THE COMPRESSIBLE REYNOLDS LUBRICATION EQUATION

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THE COMPRESSIBLE REYNOLDS LUBRICATION EQUATION

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Abstract

We give a general derivation of the Reynolds lubrication equation. We then state and sketch the proofs of some of the authors' recent results concerning the existence, uniqueness, and qualitative behavior of solutions to the compressible Reynolds lubrication equation. Finally, we give an application of our results to a problem in elastohydrodynamics.

1. Introduction

The Reynolds lubrication equation for the pressure, $P = P(x_1,x_2,t)$, that develops in a layer of fluid of thickness, $h = h(x_1,x_2,t)$, which is confined between two solid bodies when the sum of the velocities of the upper and lower bodies is $\underline{V} = (V_1,V_2)$ is [8, p. 60]

1)
$$12\mu \frac{\partial}{\partial t} (\rho h) + 6\mu\nabla \cdot (\rho h\underline{V}) = \nabla \cdot (h^3\rho\nabla P), \qquad x = (x_1,x_2) \in \Omega,\ t \in \mathbb{R},$$

$$P = P_a , \qquad x \in \partial\Omega,\quad t \in \mathbb{R},$$

where ρ is density, μ is the dynamic viscosity, $\Omega \subseteq \mathbb{R}^2$ is the region (with boundary, $\partial\Omega$) where the upper and lower boundaries are in proximity, and $P_a > 0$ is the ambient pressure.

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If the lubricating film is incompressible (liquid films), then the assumption ρ = constant is valid. In this case, we obtain the classical Reynolds lubrication equation

2) $$12\mu \frac{\partial h}{\partial t} + 6\mu \nabla \cdot (h\underline{V}) = \nabla \cdot (h^3 \nabla P), \qquad x \in \Omega,\ t \in \mathbb{R},$$

$$P = P_a, \qquad x \in \partial\Omega,\ t \in \mathbb{R},$$

which is a linear elliptic equation for the pressure.

Gas films such as those used to lubricate disk and tape magnetic recording systems [6,7,11] are usually modeled by assuming the gas to be compressible. The temperature of the surfaces of the confining solid bodies (the disk and the head, for example) are assumed to be equal and constant. It is further assumed that the temperature of the gas film remains equal to the temperature of the confining solid bodies (the gas is assumed to be isothermal). The ideal gas relationship

3) $$P/\rho = \text{constant}$$

is then utilized in 1) to derive the compressible Reynolds lubrication equation

4) $$12\mu \frac{\partial}{\partial t} (Ph) + 6\mu \nabla \cdot (Ph\underline{V}) = \nabla \cdot (h^3 P \nabla P),$$
$$x \in \Omega, \quad t \in \mathbb{R},$$
$$P = P_a, \qquad x \in \partial\Omega,\ t \in \mathbb{R},$$

which is a <u>nonlinear</u> parabolic equation for the pressure.

2. Derivation of the equation

We will now give a derivation of the Reynolds lubrication equation. We represent our upper surface by

$$\{(x_1, x_2, h_2(x_1,x_2,t)) \mid (x_1,x_2) \in \Omega,\ t \in \mathbb{R}\},$$

and we suppose that it moves with velocity

$$\underline{V}_2 = (V_{2,1}, V_{2,2}, 0).$$

We also represent our lower surface by

$$\{(x_1,x_2,h_1(x_1,x_2,t))\,|\,(x_1,x_2) \in \Omega,\ t \in \mathbb{R}\},$$

and we suppose that it moves with velocity

$$\underline{V}_1 = (V_{1,1},\ V_{1,2},\ 0).$$

Thus, $h = h_2 - h_1$ and

$$V_i = V_{1,i} + V_{2,i}$$

for $i = 1,2$.

