we obtain sharper versions, others are refuted by "computable counterexamples." Our assertions and counterexamples can also be interpreted as possibility or impossibility results for economic data.

2. BOUNDED RATIONALITY OVERVIEW

Usable magnitudes. What numbers can agents perceive? Think about? Choose? Communicate? We would want most of our models to include the integers. And rational numbers are used to solve many practical economics problems that consumers and producers face in the real world. Beyond those, the numbers we allow in our model will help determine what particular type of bounded rationality we obtain.

Whatever numbers we use, we expect them to satisfy these minimal properties:

U.a) Each magnitude must be specified by a simple rule that allows it to be used, thought of, and communicated by finitistic means, without an uncountable vocabulary. The natural numbers, for example, would satisfy this requirement.

U.b) The system of magnitudes must allow divisibility. While the natural number magnitudes satisfy (a), they do not allow the divisibility required for commodities and prices in most economic models.

U.c) The set of magnitudes must include the results of all simple calculations and operations. For example, if x and y are allowable magnitudes, then $x+y$, $x \cdot y$, $x-y$ and x/y (for $y \neq 0$) should also be allowable. Similarly, the results of other

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[2] The computability models of this paper, and the definability models in Richter and Wong (1996b).

simple operations that we expect economic agents to perform: calculation of square of roots (or more generally, extraction of roots of polynomials), calculation of circular areas, exponentiation, etc. While the rational numbers satisfy (a) and (b), they do not admit extraction of the roots, or the use of π required here.

U.d) The system of magnitudes must allow comparability of magnitudes in a simple manner. In particular, each magnitude must have a simple rule that allows determination of whether any rational number above it is above it, and whether any rational number below it is below it. (A consumer has to determine whether a particular bundle is affordable within her budget; a producer has to determine whether one activity gives more profit than another.)

Different interpretations of "simple" in (U.a,c,d) will give different bounded rationality types. In Richter and Wong (1996b), for example, "simple" means definable by a formula in a certain kind of language. In Sections 3, 4, and 5 below, "simple" means Turing computable.

Usable operations. Definability restrictions and computability restrictions on magnitudes are examples that capture different aspects of human perceptual, cognitive, decision making, and communicating limitations. But if we impose such limitations on the numbers our agents can use, then we should expect similar limitations to hold for consumer preference relations and producer technology relations, as well as for any functions that consumers and producers work with. In the opposite direction, many writers have discussed the need to impose limitations on consumer and producer relations and maximand functions. And we should expect these arguments to apply to the number system as well as to relations and functions.

We will therefore require a uniformity principle for our models: the same limitations must apply to both magnitudes and operations.

What do we mean when we say that the "same" limitations should be imposed on the number system as on the relations and functions? Relations and functions, after all, seem like very different objects than numbers. Nevertheless, numbers and functions can both be described as types of relations,[3] so any restriction on the allowable relations in our model can be viewed as applying to numbers and functions as well.

If, for example, we require all relations to be "definable" (by some formula of some specified language), this will restrict the numbers in our model to be definable numbers, and the functions to be definable functions. If we require all

(3) See Footnotes 13 and 19 below.

relations to be "computable," this will restrict the numbers and the functions to be computable. In this paper, we study the computability restriction. In Richter and Wong (1996b) we examine the definability restriction.

We are not claiming that restrictions as broad as the computability restriction of this paper, or the definability restriction in our paper (1996b)) capture all aspects of "bounded rationality" as the term has been used by other writers. As they emphasize, it is important to discover what rules of thumb and practical guidelines people and institutions use for making decisions in everyday life. Nevertheless, we hope that our framework and models show how some of the unrealistic properties of classical models can be avoided.

3. COMPUTABILITY OVERVIEW

In this part of this paper, we interpret "simple" in our usability properties (U.a,c,d) to mean computable by an algorithm or Turing machine. One justification for this interpretation is the widely accepted philosophical assertion, known as the Church-Turing Thesis (cf. Turing (1950)): any formal calculation possible for a human can be performed by a finite algorithm (Turing machine).

When we interpret "simple" in this way, properties (U.a,b,c,d) will force us to work with what Turing (1936) called the computable numbers.[4] And then our uniformity principle requires us to work with computable relations and functions. So a computable operation is usable by an economic agent, who can execute it through the simple rules of a finite algorithm.[5]

Thus we replace the usual real field $\mathbb{R}$ by the field $\mathbb{R}_c$ of computable real numbers. In our model, then, all prices, quantities, preference relations, utility functions, demand functions, etc. are required to be computable by finite algorithms. We note several useful consequences of this change from $\mathbb{R}$ to $\mathbb{R}_c$:

i) Every computable real is usable by our economic agents, who can use an algorithm to calculate its decimal digits to any desired degree of accuracy in a known amount of time, so the computable reals satisfy usability requirement (U.a).

ii) The set of computable reals is rich, containing, for example, all natural numbers, all rational numbers, all algebraic numbers, and many irrational numbers such as π, $\log 10$, $\sqrt{2}$, e, etc. (Cf. Pour-El and Richards (1989), p. 21.) Thus

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[4] **We'll see that, in fact, property (U.d) itself forces the computable reals on us.**

[5] **This does not restrict the length of the calculations, so it is a rather weak restriction.**

it contains almost all numbers that we use in practical mathematical analysis, applied economics, and the calculations of our daily lives. It more than satisfies usability requirement (U.b).

The computable real number field $(\mathbb{R}_c, 0, 1, <, +, \cdot)$ possesses many of the same natural field properties as the real number field; it is closed under natural operations such as summation, division, taking roots for polynomials of odd degrees, etc. That is, the computable real number field is real closed (cf. Rice (1954)).[6] It satisfies usability requirement (U.c). We'll see later that it also satisfies (U.d).

iii) Allowing only computable reals instead of arbitrary reals, we will avoid the impracticality of requiring agents to have communication and decision making vocabularies of uncountable size. Since there are only countably many computable reals (because there are only countably many simple rules (Turing machines)), they can be coded into integers. Indeed, there exist simple rules for coding them (cf. Moschovakis (1964a); see Definition 2 below). This permits our agents to communicate their trading terms by finite means, and allows the agents to use realistic trading processes for the agents. If an agent wants to buy $\pi = 3.1415\cdots$ amount of a commodity, for example, he only need tell the seller an integer code of π rather than attempt to state all the digits of $3.1415\cdots$.

iv) For applications to computational economics, it is often important to know whether a certain magnitude (e.g. equilibrium price) is computable by computer programs, and hence by finite algorithms. This amounts to knowing whether it belongs to $\mathbb{R}_c$.

So our commodity space will be the finite dimensional space $\mathbb{R}_c^l$ of computable reals $\mathbb{R}_c$, and our price space will be the nonnegative orthant $\mathbb{R}_{c+}^l$ of $\mathbb{R}_c^l$. We endow these spaces with the topology induced by the Euclidean norm. Similarly, we will assume agents' preferences to be computable, i.e. there exist finite algorithms that compute their (strict) preferences between pairs of commodity bundles. Also, we will assume (utility, demand, production, profit, etc.) functions f to be computable, i.e. there exist finite algorithms that transform elements x into $f(x)$. This formalizes the idea that these preferences and functions are generated by the operations that are usable by the bounded rational agents in our models.

Despite the useful properties (i) – (iv), our choice of computable reals car-

[6] This permits us to use Tarski's quantifier elimination method (1951), which carries over all the properties of the real number field that can be stated by first order logic sentences for the computable real number fields (cf. the discussion in Richter and Wong (1996b)).

ries some mathematical difficulties for proving theorems. Many familiar results from standard economic models rely for their proofs on the basic (topological) completeness property of the reals. But our number system, the computable reals, is *not* complete, when we endow it with the topology induced by the Euclidean distance. And that incompleteness extends to our commodity and price spaces: a Cauchy sequence of vectors in our price space or commodity space is not necessarily convergent. Our spaces do satisfy a weaker property, called recursive completeness (Rice (1954); see Fact 1, page 52),[7] but that is too weak for many fundamental theorems to hold in the context of recursive analysis: Weierstrass' Maximizer Existence Theorem fails (cf. Kreisel (1958)), and Brouwer's Fixed-Point Theorem fails (cf. Orevkov (1964) and Baigger (1985)). New assumptions are therefore required to obtain the classical analogies in our computability context.

Our main results have two interpretations. For computable economies, they sharpen or refute classical assertions for classical models; they also give possibility and impossibility results for computability of economic data:

1) We show that a preference is computable if and only if it can be represented by a computable real-valued function. Intuitively, this means that assuming a preference is derived from a logical-reasoning algorithm is equivalent to assuming that it is derived from a utility-computing algorithm.[8]

2) We find a counterexample to the general existence of a (computable) demand (i.e. preference-maximizing) bundle in a budget set (based on Kreisel's computable counterexample to Weierstrass' maximizer existence theorem). However, when the computable preference is convex, we re-establish the existence of a demand bundle. Moreover, we show computability of the demand functions generated by convex computable preferences. These positive computability results show that the demand behavior of a computable consumer can be generated by a purchase-algorithm.

3) We prove the two classical welfare theorems for computable economies. The first proof follows standard lines; the second proof employs a computable supporting hyperlane theorem (Theorem C-3, page 61 in Appendix C; cf. Wong (1996a, Theorem 2)) from recursive analysis. Thus we strengthen the standard

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(7) **Convergence holds for sequences that are generated by finite algorithms and that satisfy a computable analogue of the Cauchy criteria.**

(8) **Our proof for this result (1) is provided in Richter and Wong (1996a, Theorem 1).**

conclusion of the second welfare theorem with a computable supporting price. Such a price can be computed by a social planner, using an algorithm.

4) We prove a theorem characterizing computable market excess demand functions. This result is a computable analogue of Mas-Colell's theorem (1977). Our proof uses a simple effectivization of Wong's (1996b) refinement of Geanakoplos' (1984) method of constructing utility functions for a class of (Debreu's) excess demand functions.

5) We provide two "computable counterexamples" to equilibrium existence for both market excess demand functions and economies. The first example is constructed by using a "computable counterexample" to Brouwer's Fixed-Point Theorem in recursive analysis (Proposition 2, page 66; cf. Orevkov (1964), Baigger (1985)). The second example is obtained by combining the first one with our characterization result (4) for computable market excess demand functions. These examples raise doubts about the existence of general equilibrium in the context of bounded rationality.

These results have obvious implications in computational economics. For example, they verify computability for many processes: finding utility representation for preferences, optimizing satisfaction given convex preferences, finding supporting prices, finding economies for rationalizing given excess demand functions, etc. They also demonstrate nonexistence of algorithms for finding general equilibrium.

While we concentrate on exchange economies with finitely many commodities, our uniformity principle and computability restriction apply to more comprehensive models and to many other areas of economics and game theory.

Computability approaches to modeling bounded rationality have been increasingly used in economic theory (cf. Kramer (1974), Lewis (1985, 1987, 1988), Binmore (1987), Spear (1989), Anderlini (1990), Rustem and Velupillai (1990), Canning (1992) and Mihara (1994)). (For other approaches to modeling bounded rationality, see Radner (1981), Neyman (1985), Rubinstein (1986), Radner (1993), Cho (1995), etc.)

Among the earlier applications of recursive (computable) mathematical tools to economic questions, Lewis' general equilibrium framework (1987) is most closely related to ours: both of us take into account the computability of economic magnitudes (by using computable reals) and agents' preferences. However, there are two differences. First, Lewis uses computable notions to study decidability problems for general equilibrium theory, while we use them to model bounded rationality. Second, our computability notion for preference is weaker than Lewis', which is too restrictive (cf. Lewis (1987), Wong (1994)).

The following sections are organized as follows. In Section 4, we motivate and define our computability notions. Section 5 discusses our results (1-5). Section 6 discusses some implications of our results in computational economics. We conclude this paper in Section 7. Some proofs are given in Appendix A. Appendix B provides a double (classical and computable) characterization of aggregate excess demand functions. The recursive mathematics used here is self-contained; basic definitions and useful tools are given in Section 4 and Appendix C.

4. ALGORITHMS AND COMPUTABILITY NOTIONS

Here we will motivate and introduce the computability notions used in our framework. These notions are standard in recursive mathematics.

4.1 Turing Machines, Recursive Functions, and Algorithms

We need to define (i) the "computability" magnitudes and (ii) the "computability" operations on them that we allow the bounded rational agents in our model to employ. We will base both definitions on that of a Turing machine, or a partial recursive function. This widely accepted formalization of computability for calculations using natural numbers was developed by mathematical logicians (Church, Kleene, Post, Turing and others) in the 1930s. For applications in economics, see Kramer (1974), Lewis (1985, 1987, 1988), Binmore (1987), Spear (1989), etc.

Intuitively, a Turing machine is an unlimited memory digital computer that performs symbolic computations according to an in-stored program of finite-length. The notion of a Turing machine has been a standard model of human thinking in cognitive science; it has also been increasingly used in the formal theory of economic analysis. This use has been justified by the widely accepted philosophical assertion, known as the Church-Turing Thesis: any formal calculation possible for a human can also be performed by a Turing machine (cf. Turing (1950)). We accept this thesis, and use the computability of Turing machines, or partial recursive functions to define usability for the magnitudes and for the operations on the magnitudes.

We now give a brief intuitive description of a Turing machine. (For detailed descriptions, see e.g. Turing (1936), Kleene (1952) or Davis (1983).) A Turing machine M has a finite set Q of internal states, a blank tape (memory) that is divided into cells and is infinitely long on both sides, and a read-write head that scans the symbols drawn from a finite alphabet set and recorded on the tape. It performs computations as follows. An input is any string (i.e. finite sequence) of symbols from the alphabet set, and it is first written on the tape. The machine M starts in the designated initial state in Q and reads the leftmost symbol of the input string recorded on the tape. In each step, given the current internal state q and the symbol a currently read (treating "blank" as a "blank symbol"), M will (according to its in-stored program) transit to a new internal state q', replace a by a new symbol a', and move the read-write head left or right one cell. The computation halts if the machine enters the designated halting state, and the string of symbols recorded in the tape is called the output; otherwise the computation does not halt.

According to the Church-Turing Thesis, Turing machines capture the intuitive idea of computability on natural numbers. Now we note that Turing machines correspond to certain kinds of integer functions. Since inputs and outputs are finite strings from a finite alphabet set, they can be identified with natural numbers; conversely natural numbers can be represented by those finite strings. So without loss of generality we begin by restricting attention to computability of natural number functions. Thus we use the following standard definitions.

Let $\mathbb{N} = \{0, 1, 2, \cdots\}$. A *partial function* is a mapping $\phi : \mathrm{dom}(\phi) \to \mathbb{N}$, where $\mathrm{dom}(\phi)$ is contained in a l fold product $\mathbb{N}^l$ of $\mathbb{N}$. Although we only have: $\mathrm{dom}(\phi) \subseteqq \mathbb{N}^l$, it is customary to write "$\phi : \mathbb{N}^l \to \mathbb{N}$," and write "$\phi(n) \downarrow$" ("$\phi(n) \uparrow$") to indicate that ϕ is defined at n, i.e. "$n \in \mathrm{dom}(\phi)$" ("$n \notin \mathrm{dom}(\phi)$"). A *total function* is a partial function $\phi : \mathbb{N}^l \to \mathbb{N}$ that is defined on all of $\mathbb{N}^l$, i.e. $\phi(n) \downarrow$ for every $n \in N^l$. A *partial recursive function* is a partial function $\phi : \mathbb{N}^l \to \mathbb{N}$ that is *computable* by some Turing machine M, i.e. for every input $n \in \mathbb{N}^l$, the machine M will not halt on n if $\phi(n) \uparrow$, and M will halt and yield output $\phi(n)$ if $\phi(n) \downarrow$. A *recursive function* is a partial recursive function that is total. (There are other characterizations for recursive and partial recursive functions, e.g. Church's λ-calculus (1935), Kleene's recursion approach (1936), etc., which are known to be equivalent to the Turing-computability (cf. Kleene (1952)).)

We accept the Church-Turing Thesis that any algorithm a person can execute can be performed by a Turing machine, hence can be calculated by a partial recursive function. Therefore, we consider partial recursive functions as formalizations of algorithms that our bounded rational agents can use.

4.2 Computability for Economic Magnitudes and Operations

Computable Real Numbers

The definitions of Turing machines and partial recursive functions above are widely accepted notions of computability on natural numbers. But for economic modeling we need to apply computability notions to real numbers. For this, we follow Turing (1936), who based the concept for reals on the concept for natural numbers, in the following way.

According to Turing (1936), a real number x is *computable* if its decimal (hence dyadic, triadic etc.) expansions are computable by a Turing machine. In other words:

Definition 1″ (Turing (1936)). A real number x is *computable* if there is a Turing machine that, for any input n, has output d_n, the n-th decimal digit of x. Equivalently, there is a recursive function $\phi : \mathbb{N} \to \mathbb{N}$ for which $\phi(n) = d_n$ for every $n \in \mathbb{N}$.

We will consider any number computable in this sense to be usable for practical computations by economic agents: they can apply the simple rules of a Turing machine to calculate the number to any desired degree. That seems to be sufficient to fulfill our computability requirement (U.a).

What about the converse? Are these the only numbers we should consider to usable, or might there be others? In other words, is the decimal digit property of Definition 1″ necessary, as well as sufficient for usability? In view of our comparability requirement (U.d), no other real numbers are usable. That is because the following (Dedekind cut) definition of computable real number is equivalent to Definition 1″.

Definition 1′ (cf. Rice (1954)). A real number x is *computable* if there is a Turing machine that, for any rational number $r \neq x$, has output "yes" if $x > r$, and output "no" if $x < r$.

So if our numbers are to have the comparability property of condition (U.d), they must be computable in the sense of Definition 1′ as well.

For technical convenience, we will make use of still another equivalent (Cauchy sequence) definition of computable real number.

Definition 1 (Specker (1949)).

a) A real number x is *computable* if there are recursive function ϕ_1, ϕ_2, ϕ_3 :

$\mathbb{N} \to \mathbb{N}$ such that $\phi_3(k) \neq 0$ for every $k \in \mathbb{N}$ and

$$|x - (-1)^{\phi_1(k)} \frac{\phi_2(k)}{\phi_3(k)}| \leqq 2^{-k} \tag{1}$$

for every $k \in \mathbb{N}$.[9]

b) A vector $x = (x_1, \cdots, x_l) \in \mathbb{R}^l$ is *computable* if each x_i is computable.

In our model the commodity space is the l-dimensional space $\mathbb{R}^l_c$ of computable reals, the price and consumption spaces for our agents are the nonnegative orthant $\mathbb{R}^l_{c+}$ of $\mathbb{R}^l_c$. We endow these spaces with the metric topology induced by the Euclidean norm $\|\cdot\|$.

We now state several basic properties for computable reals (cf. the introduction). First, as an immediate consequence of (1), a real number is computable if and only if there is an algorithm for calculating it to any desired accuracy within a known time period. Second, the set $\mathbb{R}_c$ of computable real is countable, containing all natural numbers, rational numbers, algebraic numbers, etc. Third, the computable real number field $(\mathbb{R}_c, 0, 1, <, +, \cdot)$ is a real closed (ordered) subfield of the real number field (cf. Rice (1954)). Fourth, the metric space $(\mathbb{R}_c, |\cdot|)$ is not complete but is only recursively complete (Rice (1954); see Fact 1, page 52) and similarly for the metric spaces $(\mathbb{R}^l_c, \|\cdot\|)$ and $(\mathbb{R}^l_{c+}, \|\cdot\|)$.

Computable Operations

We want to define computability not only for numbers that represent commodity quantities and prices, etc., but also for (preference) relations over computable numbers and for (utility and demand) functions from computable numbers to computable numbers. Again, we will reduce computability of such operations to computability on natural numbers. Our first step is to reduce the basic objects (computable reals and vectors of computable reals) to natural numbers.

Reduction of computable objects to natural numbers. We can identify computable real numbers by the Turing machines or recursive functions that compute them; so utility and demand functions become higher level operators, carrying over functions to functions; and similarly for preferences. It is not immediately obvious how to define computability for these operators. There are, however, standard ways of reducing the higher level operators to ordinary functions on natural numbers, where we can apply the previous notion of computability via partial recursiveness.

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(9) Then we say x is the *effective limit* of the (recursive) sequence of rational numbers $v_k = (-1)^{\phi_1(k)} \phi_2(k)/\phi_3(k)$; and we call $\{v_k\}$ a *sequence of approximations* of x.

One standard way of making such a reduction is to use Definitions 1″, by which every every computable real number is computable by a Turing machine that approximates it, hence by a partial recursive function that computes it. Then we note that every Turing machine or partial recursive function is representable by a natural number or *Gödel number*. There are standard ways of doing this representation algorithmically. For the methods of assigning Gödel numbers to Turing machines, see Kleene (1952). For functions, Kleene's Normal Form Theorem (Theorem C-1, page 49 in Appendix C) in recursion theory yields a "universal" partial recursive function, i.e. a partial recursive function $\Phi(\cdot,\cdot)$ such that for every partial recursive function $\varphi : \mathbb{N} \to \mathbb{N}$, there is at least one $n \in \mathbb{N}$ such that:

$$\varphi(k) = \Phi(n, k) \qquad \text{for every } k \in \mathbb{N}; \tag{2}$$

then we can simply represent φ by the *Gödel number* n. [10]

For later use, we note a second standard way of reducing computable reals to natural numbers (cf. Moschovakis (1964a)). First, we fix any one-to-one recursive function $\langle\cdot,\cdot\rangle$ from $\mathbb{N} \times \mathbb{N}$ onto $\mathbb{N}$, say the Gödel

Definition 2. We say an integer $n \in \mathbb{N}$ is a *code* of a computable number x, and write $A(n, x)$), if $n = \langle\langle n_1, n_2\rangle, n_3\rangle$ for some Gödel numbers n_1, n_2, n_3 of some recursive functions ϕ_1, ϕ_2, ϕ_3 satisfying equation (1). Similarly, we say an integer $n \in \mathbb{N}$ is a *code* of a computable vector $x = (x_1, \cdots, x_l) \in \mathbb{R}_c^l$ if $n = \langle\cdots\langle\langle n_1, n_2\rangle, n_3\rangle\cdots n_l\rangle$ for some codes n_i of x_i; then we write $A(n, x)$.

As Gödel numbers represent partial recursive functions through equation (2), the codes n of computable reals x represent the partial recursive functions for computing x. In fact, by decoding n into n_1, n_2, and n_3, and feeding them into the "universal" partial recursive function Φ, we can recover x through equation (1). We can also recover computable vectors from their codes in a similar fashion.

As mentioned in the introduction, this algorithmic "coding" method permits realistic modeling of trading processes: agents can communicate their trading terms with "small" vocabulary. They can use the codes to communicate prices and quantities, rather than stating infinitely long sequences of decimal (dyadic, triadic, etc.) expansions.

Having reduced computable objects (computable numbers and vectors) to natural number representatives in an algorithmic manner, we can next define

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[10] Although each Turing machine will have a unique Gödel number, there will be multiple Gödel numbers representing any single partial recursive function.

computability for the operations (e.g. functions of computable reals, preference relations over computable reals) by applying the natural-number notion of computability to the integer representatives of the computable objects.

Computable Preferences. Classical equilibrium theory views agents' behavior as generated by constrained maximization of preferences. For bounded rationality, we require more: preferences are the outcomes of algorithmic-reasoning processes. For example, an agent may ranking alternatives according to some simple goals (Simon (1978)), or according to some simple evaluation criteria. What do we mean by a "simple" goal, or a "simple" evaluation? Certain calculations are undeniably simple: adding two natural numbers, for example. Our definition of "simple" will also allows the other operations on natural numbers that can be performed by Turing machines and partial recursive functions. (For other interpretations of "simple," see Richter and Wong (1996b).) Our justification is once again the Church-Turing Thesis. These simple operations are the ones that we say are usable by the agents in our models.

Thus, we require each agent's preference to be computable by an algorithm. In particular, we assume each agent has a strict preference $\succ$ on $\mathbb{R}^l_{c+}$ (i.e. $\succ$ is an asymmetric and negatively transitive[(11)] binary relation on $\mathbb{R}^l_{c+}$) that is computable in the following sense.

Definition 3 (cf. Moschovakis (1964a)). A binary relation $\succ$ on $\mathbb{R}^l_{c+}$ is *computable*[(12)] if there is a partial recursive function $\psi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that for all $n, n' \in \mathbb{N}$ and all $x, x' \in \mathbb{R}^l_{c+}$:

$$\text{if } A(n,x) \text{ and } A(n',x'), \text{ then: } x \succ x' \Leftrightarrow \psi(n,n') = 1. \tag{3}$$

This definition has obvious extensions to other (unary, ternary, quaternary, etc.) relations.

The partial recursive function f represents an algorithm that computes whether $x \succ x'$ for any pair of bundles x and x' in the following sense: the computation uses any codes for x and x', and halts and returns "yes" when it has determined that $x \succ x'$, if that ever occurs. Thus the computation result is independent of which particular codes of x and x' are used.

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(11) By negative transitivity, we mean $(\neg x \succ y \;\&\; \neg y \succ z) \Rightarrow \neg x \succ z$, for all x, y, z.

(12) Equivalently, if there is a Turing machine M that, for any pair of (Gödel numbers of) computable sequences $\{v_k\}$ and $\{v'_k\}$ of approximations (see Footnote 9) of some pair of x, x' in $\mathbb{R}^l_{c+}$ with $x \not\sim x'$, has output "yes" if $x \succ x'$, and output "no" if $x \prec x'$. (Compare Definition 1'.)

Remark 1.

1) Many strict preference relations are computable. For example, the strictly greater than relation $>$ (cf. Moschovakis (1964a)) on $\mathbb{R}_c$, the strict preferences corresponding to the utility functions $\log(x)$, $\min\{x, y\}$, polynomials $g(x)$ with computable coefficients, etc. (see Theorem 1 below).[13]

2) Examples of non-computable strict preferences include: lexicographic preferences (cf. (4) in this Remark, and Fact 1, page 52), strict preferences given by non-computable utility functions such as $u(x, y) = x + \alpha y$, where α is a non-computable real (cf. Theorem 1 below), etc.