We can integrate the continuity equation to obtain

$$5)\qquad \int_{h_1}^{h_2} \frac{\partial \rho}{\partial t}\, dx_3 + \sum_{i=1}^{3} \int_{h_1}^{h_2} \frac{\partial}{\partial x_i}\, (\rho u_i)\, dx_3 = 0$$

where $\underline{u}(x_1,x_2,x_3,t) = (u_1,u_2,u_3)$ is the fluid velocity.

Now

$$6)\qquad \int_{h_1}^{h_2} \frac{\partial \rho}{\partial t}\, dx_3 = \frac{\partial}{\partial t} \int_{h_1}^{h_2} \rho\, dx_3 - \sum_{j=1}^{2} (-1)^j\, \frac{\partial h_j}{\partial t}\, (x_1,x_2,t)\rho(x_1,x_2,h_j(x_1,x_2,t),t),$$

$$7)\quad \int_{h_1}^{h_2} \frac{\partial}{\partial x_i} (\rho u_i)\, dx_3 = \frac{\partial}{\partial x_i} \int_{h_1}^{h_2} \rho u_i dx_3 - \sum_{j=1}^{2} (-1)^j [\frac{\partial h_j}{\partial x_i}\, (x_1,x_2,t)][(\rho u_i)(x_1,x_2,h_j(x_1,x_2,t),t)]$$

for $i = 1,2$, and

$$8)\qquad \int_{h_1}^{h_2} \frac{\partial}{\partial x_3}\, (\rho u_3)\, dx_3 = \sum_{j=1}^{2}\, (-1)^j (\rho u_3)(x_1,x_2,h_j(x_1,x_2,t),t).$$

The kinematic boundary conditions require that

$$9)\qquad \frac{\partial h_j}{\partial t}\, (x_1,x_2,t) =$$

$$- \sum_{i=1}^{2} \frac{\partial h_j}{\partial x_i}\, (x_1,x_2,t)u_i(x_1,x_2,h_j(x_1,x_2,t),t) + u_3(x_1,x_2,h_j(x_1,x_2,t),t)$$

for $j = 1,2$. Thus, it follows from 5) - 9) that

10)
$$\frac{\partial}{\partial t}\int_{h_1}^{h_2} \rho \, dx_3 + \sum_{i=1}^{2} \frac{\partial}{\partial x_i}\int_{h_1}^{h_2} (\rho u_i)\, dx_3 = 0.$$

We now make the assumptions that $P = P(x_1,x_2,t)$, that inertial effects may be neglected, and that viscous effects due to changes in u_1 and u_2 in the x_1 and x_2 directions may be neglected. We then obtain from the Navier-Stokes equations that

11)
$$\frac{\partial^2 u_i}{\partial x_3^2}(x_1,x_2,x_3,t) = \frac{1}{\mu}\frac{\partial P}{\partial x_i}(x_1,x_2,t),$$

$$h_1(x_1,x_2,t) < x_3 < h_2(x_1,x_2,t),$$

for $i = 1,2$. The no-slip boundary conditions give that

12)
$$u_i(x_1,x_2,h_j(x_1,x_2,t),t) = V_{j,i}$$

for $i = 1,2$; $j = 1,2$. The solution of 11) and 12) is

13)
$$u_i(x_1,x_2,x_3,t) = \left[\frac{1}{2\mu}\frac{\partial P}{\partial x_i}(x_1,x_2,t)(x_3-h_1(x_1,x_2,t)) - \frac{(V_{1,i}-V_{2,i})}{h(x_1,x_2,t)}\right] \cdot \left[x_3 - h_2(x_1,x_2,t)\right] + V_{2,i}$$

for $i = 1,2$.

In either the incompressible case, ρ = constant, or in the compressible case, P/ρ = constant, the density, ρ, is independent of x_3. Thus, it follows from 13) that

14)
$$\int_{h_1}^{h_2} \rho u_i dx_3 = \rho\int_{h_1}^{h_2} u_i dx_3 = -\frac{\rho h^3}{12\mu}\frac{\partial P}{\partial x_i} + \frac{\rho h V_i}{2}$$

for $i = 1,2$. Substitution of 14) in 10) gives the Reynolds lubrication equation 1).

When the thickness of a gaseous fluid layer is of the order of the molecular mean-free path of the gas, then the compressible Reynolds equation 4) becomes a poor model for the pressure in the fluid layer. A better model results when we allow the slip flow at the boundary proposed by Maxwell for perfectly diffuse reflection [1,9]

15) $$(u_i(x_1,x_2,h_j(x_1,x_2,t),t) - V_{j,i}) + (-1)^j \lambda \frac{\partial u_i}{\partial x_3}(x_1,x_2,h_j(x_1,x_2,t),t) = 0 ,$$

for $i = 1,2$; $j = 1,2$, where λ is the molecular mean-free path of the gas. To derive a modified Reynolds equation, we now solve 11) with the boundary conditions 15) and we then compute that

16) $$\int_{h_1}^{h_2} u_i dx_3 = -\frac{1}{12\mu}\frac{\partial P}{\partial x_i} h^3\left(1 + \frac{6\lambda}{h}\right) + \frac{V_i h}{2}$$

for $i = 1,2$. We substitute 16) into 10) to obtain the equation

17) $$12\mu \frac{\partial}{\partial t}(\rho h) + 6\mu\ \nabla\cdot(\rho h \underline{V}) = \nabla\cdot\left(h^3\rho\left(1 + \frac{6\lambda}{h}\right)\nabla P\right).$$

Finally, we recall that

$$\lambda\rho = \text{constant} = \lambda_a \rho_a$$

where ρ_a (resp. λ_a) is the ambient density (resp. ambient molecular mean-free path). Thus, we finally obtain the compressible Reynolds lubrication equation modified for slip flow

18) $$12\mu \frac{\partial}{\partial t}(Ph) + 6\mu\ \nabla\cdot(Ph\ \underline{V}) = \nabla\cdot\left(h^3 P\left(1 + \frac{6\lambda_a P_a}{hP}\right)\nabla P\right), \quad x \in \Omega,\ t \in \mathbb{R},$$

$$P = P_a, \quad x \in \partial\Omega,\ t \in \mathbb{R}.$$

In most magnetic recording systems, the medium (disk or tape) is stretched around the head so that the thickness of the gaseous fluid layer is very small at one or more regions. We usually find internal layers (where the pressure has large derivatives) near the points of minimum thickness in the fluid layer [6,7]. The accurate numerical resolution of these internal layers is much improved by appropriate mesh refinements around the internal layers. Now the efficiency and accuracy of mesh refinement algorithms can be greatly increased by *a priori* qualitative information about the solution such as the location of the internal layers and estimates on the magnitude of the layers. Thus, the following results on the qualitative nature of solutions to the compressible Reynolds lubrication equation are the start of a program which we hope will result in improved mesh refinement algorithms for solutions to the compressible Reynolds equation. The first step in this direction is to study the steady state equation, i.e., to look for a solution $P = P(x_1,x_2)$ which does not depend on t and to assume that $h = h(x_1,x_2)$ also does not depend on t. This is the program of the next section.

3. The steady state equation

If we set $\underline{\Lambda} = 6\mu\underline{V}$ and $\lambda = 6\lambda_a P_a$, then the steady state equation corresponding to 18) reads:

$$\text{19)} \quad \begin{cases} \nabla\cdot\left(h^3 P\left(1 + \frac{\lambda}{hP}\right)\nabla P\right) = \nabla\cdot(hP\underline{\Lambda}), & x \in \Omega, \\ P = Pa, & x \in \partial\Omega, \end{cases}$$