3) Each strict preference $\succ$ on $\mathbb{R}^l_{c+}$ corresponds to a unique weak preference $\succeq$ (i.e. $\succeq$ is a reflexive, transitive, and total binary relation) on $\mathbb{R}^l_{c+}$) for which $\succ$ is the asymmetric part (thus $x \succ y \Leftrightarrow \neg y \succeq x$).

However, the computability of $\succ$ does not imply the computability for $\succeq$. For example, it is well-known in recursive mathematics (Turing (1936)) that the strictly greater relation $>$ on $\mathbb{R}_c$ is computable,[14] but the weakly greater than relation $\geqq$ is not computable.[15] (For economic applications, see Wong (1994, Thm. IV.4.3.) and Lewis (1987).)

4) The computability restriction automatically yields a useful topological property: By a well-known theorem (Moschovakis (1964b); see Fact 2, page 53) in recursive analysis, every computable strict preference $\succ$ on $\mathbb{R}^l_{c+}$ is *continuous*:[16]

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(13) We can view a computable number as a computable relation on the set Q of rational numbers. Each real number x can be identified with the binary relation $\succ$ on Q defined by $y \succ z \Leftrightarrow y > x > z$. It can be shown that the number x is computable if and only if the relation $\succ$ is computable.

(14) An algorithm for computing $>$ is to compare the decimal expansions (rational number approximations, etc.) of pairs of computable reals (cf. Pour-El and Richards (1989, p. 14)). More precisely, one can use the following partial recursive function $\psi(n, m) = \min\{k : (\bigwedge_{1 \leqq i \leqq 3, 1 \leqq k' \leqq k} \Phi(n_i, k') \downarrow) \& (\bigwedge_{1 \leqq i \leqq 3, 1 \leqq k' \leqq k} \Phi(m_i, k') \downarrow) \& [(-1)^{\Phi(n_1, k)} \Phi(n_2, k)/\Phi(n_3, k) - (-1)^{\Phi(m_1, k)} \Phi(m_2, k)/\Phi(m_3, k) > 2 \cdot 2^{-k}]\}$, where n_i and m_i are the unique nonnegative integers such that $n = \langle\langle n_1, n_2\rangle, n_3\rangle$ and $m = \langle\langle m_1, m_2\rangle, m_3\rangle$.

(15) Because the computability of $\geqq$ would imply computability for the equality relation $=$, which is known (Turing (1936)) to be not computable.

(16) In fact, for computable relations, the ϵ can be found algorithmically in (4). Thus computable relations have a stronger continuity property than other continuous relations in $\mathbb{R}^l_{c+}$: they are *computably continuous.*

$$\text{for every } x, y \in \mathbb{R}^l_{c+}, \text{ if } x \succ y, \text{ then there is an } \epsilon > 0 \in \mathbb{R}_c \text{ such that } x' \succ y \text{ and } x \succ y' \text{ for all } x', y' \in \mathbb{R}^l_{c+} \text{ with } \|x - x'\|, \|y - y'\| < \epsilon. \tag{4}$$

(An arbitrary computable binary relation may not be continuous; cf. Remark 13(4) in Appendix C).

Computable Functions. Now we need to model bounded rationality for utility functions, demand functions, production functions, profit functions, etc. We require a definition of computability for such functions; the definition we use is similar in spirit to that for relations.

Definition 4 (cf. Moschovakis (1964a)). Let $X \subseteqq \mathbb{R}^l_c$. A function $f : X \to \mathbb{R}^m_c$ is a *computable function*[17] [18] if there is a partial recursive function $\phi : \mathbb{N} \to \mathbb{N}$ such that for every $x \in X$ and every $n \in \mathbb{N}$:

$$A(n, x) \qquad \text{implies} \qquad \phi(n)\downarrow \ \& \ A(\phi(n), f(x)). \tag{5}$$

Since a partial recursive function ϕ corresponds to an algorithm, this essentially states that f is determined by an algorithm that transforms codes of $x \in X$ into codes of $f(x)$. Hence f is usable by our bounded rational agents, who can apply an algorithm to compute it.[19]

Remark 2.

1) Most functions that we use in applied economics are computable: $\exp\{x\}$, $\log(x)$, $\max\{x, y\}$, $x \cdot y$, polynomials with computable coefficients, etc.

2) It is clear from the definition that a composition of computable functions is computable.

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[17] **Equivalently, there is a Turing machine that transforms (a Gödel number of) each sequence of computable approximations of $x \in X$ into (a Gödel number of) a sequence of approximations of $f(y)$.**

[18] **In the literature of recursive analysis, what we call "computable function" has several different names: "effective operation" (cf. Kreisel at al (1959)), "recursive operator" (cf. Moschovakis (1964a)) "constructive operator" (cf. Šanin (1956)), etc.). The notion of a computable function is weaker than the one (Definition 7, page 56) developed by Lacombe (1955a,b) and Grzegorczyk (1955, 1957) (see also Pour-El and Richards (1989)), where an additional condition of effective uniform continuity is required (cf. Remark15, page 58).**

[19] **We can characterize computability of functions in terms of the earlier computability notion for relations. Let $X \subseteqq \mathbb{R}^l_c$. A function $f : X \to \mathbb{R}^m_c$ can be identified with the unary relation $\succ$ on $X \times \mathbb{R}^m_c \times \mathbb{R}^m_c$ defined by $(x, y, z) \in \succ \Leftrightarrow y > f(x) > z$. Then it can be shown that the function f is computable if and only if the relation $\succ$ is computable.**

3) As with computable strict preferences, a computable function $f : \mathbb{R}^l_{c+} \to \mathbb{R}^m_c$ is *continuous*:

$$\text{for every } \epsilon > 0 \in \mathbb{R}_{c+} \text{ and every } x \in \mathbb{R}^l_{c+}, \text{ there is a } \delta > 0 \in \mathbb{R}_c \text{ such that } \|x-y\| \leqq \delta \text{ implies } \|f(y)-f(x)\| \leqq \epsilon \tag{6}$$

for every $y \in \mathbb{R}^l_{c+}$. This continuity property holds also for computable functions defined on many other domains, for example on $\mathbb{R}^l_{c+}$, on $S_c = \{p \in \mathbb{R}^l_c : p \gg 0, \|p\| = 1\}$, and on $B_c(p,w) = \{x \in \mathbb{R}^l_{c+} : p \cdot x \leqq w\}$, where $(p,c) \in \mathbb{R}^l_{c++} \times \mathbb{R}_{c++}$, etc. (See Fact 2, page 53)[20] [21]

5. COMPUTABLE ECONOMIES AND STATEMENT OF RESULTS

In the previous section, we have provided computability notions in modeling procedural restrictions for our bounded-rational agents. Here we will present our computable equilibrium framework and discuss our main results.

5.1 Notation

Economic magnitudes will be measured only by computable real numbers. We list some notation.

$\mathbb{R}_c$ is the set of computable numbers,

$\mathbb{R}^l_c$ is the l-dimensional space of computable reals,

$x \geqq y$ means "$x_i \geqq y_i$ for all $i = 1, \cdots, l$" for all $x, y \in \mathbb{R}^l_c$,

$x \gg y$ means "$x_i > y_i$ for all $i = 1, \cdots, l$" for all $x, y \in \mathbb{R}^l_c$,

$\mathbb{R}_{c+} = \{x \in \mathbb{R}_c : x \geqq 0\}$,

$\mathbb{R}_{c++} = \{x \in \mathbb{R}_c : x > 0\}$,

$\mathbb{R}^l_{c+} = \{x \in \mathbb{R}^l_c : x \geqq 0\}$,

$\mathbb{R}^l_{c++} = \{x \in \mathbb{R}^l_c : x \gg 0\}$,

$e = (1, 1, \cdots, 1) \in \mathbb{R}^l_c$,

$\|\cdot\|$ is the Euclidean norm, i.e. $\|x\| = (\sum_{i=1}^l x_i^2)^{1/2}$ for all $x \in \mathbb{R}^l_c$,

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(20) **Furthermore, for computable functions on such domains, the δ in (6) also be found algorithmically. Thus they satisfy a stronger property then continuous functions: they are *effectively continuous* (cf. Footnote 16, and Moschovakis (1964b)). It should be clear that this term "effectively continuous" is different from the term "computable-continuous" given in Definition 7, page 56.**

(21) **However, $\mathbb{R}^l_c$ is not locally-compact, so continuity of $f : X \to \mathbb{R}^m_c$ does necessarily not imply locally uniform continuity even when X is bounded and is closed in $\mathbb{R}^l_c$. (Cf. Beeson (1985, p. 70).))**

$(0,1)_c = \{\lambda \in I\!R_c : 0 < \lambda < 1\}$,
$[0,1]_c = \{\lambda \in I\!R_c : 0 \leqq \lambda \leqq 1\}$,
$[x,y]_c = \{\lambda x + (1-\lambda)y : \lambda \in [0,1]_c\}$ for all $x, y \in I\!R^l_c$,
$S_c = \{p \in I\!R^l_{c++} : ||p|| = 1\}$,
$S_{c,\epsilon} = \{p \in S_c : p \geqq \epsilon e\}$ for all $\epsilon > 0 \in I\!R_c$,
$\mathcal{B}_c = I\!R^l_{c++} \times I\!R_{c++}$,
$B_c(p,w) = \{x \in I\!R^l_{c+} : p \cdot x \leqq w\}$ for all $(p,w) \in I\!R^l_{c+} \times I\!R_{c+}$.

5.2 Preference and Utility

In our bounded rationality framework, we assume each agent has a computable strict preference $\succ$ on the consumption space $I\!R^l_{c+}$. The following result shows that this approach is equivalent to assume each agent has a computable utility function.[22]

Theorem 1 (Computable Utility Functions and Computable Preferences (Richter and Wong (1996a, Theorem 1)). *If $\succ$ is a computable strict preference on $I\!R^l_{c+}$, then there is a computable function $u : I\!R^l_{c+} \to I\!R_c$ such that:*

$$u(x) > u(y) \;\Leftrightarrow\; x \succ y \qquad \text{for every } x, y \in I\!R^l_{c+}. \tag{7}$$

Conversely, if $u : I\!R^l_{c+} \to I\!R_c$ is a computable function, then there is a computable strict preference on $I\!R^l_{c+}$ satisfying (7).

5.3 Consumer Demands

We now discuss the demand behavior of a bounded rational agent. We find sufficient conditions for the existence of a computable demand bundle and the existence of a computable demand function. We provide a revealed preference characterization of computable rationality for the finite case.

Let $\succ$ be a computable strict preference on consumption space $I\!R^l_{c+}$. Let the vectors $(p,w) \in \mathcal{B}_c$, so the price vector p and income m are computable and positive. A *computable budget set* is a set $B_c(p,w) = \{x \in I\!R^l_{c+} : p \cdot x \leqq w\}$, where $(p,w) \in \mathcal{B}_c$. A *$\succ$-maximal element* of $B_c(p,w)$ is a $x \in B_c(p,w)$ such that if $y \in B_c(p,w)$, then $\neg y \succ x$.

---

[22] The theorem does not require the usual explicit assumption of continuity for preference, because continuity is automatically implied by the computability of preferences.

First, we study the existence of a computable *demand bundle,* i.e. a computable $\succ$-maximal element of a computable budget set $B_c(p, w)$. Similar to many optimization problems in recursive analysis (cf. Beeson (1985, pp. 71-73)), the major difficulty is that a computable budget set is not topologically complete (hence not compact), but only satisfies a weak completeness property (recursive completeness). The recursive completeness condition is too weak for the existence of an extremum in many cases. For example, Kreisel (1959) found a computable function $f : [0,1]_c \to I\!R_c$ that does not admit a maximizer, i.e. $\sup f > f(x)$ for all $x \in [0,1]_c$.[23] Considering this f as a utility function and consider $[0,1]_c$ as a computable budget set,[24] then Kreisel's example easily gives a counterexample to the existence of a computable demand bundle in our computability context (cf. Theorem 1).

One might surmise that the existence of a demand bundle requires certain strong restrictions on a computable preference $\succ$. On the contrary, a fairly natural and simple condition will be sufficient. A strict preference $\succ$ on $I\!R^l_{c+}$ is said to be *c-convex* if each weakly preferred set $\mathcal{R}_z = \{x \in I\!R^l_{c+} : \neg z \succ x\}$ is c-convex in $I\!R^l_{c+}$ (cf. Section C.6 in Appendix C). In other words, for every $x, y, z \in I\!R^l_{c+}$ and every computable real number $\lambda \in [0,1]_c$, if $\neg z \succ x$ and $\neg z \succ y$, then $\neg z \succ \lambda x + (1-\lambda)y$.

Theorem 2 (Existence of a $\succ$-Maximal Element). [25] *Let $\succ$ be a computable strict preference on $I\!R^l_{c+}$. Assume $\succ$ is c-convex. Then for every $(p, w) \in \mathcal{B}_c$, there exists a $\succ$-maximal element of $B_c(p, w)$.*

Proof. From Theorem 1, it is clear that $\succ$ can be represented by a computable function $u : I\!R^l_{c+} \to I\!R_c$ (i.e. (7) holds) that is *c-quasiconcave*, i.e. for all $x, y \in I\!R^l_{c+}$, $\lambda \in [0,1]_c$ implies $u(\lambda x + (1-\lambda)y) \geqq \min\{x, y\}$. Then Theorem 2 follows immediate from Richter and Wong (1996a, Theorem 2), which states that every c-quasiconcave computable function $u : I\!R^l_{c+} \to I\!R_c$ admits a computable maximizer on every computable budget set $B_c(p, w)$. Q.E.D.

Next, we prove computability of demand functions. A (demand) function $h : \mathcal{B}_c \to I\!R^l_{c+}$ is *generated* by a strict preference $\succ$ defined on $I\!R^l_{c+}$ if for all $(p, w) \in \mathcal{B}_c$, the vector $h(p, w)$ is the unique $\succ$-maximal element of $B_c(p, w)$. The

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(23) Such f may even be required to be uniformly continuous, e.g. $|f(x) - f(y)| \leqq |x - y|$ for all $x, y \in [0,1]_c$.

(24) See also Lewis (1988).

(25) Since a computable strict preference $\succ$ is continuous, we do not need to make the usual explicit assumption of continuity (or upper semicontinuity) for $\succ$.

following theorem proves that the function $h(\cdot)$ is computable if the preference $\succ$ is c-convex. This essentially shows that the demand functions of our bounded-rational agents can be determined by algorithms.

Theorem 3 (Computability of Demand Functions). *Let $\succ$ be a c-convex and computable strict preference on $\mathbb{R}^l_{c+}$. Let $h : \mathcal{B}_c \to \mathbb{R}^l_{c+}$ be a demand function generated by $\succ$. Then h is computable.*

Proof. By Theorem 1, Theorem 3 is equivalent to Richter and Wong (1996a, Theorem 3), which states that if $u : \mathbb{R}^l_{c+} \to \mathbb{R}_c$ is a c-quasiconcave and computable function, and if a function $h : \mathcal{B}_c \to \mathbb{R}^l_{c+}$ satisfies $h(p, w) = \text{argmax}_{x \in B_c(p,w)} u(x)$ for all $(p, w) \in \mathcal{B}_c$, then h is computable. Q.E.D.

Remark 3. Consider a computable strict preference $\succ$ on $\mathbb{R}^l_{c+}$ that is *strictly c-convex*, i.e. for every $x, y \in \mathbb{R}^l_{c+}$, if $x \neq y$ and $\neg x \succ y$, then $\lambda x + (1-\lambda)y \succ y$ for every $\lambda \in (0,1)_c$.[26] (In other words, each weakly-preferred set $\mathcal{R}_y = \{x \in \mathbb{R}^l_{c+} : \neg y \succ x\}$ is "strictly c-convex" in $\mathbb{R}^l_{c+}$.) Then it is c-convex, so Theorem 2 ensures the existence of a $\succ$-maximal element x of $B_c(p, w)$ for any $(p, w) \in \mathcal{B}_c$. In fact it is unique. Suppose, by contradiction, that y is another $\succ$-maximal element of $B_c(p, w)$. Then for any $\lambda \in (0,1)_c$, the bundle $\lambda x + (1-\lambda)y \in B_c(p, w)$. The strict c-convexity of $\succ$ implies $\lambda x + (1-\lambda)y \succ x$, contracting the $\succ$-maximality of x. Therefore, the preference $\succ$ generates a demand function $h : \mathcal{B}_c \to \mathbb{R}^l_{c+}$, and by Theorem 3 the function h is computable.

Finally, we ask whether there are any consequences of computable preference maximization, beyond computability of a demand function h. Equivalently, can we tell from observing h, whether it was generated by a computable preference. Certainly, h must satisfy Houthakker's Strong Axiom of Revealed Preference (cf. Richter (1966)). Are there further requirements? For the case of a finite set of observations with exhaustiveness (i.e. $p \cdot h(p, w) = w$ for all observed (p, w)),[27] these conditions characterize computable rationality For a proof see Richter and Wong (1966a).

However, for the case of an infinite set of observations, the computability and strong axiom properties of h do not suffice for computable rationality. For a proof see Richter and and Wong (1996a, Remark 4).

---

[26] The strict c-convexity of $\succ$ on $\mathbb{R}^l_{c+}$ does not imply the strict convexity of the extension of $\succ$ to all of $\mathbb{R}^l_{+}$. This is true even when $\succ$ has a unique continuous extension to $\mathbb{R}^l_{+}$. (See Appendix II in Richter and Wong (1996a).

[27] It is clear that this exhaustiveness property is satisfied when h is generated by a locally insatiable preference (as defined in Section 5.4 below).

5.4 Computable Economy and the Two Welfare Theorems

In the previous sections, we have developed a bounded rationality approach to consumer theory. Here we will give a general equilibrium framework. We will also carry over the two classical welfare theorems (Arrow (1951)) to our computable context.

We define a formal notion of economy with bounded rational agents. The commodity space is $\mathbb{R}_c^l$, the price space is $\mathbb{R}_{c+}^l$, and the consumption space for each agent is $\mathbb{R}_{c+}^l$. For simplicity, we will consider only trading activities, i.e. we will not model production activities. A *computable economy* is an m-tuple $E = \{(\succ_i, \omega_i)\}_{i=1}^m$, where $\succ_i$ is a computable strict preference on $\mathbb{R}_{c+}^l$ and ω_i is a computable endowment (i.e. $\omega_i \in \mathbb{R}_{c+}^l$) for each agent i.

We now carry over some standard notions to our context. Given a computable economy $E = \{(\succ_i, \omega_i)\}_{i=1}^m$, a *computable allocation* is a tuple $(x_i)_{i=1}^m$, where $x_i \in \mathbb{R}_{c+}^l$. A computable allocation $(x_i)_{i=1}^m$ is *feasible* if $\sum_{i=1}^m x_i \leqq \sum_{i=1}^m \omega_i$. A computable allocation $(x_i)_{i=1}^m$ is *Pareto optimal* if it is feasible and if there does not exist any feasible computable allocation $(\tilde{x}_i)_{i=1}^m$ such that:

$$
\begin{aligned}
&\text{a)} \;\; \neg x_i \succ_i \tilde{x}_i \text{ for every } i,\\
&\text{b)} \;\; \tilde{x}_i \succ_i x_i \text{ for some } i.
\end{aligned}
\tag{8}
$$

A *computable equilibrium* is a tuple $(\bar{p}, (\bar{x}_i)_{i=1}^m)$ of a computable price and a computable allocation such that:

$$
\begin{aligned}
&\text{a)} \;\; \textstyle\sum_{i=1}^m \bar{x}_i \leqq \sum_{i=1}^m \omega_i \;\text{ and }\; \bar{p} \geqq 0 \;\text{ and}\\
&\qquad \bar{p}(\textstyle\sum_{i=1}^m \bar{x} - \sum_{i=1}^m \omega_i) = 0\\
&\text{b)} \;\; \bar{x}_i \text{ is a } \succ_i\text{-maximal element of the computable budget}\\
&\qquad \text{set } B_c(\bar{p}, \bar{p}\cdot\omega_i) \text{ for each agent } i = 1, \cdots, m.
\end{aligned}
\tag{9}
$$

Then we call $\bar{p}$ a *computable equilibrium price,* and we call $(\bar{x}_i)_{i=1}^m$ a computable equilibrium allocation.

Condition (9a) is the natural condition of market clearing; and (9b) is the condition of consumer maximization.

We now study the optimality of computable equilibria. As in the standard version, the following theorem requires each consumer's preference $\succ_i$ to be *locally insatiable*, i.e. for every $x \in \mathbb{R}_{c+}^l$ and $\alpha > 0 \in \mathbb{R}_c$, there is a $y \in \mathbb{R}_{c+}^l$ such that $||x - y|| < \alpha$ and $y \succ_i x$.

Theorem 4 (Computable First Welfare Theorem). *Let $E = \{(\succ_i, \omega_i)\}_{i=1}^m$ be a computable economy, and let $(\bar{p}, (\bar{x}_i)_{i=1}^m)$ be a computable equi-*

librium for E. Suppose each $\succ_i$ is locally insatiable. Then $(\bar{x}_i)_{i=1}^m$ is Pareto optimal.

Proof. This can be proved along standard lines (Arrow (1951)); and we omit the details.

As in the standard case, the condition of locally insatiability is indispensable for Theorem 4.

We now turn to the converse problem: implementing a Pareto optimal computable allocation $(\bar{x}_i)_{i=1}^m$ as a computable equilibrium allocation. As in the standard version, we require each $\succ_i$ to be *(weakly) monotone,* i.e. for every $x, y \in \mathbb{R}^l_{c+}$: if $x \gg y$, then $x \succ_i y$. Convexity for $\succ_i$ and strict-positivity for $\bar{x}_i$ will also be required. The usual continuity assumption of $\succ_i$ will be implied by the computability of $\succ_i$.

Theorem 5 (Computable Second Welfare Theorem). *Let $E = \{(\succ_i, \omega_i)\}_{i=1}^m$ be a computable economy, and let $(\bar{x}_i)_{i=1}^m$ be a Pareto optimal (computable) allocation. Assume each computable preference $\succ_i$ is c-convex and (weakly) monotone, and each $\bar{x}_i \gg 0$. Then there exists a computable price $\bar{p}$ and a vector $(\bar{\omega}_i)_{i=1}^m$ of computable endowments with $\sum_{i=1}^m \bar{\omega}_i \leqq \sum_{i=1}^m \omega_i$ and such that $(\bar{p}, (\bar{x}_i)_{i=1}^m)$ is a computable equilibrium for the computable economy $\bar{E} = \{(\succ_i, \bar{\omega}_i)\}_{i=1}^m$.*

Proof. This can be proved along standard lines (Arrow (1951)) with a new computable separation theorem (Theorem C-3, page 61); cf. Wong (1996a, Theorem 2)). See Appendix A for details.

As is standard, if all $\succ_i$ in Theorem 5 are *strictly monotone* (i.e. $x \succ_i y$ for all $x, y \in \mathbb{R}^l_{c+}$ with $x \geqq y$ and $x \neq y$),[28] then the Pareto optimality of $(\bar{x}_i)_{i=1}^m$ forces the inequality in "$\sum_{i=1}^m \bar{\omega}_i \leqq \sum_{i=1}^m \omega_i$" to become equality.

Compared to its standard version, Theorem 5 allows a weaker topological property for the price space (recursive completeness is weaker than topological completeness). It also strengthens the standard conclusion by obtaining computability for supporting prices, which therefore can be computed by a social planner through algorithms.

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[28] As with strict c-convexity (cf. Footnote 26) the strict monotonicity property fails for continuous extension from $\mathbb{R}^l_{c+}$ to $\mathbb{R}^l_{+}$.

5.5 Market Excess Demands

In this section, we characterize aggregate excess demand functions for a computable economy.

Sonnenschein (1972, 1973) asked whether individual rationality imposes any restrictions on aggregate excess demand functions. A sequence of results by Sonnenschein (1972, 1973), Mantel (1974), Debreu (1974), Mas-Colell (1977) and others essentially showed that no rationality properties beyond Walras' Law (WL), Boundedness from Below (BB), and a boundary condition (BC). Now that we are imposing computability restrictions on the rationality of our individual agents, are there any new implications for aggregate demand functions? Theorem 6 will show that the only new consequence is computability of aggregate demand functions.

Beyond showing the consequences of computable rationality, Theorem 6 will be used in the next section to prove non-existence of computable equilibrium.

Consider a computable economy $E = \{(\succ_i, \omega_i)\}_{i=1}^{m}$ such that for each $i = 1, \cdots, m$:

$$\text{the computable strict preference } \succ_i \text{ is strictly monotone and strictly c-convex, and the computable endowment } \omega_i \gg 0. \tag{10}$$

Let the price set be $S_c = \{p \in \mathbb{R}^l_{c++} : \|p\| = 1\}$. It follows immediately from Theorem 3 (even dropping strict monotonicity) that the *excess demand function* $f : S_c \to \mathbb{R}^l_c$ defined by

$$f(p) = \sum_{i=1}^{m} \{x_i \in B_c(p, p \cdot \omega_i) : \forall y_{y \in B_c(p, p \cdot \omega_i)} \neg y \succ_i x\} - \sum_{i=1}^{m} \omega_i \tag{11}$$

is a computable function. Being computable, f is also continuous. It is easily shown that f satisfies:

WL) (Walras Law) $p \cdot f(p) = 0$ for every $p \in S_c$.

Also, it is clear from (11) that f also satisfies:

BB) (Bounded from Below) there is an $M \in \mathbb{R}_c$ such that $f(p) \geqq Me$ for every $p \in S_c$.

However, since the consumption space $\mathbb{R}^l_{c+}$ is not topologically complete (in fact bounded closed balls are not compact), f need not satisfy the following condition:[29]

BC) (Boundary Condition) for every sequence of vectors $p_k \in S_c$, if $(p_k)_i \to 0$ as $k \to \infty$ for some $i = 1, \cdots, l$, then $\|f(p_k)\|$ converges to ∞.

We now study the converse. As with current proofs of the standard analogues, we are only able to provide a partial (ϵ-qualified) answer to our problem. For simplicity of exposition, we also require the given function $f : S_c \to \mathbb{R}^l_c$ to be *twice-continuously differentiable* (C^2), i.e. f can be extended to a twice-continuously differentiable function from some open set in the l-dimensional space $\mathbb{R}^l$ of real numbers into $\mathbb{R}^l$. For such f satisfying (WL), we find a computable economy E generating f on a large portion $S_{\delta,c} = \{p \in S_c : p \geqq \delta e\}$ of S_c. We can also obtain the following property for E:

(strict concavity) each $\succ_i$ is represented by a strictly c-concave utility function $u^i : \mathbb{R}^l_{c+} \to \mathbb{R}_c$. (12)

(By the *strict c-concavity* of u^i, we mean that $u^i(\lambda x + (1-\lambda)y) > \lambda u^i(x) + (1-\lambda)u^i(y)$ for all pairs of distinct $x, y \in \mathbb{R}^l_{c+}$ and all $\lambda \in (0,1)_c$.) Moreover, when f satisfies also (BB) and (BC), we can also choose E so that it has the same set of equilibrium prices as f.

Theorem 6 (A Computable Characterization of Excess Demand Functions). *Let positive $\epsilon \in \mathbb{R}_c$ and let $f : S_c \to \mathbb{R}^l_c$ be C^2 and computable.*

I) If f satisfies (WL), then there exists a positive $\delta \in \mathbb{R}_c$ with $\delta < \epsilon$, and there exists an excess demand function $\tilde{f}$ of some computable economy $E = \{(\succ_i, \omega_i)\}_{i=1}^l$ satisfying (10) and (12) for all $i = 1, \cdots, l$, and such that:

1) $\tilde{f}|_{S_{c,\delta}} = f|_{S_{c,\delta}}$.

II) Moreover, if f also satisfies (BB) and (BC), then the δ and $\tilde{f}$ can be chosen so that:

2) $\tilde{f}^{-1}(0) = f^{-1}(0) \subseteqq S_{c,\delta}$.

Proof. This follows from Theorem B-1, page 33 in Appendix B; see Remark 6, page 34.

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[29] Indeed, by using a "singular covering" lemma in recursive analysis (e.g. Beeson (1985, p. 69, Theorem 6.1); Fact 5, page 67), one can find a computable strict preference $\succ$ on $\mathbb{R}^3_{c+}$ with the property that it is strictly monotone and strictly c-convex on $\mathbb{R}^3_{c+}$, has a continuous extension on $\mathbb{R}^3_+$ that is only weakly monotone (not strictly monotone), and has an excess demand function that does not satisfy (BC). See Richter and Wong (1996a, Appendix II, Remark 9).

Remark 4.

a) In the economy E the number l of agents and the number of commodities are equal.

b) If we drop the strict concavity conclusion (12), then Theorem 6(I) is a computable version of Geanakoplos' Theorem (1984), and Theorem 6(II) is a computable version of Mas-Colell's Theorem (with stronger C^2 assumption).

c) The C^2 assumption in Theorem 6 can be relaxed to an effective condition of local uniform continuity (Wong (1994, Theorem 2.5, p. 127)) by using effectivizations of both the methods of Debreu (1974) and Mas-Colell (1977). That method, however, does not obtain the strict concavity conclusion.

5.6 Non-existence of Computable Equilibrium

Theorems 1 through 6 established some positive results in our bounded rationality framework. We prove negative results for the existence of computable (general) equilibrium in our bounded rationality context. These negative results are rooted in a "recursive analysis counterexample" (Remark 20 and Proposition 2, page 66; cf. Orevkov (1964), Baigger (1985)) to Brouwer's Fixed-Point Theorem. They suggest new evidence on the observation (cf. Uzawa (1962), Sonnenschein (1972)) that Brouwer's theorem or one of its immediate family is needed for proving existence of a general equilibrium.

Consider a computable (hence continuous) $f : S_c \to \mathbb{R}^l_c$ satisfying (WL), (BB) and (BC). Must there exist an *equilibrium* (i.e. a $p \in S_c$ with $f(p) = 0$)? As is well-known, (Gale (1954), Nikaido (1955), Debreu (1956)) the answer for the conventional version is positive, and can be proved by using Brouwer's Fixed-Point Theorem (cf. Border (1985, Section 9.12 and Section 18.11)). However, Brouwer's theorem fails to hold in our computable (recursive analysis) context. Does this imply the failure of the equilibrium existence claim for functions f?

The existence failure is an immediate consequence of the following result, which transforms computation of fixed-points to computation of equilibria. In what follows, for any $k = 1, 2, \cdots, \infty$ and any $X \subsetneqq \mathbb{R}^l_c$, we say a function $f : X \to \mathbb{R}^m_c$ is C^k if f can be extended to a k times continuously differentiable function $\tilde{f} : U \to \mathbb{R}^m$ for some open set $U \subsetneqq \mathbb{R}^l$, where $\mathbb{R}^l$ and $\mathbb{R}^m$ are the l-dimensional and m-dimensional spaces of real numbers.

Theorem 7 (Computable Fixed-Points and Computable Equilibria). *Let positive $\epsilon \in \mathbb{R}_c$ and let $S_{c,\epsilon} = \{p \in S_c : p \geqq \epsilon e\}$ be non-empty, and let $g : S_{c,\epsilon} \to S_{c,\epsilon}$ be computable. Then:*

1) there is a computable function $f : S_c \to \mathbb{R}^l_c$ satisfying (WL), (BB), and (BC), *and such that* $f^{-1}(0) = \{p \in S_{c,\epsilon} : g(p) = p\}$,

2) for any $k = 1, 2, \cdots, \infty$, *if* g *is also* C^k *and* $g(S_{c,\epsilon}) \subsetneqq S_{c,\epsilon'}$ *for some* $\epsilon' \in \mathbb{R}_c$ *with* $\epsilon' > \epsilon$, *then the function* f *claimed in (1) can be chosen to be* C^k.

Proof. See Appendix B.

Combining Theorem 7 with a "computable counterexample" to Brouwer's Fixed-Point Theorem (Proposition 2 and Remark 20, page 66), the following corollary is immediate.

Corollary 1 (Non-existence of Equilibrium for a Market Excess Demand). *There exists a computable function* $f : S_c = \{p \in \mathbb{R}^3_{c++} : ||p|| = 1\} \to \mathbb{R}^3_c$ *that is* C^2, *satisfies (WL), (BB) and (BC), but has* $f^{-1}(0) = \emptyset$.

The negative result in Corollary 1 can be easily extended to the case where $l \geqq 3$. However, for $l \leqq 2$, the results are positive; every computable function $f : S_c = \{p \in \mathbb{R}^2_{++} : ||p|| = 1\} \to \mathbb{R}^2_c$ satisfying (WL), (BC) and (BB) does admit an equilibrium. This follows easily from an intermediate value theorem[(30)] for computable functions. Thus for $l \leqq 2$, there exists a computable equilibrium for any computable economy $E = \{(\succ_i, \omega_i)\}_{i=1}^m$ (with no more than two commodities) satisfying (10) for all $i = 1, \cdots, m$.[(31)] However, for $l \geqq 3$, computable equilibrium does not necessarily exist. This negative result follows easily from the following immediate consequence of Corollary 1 and Theorem 7.

Corollary 2 (Non-existence of Computable Equilibrium for a Computable Economy). *There exists a computable economy* $E = \{(\succ_i, \omega_i)\}_{i=1}^3$ *with commodity space* $\mathbb{R}^3_c$, *satisfying (10) for all* $i = 1, 2, 3$, *and such that there does not exist any computable equilibrium.*

Without computability restrictions on prices and trading quantities (i.e. permitting arbitrary reals), Arrow-Debreu (1954, Theorem I) proved the existence of a (not-necessarily-computable) competitive equilibrium for an economy. Corollary 2 shows that existence can fail when we impose computability restrictions on

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[(30)] **This states that if $h : [0,1]_c \to \mathbb{R}_c$ is computable, then for every computable real α with $h(0) \leqq \alpha \leqq h(1)$ there exists a $x \in [0,1]_c$ with $h(x) = \alpha$. This theorem extends Pour-El and Richards (1989, Theorem 8, p. 41) by relaxing their assumption of effective uniform continuity (EUC) on f; it can be shown by following their proof, in which they actually use only the continuity property for f (which is implied by the computability of f) rather than the stronger property of EUC.**

[(31)] **This existence claim still holds even when one weakens strict c-convexity in (10) to c-convexity.**

prices and quantities, even when preferences and endowments are computable. We cannot rely, then, on the Arrow-Debreu theorem to guarantee existence of computable equilibrium — whether we assume computable preferences on $\mathbb{R}^l_{c+}$ and computable endowments in $\mathbb{R}^l_{c++}$, or whether we weaken the assumption to classical preferences on $\mathbb{R}^l_+$ and classical endowments in $\mathbb{R}^l_{++}$.

6. APPLICATIONS IN COMPUTATIONAL ECONOMICS

Beyond modeling bounded rationality, our computability tools and results of Section 5 have several implications for computational economics. In particular, they show the existence or non-existence of algorithms to compute economic data. That is because by our definition of computability, existence of computable prices, quantities, etc. is equivalent to existence of algorithms for computing them, i.e. for calculating digital approximations to any desired degree of accuracy within known time.

1) Computing Utilities. Theorem 1 implies that given a preference that is computable by an algorithm, it is possible to find a utility representation that is computable by an algorithm. Indeed, as shown in the proof of Theorem 1 of Richter and Wong (1996a), such a utility representation can be found by a uniform algorithm — i.e., there is an algorithm that, for every computable preference, computes a computable utility function.

2) Computing Demands. The computable counterexample to the existence of a demand bundle implies the general impossibility of using algorithms to find a preference-maximization bundle in a budget set.

Theorem 2 shows that if a computable preference $\succ$ is convex, then for every computable budget there exists a computable maximizer, and by definition there exists an algorithm to compute it. It remains an open question whether there exists a uniform algorithm for this — i.e., a single algorithm that computes a computable demand for every computable budget. However, as shown in Richter and Wong (1996a, proof of Proposition 1), such a uniform algorithm exists that computes unique maximizers of strictly convex preferences.

3) Computing Supporting Prices. Theorem 5 implies that given a model of exchange economy $E = \{(\succ_i, \omega_i)\}_{i=1}^m$ satisfying the standard assumptions and a strictly positive Pareto optimal allocation $(x_i)_{i=1}^m$ in E, there exists an algorithm to compute a price p supporting $(x_i)_{i=1}^m$ as a competitive equilibrium (with transfers of endowments). Whether there exists a uniform algorithm for this remains an open question.

4) Computing Economies for Rationalizing given Market Excess Demands. Theorem 6 ensures that given a C^2 computable market excess demand f, there exists an algorithm to find an economy E that generates f on a given sphere $S_{c,\epsilon}$ and with the same set of equilibria as f. In fact, as shown in the proofs of Theorem B-1, page 33, and Proposition 1, page 38, there is a uniform algorithm that finds such an economy for those f and S_ϵ.

5) Computing Fixed-Points and Computing Equilibria. Theorem 7 shows that a fixed-point computation can be transformed into a computation of equilibrium for an excess demand function. Indeed, our proof of Theorem 7 gives a uniform algorithm for performing such transformations.

Corollary 1 asserts the general impossibility of computing equilibria for excess demand functions; and Corollary 2 asserts the general impossibility for computing equilibria for exchange economies. These corollaries show that even when all elements (e.g. consumers' preferences, endowments) in a given economy generated by an algorithm, there may not exist any algorithm to compute an equilibrium price. Therefore, when an economist attempts to construct a model of an economy and to compute an equilibrium, it may be necessary to impose more restrictions on the model, beyond the standard assumptions (such as those in the Arrow-Debreu theorem).

7. CONCLUSION

In this paper we have proposed a framework with which many models of bounded rationality can be developed. We have also examine a particular model — computable economies. Although the computable real number system and computable operations have the attraction of realism, they lack the technical convenience of classical models. In particular, the commodity and price spaces have very weak topological completeness properties.

Despite the technical inconvenience of the computable model, our positive results have shown that many important conclusions from the classical model remain valid. In particular, many of them, from utility representation to characterization of excess demand functions, can be transported to the computable (boundedly rational) environment developed in this paper.

On the other hand, we also refute the existence of general equilibrium in our computability context by finding "computable counterexamples." Some implications of our results for computational economics have also been discussed.

It has been shown in Richter and Wong (1996b), by using model-theoretic methods, that equilibrium existence can be obtained if we replace computability by definability, i.e., if we require excess demand functions or economies to be definable in a language of first order logic. The definable environment of that paper represents an alternative framework for bounded rationality. It remains to investigate other coherent models — i.e., models satisfying the uniformity principle.

Our computability approach can be extended to model economies with production, with uncertainties, with infinitely many commodities, infinitely many agents, etc. The topological tools (see e.g. Moschovakis (1964b), Zhou (1992), etc.) and integration tools (see e.g. Beeson (1985), Pour-El and Richards (1989), etc.) from recursive analysis are useful. Also, our computability approach and general bounded rationality framework can also be applied in the context of game theory, mechanism design, institutional economics, financial economics, and other areas in economic theory.

APPENDIX A: PROOFS OF THEOREMS 5 AND 7

Here we will prove Theorems 5, page 22, and 7, page 26. We require some standard notions and facts in recursive analysis, which can be found in Appendix C.

Proof of Theorem 5. Let $E = \{(\succ_i, \omega_i)\}_{i=1}^m$ and $(\bar{x}_i)_{i=1}^m$ be as given in Theorem 5.

We will first obtain a computable supporting price. For each i, Theorem 1, page 18, yields a computable function $u^i : \mathbb{R}_{c+}^l \to \mathbb{R}_c$ represents $\succ_i$, and so we can (see Remark 12(2), page 51) pick a recursive sequence (Definition 5, page 50) $\{v_k^i\}_{k \in \mathbb{N}}$ that enumerates all rational vectors $v \in \mathbb{R}_{c+}^l$ with the property that $u^i(v) > u^i(\bar{x}_i)$, or equivalently that $v \succ_i \bar{x}_i$. Each $\succ_i$ is computable, so it is continuous (Fact 2, page 53); therefore, the set $\{v_k^i : k \in \mathbb{N}\}$ is a dense subset of the set $C^i = \{y \in \mathbb{R}_{c+}^l : y \succ_i \bar{x}_i\}$. We can pick a recursive (hence computable; Remark 12(2), page 51) sequence $\{v_n\}_{n \in \mathbb{N}}$ that enumerates the set $\{v_k^1 : k \in \mathbb{N}\} + \cdots + \{v_k^m : k \in \mathbb{N}\}$. The set $\{v_n : n \in \mathbb{N}\}$ is dense in the set $C = C^1 + \cdots + C^m$; therefore C is recursively separable (see the paragraph immediately above Fact 2, page 53). The c-convexity of $\succ_i$ implies that each C_i is c-convex in $\mathbb{R}_c^l$ (as defined in Section C.6 in Appendix C), and so the set C is also. The Pareto optimality of $(\bar{x}_i)_{i=1}^m$ implies that $\sum_{i=1}^l \omega_i \notin C$. We apply a computable separating hyperplane theorem (Theorem C-3, page 61 in Appendix C), which yields a computable supporting vector $\bar{p}$ such that $\bar{p} \cdot \sum_{i=1}^m \omega_i \leqq \bar{p} \cdot y$ for all $y \in C$.

As standard, we define $\bar{\omega}_i = \bar{x}_i$ for all i, so $\sum_{i=1}^m \bar{\omega}_i \leqq \sum_{i=1}^m \omega_i$. Then the monotonicity and continuity of the preferences $\succ_i$ implies (cf. Arrow (1951)) that $(\bar{p}, (\bar{x}_i)_{i=1}^m)$ is a computable equilibrium for the computable economy $\bar{E} = \{(\succ_i, \bar{\omega}_i)\}_{i=1}^m$. Q.E.D.

Proof of Theorem 7. Let $S_{c,\epsilon}$ and $g : S_{c,\epsilon} \to S_{c,\epsilon}$ be as given in the theorem.

First, we will prove (1) in the theorem along the lines of Wong (1995). We pick (by Remark 16(5), page 58) the computable function $h : S \to S_{c,\epsilon}$ such that $\|p - h(p)\| = \min\{\|p - p'\| : p' \in S_{c,\epsilon}\}$ for all $p \in S_c$. Then $h|_{S_{c,\epsilon}}$ is the identity mapping. We define a computable function $\tilde{f} : S_c \to S_c$ by

$$\tilde{f}(p) = g(h(p)) - (g(h(p)) \cdot p)p, \tag{13}$$

which satisfies (WL), (BB) and such that $\tilde{f}(p) = 0 \Leftrightarrow [p \in S_{c,\epsilon} \ \&\ g(p) = p]$ for all $p \in S_c$. However $\tilde{f}$ violates (BC).

We define the computable function $\hat{f}(p) = (1/(lp_1), \cdots, 1/(lp_l)) - (1/(p \cdot e))e$, where $e = (1, \cdots, 1)$. Clearly, $\hat{f}$ satisfies (WL), (BB) and (BC). We pick a $\mu \in I\!R_c$ with $0 < \mu \leqq \epsilon$, and $\mu \leqq 1/l$, and $0 < \epsilon - \mu \sup\{\tilde{p} \cdot p : \tilde{p} \in S_{c,\epsilon}\ \&\ p \in S_c\}$. Then we can (Remark 19(2), page 64) pick a (C^∞) computable function $d : S_c \to [0,1]_c$ such that $d|_{S_{c,\mu}} = 0$, $d|_{S_c \setminus S_{c,\mu}} > 0$, and $d|_{S_c \setminus S_{c,\mu/2}} = 1$ Thus the function $f : S_c \to I\!R^l_c$ defined by

$$f(p) = (1 - d(p))\tilde{f}(p) + d(p)\hat{f}(p) \tag{14}$$

is computable and satisfies (WL), (BB) and (BC). It is easy to check that that $f^{-1}(0) = \{p \in S_c : p \in S_{c,\epsilon}\ \&\ g(p) = p\}$. This shows (1) in the theorem.

Now, we prove (2) in the theorem. We consider any $k = 1, 2, \cdots, \infty$. We assume that g is C^k and such that $g(S_{c,\epsilon}) \subsetneqq S_{c,\epsilon'}$ for some $\epsilon' \in I\!R_c$ with $\epsilon' > \epsilon$. The function f defined above is not differentiable because the function h used above is not differentiable, being kinked at the boundary $\partial S_{c,\epsilon}$ of $\partial S_{c,\epsilon}$.

To obtain differentiability, we pick a $\delta \in I\!R_c$ with $\epsilon < \delta < \epsilon'$, pick a large positive $M \in I\!R_c$ and pick (cf. Remark 19(3), page 64) a C^∞ function $\alpha : S_c \to [0, M]_c$ such that $\alpha|_{S_{c,\epsilon'}} = 0$ and $\alpha|_{S_c \setminus S_{c,\delta}} = M$. We define a C^∞ and computable function $\tilde{h} : S_c \to S_c$ by $\tilde{h}(p) = (p + \alpha(p)q)/\|p + \alpha(p)q\|$, where q is the vector $(1/\sqrt{l}, \cdots, 1/\sqrt{l})$. Then $\tilde{h}|_{S_{c,\epsilon'}}$ is the identity mapping. By picking a larger M, we can assume $\tilde{h}(S_c) \subsetneqq S_\epsilon$. We replace h by this $\tilde{h}$, and define f as above. Then this f is C^∞ and satisfies the properties proved above. This shows assertion (2) in the theorem. Q.E.D.

APPENDIX B: CHARACTERIZATIONS OF EXCESS DEMANDS

B.1 Introduction

We will prove a theorem characterizing market excess demand functions, which covers both the classical context (i.e. permitting arbitrary reals, and without computability restrictions) and our computability context. In particular, we will construct classical economies for given classical market excess demand functions; and we will construct computable economies for given computable market excess demand functions.

For the classical case, Theorem B-1 extends Geanakoplos' theorem (1984) by obtaining a stronger concavity property for agents' utility in the economies we construct. We also obtain an additional equilibrium invariance property as in Mas-Colell (1978, theorem): our economies have the same set of equilibria as the given excess demand functions (with a standard boundary condition). Compared to Mas-Colell's proposition, we obtain the stronger concavity utility conclusion, but make an additional differentiability (C^2) assumption on demand assumption.

For the computability case, Theorem B-1 gives Theorem 6, page 24 in Section 5.

B.2 Statement of Results

We use $\mathbb{R}^l$ to denote the l-dimensional space of real numbers; 0 denotes the origin of $\mathbb{R}^l$, and $e = (1, 1, \cdots, 1) \in \mathbb{R}^l$. For all $x, y \in \mathbb{R}^l$, we write $x \geqq y$ if $x_i \geqq y_i$ for all i; and write $x \gg y$ if $x_i > y_i$ for all i; and $x \cdot y = \sum_{i=1}^{l} x_i y_i$. We denote $\mathbb{R}^l_+ = \{x \in \mathbb{R}^l : x \geqq 0\}$; we use $\|\cdot\|$ to denote the Euclidean norm, i.e. $\|x\| = (x \cdot x)^{1/2}$. We define $B(p, p \cdot \omega) = \{x \in \mathbb{R}^l_+ : p \cdot x \leqq p \cdot \omega\}$ for all positive $p, \omega \in \mathbb{R}^l$, and define $B(p) = \{x \in \mathbb{R}^l : p \cdot x \leqq 0\}$ for all positive $p \in \mathbb{R}^l$. We denote $S = \{p \in \mathbb{R}^l : p \gg 0 \ \& \ \|p\| = 1\}$ and $S_\delta = \{p \in S : p \geqq \delta e\}$ for all positive $\delta \in \mathbb{R}$.

Consider a *classical economy* $E = \{(\succ_i, \omega_i)\}_{i=1}^m$, where for each consumer i, the consumption space is $\mathbb{R}^l_+$, and the endowment is $\omega_i \in \mathbb{R}^l$ with $\omega_i \gg 0$, and $\succ_i$ is a strictly monotone, strictly convex, and continuous strict preference relation on $\mathbb{R}^l_+$. Let the price space be S. It is well-known that the *classical excess demand function* $f : S \to \mathbb{R}^l$ of E, which is defined by

$$f(p) = \sum_{i=1}^{m} \{x \in B(p, p \cdot \omega_i) : \forall y_{y \in B(p, p \cdot \omega_i)} \neg y \succ_i x\} - \sum_{i=1}^{m} \omega_i, \qquad (15)$$

is continuous and satisfies:

WL) (Walras Law) $p \cdot f(p) = 0$ for every $p \in S$;

BB) (Bounded from Below) there is an $M \in I\!R$ such that $f(p) \geqq Me$ for every $p \in S$;

BC) (Boundary Condition) for every sequence of vectors $p_k \in S$, if $(p_k)_i \to 0$ as $k \to \infty$ for some $i = 1, \cdots, l$, then $\|f(p_k)\|$ converges to ∞.

(Cf. Arrow and Hahn (1971, ch. 4); compare (11) in Section 5.)

We now study the converse. In particular, we will construct classical economies for (classical) functions $f : S \to I\!R^l$ satisfying (WL), (BB) & (BC). Our method requires f to be twice-continuously differentiable (C^2), and gives economies E consisting of l agents, and with the property that:

SC) (Strict Concavity) $\succ_i$ is representable by a strictly concave utility function u^i for all $i = 1, \cdots, l$,

and such that E rationalizes f on the large compact portion S_δ of S, where $S_\delta = \{p \in I\!R^l : p \geqq \delta e\}$, and $\delta > 0$. Moreover, the economies E also have the same sets of equilibria as the sets of zero of the functions f.

We also notice that our construction method is in fact an algorithm, which can be applied in our computability context. In case that the given (classical) function f has a computable restriction $f|_{S_c}$ on the computable price set S_c, the restriction $E|_{I\!R^l_{c+}}$ of the classical economy E is also a computable economy (as defined in Section 5.4).

Theorem B-1 (Characterizations of Excess Demand Functions). *Let $f : S \to I\!R^l$ be a C^2 function satisfying (WL).*

1) If $0 < \delta \in I\!R$, then there is a classical economy $E = \{(\succ_i, \omega_i)\}_{i=1}^l$ satisfying (SC) and such that its excess demand function $\tilde{f}$ satisfies:

a) $\tilde{f}|_{S_\delta} = f|_{S_\delta}$.

2) If f also satisfies (BB) and (BC), and $0 < \delta \in I\!R$ is also sufficiently small, then there is a classical economy $E = \{(\succ_i, \omega_i)\}_{i=1}^l$ satisfying (SC) and such that its excess demand function $\tilde{f}$ satisfies (a) and also

b) $\tilde{f}^{-1}(0) = f^{-1}(0) \subseteqq S_\delta$.

3) If the restriction $f|_{S_c}$ is a computable function from S_c into $I\!R^l_c$, then the economies E claimed in (1) and (2) above can be chosen so that the restriction $E|_{I\!R^l_{c+}} = \{(\succ_i |_{I\!R^l_{c+}}, \omega_i)\}_{i=1}^l$ of E is a computable economy and that $\succ_i |_{I\!R^l_{c+}}$ is representable by a strictly c-concave utility function $u^i : I\!R^l_{c+} \to I\!R_c$.

Remark 5.

a) If we drop the (SC) conclusion, then Theorem B-1(1) becomes Geanakoplos' theorem (1984).

b) If we drop the (C^2) assumption and the (SC) conclusion, then Theorem B-1 becomes Mas-Colell's (1977).

c) Theorem B-1(2) was stated earlier in Wong (1996b, Proposition).

Remark 6.