(In the case $\lambda = 0$, 19) is the steady state equation for 1). So we can handle both equations at the same time.). Since P denotes the pressure, we are looking for a positive solution, P, of 19). Introduce the dependent variable

$$u = \frac{1}{2} P^2 + \frac{\lambda P}{h} .$$

If P is a positive solution of 19) - for instance in the classical sense - then it is easy to check that u is a solution of

$$\text{20)} \quad \begin{cases} \nabla\cdot(h^3 \nabla u) = \nabla\cdot\underline{\alpha}(x,u), & x \in \Omega, \\ u = \phi(x) = \frac{1}{2} P_a{}^2 + \frac{\lambda P_a}{h(x)} , & x \in \partial\Omega, \end{cases}$$

where $\underline{\alpha}(x,u)$ is defined by

$$\text{21)} \quad \underline{\alpha}(x,u) = \left(-\lambda + \sqrt{\lambda^2 + 2h^2(x)u}\right) (\underline{\Lambda} - \lambda\nabla h(x)).$$

Conversely, if u is a positive solution of 20), then

$$P = -\frac{\lambda}{h} + \sqrt{\frac{\lambda^2}{h^2} + 2u}$$

is a positive solution of 19). So, instead of solving 19), we can solve 20). The advantage of considering 20) lies in the fact that the degeneracy (when $\lambda = 0$, $P = 0$) of the elliptic part of the operator in 19) has now disappeared. Of course, when $\lambda = 0$ this is at the expense of introducing a singularity in $\underline{\alpha}(x,u)$ (see 21)). Now, since we don't know a priori if, for $\phi > 0$, the solution of 20) is non negative, we change the definition of $\underline{\alpha}(x,u)$ to:

22) $$\underline{\alpha}(x,u) = \begin{cases} (-\lambda + \sqrt{\lambda^2+2h^2u})\ (\underline{\Lambda}-\lambda\nabla h(x)) & \text{when } u>0 \\ 0 & \text{when } u<0. \end{cases}$$

Assuming that Ω is a smooth bounded open set in $\mathbb{R}^2$ we will denote by $L^2(\Omega)$ the space of square integrable functions with the norm

$$|v|_{L^2(\Omega)} = (\int_\Omega |v|^2 dx)^{1/2}$$

and by $H^1(\Omega)$, $H_0^1(\Omega)$ the usual Sobolev spaces defined by

$$H^1(\Omega) = \{v \in L^2(\Omega) \mid \nabla v \in (L^2(\Omega))^2\},$$
$$H_0^1(\Omega) = \{v \in H^1(\Omega) \mid v = 0 \text{ on } \partial\Omega\}.$$

We will assume that these two spaces are given the norm:

$$\|v\|^2_{H^1(\Omega)} = |v|^2_{L^2(\Omega)} + ||\nabla v||^2_{L^2(\Omega)}$$

where $|\ |$ denotes the Euclidean norm in $\mathbb{R}^2$.

We will also assume that the function $h = h(x)$ is Lipschitz continuous and satisfies,

23) $$0 < h_1 \leq h(x) \leq h_2, \quad \text{a.e. } x \in \Omega,$$
$$|\nabla h(x)| \leq h_3, \quad \text{a.e. } x \in \Omega,$$

where h_1, h_2, h_3 are strictly positive constants.

For $\phi \in H^1(\Omega)$, we will say that u is a weak solution to 20) if

$$u - \phi \in H_0^1(\Omega) \ ,$$

24)

$$\int_\Omega h^3 \nabla u \cdot \nabla \xi dx = \int_\Omega \underline{\alpha}(x,u) \cdot \nabla \xi dx, \quad \forall \ \xi \in H_0^1(\Omega).$$

Note that the above integrals make sense for $u, \xi \in H^1(\Omega)$. For the first one this is by 23), for the second one this is due to the estimate:

25) $$|\underline{\alpha}(x,u)| < C(-\lambda + \sqrt{\lambda^2 + 2h^2|u|}) < C\sqrt{2h^2|u|} < C\sqrt{|u|}$$

where C denotes some constants (see 23)). Indeed, if $u \in L^2(\Omega) \subset L^1(\Omega)$, then the above estimate implies that $|\alpha(x,u)| \in L^2(\Omega)$ and thus $|\underline{\alpha}(x,u)| \cdot |\nabla \xi| \in L^1(\Omega)$. Under the above assumptions we first can prove:

<u>Theorem</u> 1 [3] If $\phi \in H^1(\Omega)$, then there exists a weak solution to 24).