Theorem 6, page 24, is an immediate corollary of the theorem. To see this, let $0 < \epsilon \in I\!R_c$ and let $g : S_c \to I\!R^l_c$ be a C^2 computable function satisfying (WL), (BB) and (BC) on S_c. Then by our definition of a C^2 computable function, the function g can be extended to a C^2 function $f : S \to I\!R^l$. It is clear that f satisfies (WL), (BB) and (BC) on S. Then we can pick a positive computable δ and a classical economy E, so that the (classical) excess demand function $\tilde{f}$ of E satisfies (a,b) in Theorem B-1 for f. By Theorem B-1(3), we can assume that the restriction $E|_{I\!R^l_{c+}} = \{(\succ_i |_{I\!R^l_{c+}}, \omega_i)\}$ is a computable economy, which clearly satisfies (10), page 23. It is easily shown that the excess demand function (as given in (11), page 23) of the computable economy $E|_{I\!R^l_{c+}}$ is the function $\tilde{f}|_{S_c}$. Notice that $g = f|_{S_c}$; then by (a) in Theorem B-1 we have $\tilde{f}|_{S_{c,\delta}} = g|_{S_{c,\delta}}$, and by (b) we have $\tilde{f}|^{-1}_{S_c}(0) = g^{-1}(0) \subseteqq S_{c,\delta}$. This proves Theorem 6.

In our proof of Theorem B-1, we will make use of the *Debreu individual excess demand functions* $f^i : S \to I\!R^l$, defined by

$$f^i(p) = \beta_i(p)\pi^i(p) \tag{16}$$

where $\beta_i : S \to I\!R$ is C^2 and positive on S, and and π^i is the orthogonal projection from the i-th standard coordinate vector e^i into the plane orthogonal to p, i.e.

$$\pi^i(p) = e^i - (e^i \cdot p)p. \tag{17}$$

The functions β^i may also satisfy the property that there is a $\sigma > 0$ such that:

$$\beta_i(p) = \frac{1}{l}\frac{1}{p_i} \qquad \text{for all } p \in S \backslash S_\sigma. \tag{18}$$

B.3 Proof of Theorem B-1

We will first focus on part 2 of the theorem. Then we will discuss how to obtain the other assertions of the Theorem by modifying our proof of part (2).

Proof of Theorem B-1(2).

I) Boundary Step. Let $f : S \to \mathbb{R}^l$ be C^2 and satisfy (WL). We consider a $\delta > 0$ and a C^2 (even C^∞) function $d : S \to [0,1]$ with the properties that:

$$\text{a)}\ \ d|_{S_\delta} = 1, \qquad \text{b)}\ \ d|_{S \setminus S_{\delta/2}} = 0. \tag{19}$$

We define a function $\hat{f} : S \to \mathbb{R}_c$ by

$$\hat{f}(p) = d(p)f(p) + (1 - d(p))g(p), \tag{20}$$

where

$$g(p) = (1/l)(1/p_1, \cdots, 1/p_l) - p. \tag{21}$$

By (19) and (20) we have $\hat{f}|_{S_\delta} = f|_{S_\delta}$, and $\hat{f}|_{S \setminus S_{\delta/2}} = g|_{S \setminus S_{\delta/2}}$. It is clear that the function $\hat{f}$ is C^2 and satisfies (WL) and (BB) and (BC). Now, as the hypothesis of Theorem B-1(2) requires, we further assume that f satisfies (BB) and (BC). Then by Mas-Colell (1977, p. 120 (Lemma 1) and p. 125), for δ sufficiently small, we have: $f^{-1}(0) = \hat{f}^{-1}(0) \subseteqq S_\delta$. We pick any such δ and fix a function d as given above.

II) Decomposition Step. We will now decompose the function $\hat{f}$ into Debreu individual excess demand functions. Since $\hat{f}$ satisfies (BB), we can pick a $K > 1$ with $\hat{f}(S) + (K\delta/2)e \gg 0$. We can pick a C^2 (even C^∞) function $\theta : S \to [1, K]$ such that

$$\text{a)}\ \theta|_{S_{\delta/2}} = K, \qquad \text{b)}\ \theta|_{S \setminus S_{\delta/3}} = 1. \tag{22}$$

For all $p \in S_{\delta/2}$ we have $\hat{f}(p) + \theta(p)p \geqq \hat{f}(p) + (K\delta/2)e \gg 0$. For all $p \in S \setminus S_{\delta/2}$, by (19*b*) and (20) we have $\hat{f}(p) + \theta(p)p = g(p) + \theta(p)p \geqq g(p) + p \gg 0$ because

$$g(p) + p = (1/l)(1/p_1, \cdots, p_l). \tag{23}$$

Thus $\hat{f}(p) + \theta(p)p \gg 0$ for all $p \in S$, and we can write

$$\hat{f}(p) + \theta(p)p = \sum_{i=1}^{l} \beta_i(p)e^i \qquad \text{for all } p \in S, \tag{24}$$

so that the functions $\beta_i : S \to \mathbb{R}$ are C^2 and positive on S. By (24) we have: $\hat{f} = \sum_{i=1}^{l} f^i$, where f^i is the Debreu individual excess demand function defined by (16). Also by (19b) we have $\hat{f}|_{S\setminus S_{\delta/3}} = g|_{S\setminus S_{\delta/3}}$; so by (22$b$), (23), and (24), the functions β_i satisfy (18) with $\sigma = \delta/3$.

III) Economy Construction Step. By Proposition 1(2) below, we can pick $M, \mu, \tilde{\mu} > 0$ with $\tilde{\mu} < \mu < \delta/3$ such that there exist positive vectors $\omega_i, \cdots, \omega_l \in \mathbb{R}^l$, and strictly monotone and strictly concave functions $u^1, \cdots, u^l : \mathbb{R}^l_+ \to \mathbb{R}$ such that the functions $\tilde{f}^i : S \to \mathbb{R}^l$ defined by (25) below satisfy the properties (I-II) in Proposition 1 below. Then $\tilde{f}(p) = \sum_{i=1}^{l} \tilde{f}^i(p)$ is the excess demand function of the classical economy $E = \{(\succ_i, \omega_i)\}_{i=1}^{l}$ that satisfies (SC) for all $i = 1, \cdots, l$, where each $\succ_i$ is defined by $x \succ_i y \Leftrightarrow u^i(x) > u^i(y)$.

We claim that $\tilde{f}^{-1}(0) \subseteqq S_\mu$. Consider any $p \in S\setminus S_\mu$. We have either $p_i < \tilde{\mu}$ for some i or $p \in S_{\tilde{\mu}}$, so by (II.a) and (II.b) in Proposition 1 for some coordinate i, we have $\sum_{j\neq i}(\tilde{f}^j(p))_i + (\tilde{f}^i(p))_i \geqq M$, hence $\tilde{f}(p) \neq 0$.

We now show that the function $\tilde{f}$ satisfies the properties (a-b) in the theorem. By (I) in Proposition 1 below (with $\bar{\mu} = \mu$), we have $\tilde{f}|_{S_\mu} = \hat{f}|_{S_\mu}$. Notice that $\mu < \delta/3 < \delta$ and $\hat{f}|_{S_\delta} = f|_{S_\delta}$; therefore $\tilde{f}|_{S_\delta} = \hat{f}|_{S_\delta} = f|_{S_\delta}$; this shows (a). Also, since $\tilde{f}^{-1}(0) \subseteqq S_\mu$ and $\hat{f}^{-1}(0) \subseteqq S_\delta \subseteqq S_\mu$, it follows that $\tilde{f}^{-1}(0) = \hat{f}^{-1}(0)$. Since $\hat{f}^{-1}(0) = f^{-1}(0) \subseteqq S_\delta$ by Step I, we have $\tilde{f}^{-1}(0) = f^{-1}(0) \subseteqq S_\delta$; this shows (b). This completes our proof of Theorem B-1(2).

Proof of Theorem B-1(1). Let $f : S \to \mathbb{R}^l$ be C^2 and satisfies (WL), and let $0 < \delta \in \mathbb{R}$. We give two different methods for finding a classical economy E satisfying (a).

A) As in the beginning of Step I above, we can pick a C^2 function $d : S \to [0, 1]$ satisfying (19). Then the function $\hat{f}$ defined by (20) is C^2, satisfies (WL), (BB), and (BC). We can apply Theorem B-1(2) and find a positive $\delta' < \delta$ and a classical economy E whose excess demand function $\tilde{f}$ agrees with $\hat{f}$ on $S_{\delta'} \supseteq S_\delta$, and so $\tilde{f}|_{S_\delta} = \hat{f}|_{S_\delta} = f|_{S_\delta}$ by (19). This shows condition (a) in Theorem B-1(1) holds.

B) Alternatively, we can use a simpler version of the proof of Theorem B-1(2). As in Geanakoplos (1984), we can simply pick a C^2 function $\theta : S_c \to \mathbb{R}$ satisfying (24) with $\hat{f} = f$ and define function β^i by (24) with $\hat{f} = f$, and define Debreu individual demand functions f^i as in Step II. We can apply Proposition 1(1) to the functions f^i; we obtain individual utility functions u^i and ω_i as given in Proposition 1(1). Then we can construct a classical economy E as in the first paragraph of Step III; its excess demand function $\tilde{f}$ clearly satisfies (a) in Theorem B-1.

Proof of Theorem B-1(3).

A′) The computability case with (BB) & (BC). Let the given function $f : S \to I\!R^l$ be C^2 and satisfy (WL), (BB) and (BC), and also the property that $f|_{S_c}$ is a computable function. We now show that with a more delicate proof, we can ensure that the economy E of Theorem B-1(2) satisfies the property claimed in Theorem B-1(3).

1) We can assume that the restriction $d|_{S_c}$ of the function d chosen in Step I is a computable function (by Remark 19(2), page 64). It is clear that $g|_{S_c}$ is computable function from S_c into $I\!R^l_c$, and so if $f|_{S_c}$ is a computable function from S_c into $I\!R^l_c$, then $\hat{f}|_{S_c}$ (see (20)) is a computable function from S_c into $I\!R^l_c$ (cf. Remark 2, page 16).

2) We can clearly assume the K chosen in Step II to be computable. By replacing θ (see (22)) we can also assume that $\theta|_{S_c}$ is a computable function from S_c into $I\!R_c$ (by Remark 19(2), page 64).

Then by the modifications (1) and (2), we can assume the $\beta_i|_{S_c}$ are also computable functions from S_c into $I\!R_c$.

3) We can assume that the M and μ chosen in Step III are computable numbers. Then by Proposition 1(3) we can assume the ω_i and u^i chosen in Step III satisfy property (III) in Proposition 1(3). Notice that for all consumers i in the economy E defined in Step III, the preference $\succ_i|_{I\!R^l_{c+}}$ is represented by the computable utility function $u^i|_{I\!R^l_{c+}} I\!R^l_{c+} \to I\!R_c$, and so the preference $\succ_i|_{I\!R^l_{c+}}$ is computable (by Theorem 1). Thus $E|_{I\!R^l_{c+}}$ is a computable economy. Also, since u^i strictly concave, the function u^i is strictly c-concave. Thus the economy E satisfies the computability requirements in Theorem B-1(3).

B′) The computability case without (BB) & (BC). Suppose $f : S \to I\!R^l$ is C^2 and satisfies (WL), and $f|_{S_c}$ is a computable function. As in method A in the proof of Theorem B-1(1), we can pick a computable $\delta' \leqq \delta$ and we can pick a C^2 (even C^∞) function $d : S \to [0,1]$ satisfying (19) for δ' and such that $d|_{S_c}$ is a computable function (by Remark 19(2), page 64). Then the function $\hat{f}$ defined by (20) is C^2, satisfies (WL), (BC) and (BB) and $\hat{f}|_{S_c}$ is computable. By the case (A′) we can pick a classical economy E as claimed in Theorem B-1(3) and such that its excess demand function $\tilde{f}$ agrees with $\hat{f}$ on $S_\delta \subsetneqq S_{\delta'}$. Since $f|_{S_{\delta'}} = \hat{f}|_{S_{\delta'}}$, we have $\tilde{f}|_{S_\delta} = f|_{S_\delta}$, i.e. (a) in Theorem B-1 holds. This proves case (B′). (Of

course, one can also give a computable analogue of method B discussed above.)
Q.E.D.

B.4 Utility functions for Debreu excess demands

We will construct a *strongly classical consumer* (u^i, ω_i) for a function f^i as defined in Step I, i.e. $0 < \omega_i \in \mathbb{R}^l$ and $u^i : \mathbb{R}^l_+ \to \mathbb{R}$ is strictly concave, strictly monotone and continuous. In particular, the *excess demand function* $\tilde{f}^i : S \to \mathbb{R}^l$ of the consumer (u^i, ω_i), which is defined by (cf. (15)):

$$\tilde{f}^i(p) = \{x \in B(p, p \cdot \omega_i) : u^i(x) \geqq u^i(y) \text{ for all } y \in B(p, p \cdot \omega_i)\} - \omega_i, \quad (25)$$

agrees with f^i on a large compact portion $S_\mu \subseteqq S_c$. In additional, our method permits us to obtain a boundary condition for $\tilde{f}^i$ (see properties II in Proposition 1) and a computability condition for (u^i, ω_i) (see assertion 3 in Proposition 1).

Proposition 1. *Consider any $i = 1, \cdots, l$. Let $\beta_i : S \to \mathbb{R}$ be C^2 and positive on S. Let $f^i : S \to \mathbb{R}^l$ be defined by (16).*

1) *(Strictly concave rationalization) Assume $0 < \bar{\mu} \in \mathbb{R}$. Then there exists a strongly classical consumer (u^i, ω_i) whose excess demand function $\tilde{f}^i : S_c \to \mathbb{R}^l_c$ satisfies:*

   I) $\quad \tilde{f}^i|_{S_{\bar{\mu}}} = f^i|_{S_{\bar{\mu}}}$.

2) *(Rationalization with boundary properties) Assume $0 < \sigma \in \mathbb{R}$ and (18) holds. For all sufficiently large real M and all sufficiently small real $\mu > 0$, the following is true: for all sufficiently small positive real $\tilde{\mu} \leqq \mu$, there exists a strongly classical consumer (u^i, ω_i) whose excess demand function $\tilde{f}^i$ satisfies (I) for μ, and has the "boundary" properties:*

   II.a) $\tilde{f}^i(p) \geqq -Me$ *for all $p \in S$,*
   II.b) $(\tilde{f}^i(p))_i \geqq lM$ *for all $p \in S$ with $p_i \leqq \tilde{\mu}$ and all $p \in S_{\tilde{\mu}}$ with $p_i \leqq \mu$.*

3) *(Computable rationalization) Suppose $\beta^i|_{S_c}$ is also a computable function from S_c into $\mathbb{R}_c$.*

   a) Then the consumer (u^i, ω_i) claimed in (1) above can be chosen so that:

   III) $\quad \omega_i \in \mathbb{R}^l_c$ *and $u^i|_{\mathbb{R}^l_{c+}}$ is a computable function from $\mathbb{R}^l_{c+}$ into $\mathbb{R}_c$;*

   b) If, in addition to the hypotheses of (2) above, one has $M, \mu \in \mathbb{R}_c$, then the consumer claimed in (2) can also be chosen with property (III).

Remark 7.

a) Proposition 1(1) is a sharper version of Geanakoplos' result (1984) with strictly concavity (rather than just concave) utility conclusion.

b) Proposition 1(2) was stated earlier in Wong (1996b, Lemma).

B.5 Proof of Proposition 1(1)

This lemma is trivial for $l = 1$. We now assume $l > 1$.

Proof of Proposition 1(1).

Step 1: Finding a utility function V on a rectangle X (Geanakoplos (1984)).)

Since f^i is C^2 (hence continuous), we can choose rectangle $X = \Pi_{j=1}^{l}[a_j, b_j]$ that contains the compact set $f^i(S_{\bar{\mu}})$ in its interior. For all $p \in S$, by (16) and (17) we have $(f^i(p))_i > 0$ and $(f^i(p))_j < 0$ for all $j \neq i$; therefore, we can choose the vertices of the rectangle so that $a_i, b_i > 0$ and $0 > a_j, b_j$ for all $j \neq i$.[32]

We can define the C^2 real function $q : X \to S$ by

$$q(x) = e^i - x_i(x/x \cdot x); \tag{26}$$

thus $q(x)$ is the unique $p \in S$ such that x is on the ray $\{\lambda\pi^i(p) : 0 \leqq \lambda \in I\!R\}$ (cf. Geanakoplos (1984, p. 6)). By Geanakoplos (p. 9, 2nd paragraph), if $\alpha_1 > 0$ is small enough, and $n_1 > 0$ is big enough, then there is a positive $\bar{N} \in I\!R$ such that for all $n_2 \geqq \bar{N}$, the function $V : X \to I\!R$ defined by[33]

$$V(x) = -\exp\{n_2(\|\pi^i(q(x)) - e^i\|^2 + \alpha_1 \exp\{n_1\|x - \beta_i(q(x))\pi^i(q(x))\|^2\})\}. \tag{27}$$

is a strictly monotone and strictly concave C^2 function with negative definite matrix $D^2V(x)$ for all $x \in X$, and has the property that $f^i(p)$ uniquely maximizes $V(x)$ on the set $B(p) \cap X$ for all $p \in S_{\bar{\mu}}$, where $B(p) = \{x \in I\!R^l : p \cdot x \leqq 0\}$. We fix such α_1, n_1, n_2.[34]

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[32] We can even require these $a_1, b_1, \cdots, a_l, b_l$ to be rational or computable numbers.

[33] This modifies the second displayed formula in p. 7 of Geanakoplos (1984), in order to obtain differentiability of $V(\cdot)$ and negative definiteness of its Hessian. Corresponding changes in pages 7–9 there support the proof of the rationality, strict monotonicity, strict concavity, and negative definiteness properties claimed for $V(\cdot)$. (Although the concavity and negative definiteness claims in lines 9 and 11 of his page 8 are not justified, the negative definiteness claim in his line 12 is correct, as a direct calculation can show.)

[34] It is clear that these α_1, n_1, n_2 can be chosen to be rational or computable numbers.

Step 2) Extending V to a strictly concave function U on all of $\mathbb{R}^l$.

We pick any small positive $\alpha_2 \in \mathbb{R}$. For all $x \in X$ and all $y \in \mathbb{R}^l$, we define

$$L_x(y) = V(x) + DV(x)(y-x)^T + \sum_{j=1}^{l}(1 - \exp\{-\alpha_2(y_j - x_j)\}). \qquad (28)$$

(For any fixed $x \in X$, Geanakoplos' formula $V(x) + DV(x-y)^T$ gives the affine tangent plane H_x approximation to the V-surface (as a function of y) near x. We have added the strictly concave sum in order to obtain (in (29) below) a strictly concave function having the same tangent plane as L_x. We will see below (32) that, by picking α_2 small enough, we can also assume that the $L_x(\cdot)$-surface lies between the tangent plane H_x and the V-surface.)

For all x, the function $L_x(y)$ is clearly strictly concave in y. By picking a smaller $\alpha_2 \in \mathbb{R}$, we can assume $DV(x) \gg (\alpha_2, \cdots, \alpha_2)$ for all x in the compact set X; so for all $x \in X$, the function $L_x(y)$ is also strictly monotone in y. For all $y \in \mathbb{R}^l$, the mapping $x \mapsto L_x(y)$ a is continuous function on the compact set X; then it follows that the function $U : \mathbb{R}^l \to \mathbb{R}$ defined by

$$U(y) = \min_{x \in X} L_x(y) \qquad (29)$$

is strictly monotone and strictly concave.(35)

To ensure that U coincides with V on X, notice that V is C^2 and the matrix $D^2V(\tilde{x})$ is negative definite for all $\tilde{x}$ in the compact set X; so by picking a still smaller $\alpha_2 \in \mathbb{R}$ we can assume:

$$-zD^2V(\tilde{x})z^T > (\alpha_2)^2 z \cdot z \exp\{\alpha_2 K\} \qquad (30)$$

for all $\tilde{x} \in X$ and all non-zero vectors $z \in \mathbb{R}^l$, where $K = \max_{x,y \in X} ||x - y||$.(36)

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(35) **Notice that U and V depend on $\bar{\mu}$ (through X).**

(36) **It is clear that we can choose α_2 to be a rational or computable number.**

Then for all pairs of distinct $x, y \in X$, we have $L_x(y) > V(y)$ because:

$$\begin{aligned} L_x(y) - V(y) &= L_x(y) - V(y) - L_x(x) - V(x) \\ &= \int_0^1 \int_0^1 \frac{d^2}{dt^2}(L_x(ty + (1-t)x) - V(ty + (1-t)x))dt\ dt \\ &= \int_0^1 \int_0^1 (-(\alpha_2)^2 \sum_{j=1}^{l} (y_j - x_j)^2 \exp\{-\alpha_2 t(y_j - x_j)\} \\ &\qquad - (y-x)D^2V(ty + (1-t)x)(y-x)^T)dt\ dt \\ &\geqq \int_0^1 \int_0^1 (-(\alpha_2)^2 \sum_{j=1}^{l} (y_j - x_j)^2 \exp\{\alpha_2 K\} \\ &\qquad - (y-x)D^2V(ty + (1-t)x)(y-x)^T)dt\ dt \\ &> 0, \end{aligned} \tag{31}$$

where the first equality holds because $L_x(x) = V(x)$ by (28), and the last inequality is obtained by substituting $z = (x - y)$ and $\tilde{x} = (ty + (1-t)x)$ into (30). Also, notice that $V(y) = L_y(y)$ when $y \in X$; so we have:

$$V(y) \leqq L_x(y) \qquad \text{for all } x, y \in X. \tag{32}$$

Then by (29) we have:

$$U|_X = V|_X. \tag{33}$$

For all $p \in S_{\bar{\mu}}$, by Step 1 the the vector $f^i(p)$ uniquely maximizes $V(x)$ on $B(p) \cap X$, so by (33) $f^i(p)$ uniquely maximizes $U(x)$ on $B(p) \cap X$, hence also on $B(p)$ since U is (strictly) concave.

Step 3) Choosing (u^i, ω_i).

We can pick any $M > 0$ with $f^i(p) > -Me$ for all p in the compact set $S_{\bar{\mu}}$.(37) We define $\omega_i = (M, \cdots, M)$ and define $u^i : I\!R^l_+ \to I\!R$ by $u^i(x) = U(x - \omega_i)$. Obviously, the function u^i is C^2, strictly monotone and strictly concave. For any $p \in S_{\bar{\mu}}$, the vector $f^i(p) + \omega_i$ is nonnegative, belongs to $B(p, p \cdot \omega_i)$; it also uniquely maximizes $u^i(x)$ on $B(p, p \cdot \omega_i)$ since $f^i(p)$ uniquely maximizes $U(x)$ on $B(p)$. Therefore, we have $f^i(p) = \tilde{f}^i(p)$ for all $p \in S_{\bar{\mu}}$, where $\tilde{f}^i$ is the excess demand function (see (25)) of the classical consumer (u^i, ω_i). Thus $f^i|_{S_{\bar{\mu}}} = \tilde{f}^i|_{S_{\bar{\mu}}}$, as we desire.

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(37) It is clear that we can choose M to be a rational or computable number.

Proof of Proposition 1(2). By (16), (17), and (18), we have:

$$(f^i(p))_i = (1/l)((1/p_i) - p_i) \qquad (34a)$$

$$(f^i(p))_j = -(1/l)p_j \qquad \text{for } j \neq i \qquad (34b)$$

for all $p \in S \backslash S_\sigma$. Since f^i is C^2, if $M > 1$ is sufficiently large, we have $f^i(p) \geqq -Me$ for all p in the compact set S_σ, and so by (34) we have:

$$f^i(p) \geqq -Me \qquad (35)$$

for all $p \in S$. We fix such an M.

By (34) it is clear that if a positive μ is sufficiently small, then:

$$p_i \leqq \mu \;\Rightarrow\; (f^i(p))_i \geqq 2lM \text{ for all } p \in S. \qquad (36)$$

We fix such a $\mu < \sigma$. Notice that (34) holds for all $p \in S$ with $p_i \leqq \mu$ (or when p_i is near μ).

We will make use of the two vectors $\bar{p}, \hat{p} \in S_\mu$ such that

$$\begin{array}{ll} 1) & \text{a) } \bar{p}_i = \mu, \quad \text{b) } \bar{p}_j = ((1-\mu^2)/(l-1))^{1/2} \text{ for } j \neq i; \\ 2) & \text{a) } \hat{p}_i = \mu, \quad \text{b) } \hat{p}_{i'} = (1-(l-1)\mu^2)^{1/2}, \\ & \qquad \text{c) } \hat{p}_j = \mu \text{ for } j \neq i, i', \end{array} \qquad (37)$$

where i' is a coordinate different from i. Notice that $\hat{p}$ (uniquely) maximizes $p_{i'}$ on S_μ, i.e.

$$\hat{p}_{i'} - p_{i'} \geqq 0 \qquad \text{for all } p \in S_\mu. \qquad (38)$$

Moreover, it can be checked that:[38]

$$\hat{p}_{i'} - p_{i'} \geqq (\mu/(2\hat{p}_{i'}))(p_j - \hat{p}_j) \qquad \text{for all } p \in S_\mu \text{ and all } j \neq i'. \qquad (39)$$

We now define $\bar{x} = f^i(\bar{p})$. For any $p \in S_c$, by substituting (37(1)) into (34) we have:

$$\begin{aligned} p \cdot \bar{x} &= p \cdot f^i(\bar{p}) \\ &= (1/l)[p_i(1/\mu - \mu) - \sum_{j \neq i} p_j((1-\mu^2)/(l-1))^{1/2}] \\ &\leqq (1/l)[p_i(1/\mu - \mu) - (1-p_i)((1-\mu^2)/(l-1))^{1/2}], \end{aligned} \qquad (40)$$

[38] To see (39), consider any $p \in S_\mu$. Notice that $(\hat{p}_{i'} - p_{i'})^2 \geqq 0$, so $2\hat{p}_{i'}(\hat{p}_{i'} - p_{i'}) \geqq (\hat{p}_{i'})^2 - (p_{i'})^2$; hence $2\hat{p}_{i'}(\hat{p}_{i'} - p_{i'}) \geqq \sum_{j \neq i'}((p_j)^2 - (\hat{p}_j)^2) = \sum_{j \neq i'}((p_j)^2 - (\mu)^2)$. Also, since $p \in S_\mu$, we have $p_j \geqq \mu$ for all j. Hence for all $j \neq i$, we have $2\hat{p}_{i'}(\hat{p}_{i'} - p_{i'}) \geqq (p_j)^2 - \mu^2 \geqq \mu(p_j - \mu) = \mu(p_j - \hat{p}_j)$; so (39) follows immediately.

where the last inequality holds because $\sum_{j \neq i} p_j \geqq 1 - p_i$ for all $p \in S$. Therefore, for all sufficiently small $\tilde{\mu} > 0$, we have:

$$p_i \leqq \tilde{\mu} \Rightarrow p \cdot \bar{x} \leqq 0 \qquad \text{for all } p \in S. \tag{41}$$

We now fix any such $\tilde{\mu}$ with $\tilde{\mu} < \mu$.

We now show that it suffices to find a continuous, strictly monotone and strictly concave function $\tilde{u}^i : I\!R^l \to I\!R$ such that

$$\begin{aligned} &1)\ f^i(p) \text{ uniquely maximizes } \tilde{u}^i \text{ on } B(p) \text{ for all } p \in S_\mu, \\ &2)\ \tilde{u}^i(y) \geqq \tilde{u}^i(\bar{x}) \text{ implies } y_i \geqq lM \text{ for all } y \in I\!R^l, \\ &3)\ \text{if } p \in S_{\tilde{\mu}} \text{ with } p_i \leqq \mu, \text{ and } y \in I\!R^l \text{ with } \tilde{u}^i(y) \geqq \\ &\qquad \tilde{u}^i(f^i(p)), \text{ then } y_i \geqq lM, \end{aligned} \tag{42}$$

where $B(p) = \{x \in I\!R^l : p \cdot x \leqq 0\}$ (as defined in Step 1 above).

For such a $\tilde{u}^i$, we define $\omega_i = (M, \cdots, M)$ and define $u^i : I\!R^l_+ \to I\!R$ by $u^i(x) = \tilde{u}^i(x - \omega_i)$. The function u^i is continuous, strictly monotone and strictly concave. We consider the excess demand function $\tilde{f}^i$ of the strongly classical consumer (u^i, ω_i) (see (25)). Conditions (35) and (42(1)) imply, as in the last paragraph of Step 3 above, that $\tilde{f}^i$ satisfies property (I) in Proposition 1 for μ. Also, $\tilde{f}^i(S) \geqq -\omega_i = -Me$, i.e. $\tilde{f}^i$ satisfies property (II(a)). To see property (II(b)), we first suppose $p \in S$ with $p_i \leqq \tilde{\mu}$. By (35) and (41), the vector $\bar{x} + \omega_i \in B(p, p \cdot \omega_i)$, so $u^i(\tilde{f}^i(p) + \omega_i) \geqq u^i(\bar{x} + \omega_i)$, hence $\tilde{u}^i(\tilde{f}^i(p)) \geqq \tilde{u}^i(\bar{x}^i)$; then by (42(2)) we have $(\tilde{f}^i(p))_i \geqq lM$. Now we suppose $p \in S_{\tilde{\mu}}$ with $p_i \leqq \mu$. By (35) the vector $f^i(p) + \omega_i \in B(p, p \cdot \omega_i)$, so it follows that $\tilde{u}^i(\tilde{f}^i(p)) \geqq \tilde{u}^i(f^i(p))$, hence by (42(3)) we have $(\tilde{f}^i(p))_i \geqq lM$. Thus $\tilde{f}^i$ also satisfies (II(b)).

It remains to find such function $\tilde{u}^i$. We pick functions V and U as defined in Steps 1 and 2 (of proof of Proposition 1(1)) above with $\bar{\mu} = \tilde{\mu}$.

We define $\hat{x} = f^i(\hat{p})$, where $\hat{p}$ is defined by (37(2)). For every $\lambda \in I\!R$ with $0 \leqq \lambda < 1$, we define the function $g_\lambda : I\!R^l \to I\!R$ by

$$\begin{aligned} g_\lambda(x) = V(\hat{x}) &+ (1 - \exp\{-\lambda(x_i - \hat{x}_i)\}) \\ &+ \lambda(1 - \exp\{-\lambda^2(x_{i'} - \hat{x}_{i'})\}) \\ &\quad + \lambda \sum_{j \neq i, i'} (1 - \exp\{-\lambda^3(x_j - \hat{x}_j)\}). \end{aligned} \tag{43}$$

It is clear that for all $\lambda > 0$, the function g_λ is C^2, strictly monotone, and strictly concave.

In order to ensure (42(2)) and (42(3)) for the function $\tilde{u}^i$ that we seek, it will be useful to show that if $\lambda > 0$ is sufficiently small then:

$$[p_i \leqq \mu \;\;\&\;\; g_\lambda(y) \geqq g_\lambda(f^i(p))] \text{ implies } y_i > lM \tag{44}$$

for all $y \in \mathbb{R}^l$ and all $p \in S$. (So when we view g_λ as a "utility" function, (44) ensures that, if p_i is "small," then the component y_i of every bundle y in the upper contour C of $f^i(p)$ is "large" — i.e. C lies in the upper half space $\{x \in \mathbb{R}^l : x_i > lM\}$.)

To verify (44), consider any $p \in S$ with $p_i \leqq \mu = \hat{p}_i$. By (34a) we have $(f(p))_i = (1/l)(1/p_i - p_i) \geqq (1/l)(1/\mu - \mu) = f^i(\hat{p})_i = \hat{x}_i$; so $(f^i(p))_i - \hat{x}_i \geqq 0$. For all $j \neq i$, we have $(f^i(p))_j \geqq -M$ by (35), and $\hat{x}_j = (f^i(\hat{p}))_j \leqq 0$ by (34b) & (37(2.a,b)); so $f^i(p)_j - \hat{x}_j \geqq -M$. Also notice that $\lambda < 1$, so by (43) have:

$$g_\lambda(f^i(p)) \geqq V(\hat{x}) + \lambda(l-1)(1 - \exp\{\lambda M\}). \tag{45}$$

Now consider any $y \in \mathbb{R}^l$ with $y_i \leqq lM$. We have $\hat{x}_i = (f^i(\hat{p}))_i \geqq 2lM$ by (36) & (37(2.a)), so $y_i - \hat{x}_i \leqq -lM$. Also, the sum of the last two lines in (43) is bounded above by $\lambda(l-1)$; so

$$g_\lambda(y) \leqq V(\hat{x}) + 1 - \exp\{\lambda lM\} + \lambda(l-1). \tag{46}$$

By (45) and (46), we have:

$$g_\lambda(f^i(p)) - g_\lambda(y) \geqq H(\lambda), \tag{47}$$

where $H(\lambda) = -\lambda(l-1)\exp\{\lambda M\} - (1 - \exp\{\lambda lM\})$. Notice that $H(0) = 0$ and $DH(0) = -(l-1) + lM > 0$. Therefore, if $\lambda > 0$ is sufficiently small, then we have $H(\lambda) > 0$, so by (47) we have (44) for all $y \in \mathbb{R}^l$ and all $p \in S$.

In order to ensure property (42(1)) for the function $\tilde{u}^i$ that we seek, it will be useful to show that if λ is sufficiently small, then:[39]

$$g_\lambda(f^i(p)) \geqq V(f^i(p)) \text{ for all } p \in S_\mu. \tag{48}$$

(The following proof of (48) ends in the paragraph preceding (61).)

We suppose by contradiction that there exists a sequence of $\lambda_k > 0$ convergent to 0 and a sequence of $p_k \in S_\mu$ such that

$$g_{\lambda_k}(f^i(p_k)) < V(f^i(p_k)) \qquad \text{for all } k. \tag{49}$$

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[39] This need not hold for p outside S_μ, or for bundles not chosen by f^i.

Notice that for the function q defined in (26), we have $q(f^i(p)) = p$ for all $p \in S_{\bar{\mu}}$ (as in Geanakoplos (1984, p. 7)). Thus for $x = f^i(p)$ and $p \in S_\mu$, the argument of the second exponential in (27) vanishes, and therefore:

$$V(f^i(p)) = -\exp\{n_2((p_i)^2 + \alpha_1)\} \tag{50}$$

for all $p \in S_\mu$. We can pick a positive $N > 0$ such that $-\exp\{n_2(p_i)^2 + \alpha_1\} \leqq -\exp\{n_2(\mu)^2 + \alpha_1\} - (1/N)(p_i - \mu)$ for all $p \in S_\mu$; so when $\hat{x} = f(\hat{p})$ and $\hat{p}_i = \mu$, then:

$$V(f^i(p)) \leqq V(\hat{x}) - (1/N)(p_i - \mu) \tag{51}$$

for all $p \in S_\mu$.

Since S_μ is compact, by picking a subsequence we can assume $p_k \to p^*$ for some $p^* \in S_\mu$. There are two possibilities:

Case 1) Suppose $p_i^* > \mu$. Then we have $p_i^* > \hat{p}_i = \mu$. By (50) we have $V(f^i(p^*)) < V(f^i(\hat{p})) = V(\hat{x})$. Since $V(f^i(p_k)) \to V(f^i(p^*))$ and $g_{\lambda_k}(f^i(p_k)) \to g_0(f^i(p^*)) = V(\hat{x})$ as $k \to \infty$, it follows that $g_{\lambda_k}(f^i(p_k)) > V(f^i(p_k))$ for all sufficiently large k; this contradicts (49).

Case 2) Suppose $p_i^* = \mu$. Since $\sigma > \mu = \lim_{k\to\infty}(p_k)_i$, by picking a subsequence we can assume for all k that $(p_k)_i < \sigma$, and so (34) holds for all the vectors p_k, hence by (34) and (37(1.b)) we have: $(f^i(p_k))_i - \hat{x}_i = (1/l)[(1/(p_k)_i - (p_k)_i)) - (1/\mu - \mu)]$. Therefore by picking a larger N we can assume (in addition to (51)):

$$|(f^i(p_k))_i - \hat{x}_i| \leqq N((p_k)_i - \mu). \tag{52}$$

By (34b) we have

$$f^i(p_k)_j - \hat{x}_j = (1/l)(\hat{p}_j - (p_k)_j) \qquad \text{for all } j \neq i. \tag{53}$$

Then by picking a still larger N we can assume $N > (1/l)2\hat{p}_{i'}/\mu$; so by (38) and (39) and (53) we have:

$$f^i(p_k)_j - \hat{x}_j \geqq -N(\hat{p}_{i'} - p_{i'}) \qquad \text{for all } j \neq i, i'. \tag{54}$$

Substituting into (43) the inequalities (52), (53) for $j = i'$, and (54), we obtain

$$\begin{aligned} g_{\lambda_k}(f^i(p_k)) \geqq V(\hat{x}) + 1 - \exp\{\lambda_k N((p_k)_i - \mu)\} \\ \lambda_k[1 - \exp\{-(\lambda_k)^2(1/l)(\hat{p}_{i'} - (p_k)_{i'})\} \\ + (l-2)(1 - \exp\{(\lambda_k)^3 N(\hat{p}_{i'} - (p_k)_{i'})\})]. \end{aligned} \tag{55}$$

By (51) and (55), we have:

$$g_{\lambda_k}(x)(f^i(p_k)) - V(f^i(p_k)) \geqq G(\lambda_k;(p_k)_i - \mu) + \lambda_k K(\lambda_k;\hat{p}_{i'} - (p_k)_{i'}), \quad (56)$$

where

$$G(\lambda_k;(p_k)_i - \mu) = 1 - \exp\{\lambda_k N((p_k)_i - \mu)\} + (1/N)((p_k)_i - \mu) \quad (57)$$

$$K(\lambda_k;\hat{p}_{i'} - (p_k)_{i'}) = 1 - \exp\{-(\lambda_k)^2(1/l)(\hat{p}_{i'} - (p_k)_{i'})\} \quad (58)$$
$$+ (l-2)(1 - \exp\{(\lambda_k)^3 N(\hat{p}_{i'} - (p_k)_{i'})\}.$$

In order to contradict (49), we will now show that the G and K values are nonnegative for large k.

Notice that for all nonnegative $s \leqq 1$, we have

$$\begin{aligned} G(\lambda_k, s) &= \int_0^s [-\lambda_k N \exp\{\lambda_k N t\} + (1/N)]\, dt \\ &\geqq \int_0^s [-\lambda_k N \exp\{N\} + (1/N)]\, dt, \end{aligned} \quad (59)$$

(where the inequality holds since $0 < \lambda_k < 1$ and $0 \leqq t \leqq s \leqq 1$). Recall that $\lambda_k \to 0$; therefore for all sufficiently large k, we have $1/N \geqq \lambda_k N \exp\{N\}$, so $G(\lambda_k, s) \geqq 0$ for all nonnegative $s < 1$, hence $G(\lambda_k;(p_k)_i - \mu) \geqq 0$ (since $0 \leqq (p_k)_i - \mu \leqq 1$).

Similarly, for all nonnegative $s \leqq 1$, we have

$$\begin{aligned} K(\lambda_k, s) &= \int_0^s [(\lambda_k)^2(1/l)\exp\{-(\lambda_k)^2(1/l)t\} - (l-2)(\lambda_k)^3 N \exp\{(\lambda_k)^3 N t\}]dt \\ &\geqq \int_0^s [(\lambda_k)^2(1/l)\exp\{-(\lambda_k)^2(1/l)\} - (l-2)(\lambda_k)^3 N \exp\{N\}]dt \quad (60) \\ &= (\lambda_k)^2 \int_0^s [(1/l)\exp\{-(\lambda_k)^2(1/l)\} - (l-2)\lambda_k N \exp\{N\}]dt, \end{aligned}$$

(where the inequality holds since $0 < \lambda_k < 1$ and $0 \leqq t \leqq s \leqq 1$). Notice that for the function $J(\lambda) = (1/l)\exp\{-(\lambda)^2(1/l)\} - (l-2)\lambda N \exp\{N\}$, we have $J(0) > 0$, so $J(\lambda) > 0$ for small λ. Recall that $\lambda_k \to 0$; therefore for all sufficiently large k we have $J(\lambda_k) > 0$, so $K(\lambda_k, s) \geqq 0$ for all nonnegative $s \leqq 1$, and hence $K(\lambda_k, \hat{p}_{i'} - (p_k)_{i'}) \geqq 0$ (since $0 \leqq \hat{p}_{i'} - (p_k)_{i'} \leqq 1$, see (38)).

As the last two paragraphs show, the G and K values are nonnegative for large k, so by (56) we have $g_{\lambda_k}(f^i(p_k)) \geqq V(f^i(p_k))$ for all sufficiently large k. This contradicts (49).

Thus we have shown that if λ is sufficiently small, then (48) holds for all $p \in S_\mu$. Pick such $\lambda > 0$. By picking a smaller $\lambda > 0$, we can also assume (44) holds for all $p \in S_\mu$.[40]

Now we define the function $\tilde{u}^i : \mathbb{R}^l \to \mathbb{R}$ by

$$\tilde{u}^i(x) = \min\{U(x), g_\lambda(x)\}. \tag{61}$$

It is clear that $\tilde{u}^i$ is continuous, strictly monotone and strictly concave.

It remains to show that the function $\tilde{u}$ satisfies (42(1-3)).

To see (42(1)), consider any $p \in S_\mu$. Suppose $y \in B(p)$ with $y \neq f^i(p)$. Since $f^i(p)$ uniquely maximizes $U(x)$ on $B(p)$, we have $U(f^i(p)) > U(y)$. By (48) and (61) we have

$$\tilde{u}^i(f^i(p)) = U(f^i(p)); \tag{62}$$

so $\tilde{u}^i(f^i(p)) > U^i(y) \geqq \tilde{u}^i(y)$. Thus $f^i(p)$ uniquely maximizes $\tilde{u}^i(x)$ on $B(p)$ for all $p \in S_\mu$. This proves (42(1)).

To see (42(2)), notice that by (37(1a,2a)) and (50) we have $V(f^i(\hat{p})) = V(f^i(\bar{p}))$, so $U(f^i(\hat{p})) = U(f^i(\bar{p}))$ by (33), hence $\tilde{u}^i(f^i(\hat{p})) = \tilde{u}^i(f^i(\bar{p}))$ by (62); i.e. $\tilde{u}^i(\hat{x}) = \tilde{u}^i(\bar{x})$. Also by (43) we have $g_\lambda(\hat{x}) = V(\hat{x})$, so $g_\lambda(\hat{x}) = U(\hat{x})$ by (33), hence $\tilde{u}^i(\bar{x}) = g_\lambda(\hat{x})$ by (62). Now consider any $y \in \mathbb{R}^l$ with $\tilde{u}^i(y) \geqq \tilde{u}^i(\bar{x})$. Then $g_\lambda(y) \geqq \tilde{u}^i(y) \geqq \tilde{u}^i(\bar{x}) = g^i_\lambda(\hat{x})$. Then by (44) we have $y_i > lM$. This proves (42(2)).

To see (42(3)), consider any $p \in S_{\bar{\mu}}$ with $p_i \leqq \mu$, and any $y \in \mathbb{R}^l$ with $\tilde{u}^i(y) \geqq \tilde{u}^i(f^i(p))$. Suppose $y = f^i(p)$, then by (36) we have $y_i \geqq 2lM \geqq lM$. Now suppose $y \neq f^i(p)$. Then we have: $\tilde{u}^i(f^i(p)) = g_\lambda(f^i(p))$, because otherwise, we would have: $\tilde{u}^i(f^i(p)) = U(f^i(p)) > U(y) \geqq \tilde{u}^i(y)$, contradicting the hypothesis that $\tilde{u}^i(y) \geqq \tilde{u}^i(f^i(p))$. Thus we must have $g_\lambda(y) \geqq \tilde{u}^i(y) \geqq \tilde{u}^i(f^i(p)) = g^i_\lambda(f^i(p))$, and so by (44) we have $y_i > lM$. Hence (42(3)) holds. This completes our proof of Proposition 1(2). Q.E.D.

Proof of Proposition 1(3a) Suppose the given function $\beta_i : S \to \mathbb{R}$ also satisfies the property that $\beta_i|_{S_c}$ is a computable function from S_c into $\mathbb{R}_c$. We will show that we can ensure that the consumer (u^i, ω_i) defined in Step 3 of our proof of Proposition 1(1) also satisfies property (III).

We can assume the vertices of the box $X = \Pi_{i=1}^l [a_i, b_i]$ chosen in the beginning of Step 1 in the above proof of Proposition 1(1) to be rational (hence computable) vectors, i.e. a_i, b_i are rational numbers (as noted in Footnote 32).

[40] It is clear that this λ can be chosen to be a rational or computable number.

Now suppose the given function $\beta_i : S \to \mathbb{R}$ also satisfies the property that $\beta_i|_{S_c}$ is a computable function from S_c into $\mathbb{R}_c$. It is clear from (26) that the $q|_{X_c}$ is a computable function from X_c into S_c, where X_c is the set of all computable vectors in X. Also, it is clear from (17) that the function $\pi^i|_{S_c}$ is also a computable function from S_c into $\mathbb{R}^l_c$. We can assume the numbers α_1, n_1, n_2 chosen in (27) to be rational (hence computable) number (as noted in Footnote 34). Then, $V|_{X_c}$ is a computable function from X_c into $\mathbb{R}_c$ (cf. Remark 2(2), page 16). Since V is C^2 (hence C^1) on the rational box X, (by Remark 17(1), page 60(1)) it is computable-continuous. It is also clear that we can assume the α_2 chosen in (28) is a rational (hence computable) number. Then (by Remark 17(3), page 60) the function U defined by (29) is computable-continuous.

Since the function $U : \mathbb{R}^l \to \mathbb{R}$ is computable-continuous, its restriction $U|_{\mathbb{R}^l_c}$ is a computable function from $\mathbb{R}^l_c$ into $\mathbb{R}_c$ (by Remark 15(1), page 58). It is also clear that we can assume the vector M chosen in Step 3 of our proof of Proposition 1(1) to be rational (hence computable). Then it is immediate that the function u^i and the endowment ω_i (again defined in Step 3) also satisfy property (III) in Proposition 1(3), and this proves Proposition 1(3a).

Proof of Proposition 1(3b). Let the given function β_i have a computable restriction $\beta_i|_{S_c} \to \mathbb{R}_c$, and let $M, \mu \in \mathbb{R}_c$. We will show that we can ensure that the consumer (u^i, ω_i) constructed in the proof of Proposition 1(2) also satisfies property (III).

As in the proof of Proposition 1(3a) above, we can assume the box X has rational vertices, and that the numbers $\alpha_1, n_1, n_2, \alpha_2$ are rational, and so the function $U : \mathbb{R}^l \to \mathbb{R}$ defined by (29) is computable-continuous. Since μ is computable, then by (37(2)) the vector $\hat{p}$ is computable, and by (34) so is the vector $\hat{x} = f^i(\hat{p})$; also by (50) we have $V(\hat{x}) = -\exp\{n_2(p_i)^2 + \alpha_1\}$, which is a computable number. It is clear that we can assume the λ used in (61) to be rational (as noted in Footnote 40), so the function g_λ is computable-continuous (cf. Remark 14(1,2), page 56). Then the utility $\tilde{u}^i$ defined by (61) is computable-continuous (cf. Remark 14(1,2)).

Since $M \in \mathbb{R}_c$, the endowment $\omega_i = (M, M, \cdots, M)$ defined in the paragraph following (42) is computable. Also, since the function $\tilde{u}^i$ is computable-continuous, the function $u^i = \tilde{u}^i(x - \omega_i)$ is also computable-continuous, and hence $u^i|_{\mathbb{R}^l_{c+}}$ is a computable function from $\mathbb{R}^l_{c+}$ into $\mathbb{R}_c$ (by Remark 15(1), page 58). Thus the consumer (u^i, ω_i) also satisfies the property (III).

APPENDIX C: A REVIEW OF RECURSIVE ANALYSIS

We discuss some recursive analysis tools that we have applied in our economic analysis. For general background, see Moschovakis (1964a, 1964b), Beeson (1985) and Pour-El and Richards (1989).

We take as given the notions of a recursive function and a partial recursive function from $\mathbb{N}$ into $\mathbb{N}$. (Cf. Section 4.1.)

We use the notation of Section 5.1.

C.1 Two theorems in recursion theory

In recursive analysis, recursion theory is applied to studying problems in mathematical analysis. The following two theorems are fundamental in recursion theory. Their proofs can be found in standard references, e.g. Kleene (1952) and Davis (1983).

Theorem C-1 (Kleene's Normal Form Theorem). *There is a recursive function $U : \mathbb{N} \to \mathbb{N}$ and recursive functions $R^k : \mathbb{N}^{k+2} \to \mathbb{N}$, where $k = 1, 2, \cdots$, such that for every partial recursive function $\varphi : \mathbb{N}^k \to \mathbb{N}$, there exists at least one $n \in \mathbb{N}$ (called a Gödel number of φ) such that:*

$$\varphi(x_1, \cdots, x_k) = \Phi^k(n, x_1, \cdots, x_k) \tag{63}$$

for all $(x_1, \cdots, x_k) \in \mathbb{N}^k$, where

$$\Phi^k(n, x_1, \cdots, x_k) = U(\min\{t \in \mathbb{N} : R^k(n, x_1, \cdots, x_k, t) = 0\}). \tag{64}$$

Remark 8.

We choose any such hierarchy of functions Φ^k and treat them fixed through this appendix. The function Φ in (2) in Section 4 is understood to be Φ^1.

Theorem C-2 (Kleene's S-m-n Theorem). *There are recursive functions $S^n_m(y, z_1, \cdots, z_n)$ such that:*

$$\Phi^{n+m}(y, z_1, \cdots, z_n, x_1, \cdots, x_m) = \Phi^m(S^n_m(y, z_1, \cdots, z_n), x_1, \cdots, x_m) \tag{65}$$

C.2 Computability for sequences of computable real vectors

The classical notion of recursive function applies to functions taking values in $I\!N$. The "recursive" notion easily extends to functions (sequences) taking values in the set Q of rational numbers (by applying the classical recursive definition of numerators and denominators). To extend to functions (or sequences), taking other real values, we will take limits (in a recursive/effective way); then we use the terminology "computable" rather than "recursive."

Definition 5 (Specker (1949), Rice (1954); cf. Pour-El and Richards (1989)). A sequence $\{v_k\}_{k\in I\!N}$ of q-dimensional vectors of rational numbers is *recursive* if it is generated by an algorithm, i.e. if there are recursive functions $\phi_1^1, \phi_2^1, \phi_3^1, \cdots, \phi_1^q, \phi_2^q, \phi_3^q : I\!N \to I\!N$ such that $\phi_3^i(k) \neq 0$ for all $k \in I\!N$ and all $i = 1, \cdots, q$, and

$$v_k = ((-1)^{\phi_1^1(k)} \frac{\phi_2^1(k)}{\phi_3^1(k)}, \cdots, (-1)^{\phi_1^q(k)} \frac{\phi_2^q(k)}{\phi_3^q(k)}) \text{ for all } k \in I\!N. \tag{66}$$

Similarly, a double sequence $\{v_{nk}\}_{k,n\in I\!N}$ of rational vectors is *recursive* if there are recursive functions $\phi_1^1, \phi_2^1, \phi_3^1, \cdots, \phi_1^q, \phi_2^q, \phi_3^q : I\!N \times I\!N \to I\!N$ such that $\phi_3^i(n,k) \neq 0$ for all $n, k \in I\!N$ and all $i = 1, \cdots, q$, and

$$v_{nk} = ((-1)^{\phi_1^1(n,k)} \frac{\phi_2^1(n,k)}{\phi_3^1(n,k)}, \cdots, (-1)^{\phi_1^q(n,k)} \frac{\phi_2^q(n,k)}{\phi_3^q(n,k)}) \text{ for all } n, k \in I\!N \tag{67}$$

The notion of a recursive triple sequence of rational vectors is defined in a similar manner.

Going beyond rational numbers, to arbitrary reals, we define the notion of computable real numbers and vectors as in Definition 1, page 11.

Remark 9.