<u>Sketch of proof</u>. For $v \in L^2(\Omega)$ we consider $u = T(v)$ the solution of

$$u - \phi \in H_0^1(\Omega),$$

$$\int_\Omega h^3 \nabla u \cdot \nabla \xi dx = \int_\Omega \underline{\alpha}(x,v) \cdot \nabla \xi dx, \quad \xi \in H_0^1(\Omega).$$

Due to the growth of $\underline{\alpha}(x,u)$ at infinity in u (see 25)) we can show that for R large enough T maps compactly the ball

$$B_R = \{v \in L^2(\Omega) \mid |v|_{L^2(\Omega)} < R\}$$

into itself. It results then, by the Schauder fixed point theorem, that T has a fixed point in B_R. But such a fixed point $u = T(u)$ is a solution to 24). This concludes the proof. We refer the reader to [3] for the details.

It is physically reasonable that P (and thus u) is increasing when its boundary values increase, and thus it is not surprising that we have:

Theorem 2 [3] Assume that u_1 is a weak solution to 24) corresponding to boundary data ϕ_1 and that u_2 is a weak solution to 24) corresponding to boundary data ϕ_2. If $\phi_1 \geq \phi_2$ a.e. on $\partial\Omega_1$, then $u_1 \geq u_2$ a.e. on $\partial\Omega$.

Proof. We refer to [3].

As a consequence we have:

Theorem 3 [3] For $\phi \in H^1(\Omega)$ there exists a unique solution to 24). Moreover, if $\phi \geq 0$ a.e. on $\partial\Omega$, then $u \geq 0$ a.e. on Ω.

Proof Let u,v be two solutions of 24). Applying Theorem 2 with $\phi_1 = \phi_2 = \phi$ we get $u \geq v$ and $v \geq u$ so that $u = v$ in Ω and thus the problem 24) has a unique solution. For $\phi \equiv 0$ it is clear (since $\underline{\alpha}(x,0) = 0$) that $u \equiv 0$ is the only solution to 24). So if $\phi \geq 0$ and if u is the solution to 24), then we can apply Theorem 2 with $u_1 = u$, $u_2 = 0$ to get $u \geq 0$ a.e. in Ω. This completes the proof of the theorem.

Remark: If $\phi \geq 0$ (which is the case when $\phi = \frac{1}{2} Pa^2 + \frac{\lambda p_a}{h(x)}$), then we can define

26) $$p = -\frac{\lambda}{h} + \sqrt{\frac{\lambda^2}{h^2} + 2u}$$

and clearly p is a weak solution to 19). So we have proved that 19) has a weak solution.

Next, we would like to know when u and p defined by 26) are classical solutions to 20) and 19) respectively. First, in the general case ($\lambda \geq 0$) we have:

Theorem 4 [3] Let u be the solution to 24). If Φ is a constant such that

$$\phi \geq \Phi > 0 \text{ a.e. in } \Omega, \quad \nabla \cdot \underline{\alpha}(x,\Phi) \leq 0 \quad \text{in } \Omega$$

(this last inequality is meant for instance in the distributional sense), then we have

$$u \geq \Phi \quad \text{a.e. in } \Omega.$$

Moreover, if Ω is smooth, ϕ, $h \in C^{\infty}(\overline{\Omega})$, then u (and p defined by 26)) are in $C^{\infty}(\overline{\Omega})$. ($C^{\infty}(\overline{\Omega})$ denotes the space of functions of class C^{∞} whose derivatives are extendable in a continuous way up to the boundary).