Obviously, a q-dimensional real vector x is computable if and only if x is the *effective limit* (cf. Footnote 9) of a recursive sequence of rational vectors v_k, i.e. $\|v_k - x\| \leqq 2^{-k}$ for all $k \in I\!N$.

Remark 10.

The computable real numbers constitute a real closed ordered field $(I\!R_c, +, \cdot, 0, 1, >)$ under the usual definitions of $+, \cdot, 0, 1,$ and $>$. (Cf. Footnote 49, page 59.)

Definition 6 (Rice (1954); cf. Pour-El and Richards (1989)). A sequence $\{x_n\}_{n \in \mathbb{N}}$ of real vectors is *computable* if it is the *effective limit* of some recursive double sequence of rational vectors v_{nk}, i.e. $\|x_n - v_{nk}\| \leqq 2^{-k}$ for all $n, k \in \mathbb{N}$. Similarly, a double sequence $\{x_{nk}\}_{n,k \in \mathbb{N}}$ of real vectors is *computable* if it is the effective limit of some recursive triple sequence $\{v_{nkt}\}_{n,k,t \in \mathbb{N}}$ of rational vectors, i.e. $\|x_{nk} - v_{nkt}\| \leqq 2^{-t}$ for all $n, k, t \in \mathbb{N}$.

Remark 11.

The following facts are easily verified:

1) If $\{x_n\}$ is a computable sequence of real vectors, then each x_n is a computable vector; but the converse is not generally true, i.e. an arbitrary sequence of computable vectors is not necessarily computable.

2) A recursive sequence of rational vectors is a computable sequence; however a computable sequence of rational vectors is not necessarily recursive (cf. Pour-El and Richards (1989, p. 24)).

3) If a sequence $\{x_k\}_{k \in \mathbb{N}}$ of vectors in $\mathbb{R}^l_c$ is computable, then there is a recursive function $\varphi : \mathbb{N} \to \mathbb{N}$ generating codes $\varphi(k)$ of the vectors x_k (cf. Definition 2, page 13), i.e. $A(\varphi(k), x_k)$ for all $k \in \mathbb{N}$ (cf. our proof of Fact 3, page 54). The converse is also true (cf. Definitions 5 and 6, equation (63) and Definition 2).

Remark 12.

1) For every $X \subseteqq \mathbb{R}^l_c$, it follows immediately from Remark 11(3) that every computable[41] function $f : X \to \mathbb{R}^q_c$ is *sequential-computable* (as defined in Pour-El and Richards (1989, p. 25)), i.e.

$$f \text{ maps every computable sequence of vectors } x_k \in X \text{ into a computable sequence of vectors } f(x_k). \tag{68}$$

2) In our economic analysis (see Appendix A, proof of Theorem 5), we have used the following sequential-computability property. Let $u : \mathbb{R}^l_{c+} \to \mathbb{R}_c$ be a computable (hence sequential-computable) and monotone function, let $\bar{x} \in \mathbb{R}^l_c$, and let $\{v_n\}_{n \in \mathbb{N}}$ be a recursive sequence that enumerates all rational vectors in $\mathbb{R}^l_{c+}$. We want to enumerate algorithmically all $v_{n'}$ with the property that $u(v_{n'}) > u(\bar{x})$. The monotonicity of u implies there are infinitely many such v_n.

(41) Definition 4, page 16.

To do this, we use the sequential-computability of u, which ensures that the sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ of real numbers $\alpha_n = u(v_n) - u(\bar{x})$ is computable; so there exists a recursive double sequence $\{r_{nk}\}_{n,k\in\mathbb{N}}$ of rational numbers such that $|r_{nk} - \alpha_n| \leqq 2^{-k}$ for all $n, k \in \mathbb{N}$. Thus for all $n' \in \mathbb{N}$: $u(v_{n'}) > u(\bar{x})$ if and only if $r_{n',k+1} > 2^{-k}$ for some $k \in \mathbb{N}$. For each $k \in \mathbb{N}$, we define

$$C_k = \{n' \leqq k : (\exists k' \leqq k)[r_{n',k'+1} > 2^{-k'}]\}.$$

Then we define:

$$\zeta(0) = \min\{k : C_k \neq \emptyset\},$$
$$\gamma(0) = \min\{n' : n' \in C_{\zeta(0)}\},$$

and we define for all $n > 0$:

$$\zeta(n) = \min\{k : C_k \backslash \{\gamma(0), \cdots, \gamma(n-1)\} \neq \emptyset\},$$
$$\gamma(n) = \min\{k : k \in C_{\zeta(n)} \backslash \{\gamma(0), \cdots, \gamma(n-1)\}\}.$$

It is easy to check that the recursive sequence $\{v_{\gamma(n)}\}_{n\in\mathbb{N}}$ enumerates all $v_{n'}$ with $u(v_{n'}) > u(\bar{x})$.

The following Fact 1 asserts that $(\mathbb{R}^l_c, \|\cdot\|)$ is not a complete metric space, but is only recursively complete.

We say a sequence of vectors $x_k \in \mathbb{R}^l_c$ is *computably cauchy* (see Rice (1954)) if there is a recursive function $\psi : \mathbb{N} \to \mathbb{N}$ such that for all $N, n, n' \in \mathbb{N}$: $n, n' \geqq \psi(N) \Rightarrow \|x_n - x_{n'}\| \leqq 2^{-N}$.

Fact 1 (Specker (1949), Rice (1954)).

1) There is a bounded, nondecreasing, and recursive sequence of rational numbers that does not converge (in the usual sense) to any computable real number.

2) If a computable sequence $\{x_n\}_{n\in\mathbb{N}}$ of vectors in $\mathbb{R}^l_c$ is computably cauchy, then $x_n \to x$ for some $x \in \mathbb{R}^l_c$.

Proof. See Rice (1954).

C.3 Continuity and computability

The following Fact 2 states the continuity property for a computable preference and computable functions. It has been mentioned earlier. (See Definitions 3, 4, and Remarks 1, page 15, and 2, page 16.)

We say a set $X \subseteqq \mathbb{R}_c^l$ is *recursively separable* if it has a dense[42] and computable sequence of elements $x_k \in X$ (cf. Moschovakis (1964b, p. 223)).

Fact 2 (cf. Moschovakis (1964b)). *If $X \subseteqq \mathbb{R}_c^l$ is closed in $\mathbb{R}_c^l$ and is recursively separable then:*

1) every computable function on X is continuous.

2) every computable strict preference relation on X is continuous.

Proof. Assertion (1) is immediate from Moschovakis (1964b, Theorem 3 and Corollary 4.1).

To see (1), by Moschovakis (1964b, Corollary 4.1 and Lemma 3) we have the following fact:

$$\text{if } L_1, L_2 \subseteqq X \text{ are } \textit{listable} \text{ (in the sense of Moschovakis (p. 217)) and such that } L_1 \cap L_2 = \emptyset, \text{ then for all } x \in L_1 \text{ and all } y \in L_2, \text{ there exists a computable } \epsilon > 0 \text{ with } B_\epsilon(x) \cap L_2 = \emptyset \text{ and } B_\epsilon(y) \cap L_1 = \emptyset, \tag{69}$$

where $B_\epsilon(x) = \{z' \in X : \|z' - x\| < \epsilon\}$ and $B_\epsilon(y) = \{z' \in X : \|z' - y\| < \epsilon\}$. Also, notice that for every computable binary relation $\succ$ on X, the (strictly-preferred) set $C_z = \{z' \in X : z' \succ z\}$ and the (strictly-worsen) set $W_z = \{z' \in X : z \succ z'\}$ are listable sets for all $z \in X$.

Now we consider a computable strict preference relation on X and any $x, y \in X$ with $x \succ y$. There are two cases:

a) Suppose there exists a $z \in X$ with $x \succ z \succ y$. Then $x \in C_z$ and $y \in W_z$. Since $\succ$ is computable, the sets C_z and W_z are listable. Notice that $C_z \cap W_z = \emptyset$; so by (69) for some positive $\epsilon \in \mathbb{R}_c$, we have: $B_\epsilon(x) \cap W_z = \emptyset = C_z \cap B_\epsilon(y)$, hence $x' \succeq z \succ y$ and $x \succ z \succeq y'$ for all $x' \in B_\epsilon(x)$ and all $y' \in B_\epsilon(y)$, where $\succeq$ is the weak preference corresponding to $\succ$ (cf. Remark 1(3), page 15). Thus $\succ$ is continuous at (x, y).

b) Suppose there does not exist any z with $x \succ z \succ y$. Then we have i) $C_y \cap W_x = \emptyset$ and ii) $C_y \cup W_x = X$. By (i) and (69) for some positive $\epsilon \in \mathbb{R}_c$, we have $B_\epsilon(x) \cap W_x = \emptyset = C_y \cap B_\epsilon(y)$, and so by (ii) we have: $B_\epsilon(x) \subseteqq C_y$ and $B_\epsilon(y) \subseteqq W_x$, i.e. $\succ$ is continuous at (x, y). Q.E.D.

Remark 13.

1) The continuity property given in Fact 2 can be strengthened to effective

[42] I.e. the set $\{x_k : k \in \mathbb{N}\}$ is dense in X.

continuity (cf. Footnotes 16 and 20).

2) Many sets in $\mathbb{R}^l_c$ are closed and recursively separable, e.g. $\mathbb{R}^l_{c+}$, $\mathbb{R}^l_c$, and $B_c(p,w)$ for all $(p,w) \in \mathcal{B}_c$.

3) Also, since each computable closed ball is recursively separable, it follows from Fact 2 that every open and recursively separable set $X \subseteqq \mathbb{R}^l_c$ satisfies properties (1,2) in Fact 2.

4) An arbitrary computable relation need not be continuous. To see this, Moschovakis (1964b, Theorem 9) gives a listable set L in $\mathbb{R}_c$ that is not open (indeed is nowhere dense), so the corresponding computable unary relation on $\mathbb{R}_c$ is not continuous. A trivial extension applies for n-ary relations.

C.4 Verifying computability for functions

The following fact is useful for showing whether a given function is computable.

Fact 3. *Let $X \subseteqq \mathbb{R}^l_c$ and $f : X \to \mathbb{R}^q$. Assume there exists a recursive sequence of rational vectors $v_k \in \mathbb{R}^q_c$ and a partial recursive function $\gamma : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that for all $n \in \mathbb{N}$ and all $x \in X$:*[43]

$$A(n,x) \;\Rightarrow\; (\forall k \in \mathbb{N})[\gamma(n,k)\downarrow \;\&\; \|v_{\gamma(n,k)} - f(x)\| \leqq 2^{-k}]. \tag{70}$$

Then f is a computable function from X into $\mathbb{R}^q_c$.

Proof. First, for any $x \in X$, we can pick any code n of x; then (70) implies that $f(x)$ is the effective limit of the recursive sequence of rational vectors $w_k = v_{\gamma(n,k)}$, and so is computable.

We now construct a partial recursive function that transforms codes of $x \in X$ into codes of $f(x)$ as in (5) in Section 4. Since $\{v_k\}$ is recursive, by definition there are recursive functions $\phi^1_1, \phi^1_2, \phi^1_3, \cdots, \phi^q_1, \phi^q_2, \phi^q_3 : \mathbb{N} \to \mathbb{N}$ as given in (66). For any ϕ^i_j, by Theorem C-1 we can pick a (Gödel number) $t_{ij} \in \mathbb{N}$, so $\Phi^2(t_{ij}, n, k) = \phi^i_j(\gamma(n,k))$ for all $n, k \in \mathbb{N}$. Then by Theorem C-2 and Remark 8 we can pick a recursive function $S^1_1(\cdot)$ so that

$$\Phi(\psi^i_j(n), k) = \phi^i_j(\gamma(n,k)) \text{ for all } n, k \in \mathbb{N}, \tag{71}$$

where $\psi^i_j(n) = S^1_1(t_{i,j}, n)$, and Φ is given in (2) in Section 4. Now we define a

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(43) Recall the definition of $A(n,x)$ in Definition 2 in Section 4.

partial recursive function $\phi : \mathbb{N} \to \mathbb{N}$ by[44]

$$\phi(n) = \langle \cdots \langle \langle \langle \psi_1^1(n), \psi_2^1(n) \rangle, \psi_3^1(n) \rangle, \langle \langle \psi_1^2(n), \psi_2^2(n) \rangle, \psi_3^2(n) \rangle \rangle \cdots \rangle. \quad (72)$$

We now show that ϕ satisfies (5). Let $x \in X$ and $n \in \mathbb{N}$ with $A(n, x)$. Then $\psi_j^i(n) \downarrow$ for all $i = 1, \cdots, q$ and all $j = 1, 2, 3$, and so $\phi(n) \downarrow$. Notice that for all $i = 1, \cdots, q$ and all $k \in \mathbb{N}$, by (66) and (71) we have

$$(-1)^{-\Phi(\psi_1^i(n),k)}(\Phi(\psi_2^i(n), k)/\Phi(\psi_3^i(n), k)) = (v_{\gamma(n,k)})_i,$$

so $|(v_{\gamma(n,k)})_i - (f(x))_i| \leqq \|v_{\gamma(n,k)} - f(x)\| \leqq 2^{-k}$ by (70), and hence

$$A(\langle \langle \psi_1^i(n), \psi_2^i(n) \rangle, \psi_3^i(n) \rangle, (f(x))_i).$$

Thus we have $A(\phi(n), f(x))$. This shows that ϕ satisfies (5), and so f is computable. Q.E.D.

We give an application of Fact 3.

Fact 4. *Let $X \subseteqq \mathbb{R}_c^l$, and $f : X \to \mathbb{R}^q$. Assume f is:*

1) weakly sequential-computable,[45] i.e.

there is a computable sequence of vectors $x_k \in X$ such that $\{x_k : k \in \mathbb{N}\}$ is dense in X and such that the sequence of vectors $f(x_k)$ is computable; (73)

2) effectively locally uniformly continuous (e.l.u.c.) i.e.

there is a recursive function $\zeta : \mathbb{N} \to \mathbb{N}$ such that for all $n \in \mathbb{N}$ and all $x, y \in X$: if $\|x\|, \|y\| \leqq n$ and $\|x - y\| \leqq 2^{-\zeta(n)}$, then $\|f(x) - f(y)\| \leqq 2^{-n}$. (74)

Then f is computable function from X into $\mathbb{R}_c^q$.

Proof. For $\{x_k\}$ as in (73) the sequence of vectors $f(x_k)$ is computable, so by definition there is a recursive double sequence of rational vectors v_{km} such that $\|f(x_k) - v_{km}\| \leqq 2^{-m}$ for all $k, m \in \mathbb{N}$. Then for the recursive sequence of rational vectors $v_k = v_{k,k+1}$, we have $\|f(x_k) - v_k\| \leqq 2^{-(k+1)}$ for all $k \in \mathbb{N}$.

By using partial recursive functions that compute the Euclidean norm $\| \cdot \|$ and the greater than relation $>$ (cf. Footnote 14), it is easy to construct a partial

(44) Recall that we have fixed the recursive "coding" function $\langle \cdot, \cdot \rangle$ in Definition 2 in Section 4.

(45) This is a weakened variant of (68).

recursive function $\gamma : \mathbb{N} \times N \to \mathbb{N}$ such that for all $x \in X$ and all $n \in N$, if $A(n, x)$, then for all $k \in \mathbb{N}$, one has: $\gamma(n, k) \downarrow$, and $||x||, ||x_{\gamma(n,k)}|| < 2^{-(k+1)}$, and $||x - x_{\gamma(n,k)}|| < 2^{-\zeta(k+1)}$ for $\zeta(\cdot)$ as in (74). Then it follows that γ satisfies (70), and so Fact 3 ensures that f is computable. Q.E.D.

C.5 Computable-continuous real functions

Fact 4 connects (see Remark 15 below) the notion[(46)] of a computable function that we are using to another widely used computability notion, that of a "computable-continuous" function (Grzegorczyk (1955, 1957) and Lamcobe (1955a,b); cf. Pour-El and Richards (1989, Definition A, p. 25)). While a computable function is defined on the computable reals $\mathbb{R}_c$, a computable-continuous function is defined on the reals $\mathbb{R}$. Also, while a computable function is not necessarily locally uniformly continuous (cf. Beeson (1985, p. 70)), by definition a computable-continuous is locally uniformly continuous.

For simplicity, we will consider computable-continuous functions defined on the closed sets such as $\mathbb{R}^l$, $\mathbb{R}^l_+$, I, $I \times \mathbb{R}^l$, where: $\mathbb{R}^l$ is the l-dimensional space of real numbers, $\mathbb{R}^l_+ = \{x \in \mathbb{R}^l : x \geqq 0\}$, and I is an arbitrary *computable rectangle* in $\mathbb{R}^l$, i.e. $I = \Pi^l_{i=1}[a_i, b_i]$, where a, b are computable vectors in $\mathbb{R}^l$ with $a \leqq b$. The definition can be applied to other sets in $\mathbb{R}^l$.

Definition 7 (Grzegorczyk (1955, 1957), Lacombe (1955); cf. Pour-El and Richards (1989)). Let X be $\mathbb{R}^l$, $\mathbb{R}^l_+$, a computable rectangle I in $\mathbb{R}^l$, or the product $\mathbb{R}^l \times I$ of $\mathbb{R}^l$ with a computable rectangle I in $\mathbb{R}^l$. A function $f : X \to \mathbb{R}^m$ is *computable-continuous (c-c)* if it is both sequential-computable and effectively locally uniformly continuous, i.e. if f satisfies (68) and (74) on X. A sequence $\{f_n\}_{n \in \mathbb{N}}$ of *c-c* functions $f_n : X \to \mathbb{R}^m$ is *computable* if:[(47)]

1) $\{f_n\}$ is *sequential-computable*, i.e. for every computable sequence $\{x_k\}_{k \in \mathbb{N}}$ of elements in X, the double sequence $\{f_n(x_k)\}_{n,k \in \mathbb{N}}$ of vectors is computable.
2) $\{f_n\}$ is *effectively locally uniformly continuous (e.l.u.c.)*, i.e. there is a recursive function $\zeta : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that for all $n, k \in \mathbb{N}$ and all $x, y \in X$: if $||x||, ||y|| \leqq k$ and $||x - y|| \leqq 2^{-\zeta(n,k)}$, then $||f_n(x) - f_n(y)|| \leqq 2^{-k}$.

Remark 14.

1) Most functions we use in applied economics and applied mathematics are

(46) See Footnote 17.

(47) Usually, the abbreviation "*c-c*" will be used when we discuss a sequence of computable-continuous functions, and we will not use it for a single computable-continuous function.

computable-continuous: $x+y$, $x-y$, $\sqrt{x}$, $\exp\{x\}$, $\log(x)$, $\max\{x,y\}$, $\min\{x,y\}$ $x \cdot y$, polynomials with computable coefficients, etc. (See Remark 2(1,2) page 16, and Remark 15 below).

2) (a) A composition of computable-continuous function is computable continuous.

(b) More generally, the class of computable sequences of c-c functions is closed under effective compositions. For example, suppose $\{F_k\}_{k\in\mathbb{N}}$ is a computable sequence of c-c functions $F_k : \mathbb{R}^{rq} \to \mathbb{R}^m$, and $f : \mathbb{R}^l \to \mathbb{R}^q$ is a computable-continuous function, and $\{g_k^1\}_{k\in\mathbb{N}}, \cdots, \{g_k^r\}_{k\in\mathbb{N}}$ are computable sequences of c-c functions $g_k^1, \cdots, g_k^r : \mathbb{R}^l \to \mathbb{R}^q$, then the sequence $\{f_n\}_{n\in\mathbb{N}}$ of functions $f_n : \mathbb{R}^l \to \mathbb{R}^m$ defined by the following Scheme (I) is also a computable sequence of c-c functions;

$$\begin{aligned} &\psi : \mathbb{N} \to \mathbb{N} \text{ is a recursive function, and} \\ &f_n(\cdot) = F_{\psi(n)}(g^1_{\psi(n)}(\cdot), \cdots, g^r_{\psi(n)}(\cdot)) \quad \text{for all } n \in \mathbb{N}. \end{aligned} \tag{I}$$

And similarly, for the sequence $\{f_n\}_{n\in\mathbb{N}}$ of functions defined by Scheme (II):

$$\begin{aligned} f_0(\cdot) &= F_0(f(\cdot), g_0^1(\cdot), \cdots, g_0^{r-1}(\cdot)) \\ f_{n+1}(\cdot) &= F_{n+1}(f_n(\cdot), g^1_{n+1}(\cdot), \cdots, g^{r-1}_{n+1}(\cdot)) \qquad \text{for all } n \in \mathbb{N}. \end{aligned} \tag{II}$$

is also a computable sequence of c-c functions. (This general assertion can be proved by an easy modification of the methods in Pour-El and Richards (1989, pp. 28–32).) Also, notice that the sequence of functions defined by Scheme (III):

$$\begin{aligned} &\psi : \mathbb{N} \to \mathbb{N} \text{ is an increasing recursive function, and} \\ &f_n(\cdot) = \textstyle\sum_{k=\psi(n)}^{\psi(n+1)-1} g_k^1(\cdot), \qquad \text{for all } n \in \mathbb{N} \end{aligned} \tag{III}$$

can be defined by the effective composition schemata (I) and (II) above, and so $\{f_n\}$ is also a computable sequence of c-c functions.[48]

3) The class of computable-continuous functions is closed under effective uniform convergence (cf. Pour-El and Richards (1989, Theorem 4, p. 34)). Thus given a computable rectangle I in $\mathbb{R}^l$, and a computable sequence of computable-continuous functions $f_n : X \to \mathbb{R}^m$, if there is a recursive function $\zeta : N \to \mathbb{N}$ such that for all $k \in \mathbb{N}$ and all $n \in \mathbb{N}$:

$$n \geqq \zeta(k) \text{ implies } \|f_n(x) - f_{\zeta(k)}(x)\| \leqq 2^{-k} \qquad \text{for all } x \in I, \tag{75}$$

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[48] It is clear that scheme (I) covers the case for $f_n = g^1_{\psi(n)}$.

then the limit function $f(x) = \lim_{n\to\infty} f_n(x)$ is a computable-continuous function from I into $\mathbb{R}^m$.

Remark 15.

We notice the relationship between the notion of a computable function and that of a computable-continuous function. Let X be as given in Definition 7 and let X_c be the set of computable vectors $x \in X$.

1) If $f : X \to \mathbb{R}^m$ is computable-continuous, then $f|_{X_c}$ is a computable function from X_c into $\mathbb{R}^m_c$. To see this, we can (as in Pour-El and Richards p. 40) pick a computable sequence of vectors $x_k \in X$ that is dense in X, hence is dense in X_c. The sequential-computability of f implies that the sequence $\{f(x_k)\}$ is computable, and hence $f|_{X_c}$ is weakly sequential-computable (as defined in (73)). Since f is *e.l.u.c.* on X, the restriction $f|_{X_c}$ is also *e.l.u.c.* on X_c. Then Fact 4 implies that $f|_{X_c}$ is a computable function from X_c into $\mathbb{R}^m_c$.

2) Conversely, if $f|_{X_c}$ is a computable function from X_c into $\mathbb{R}^m$, then (by (Remark 12(1), page 51) the function $f|_{X_c}$ is sequential-computable, so the function f is also. Thus if f is also *e.l.u.c.*, then f is computable-continuous.

Remark 16.

We now summarize some known results on the optimization of computable-continuous functions. Let I be a computable rectangle I in $\mathbb{R}^l$.

1) If $f : I \to \mathbb{R}$ is computable-continuous, then the maximum value $\max_{x\in I} f(x)$ is a computable number (Grzegorczyk (1955)).

(a) More generally, for every computable sequence of *c-c* functions $f_n : I \to \mathbb{R}$, the numbers $\max_{x\in I} f_n(x)$ form a computable sequence of computable reals (cf. Pour-El and Richards (1989, p. 40, Theorem 7)).

(b) Even more generally, let $\{K_n\}_{n\in\mathbb{N}}$ be a *computable* sequence of finite unions K_n of computable rectangles, i.e. there is a computable sequence of computable vectors $(a_k, b_k) \in \mathbb{R}^l \times \mathbb{R}^l$ with $a_k \leqq b_k$ for all $k \in \mathbb{N}$ and a increasing recursive function $\psi : \mathbb{N} \to \mathbb{N}$ such that

$$K_n = \cup_{k=\psi(n)}^{\psi(n+1)-1} I_k \qquad \text{for all } n \in \mathbb{N},$$

where $I_k = \Pi_{i=1}^{l}[(a_k)_i, (b_k)_i]$. Let $\{f_n\}_{n\in\mathbb{N}}$ be a *computable* sequence of *c-c* functions $f_n : K_n \to \mathbb{R}$, i.e.

1) $\{f_n\}_{n\in\mathbb{N}}$ is *sequential-computable*: for every computable double sequence of computable vectors $x_{nk} \in K_n$, the sequence $\{f_n(x_{nk})\}_{n,k}$ is a computable sequence of computable vectors in $\mathbb{R}$;

2) $\{f_n\}_{n\in\mathbb{N}}$ is *effectively uniformly locally continuous (e.l.u.c.)*: (2) in Definition 7 holds with $X = K_n$.

Then the numbers $\max_{x\in K_n} f_n(x)$ also form a computable sequence of computable reals. (Cf. Pour-El and Richards (1989, pp. 40–41, proof of Theorem 7).)

2) However a computable-continuous function $f : I \to \mathbb{R}$ does not necessarily admit a computable maximizer (Kreisel (1958); compare our Theorem 2, page 19). A counterexample can be constructed with reference to Fact 5 below (see e.g. Beeson (1985, pp. 71–73)).

3) Nevertheless, if a computable-continuous function $f : I \to \mathbb{R}$ has exactly one maximizer, then such maximizer is a computable vector. Similarly, for a computable sequence of *c-c* functions $f_k : I \to \mathbb{R}$, if each f_k admits exactly one maximizer x_k, then the sequence of the vectors x_k is also computable. (Cf. Grzegorczyk (1955, pp. 196–198, proof of Theorem 4) and Ko (1991, p. 73 (proof of Theorem 3.1) and p. 75 (Corollary 3.2(b))).[49]

4) Also, if a computable-continuous $f : I \to \mathbb{R}$ is quasiconcave, then f has a computable maximizer (Wong (1996a)). In fact, with quasiconcavity, we only need that $f|_{I_c}$ is a computable function from $I_c = \{x \in \mathbb{R}^l_c : x \in I\}$ into $\mathbb{R}_c$. (See Theorem 1 and Theorem 2; cf. Richter and Wong (1996a, Theorem 2).)

5) As an application of (3), let $0 < \epsilon \in \mathbb{R}_c$, and let $\Delta_\epsilon = \{x \in \mathbb{R}^l : \sum_{i=1}^{l} x_i = 1 \text{ and } x_i \geqq \epsilon \text{ for all } i\}$ be non-empty. For all $x \in \mathbb{R}^l$, it is clear that there exists exactly one $h(x) \in \Delta_\epsilon$ with $\|h(x) - x\| = \min\{\|y - x\| : y \in \Delta_\epsilon\}$. We now show that the function $h : \mathbb{R}^l \to \Delta_\epsilon$ is computable-continuous. Since $\|h(x) - h(y)\| \leqq \|x - y\|$ for all $x, y \in \mathbb{R}^l$, it follows that h is *e.l.u.c.*. Now we consider any sequence of vectors $x_k \in \mathbb{R}^l$. We can consider Δ_ϵ as a computable rectangle, so the functions $d_k : \Delta_\epsilon \to \mathbb{R}$ defined by $d_k(y) = -\|y - x_k\|$ clearly form a computable sequence of *c-c* functions. Notice that for all k, the vector $h(x_k)$ is the unique maximizer of d_k; so by (3) the sequence $\{h(x_k)\}$ is a computable

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[49] In fact the real closed ordered field properties in Remark 10, page 50, can be proved from this.

sequence of computable vectors. This shows the sequential-computability of h. Hence h is computable-continuous.

Also, h is computable-continuous, so $h|_{\mathbb{R}^l_c}$ is a computable function from $\mathbb{R}^l_c$ into $\mathbb{R}^l_c$, hence into $\Delta_{c,\epsilon} = \{x \in \mathbb{R}^l_c : x \in \Delta_\epsilon\}$. Therefore for all $x \in \mathbb{R}^l_c$, the computable vector $h(x)$ uniquely minimizes $\|x - y\|$ on $\Delta_{c,\epsilon}$, and the restriction of h on any set in $\mathbb{R}^l_c$, for example the set $\Delta_c = \{x \in \mathbb{R}^l_{c+} : \sum_{i=1}^l x_i = 1\}$, is a computable function.

Remark 17.

Let I be a computable rectangle in $\mathbb{R}^l$.

1) If $f : I \to \mathbb{R}^m$ is continuously differentiable (C^1), then it is uniformly lipschitz, and hence *e.l.u.c.*. Further assume that $f|_{I_c}$ is a computable function from $I_c = \{x \in I : x \in \mathbb{R}^l_c\}$ into $\mathbb{R}^m_c$, so f is sequential-computable (by Remark 12(1), page 51). Thus f is computable-continuous.

2) If a C^2 function $f : I \to \mathbb{R}^m$ is computable-continuous, then its derivative Df is computable-continuous (cf. Pour-El and Richards (1989, Theorem 2, p. 53)). However, the C^2 condition cannot be weakened to C^1 (Myhill (1971); cf. Pour-El and Richards (1989, p. 51, Theorem 1)).

3) As an application of (2), suppose $V : I \to \mathbb{R}$ is computable-continuous and C^2. By (2), its derivative DV is also computable-continuous. Then for all $\alpha \in \mathbb{R}_c$ the equation

$$L_x(y) = V(x) + DV(x)(y - x)^T + \sum_{j=1}^{l}(1 - \exp\{-\alpha(y_j - x_j)\})$$

defines a computable-continuous function $(x, y) \mapsto L_x(y)$ from $I \times \mathbb{R}^l$ into $\mathbb{R}$ (cf. Remark 14(1), page 56, and Remark 14(2a)).

Also, the function $U : \mathbb{R}^l \to \mathbb{R}$ defined by $U(y) = \min\{L_x(y) : x \in I\}$ is also computable-continuous. First, since the function U is *e.l.u.c.*, the mapping $(x, y) \mapsto L_x(y)$ is also *e.l.u.c.*. Next, to see the sequential-computability of U, consider any computable sequence of computable vectors $y_k \in \mathbb{R}^l$. Then the functions $f_k : I \to \mathbb{R}$ defined by $f_k(x) = L_x(y_k)$ clearly form a computable sequence of computable-continuous functions, and so by Remark 16(1a), the minimum values $\min_{x \in I} f_k(x) = U(y_k)$ form a computable sequence of computable reals; hence U is sequential-computable.

4) (a) Let the functions $f, F_k, g^1_k, \cdots, g^r_k$, be as given in Remark 14(2b), page 56, where $k \in \mathbb{N}$. Assume they are C^∞, and that all the sequences of

derivatives

$$\{D^t f\}_{t\in\mathbb{N}}, \{D^t F_k\}_{t,k\in\mathbb{N}}, \{D^t g_k^1\}_{t,k\in\mathbb{N}}, \cdots, \{D^t g_k^r\}_{t,k\in\mathbb{N}} \tag{76}$$

are computable sequences[(50)] of *c-c* functions. Let $\{f_n\}$ be defined by any of the schemata (I)-(III) in Remark 14(2), page 56. Then the sequence $\{D^t f_n\}_{t,n\in\mathbb{N}}$ of derivatives $D^t f_n$ can be defined by effective compositions as described by schemata (I)-(III) on the sequences given in (76), and so (as in Remark 14(2b)) $\{D^t f_n\}_{t,n\in\mathbb{N}}$ is also a computable sequence of *c-c* functions.

(b) Suppose that $\{f_n\}$ is a computable sequence of *c-c* functions $f_n : I \to \mathbb{R}^m$ and that $\{D^k f_n\}_{n,k\in\mathbb{N}}$ is also a computable sequence of *c-c* functions, where I is a computable rectangle. Suppose $\{f_n\}$ converges effectively uniformly to a function f as in Remark 14(3). It is not necessary that $\{D^k f\}_{k\in\mathbb{N}}$ is a computable sequence of computable-continuous functions. A counterexample can be constructed easily with reference to Pour-El and Richards (1989, p. 56).

C.6 Computability of separating hyperplanes

The following theorem carries over a classical separating hyperplane theorem to the context of recursive analysis. It extends Theorem 2 in Wong (1996a) by permitting relatively convex sets in $\mathbb{R}_c^l$, instead of requiring sets to be convex in $\mathbb{R}^l$. (The proof in Wong (1996a) uses a classical separation theorem together with an existence theorem for computable maximizers (cf. Remark 16(4)).) In what follows, a set $X \subseteqq \mathbb{R}_c^l$ is *c-convex* if $\lambda x + (1-\lambda)y \in X$ for all $x, y \in X$ and all $\lambda \in [0,1]_c$.

Theorem C-3 (A Computable Supporting Hyperplane Theorem; cf. Wong (1996a)). *Let X be a recursively separable and c-convex set in $\mathbb{R}_c^l$. If X is disjoint from $\{0\}$, then there exists a nonzero $\bar{p} \in \mathbb{R}_c^l$ with $0 \leqq \bar{p}\cdot x$ for all $x \in X$.*

Proof. Let $\text{co}(X)$ be the convex hull of X in $\mathbb{R}^l$. We first show that $0 \notin \text{co}(X)$. If not, then by Carathéodory's Theorem, for some (computable) $x^0, \cdots, x^1 \in X$ there exist nonnegative real numbers $\lambda_0, \cdots, \lambda_l$ with $\sum_{i=0}^{l} \lambda_i = 1$ and $0 =$

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(50) In Definition 7, we define a computable sequence of *c-c* functions $f_n : X \to \mathbb{R}^m$ for a fixed range space $\mathbb{R}^m$. But in (76), if $f : X \to \mathbb{R}^m$, then $Df, D^2f, \cdots, D^tf \cdots$ all have different range spaces. Therefore, to formally define the notion of a computable sequence of *c-c* functions in the context of (76), we would modify Definition 7 to allow $f_n : X \to \mathbb{R}^{m_n}$, and accordingly extend Definition 5 of computable double sequence of vectors to allow y_{nk} in the varying range spaces $\mathbb{R}^{m_n}$. And similarly we can understand the definition of *e.l.u.c.* in Definition 7 to allow the norms $\|\cdot\|$ to apply to the varying ranges $\mathbb{R}^{m_n}$.

$\sum_{i=0}^{i=0} \lambda_i x^i$. Since the computable real number field is a real closed ordered subfield of the reals, it follows from Tarski's theorem (1951) on elimination of quantifiers that we can assume those λ_i are computable.[51] So the c-convexity of X implies $0 \in X$, contradicting the hypothesis. Thus we must have $\text{co}(X) \cap \{0\} = \emptyset$.

Since X is recursively separable, it has a dense and computable sequence of elements $x_n \in X$. It is clear that $\{x_n\}$ is also dense in $\text{co}(X)$. Then Wong (1996a, Theorem 2) yields a nonzero $\bar{p} \in \mathbb{R}_c^l$ with $0 \leqq \bar{p} \cdot x$ for all $x \in \text{co}(X)$, and hence $0 \leqq \bar{p} \cdot x$ for all $x \in X$. Q.E.D.

Remark 18.

As in the classical case, Theorem C-3 can be extended to a computable separating hyperplane theorem for c-convex and recursively separable sets $X, Y \subseteqq \mathbb{R}_c^l$.

C.7 Computable Urysohn's Lemma

The following Lemma 1 is a computable version of Urysohn's Lemma. It essentially extends the lemma in Zhou (1992, p. 33) by obtaining the C^∞ property for the "bump" function.

Lemma 1 focuses on an arbitrary union $K = \cup_{j=1}^m I_i$ of finitely many rational rectangles I_j in $\mathbb{R}^l$, where $I_j = \Pi_{i=1}^l [(a_j)_i, (b_j)_i]$ and a_j, b_j are rational numbers in $\mathbb{R}^l$ with $a_j \leqq b_j$.

For every computable number $r > 0$, we define the computable rectangle

$$B_r = \Pi_{i=1}^l [-r, r], \tag{77}$$

so the set $K + B_r$ contains the neighbourhood $\cup_{x \in K} \{y \in \mathbb{R}^l : \|y - x\| < r\}$ of K.

We will make use of the standard non-decreasing, C^∞ and computable-continuous "bump" function $H : \mathbb{R} \to [0,1]$ defined by:

$$H(x) = \begin{cases} 1 & \text{for } x \geqq 1 \\ 1 - \exp\{-x/(1-x)\} & \text{for } 0 < x < 1 \\ 0 & \text{for } x \leqq 0 \end{cases} \tag{78}$$

---

[51] As noted in Richter and Wong (1996a, Footnote 10), this also follows from the weaker property of model completeness for the theory of real closed ordered fields. In fact, since the present equality system is linear, it also follows from the fact that solvable linear equality systems are solvable in the subfield in which their parameters lie.

Notice that the sequence $\{D^k H(\cdot)\}_{k\in \mathbb{N}}$ of derivatives is a computable sequence of c-c functions from $\mathbb{R}$ into $\mathbb{R}$. (Cf. Pour-El and Richards (1989, p.52).)

Lemma 1 (A Computable Urysohn's Lemma; cf. Zhou (1992)). *Let $K = \cup_{j=0}^{m} I_j$ be a finite union of rational rectangles $I_j = \Pi_{i=1}^{l}[(a_j)_i, (b_j)_i]$. Let $r > 0$ be a rational number. Then there is a C^∞ computable-continuous function $G_{K,r} : \mathbb{R} \to [0,1]$ such that:*

$$\begin{aligned} &1)\ G_{K,r}(x) = 1 \qquad \text{for all } x \in K, \\ &2)\ G_{K,r}(x) = 0 \qquad \text{for all } x \in \mathbb{R}^l \backslash (K + B_r). \end{aligned} \tag{79}$$

Furthermore, the $G_{K,r}$ can be chosen so that:

$$\begin{aligned} &3)\ \{D^k G_{K,r}\}_{k\in \mathbb{N}} \text{ is a computable sequence of c-c functions,} \\ &4)\ \text{for all } k = 1, 2, \cdots, \text{ the derivatives } D^k G_{K,r}(x) = 0 \\ &\qquad \text{for all } x \in K \text{ and for all } x \text{ in the closure of } \mathbb{R}^l \backslash (K + B_r). \end{aligned} \tag{79}$$

Proof. For each rectangle I_j, we can clearly choose a finite lattice of points $v \in I_j$ such that each $x \in I_j$ is within distance $r/2$ of some v. Furthermore we can do this algorithmically. For example, for each j we define N_j to be the least $k \in \mathbb{N}$ such that

$$(b_j)_i - (a_j)_i \leqq (1/2l^{1/2})kr \qquad \text{for all } i = 1, \cdots, l, \tag{80}$$

and define:

$$N_K = \max\{N_j : 0 \leqq j \leqq m\}. \tag{81}$$

For each I_j, we define $C(I_j, N_K)$ to be the set (lattice) of all the rational vectors v such that

$$v = \left(\frac{n_1}{N_K}(a_j)_1 + \left(1 - \frac{n_1}{N_K}\right)(b_j)_1, \cdots, \frac{n_l}{N_K}(a_j)_l + \left(1 - \frac{n_l}{N_K}\right)(b_j)_l\right) \tag{82}$$

for some natural numbers $n_1, \cdots, n_l \leqq N_K$. For each $v \in \cup_{j=1}^{m} C(I_j, N_K)$, we define a C^∞ computable-continuous function $G_{v,r} : \mathbb{R} \to [0,1]$ by:

$$G_{v,r}(x) = 1 - H(\|x - v\|/r). \tag{83}$$

Consider any $x \in K$. Then $x \in I_j$ for some j, so $\|x - v\| \leqq r/2$ for some $v \in C(I_j, r)$, hence $G_{v,r}(x) = 1 - H(\|x - v\|/r) \geqq 1 - H(1/2) > 0$. Therefore, we have:

$$\sum_{v \in \cup_{j=0}^{m} C(I_j, N_K)} G_{v,r}(x) \geqq 1 - H(1/2) \qquad \text{for all } x \in K. \tag{84}$$

Now consider any $x \in \mathbb{R}^l \backslash (K + B_r)$. Then $\|x - y\| \geqq r$ for all $y \in K$; so for all $v \in \cup_{j=1}^m C(I_j, N_K)$, we have $\|x - v\| \geqq r$, hence by (78) we have $H(\|x - v\|/r) = 1$, hence $G_{v,r}(x) = 0$. Therefore, we have:

$$\sum_{v \in \cup_{j=0}^m C(I_j, N_K)} G_{v,r}(x) = 0 \qquad \text{for all } x \in \mathbb{R}^l \backslash (K + B_r). \tag{85}$$

Finally, we define the C^∞ computable function $G_{K,r} : \mathbb{R} \to [0, 1]$ by

$$G_{K,r}(x) = H(\frac{\sum_{v \in \cup_{0=1}^m C(I_j, N_K)} G_{v,r}(x)}{1 - H(1/2)}). \tag{86}$$

By (84) and (78), the function $G_{K,r}$ satisfies (79(1)); and by (85) and (78), it also satisfies (79(2)). Also, recall that $\{D^k H\}_{k \in \mathbb{N}}$ is a computable sequence of *c-c* functions; so (79(3)) follows from (86) and (83) (cf. Remark 17(4a)). The property (79(4)) is easily checked. Q.E.D.

Remark 19.

1) Instead of restricting the range of $G_{K,r}$ to the unit interval $[0, 1]$ in Lemma 1, we can restrict it to any interval $[\alpha, \beta]$ for any $\alpha, \beta \in \mathbb{R}_c$ with $\alpha \leqq \beta$, by using e.g. the function $\alpha + (\beta - \alpha) G_{K,r}$.

2) More generally, let $X, Y \subseteqq \mathbb{R}^l$. Suppose there is a finite union $K = \cup_{0=1}^l I_j$ of rational rectangles such that $X \subseteqq K$ and $Y \cap (K + B_r) = \emptyset$ (see (77)) for some rational number $r > 0$. Let $\alpha, \beta \in \mathbb{R}_c$ with $\alpha \leqq \beta$. Then by (1) there is a C^∞ computable-continuous function $G : \mathbb{R} \to [\alpha, \beta]$ such that $G|_K = \beta$ and $G|_{\mathbb{R}^l \backslash (K+B_r)} = \alpha$, so $G|_X = \beta$ and $G|_Y = \alpha$. Also, since G is computable-continuous, the restriction $G|_{\mathbb{R}^l_c}$ is a computable function from $\mathbb{R}^l_c$ into $\mathbb{R}_c$, and hence the function $G|_{D_c}$ is also computable for any $D_c \subseteqq \mathbb{R}^l_c$.

3) One can apply Lemma 1 to prove a computable analogue of the Tietze Extension Theorem; cf. Zhou (1992, Theorem 4, p. 48).

4) Lemma 1 has the following extension: Let $\{X_n\}_{n \in \mathbb{N}}$ be a *recursive* sequence of finite unions X_n of rational rectangles, i.e.

> there is a recursive sequence of rational vectors $(a_j, b_j) \in \mathbb{R}^l \times \mathbb{R}^l$ with $a_j \leqq b_j$ for all j, and there is a increasing recursive function $\phi : \mathbb{N} \to \mathbb{N}$ such that (87)
>
> $$X_n = \cup_{j=\phi(n)}^{\phi(n+1)-1} I_j \qquad \text{for all } n \in \mathbb{N},$$

where $I_j = \Pi_{i=1}^{l}[(a_j)_i, (b_j)_i]$.[52] Let $\{r_n\}$ be a recursive sequence of rational numbers $r_j > 0$. Then there is a computable (cf. Remark 16(1b), page 58) sequence of computable-continuous functions $G_n : \mathbb{R}^l \to [0,1]$ such that:

$$
\begin{aligned}
&1)\ G_n(x) = 1 \quad \text{for all } x \in X_n \text{ and all } n \in \mathbb{N} \\
&2)\ G_n(x) = 0 \quad \text{for all } x \in \mathbb{R}^l \backslash (X_n + B_{r_n}) \text{ and all } n \in \mathbb{N}, \\
&3)\ \{D^k G_n\}_{n,k \in \mathbb{N}} \text{ is a computable sequence of } c\text{-}c \text{ functions.} \\
&4)\ \text{for all } n \in \mathbb{N} \text{ and all } k = 1, 2, \cdots, \text{ the derivatives } D^k G_n(x) = 0 \text{ for all } x \in X_n \text{ and for all } x \text{ in the closure of } \mathbb{R}^l \backslash (X_n + B_{r_n}).
\end{aligned}
\tag{88}
$$

Such functions G_n can be constructed by modifying our proof of Lemma 1 as follows. In particular, regarding each X_n as K and r_n as r in that Lemma, we can construct G_n with properties (1,2,4) in (88). Since ϕ and the (a_j, b_j) sequence present the sequences I_j and X_n in a recursive manner, this clearly permits us to ensure property (3) in (88) and the computability of the G_n sequence.

In more detail, for all $j \in \mathbb{N}$, we define k_j to be the least $k \in \mathbb{N}$ satisfying (80) with $r = r_n$, where n is the unique $n \in \mathbb{N}$ with $\phi(n) \leqq j < \phi(n+1)$. For each $n \in N$, we define (as in 81) $N_{X_n} = \max\{k_j : \phi(n) \leqq j < \phi(n+1)\}$. For all $n \in \mathbb{N}$ and all j with $\phi(n) \leqq j < \phi(n+1)$, we define $C(I_j, N_{X_n})$ through (82) with $K = X_n$.

Since the sequence $\{(a_j, b_j)\}_{j \in \mathbb{N}}$ is recursive and the function $\phi : \mathbb{N} \to \mathbb{N}$ is recursive, one can clearly construct a recursive sequence $\{v_k\}_{k \in \mathbb{N}}$ of rational vectors in $\mathbb{R}^l$ and an increasing recursive function $\psi : \mathbb{N} \to \mathbb{N}$ such that for all n, the sequence $v_{\psi(n)}, \cdots, v_k \cdots, v_{\psi(n+1)-1}$ lists all the elements in $\cup_{j=\phi(n)}^{\phi(n+1)-1} C(I_j, N_{X_n})$. Then $\{G_{v_k, r_n}\}_{k,n \in \mathbb{N}}$ defined by (83) with $v = v_k$ and $r = r_n$ is a computable sequence of c-c functions; hence the functions

$$\tilde{G}_n(x) = \sum_{k=\psi(n)}^{\psi(n+1)-1} G_{v_k, r_n}(x)$$

also form a computable sequence of c-c functions (cf. Remark 14(2b(III)), and so do the functions

$$G_n(x) = H(\tilde{G}_n(x)/(1 - H(1/2))$$

---

[52] **For a sequence $\{K_n\}$ of finite unions of rational rectangles, if it recursive, then it is computable (in the sense of Remark 16(1b), page 58); but the converse is not generally true (cf. Remark 11, page 51).**

(cf. Remark 14(2b(I)). Also, since $\{D^t H\}_{t\in\mathbb{N}}$ and $\{D^t G_{v_k,r_n}\}_{t,k,n\in\mathbb{N}}$ are computable sequences of c-c functions, the sequence $\{D^k G_n\}_{k,n\in\mathbb{N}}$ satisfies 88(3) (cf. Remark 17(4a), page 60). Also notice that

$$G_n(x) = H\left(\frac{\sum_{v\in\cup_{j=\phi(n)}^{\phi(n+1)-1} C(I_j, N_{X_n})} G_{v,r_n}(x)}{1 - H(1/2)}\right);$$

so it follows easily (as in the proof of Lemma 1) that the functions G_n satisfy (88(1,2)); and (88(4)) is also easily checked.

C.8 Non-computability of fixed-points

The following result "refutes" Brouwer's Fixed-Point Theorem in the context of recursive analysis.[53] It asserts, for any $k = 1, 2, \cdots$, the existence of a C^k function $f : D = \{p \in \mathbb{R}^2 : ||p|| \leqq 1\} \to D$, that is computable-continuous (i.e. f is sequential-computable and effectively locally uniformly continuous on D) and such that $f(x) \neq x$ for all computable vectors $x \in D$. It strengthens the results of Orevkov (1964) and Baigger (1985) by obtaining the C^k property for f. For the purpose of application, we will construct f so that $f(D)$ is contained in the interior of D.

Proposition 2 (A computable counterexample to Brouwer's Fixed-Point Theorem; cf. Orevkov (1964) and Baigger (1985)). *For all $k = 1, 2, \cdots$, there exists a C^k computable-continuous function $f : D \to D$ such that:*

1) $f(x) \neq x$ for all computable vectors $x \in D$,

2) $f(D) \subsetneqq D_\epsilon = \{p \in \mathbb{R}^2 : ||p|| \leqq \epsilon\}$ for some $\epsilon \in \mathbb{R}_c$ with $0 < \epsilon < 1$.

Remark 20.

By Remark 17(9), page 60, Proposition 2 can be stated equivalently as follows: for all $k = 1, 2, \cdots$, there is a computable function $g : D_c = \{x \in D : x \in \mathbb{R}_c^2\} \to D_c$ such that g can be extended to a C^k function from D into D, and $g(x) \neq x$ for all $x \in D_c$, and $g(D_c) \subsetneqq D_{c,\epsilon} = \{x \in D_\epsilon : x \in \mathbb{R}_c^2\}$ for some $\epsilon \in \mathbb{R}_c$ with $0 < \epsilon < 1$.

We will prove Proposition 2 along the lines of Orevkov (1964), Baigger (1985), and Beeson (1985, pp. 75–76). The approach is to find an increasing

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(53) For a counterexample in the context of intuitistic mathematics, see Brouwer (1952).

and recursive sequence of finite unions $K_n \subsetneqq \mathbb{R}^2$ of rational rectangles such that the complement of $\cup_{n\in\mathbb{N}} K_n$ in D is non-empty but contains no computable points. We will use the K_n to find a C^k computable function $f : D \to D$ that moves the f-values of every $x \in \cup_{n\in\mathbb{N}} K_n$ away from x, so f has no computable fixed-points.

To find such $K_n \subsetneqq \mathbb{R}^2$, we begin with the following one dimensional Fact, which is a well-known "computable counterexample" to the Heine-Borel theorem. Our version is due to Beeson (1985, pp. 69–70), and is stated in a form convenient for our applications.

Fact 5 (Lacombe (1955), Zaslavskii and Čeitin (1962); cf. Beeson (1985)). *There are two recursive sequences of rational numbers $a_n, b_n \in [-1/2, 1/2]$ such that $a_n < b_n$ for all $n \in \mathbb{N}$, where the sequence of computable intervals $J_n = [a_n, b_n]$ satisfies:*

1) $J_1 = [-1/2, -1/4]$ and $J_2 = [1/4, 1/2]$.

2) Any two J_n are disjoint or have only one common endpoint;

3) For each computable $x \in (-1/2, 1/2)$, there exist n, n' with $x \in (a_n, b_{n'})$ and $b_n = a_{n'}$.

Remark 21.

1) The union of all J_n contains all computable reals in the interval $[-1/2, 1/2]$ (by (3)).

2) However, it does not contain all reals in the interval (since no finite union of them covers the interval (by (2))).

3) Thus $[-1/2, 1/2]\backslash \cup_{n\in\mathbb{N}} J_n$ is non-empty and contains no computable real.