Proof: (See [3]). The idea is to show that thanks to the inequality $\nabla \cdot \underline{\alpha}(x,\Phi) \leq 0$, the function $v = \Phi$ is a "subsolution" to 24). Arguing then as in Theorem 2 we can show that this implies that $u \geq \Phi$ a.e. in Ω. This proves the first part of the theorem. To prove the second, we remark that the function

$$u \to \underline{\alpha}(x,u)$$

is C^{∞} in u on $[\Phi,+\infty)$. Thus for $u \in H^1(\Omega)$, $\underline{\alpha}(x,u) \in (H^1(\Omega))^2$ (note that $\underline{\alpha}(x,u)$ is smooth in x since we assume that h is smooth). By usual elliptic regularity theory, this implies $u \in H^2(\Omega)$ and thus $\underline{\alpha}(x,u) \in (H^2(\Omega))^2$. So, $u \in H^3(\Omega)$ and we can keep going, proving that $u \in H^k(\Omega)$ for all k (see for instance [5] for a definition of these spaces). This completes the proof of the theorem.

Remark: In the physical case that we are considering we have (see 20))

$$\phi(x) = \frac{1}{2} P_a^2 + \frac{\lambda P_a}{h(x)} \geq \frac{1}{2} P_a^2 > 0 .$$

In the case where $\lambda = 0$, that is to say, in the case of the classical Reynolds equation, we can get rid of the hypothesis on $\underline{\alpha}(x,\Phi)$. More precisely we have:

Theorem 5 [2] Let u be the solution to 24) and assume that for some constant Φ we have

$$\phi \geq \Phi > 0 \quad \text{a.e. on} \quad \partial\Omega .$$

If $h \in W^{2,\infty}(\Omega)$, then there exists a constant C, depending on the data, such that

$$u \geq C > 0 \quad \text{a.e. in} \quad \Omega.$$

Moreover, if Ω is smooth, ϕ, $h \in C^{\infty}(\overline{\Omega})$, then u (and P defined by 26)) are in $C^{\infty}(\overline{\Omega})$.

Proof: We refer to [2]. The idea is again to find a positive subsolution to 24). However, in this case, the function $v = \Phi$ is not a subsolution in general and some more work is needed. The fact that $u \in C^\infty(\bar{\Omega})$ results from the same arguments that were given in Theorem 5.

As we mentioned in the preceeding section, it would be helpful for the numerical analysis of the problem to know the region of Ω where p has large derivatives. In other words, a certain knowledge of the shape of p or u is needed. We will assume in the following of this section that $\lambda = 0$, i.e., we will deal with the usual Reynolds equation, and $P = P_a > 0$ on $\partial\Omega$ where P_a is a positive constant.

When the scale of Ω in the x_1 direction is small compared to its scale in x_2, when h doesn't depend on x_2 and $\underline{V} = (V_1,0)$, we can reduce 19) to an ordinary differential equation and we are then able to deduce some information about the shape of P. We refer the interested reader to [2] for details. The problem is much more difficult for the general two dimensional problem. For the sake of simplicity, we are going to assume that $\underline{V} = (V_1,0)$ and that h and Ω are smooth so that P will be the smooth solution to

27)
$$\begin{cases} \nabla\cdot(h^3 P \nabla P) = \Lambda \dfrac{\partial}{\partial x_1}(hP) , & x = (x_1,x_2) \in \Omega , \\ P = P_1 & x = (x_1,x_2) \in \partial\Omega , \end{cases}$$

where $\Lambda = 6\mu V_1$ (see 19)).

Let us denote by $n = (n_1,n_2)$ the outward normal to $\partial\Omega$ and by $d\sigma$ the surface measure on $\partial\Omega$. Then we have:

Theorem 6 [2] Assume that

28) $$\int_\Omega h n_1 \, d\sigma = 0,$$

29) $$\exists x_0 \in \partial\Omega \text{ such that } \frac{\partial h}{\partial x_1}(x_0) > 0,$$

and

30) $$\exists x_0' \in \partial\Omega \text{ such that } \frac{\partial h}{\partial x_1}(x_0') < 0,$$

then P, the solution of 27), achieves a maximum strictly greater than P_a and a minimum strictly less than P_a in Ω.