Proposition 2 focuses on the unit disc; it will be convenient to restrict our attention to the square

$I = [-1, 1] \times [-1, 1].$

We define

$K = [-1/2, 1/2] \times [-1/2, 1/2],$

which will play a role analogous to that of the interval $[-1/2, 1/2]$ in Fact 5. We define:

$R_{-4} = [1/2, 1] \times [-1, 1], \qquad R_{-3} = [-1, -1/2] \times [-1, 1],$
$R_{-2} = [-1/2, 1/2] \times [-1, -1/2], \qquad R_{-1} = [-1/2, 1/2] \times [1/2, 1];$

$K_{-1} = R_{-4} \cup R_{-3} \cup R_{-2} \cup R_{-1};$

so K_{-1} is the closure of $I\backslash K$. Now, let the sequences $\{a_n\}$ and $\{b_n\}$ define the interval intervals J_n as in Fact 5. We will now use the one-dimensional intervals J_m as building blocks for two dimensional rectangles R_{mk}, whose unions will constitute the desired sets K_n. In particular, we define:

$R_{nk} = J_n \times J_k \qquad$ for all $n, k \in I\!N$,

$A_n = \cup_{k \leqq n}(R_{kn} \cup R_{nk}) \qquad$ for all $n \in I\!N$.

Notice that the rectangles R_{kn}, R_{nk} in A_n are newly created at stage n and do not belong to previous A_m (by Fact 5(2)). We define

$K_n = A_n \cup K_{n-1} \qquad$ for all $n \in I\!N$;

so A_n is the part of K_n newly created at stage n. We also denote

$B_r = [-r, r] \times [-r, r] \qquad$ for all rational numbers $r > 0$ (as in (77)),

$S = \{p \in I\!R^2 : \|p\| = 1\}$.

Since the sequences $\{a_k\}$ and $\{b_k\}$ are recursive, it follows that the sequence $\{K_n\}_{n \in I\!N}$ of finite unions of rational rectangles is also recursive (as defined in Remark 19(4)).

Similarly, $\{P_n\}_{n \in I\!N}$ is also a recursive sequence of finite unions of rational rectangles, where P_n = closure$(K \backslash K_n)$. To see this, first we notice that for $n = 0, 1$, the set P_n is finite union of rational rectangles in K. Next, we notice that for each $n \geqq 2$, by Fact 5(2) we can (uniquely) list all the a_k, b_k with $k \leqq n$ in a nondecreasingly sequence:

$$a_{m_0} < b_{m_0} \leqq a_{m_1} < b_{m_1} \leqq a_{m_2} < b_{m_2} \leqq \cdots, \leqq a_{m_n} < b_{m_n}.$$

By Fact 5(1) we have $b_{m_0} = -1/4$ and $a_{m_n} = 1/4$. We define C_n to be set of the $i \leqq n-1$ such that $b_{m_i} < a_{m_{i+1}}$. The set C_n is non-empty because otherwise, the set $\cup_{k=0}^n J_k$ would cover $[-1/2, 1/2]$, which is impossible (cf. the paragraph immediately below Fact 5). For each $i \in C_n$, we define the rational interval $L_i = [b_{m_i}, a_{m_{i+1}}]$, which is clearly contained in $[-1/2, 1/2]$ and intersects $\cup_{k=1}^n J_k$ only at the endpoints $b_{m_i}, a_{m_{i+1}}$. Now we define

$$\begin{aligned} I^*_{ik} &= L_i \times J_k && \text{for all } i \in C_n \text{ and all } k \leqq n, \\ I^{**}_{ki} &= J_k \times L_i && \text{for all } i \in C_n \text{ and all } k \leqq n, \\ I^{***}_{i,i'} &= L_i \times L_{i'} && \text{for all } i, i' \in C_n, \end{aligned} \tag{89}$$

and so

$$P_n = \cup\{I^*_{ik}, I^{**}_{ki}, I^{***}_{i,i'} : i, i' \in C_n \;\&\; k \leqq n\}. \tag{90}$$

Thus the sequence $\{P_n\}_{n \in I\!N}$ of sets P_n can be described as in (87), therefore is recursive.

Also, notice that for all n, the intersection $P_n \cap K_n \subseteqq K$ and is contained in the boundary of K_n.

Proof of Proposition 2(1). We pick a sequence of functions h_n as given in Lemma 2 below. We fix any $k = 1, 2, \cdots$. We will transform these functions h_n into a C^k computable-continuous "perturbation" function h, and then we will transform this h into the function f that we seek ((97) below).

Step 1) Constructing the "perturbation" h. We pick the recursive sequence $\{P_n\}_{n\in\mathbb{N}}$ of sets defined through (90), and pick a computable sequence of c-c functions $G_n : \mathbb{R}^2 \to [0,1]$ satisfying (88) with $X_n = P_n$ and $r_n = 2^{-n}$.

For all $n \in \mathbb{N}$, we define a function $\tilde{h}_n : I \to \mathbb{R}^2$ by

$$\tilde{h}_n(x) = \begin{cases} (1 - G_n(x))h_n(x) & \text{for } x \in K_n \\ 0 & \text{for } x \notin K_n. \end{cases} \tag{91}$$

Recall that each P_n is the closure of the complement of K_n in K, then by (88(1,4)) and Remark 19(4b), it follows that each $\tilde{h}_n$ is a C^∞ function. Since $\{G_n\}$ and $\{h_n\}$ are computable sequences of c-c functions, it follows that $\{\tilde{h}_n\}$ also is (cf. Remark 14(2b)). Also, by (88(3)) and Lemma 2(3), it follows that $\{D^k h_n\}$ is also a computable sequence of c-c functions (cf. Remark 17(4a)), and so is the sequence $\{||D^k h_n(\cdot)||\}_{k,n\in\mathbb{N}}$. Then Remark 16(1b) ensures that there is a computable sequence of computable reals $\lambda_n > 0$ such that

$$||D^1\tilde{h}_n(x)||, \cdots, ||D^k\tilde{h}_n(x)|| \leqq \lambda_n \qquad \text{for all } x \in I \text{ and all } n \in \mathbb{N}. \tag{92}$$

For every $x \in I$, we define

$$h(x) = \frac{1}{2}\sum_{n=0}^{\infty} \frac{1}{2^n + \lambda_n}\tilde{h}_n(x); \tag{93}$$

recall that $h_n(x) \in S$ for all n, so by (91) and (93) $||h(x)|| \leqq 1$. Therefore (93) defines a function $h : I \to D$. Since the functions $\tilde{h}_n$ are C^∞ (hence C^k), by (92) and (93) the function h is also C^k. Also, since $\{\tilde{h}_n\}$ is a computable-sequence of c-c functions, the function h is also computable-continuous (cf. Remarks 14(2b,3)).

Now consider any $x \in I$ with $1 - G_n(x) > 0$ for some $n \in N$. Then we have:

$$0 < \Phi(x) \leqq 1, \tag{94}$$

where

$$\Phi(x) = \frac{1}{2}\sum_{n=N_x}^{\infty} \frac{1}{2^n + \lambda_n}(1 - G_n(x))$$
$$N_x = \min\{n \in I\!N : 1 - G_n(x) > 0\}. \tag{95}$$

Also, since $G_{N_x}(x) < 1$, we have $x \notin P_{N_x}$ (by (88(1)) with $X_n = P_n$); so $x \in K_{N_x}$, hence $x \in K_n$ for all $n \geqq N_x$, and therefore by Lemma 2(2), we have $h|_n(x) = h|_{N_x}(x)$ for all $n \geqq N(x)$. Therefore by (91) and (93) for all $x \in I$:

$$h(x) = \Phi(x)h_{N_x}(x) \qquad \text{if } 1 - G_n(x) > 0 \text{ for some } n. \tag{96}$$

Step 2) Constructing f. We now define the function $f : D \to I\!R^2$ by

$$f(x) = x - (1/4)h(x). \tag{97}$$

Since h is C^k and computable-continuous, the function f is also C^k and computable-continuous.

We will now show that $f(D)$ is contained in the interior $\text{Int}(D)$ of D.

(Case 1) Suppose $x \in D\backslash K$. Since all the sets P_n are contained in the compact set K, it follows that $x \notin P_n + B_{1/n}$ for some n, so $1 - G_n(x) > 0$ for some n, hence by (96) we have $f(x) = x - (1/4)\Phi(x)h_{N_x}(x)$. Recall that $x \notin K$; so $x \in K_{-1} \subseteqq K_0$, hence by Lemma 2(1) we have $x/||x|| = h_0(x)$ and therefore $h_{N_x}(x) = x/||x||$ by Lemma 2(2), and consequently:

$$f(x) = x - \frac{1}{4}\frac{\Phi(x)x}{||x||}. \tag{98}$$

Since $x \in D\backslash K$, we have $1 \geqq ||x|| > 1/2 > 1/4\Phi(x) > 0$ by (94); so $||f(x)|| = (||x|| - (1/4)\Phi(x))||(x/||x||)|| < ||(x/||x||)|| \leqq 1$, hence $f(x) \in \text{Int}(D)$.

(Case 2) Suppose $x \in K$. Then $||x|| \leqq (1/2)^{1/2}$. Recall that $||h(x)|| \leqq 1$; so from (97) we have $||f(x)|| \leqq ||x|| + (1/4)||h(x)|| \leqq (1/2)^{1/2} + 1/4 < 1$, hence $f(x) \in \text{Int}(D)$.

Thus $f(D) \subseteqq \text{Int}(D)$; then (2) in Proposition 2 is immediate by the compactness of $f(D)$.

We now show property (1) in Proposition 2.

(Case I) Suppose a computable $x \in D\backslash K$. Then, as seen in Case 1 above, we have $\Phi(x) > 0$ and (98), so $f(x) - x = -(1/4)\Phi(x)(x/||x||) \neq 0$.

(Case II) Suppose a computable $x \in K$. As we will show, we have $x \in \text{Int}(K_n)$ for some n, so $x \notin P_n + B_{1/m}$ for any sufficiently large m, hence $x \notin$

$P_{n+m} + B_{2^{-(n+m)}}$, hence (by (88) with $X_n = P_{n+m}$ and $r_n = 2^{-(n+m)}$) we have $1 - G_{n+m}(x) > 0$. Then by (96) and (97) we have $f(x) - x = (1/4)\Phi(x)h_{N_x}(x)$. Recall that $h_{N_x}(x) \in S$; so by (94) we have $f(x) - x \neq 0$.

It remains to show that each computable $x \in K$ belongs to $\mathrm{Int}(K_n)$ for some n. First, suppose x is in the corners of K, then by Fact 5(1) $x \in \mathrm{Int}(K_1)$. Next, suppose x is in the boundary of K, but not in the corners of K. For example, let $x_1 = 1/2$ and $x_2 \in (-1/2, 1/2)$. Then by Fact 5(3), $x_2 \in (a_k, b_{k'})$ for some k, k' with $b_k = a_{k'}$, so $x_2 \in \mathrm{Int}(J_k \cup J_{k'})$, hence $x \in \mathrm{Int}(K_{-1} \cup J_2 \times (J_k \cup J_{k'}))$, therefore $x \in \mathrm{Int}(K_n)$ for all $n \geqq \max\{2, k, k'\}$. Finally, suppose $x \in \mathrm{Int}(K)$. Then $x_1, x_2 \in (-1/2, 1/2)$, and so by Fact 5(3) we have $x \in (a_k, b_{k'}) \times (a_{\tilde{k}}, b_{\tilde{k}'})$ for some $k, k', \tilde{k}, \tilde{k}'$ with $b_k = a_{k'}$ and $b_{\tilde{k}} = a_{\tilde{k}}$. Thus $x \in \mathrm{Int}((J_k \cup J_{k'}) \times (J_{\tilde{k}} \times J_{\tilde{k}'}))$, so $x \in \mathrm{Int}(K_n)$ for all $n \geqq \max\{k, k', \tilde{k}, \tilde{k}'\}$. Thus we have shown that any computable $x \in K$ belongs to $\mathrm{Int}(K_n)$ for some n, completing the proof of (1) in Proposition 2. Q.E.D.

Lemma 2. *There is a computable sequence of C^∞ c-c functions $h_n : K_n \to S$ satisfying:*

1) $h_n(x) = x/\|x\|$ for all $x \in K_0$,

2) $h_{n'}|_{K_n} = h_n$ for all $n, n' \in \mathbb{N}$ with $n' \geqq n$,

3) $\{D^k h_n\}_{k,n\in\mathbb{N}}$ is a computable sequence of c-c functions.

Proof of Lemma 2. In order to obtain C^∞ differentiability for the extension functions h_n, it is convenient to expand the sets K_n into the sets $K_n + B_{1/N_n}$, where the natural numbers $N_n > 0$ are defined by the following algorithm. We define $N_0 = 4$ (so that $0 \notin K_0 + B_{1/N_0}$). We continue as follows. At the n-th stage, we have defined N_n. Now consider the set $K_{n+1} = K_n \cup A_{n+1}$. Notice that A_{n+1} is contained in the "cross-shaped" region $T = J_{n+1} \times [-1/2, 1/2] \cup [-1/2, 1/2] \times J_{n+1}$. By Fact 5(2), $K_n \cap T$ is contained in the boundary of T. We can decompose the set A_{n+1} (uniquely) into its connected components $T_1^{n+1}, \cdots, T_{m_{n+1}}^{n+1}$. Each T_i^{n+1} must have at least one boundary segment interior to T,[54] because otherwise T_i^{n+1} would be equal to all of T, so $\cup_{k \leqq n+1} J_k = [-1/2, 1/2]$, contradicting Remark 21(2).[55] Therefore, for any sufficiently large

[54] This observation is due to Beeson (1985, p. 76, first paragraph).

[55] Alternatively, there are some rectangles, e.g. $R_{n+1,n+2}$, which belong to T but (by Fact 5(2)) do not belong to A_{n+1}, hence also do not belong to T_i^{n+1}.

$m' \in \mathbb{N}$, we have:

$$\begin{array}{ll} \text{a)} & \partial(T_i^{n+1} + B_{1/m'}) \subsetneqq (K_n + B_{1/m'}) \text{ for all } i \leqq m_{n+1}, \\ \text{b)} & (T_i^{n+1} + B_{1/m'}) \cap (T_j^{n+1} + B_{1/m'}) = \emptyset \text{ for all } i, j \leqq m_{n+1} \\ & \quad \text{with } i \neq j. \end{array} \tag{99}$$

where $\partial(T_i^{n+1} + B_{1/m'})$ is the boundary of $T_i^{n+1} + B_{1/m'}$. Therefore we can define N_{n+1} to be the least natural number $m' \geqq N_n$ satisfying (99). Since $\{K_n\}$ is a recursive sequence a finite union of rational rectangles, it follows that the function $n \mapsto N_n$ is also a recursive from from $\mathbb{N}$ into $\mathbb{N}$, and so $\{1/N_n\}_{n \in \mathbb{N}}$ is a recursive sequence of positive rational numbers.

The rest of the proof will be divided into three steps. First, in Step I we will choose a computable sequence $\{g_{\phi(n)}\}_{n \in \mathbb{N}}$ of rational polynomials[(56)] (hence computable-continuous) $g_{\phi(n)} : \mathbb{R}^2 \to \mathbb{R}^2$ such that:

$$\begin{array}{ll} 1) & g_{\phi(0)} \text{ is the identity mapping } Id; \\ 2) & \min\{\|g_{\phi(n)}(x)\| : x \in K_n + B_{1/N_n}\} > 1/8 \text{ for all } n \in \mathbb{N}; \\ 3) & \max\{\|g_{\phi(n)}(x) - g_{\phi(n+1)}(x)\| : x \in K_n + B_{1/N_{n+1}}\} < (1/8)2^{-(n+1)} \\ & \quad \text{for all } n \in \mathbb{N}. \end{array} \tag{100}$$

Then in Step II we will transform $\{g_{\phi(n)}\}_{n \in \mathbb{N}}$ into a computable sequence $\{f_n\}_{n \in \mathbb{N}}$ of computable-continuous functions $f_n : \mathbb{R}^2 \to \mathbb{R}^2$ such that:

$$\begin{array}{ll} 1) & f_0 = g_{\phi(0)}|_{K_0}; \\ 2) & f_{n'}|_{K_n} = f_n|_{K_n} \text{ for all } n, n' \in \mathbb{N} \text{ with } n' \geqq n; \\ 3) & \min\{\|f(x)\| : x \in K_n\} > 0 \text{ for all } n \in \mathbb{N}; \\ 4) & \{D^k f_n\}_{k,n \in \mathbb{N}} \text{ is a computable sequence of c-c functions.} \end{array} \tag{101}$$

Finally, in Step III we will define the functions $h_n : K \to \mathbb{R}^2$ by:

$$h_n(x) = f_n(x)/\|f_n(x)\|, \tag{102}$$

which will be shown to satisfy (1-3) in Lemma 2.

Step I) Choosing $\{g_{\phi(n)}\}_{n \in \mathbb{N}}$. First, we can pick a computable sequence $\{g_k\}_{k \in \mathbb{N}}$ that enumerates (with or without repetitions) all rational polynomials from $\mathbb{R}^2$

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(56) I.e. $g_{\phi(n)} = (g^1_{\phi(n)}, g^2_{\phi(n)})$, where each $g^i_{\phi(n)}$ is a polynomial with rational coefficients.

into $\mathbb{R}^2$. Then for all $n, k', k \in \mathbb{N}$, we define:

$$\begin{aligned} &1)\ s_{n,k,k'} = \max\{\|g_{k'}(x) - g_k(x)\| : x \in K_n + B_{1/N_{n+1}}\}, \\ &2)\ t_{n,k} = \min\{\|g_k(x)\| : x \in K_n + B_{1/N_n}\}. \end{aligned} \tag{103}$$

Notice that $\{K_n + B_{1/N_{n+1}}\}_{n\in\mathbb{N}}$ and $\{K_n + B_{1/N_{n+1}}\}_{n\in\mathbb{N}}$ are recursive (hence computable) sequences of finite unions of rational (hence computable) rectangles. Therefore, $\{t_{nk}\}_{n,k\in\mathbb{N}}$ and $\{s_{n,k,k'}\}_{n,k,k'\in\mathbb{N}}$ are computable sequences of computable reals (cf. Remark 16(1b), page 58), so by definition there are recursive sequences $\{c_{n,k,k',m}\}_{n,k,k',m\in\mathbb{N}}$ and $\{d_{n,k,m}\}_{n,k,m\in\mathbb{N}}$ of rational numbers such that:

$$\begin{aligned} &1)\ |s_{n,k,k'} - c_{n,k,k',m}| \leqq 2^{-m} \qquad \text{for all } n, k, k', m \in \mathbb{N}, \\ &2)\ |t_{n,k} - d_{n,k,m}| \leqq 2^{-m} \qquad \text{for all } n, k, m \in \mathbb{N}. \end{aligned} \tag{104}$$

Now we can define a function $\phi : \mathbb{N} \to \mathbb{N}$ by recursion as follows. (The function ϕ will select the desired sequence $\{g_{\phi(n)}\}_{n\in\mathbb{N}}$ from $\{g_n\}_{n\in\mathbb{N}}$.)

First, we set $\phi(0)$ to be a $k \in \mathbb{N}$ such that $g_k = Id$. Since $\|x\| > 1/8$ for all $x \in K_0 + B_{1/N_0}$, property (100(2)) holds for $n = 0$.

We continue as follows. At the n-th stage, we are given $g_{\phi(n)}$ such that $t_{n,\phi(n)} > 1/8$. First, we want to ensure that the restriction $g_{\phi(n)}|_{K_n+B_{1/N_{n+1}}}$ can be extended to a continuous function h from $K_{n+1} + B_{1/N_{n+1}} = (K_n + B_{1/N_{n+1}}) \cup (A_{n+1} + B_{1/N_{n+1}})$ into $\mathbb{R}^2$ with the property that $\min\{\|h(x)\| : x \in K_{n+1} + B_{1/N_{n+1}}\} > 1/8$. Of course, this might be impossible if the boundary of one of the connected components in the "new" part $A_{n+1} + B_{1/N_{n+1}}$ were not completely contained in the "old" domain $K_n + B_{1/N_{n+1}}$. However, (99) (with $m' = 1/N_{n+1}$) ensures that such containment cannot occur; and then standard tools of topology (e.g. the lifting lemma (cf. Munkres (1975), Section 8-4), or Hirsch (1976, Theorem 1.8, p. 126)) ensure the existence of such extension h. By the Weierstrass Approximation Theorem, there exist some g_k close enough to h on $K_{n+1} + B_{1/N_{n+1}}$ so that

$$t_{n+1,k} > 1/8 \ \ \& \ \ s_{n,\phi(n),k} < (1/8)2^{-(n+1)}. \tag{105}$$

Notice that for all $k \in \mathbb{N}$, by (98) we have: k satisfies (105) if and only if the following (106) holds for some m;

$$[d_{n+1,k,m} > 1/8 + 2 \cdot 2^{-m}] \ \& \ [c_{n,\phi(n),k,m} < (1/8)2^{-(n+1)} - 2 \cdot 2^{-m}]. \tag{106}$$

Then we can set M_{n+1} to be the least $M \in \mathbb{N}$ such that (106) holds for some natural numbers $k, m \leqq M$; and define $\phi(n+1)$ to be the least natural number

$k \leqq M$ satisfying (106) for some $m \leqq M$. Hence $g_{\phi(n+1)}$ satisfies (100(2)), and with $g_{\phi(n)}$ satisfies (100(3)).

It is clear that the function $\phi : \mathbb{N} \to \mathbb{N}$ is recursive; so $\{g_{\phi(n)}\}_{n \in \mathbb{N}}$ is a computable sequence of rational polynomials (cf. Footnote 48).

Step II) Constructing $\{f_n\}_{n \in \mathbb{N}}$. First, we pick a computable sequence of C^∞ *c-c* functions $G_n : \mathbb{R}^2 \to [0,1]$ satisfying (88(1,2,3)) with $X_n = K_n$ and $r_n = 1/N_{n+1}$.

We define the functions $f_n : \mathbb{R}^2 \to \mathbb{R}^2$ by the following recursion. First, we define the function $f_0 = g_{\phi(0)}$, so (101(1)) holds. For $n \in \mathbb{N}$, we define the function $f_{n+1} : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$f_{n+1}(x) = G_n(x) f_n(x) + (1 - G_n(x)) g_{\phi(n+1)}(x). \tag{107}$$

By (88(1)) with $X_n = K_n$, we have $G_n|_{K_n} = 1$ for all n, so $f_{n+1}|_{K_n} = f_n|_{K_n}$. It follows that $\{f_k\}_{k \in \mathbb{N}}$ will satisfy (101(2)).

Since $\{G_n\}_{n \in \mathbb{N}}$ and $\{g_{\phi(n)}\}_{n \in \mathbb{N}}$ are computable sequence of *c-c* functions, the sequence $\{f_n\}_{n \in \mathbb{N}}$ also is (cf. Remark 14(2(II)), page 56). Also, notice that the derivatives of a computable sequence of rational polynomials also form a computable sequence of rational polynomials; so the sequence $\{D^k g_{\phi(n)}\}_{k,n \in \mathbb{N}}$ is also a computable sequence of rational polynomials. Then by (88(3)), the sequence $\{D^k f_n\}_{n,k \in \mathbb{N}}$ satisfies (101(4)) (cf. Remark 17(4), page 60).

To complete this step, it remains to show (101(3)). First, notice that by definition for all $x \in K_0 + B_{1/N_0}$ we have $f_0(x) = g_{\phi(0)}(x)$, so $\|f_0(x) - g_0(x)\| = 0$. Next, we will show that for all $n \geqq 1$:

$$\|f_n(x) - g_{\phi(n)}(x)\| \leqq (1/8) \sum_{i=1}^{n} 2^{-i} \qquad \text{for all } x \in K_n + B_{1/N_n}. \tag{108}$$

To see this, first let $n = 1$. Consider any $x \in K_1 + B_{1/N_1}$. Suppose $x \notin K_0 + B_{1/N_1}$. Since (88(2)) holds for $X_n = K_0$ and $r_n = 1/N_1$, we have $G_0(x) = 0$; so by (107) we have $\|f_1(x) - g_{\phi(1)}\| = 0$. Suppose $x \in K_0 + B_{1/N_1}$. By (107) we have $\|f_1(x) - g_{\phi(1)}\| = G_n(x)\|f_0(x) - g_{\phi(1)}(x)\| \leqq \|f_0(x) - g_{\phi(0)}(x)\| + \|g_{\phi(0)}(x) - g_{\phi(1)}(x)\| \leqq 0 + (1/8)(1/2)$ (see (100(3))). Hence (108) holds for $n = 1$. Now we assume (108) holds for n, and we show (108) for $n+1$. Consider any $x \in K_{n+1} + B_{1/N_{n+1}}$. Suppose $x \notin K_n + B_{1/N_{n+1}}$. Since (88(2)) holds with $X_n = K_n$ and $r_n = N_{n+1}$, we have $G_n(x) = 0$; so by (107) we have $\|f_{n+1}(x) - g_{\phi(n+1)}(x)\| = 0$. Suppose $x \in K_n + B_{1/N_{n+1}}$. By (107) we have: $\|f_{n+1}(x) - g_{\phi(n+1)}(x)\| = G_n(x)\|f_n(x) - g_{\phi(n+1)}(x)\| \leqq \|f_n(x) - g_{\phi(n)}(x)\| + \|g_{\phi(n)}(x) - g_{\phi(n+1)}(x)\|$. Then by the induction hypothesis and (100(3)), we have: $\|f_{n+1}(x) - g_{\phi(n+1)}(x)\| \leqq (1/8)\sum_{i=1}^{n} 2^{-i} + (1/8)2^{-(n+1)}$. Hence (108) holds for $n+1$.

Then for all $n \in I\!N$ and all $x \in K_n$, we have: $\|f_n(x)\| \geqq \|g_{\phi(n)}(x)\| - \|f_n(x) - g_{\phi(n)}(x)\| > 0$ by (108) and (100(2)); i.e. (101(3)) holds.

Step III) Constructing $\{h_n\}_{n\in I\!N}$. By (101(3)), we can define the functions $h_n : K_n \to S$ by (102). Since $\{f_n\}_{n\in I\!N}$ is a computable sequence of *c-c* functions, the sequence $\{h_n\}_{n\in I\!N}$ also is. Notice that f_0 agrees with $g_{\phi(0)} = Id$ on K_0; so h_0 satisfies (1) in Lemma 2. Also, (2) in Lemma 2 is immediate from (101(2)); and (3) in Lemma 2 is also clear from (101(4)). This completes our proof for the lemma. Q.E.D.

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