Proof: Let us prove for instance that P achieves a maximum strictly greater than P_a. If not, then we have $P \leq P_a$ in Ω. This implies $\frac{\partial P}{\partial n} \geq 0$ on $\partial\Omega$. Now around x_0 we have by 27), 29):

$$\nabla\cdot(h^3 P\nabla P) - \alpha h \frac{\partial P}{\partial x_1} = \alpha \frac{\partial h}{\partial x_1} P > 0. \tag{31}$$

Let us denote by N a neighborhood of x_0 in Ω where 31) holds. If $P < P_a$ in N, then by the strong maximum principle we have

$$\frac{\partial P}{\partial n}(x_0) > 0 \tag{32}$$

Now integrating 27) over Ω and applying the divergence theorem we get

$$0 < \int_{\partial\Omega} h^3 P_a \frac{\partial P}{\partial n}\, dx = \int_{\partial\Omega} \alpha h P_a n_1 d\sigma = \alpha P_a \int_{\partial\Omega} h n_1 d\sigma = 0$$

This is impossible and thus one cannot have $P < P_a$ in N. Since $P \leq P_a$ in Ω, there is a point in N where P achieves its maximum P_a. Thanks to 31) and the strong maximum principle this would imply that $P \equiv P_a$ in N. Hence, 31) would read

$$0 = \alpha \frac{\partial h}{\partial x_1} P_a > 0$$

in N which is impossible. So, we cannot have $P \leq P_a$ in Ω and P achieves a maximum strictly greater than P_a in Ω. The proof that P achieves a minimum strictly less than P_a uses the same arguments around x_0'. This completes the proof of the theorem.

Remark: We have that 28) holds, for instance, when Ω and h are symmetric with respect to the x_2-axis. Using the maximum principle and 31) we can easily see that P can only achieve its maximum (resp., minimum) in the region

$$[\frac{\partial h}{\partial x_1} < 0] = \{(x_1,x_2) \in \Omega \,|\, \frac{\partial h}{\partial x_1}(x_1,x_2) < 0\}$$

$$(\text{resp., } [\frac{\partial h}{\partial x_1} > 0] = \{(x_1,x_2) \in \Omega \,|\, \frac{\partial h}{\partial x_1}(x_1,x_2) > 0\}) .$$

For this and complementary results we refer the reader to [2].

4. Application to a system governing the deflection of a floppy disk

In magnetic recording systems such as "floppy" disk systems and "Winchester" disk systems the following model equations for the deflection of a rotating disk by the pressure load developed between the recording head and the disk have been successfully utilized [10]:

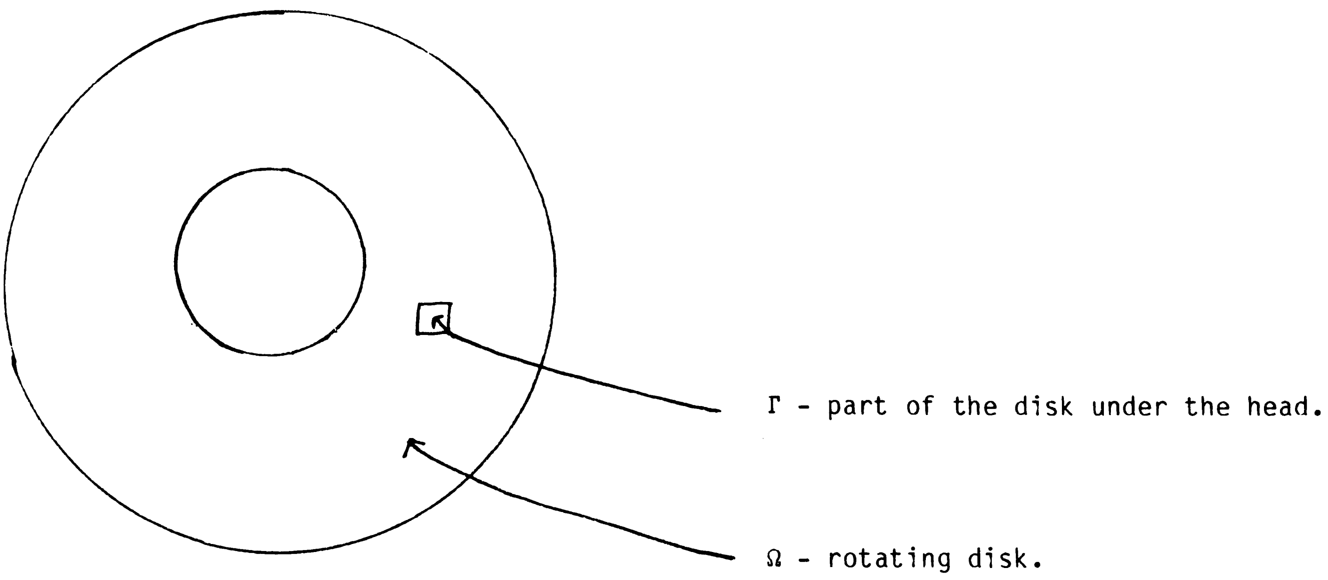
33) $$(E\Delta^2 - T\Delta + \rho\omega^2 \frac{\partial^2}{\partial\theta^2} + \alpha\omega \frac{\partial}{\partial\theta})v = P - P_a, \qquad x \in \Omega ,$$

$$v = \frac{\partial v}{\partial n} = 0, \qquad x \in \partial\Omega,$$

34) $$\nabla \cdot (h^3(v)P\nabla P) = 6\mu \underline{V} \cdot \nabla(h(v)P), \qquad x \in \Gamma,$$

$$P = P_a, \qquad x \in \partial\Gamma.$$

Here $\Omega \subset \mathbb{R}^2$ is an annulus, $\Gamma \subset \Omega$ is an open set in $\mathbb{R}^2$ which represents the region where the head and the disk are in close proximity (see the figure below),



v is the vertical displacement of the disk, and P is the pressure which develops between the head and the disk. So 33), 34) couples the equation of the deflection of the disk with the Reynolds equation under the head. We wish to solve 33), 34) for v and P when we are given the positive constants $E, T, \rho, \omega, \alpha, P_a$ and μ. Here E is the stiffness coefficient, T is the tension, ρ is the density of the disk, ω is the angular speed of rotation of the disk, α is the damping coefficient, P_a is the ambient pressure, and μ is the dynamic viscosity of the air layer between the head and the disk. Further, $\underline{V} = \underline{V}(x)$ denotes the velocity of the disk at the point $x \in \Omega$, so we have:

$$\underline{V}(x) = \omega(-x_2, x_1), \quad x = (x_1, x_2) \in \Omega.$$

Also, $h(v) = h(v)(x)$ is the thickness of the fluid layer between the head and the disk and we have

$$h(v) = \psi - v$$

where ψ is the vertical position of the head. Finally, note that 34) gives the pressure P in Γ. We assume that P is extended by P_a outside of $\underline{\Gamma}$ in such a way that 33) makes sense. We assume that ψ is a Lipschitz continuous function such that

$$0 < m < \psi(x) < M, \quad x \in \Omega.$$

We then have:

<u>Theorem 7</u> [4] If ω or $|\Gamma|$ - the measure of Γ - is small enough, then there exists a weak solution to 33), 34).

<u>Proof:</u> We refer the interested reader to [4]. Using the existence results of the preceding section the method consists in applying the Schauder fixed point theorem on a suitable convex subset of $L^2(\Gamma)$.

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