Two problems in parabolic PDEs

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Lu Li

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Abstract

In this thesis, we will study two problems in parabolic Partial Differential Equations.

In the Chapter 2 we study the problem of backward uniqueness of the linear heat equation. We briefly recall some results about backward uniqueness under the assumption of Dirichlet boundary condition and results of boundary controllability in control theory. It was shown in [9, 25] that a bounded solution of the heat equation in a half-space which becomes zero at some time must be identically zero, even though no assumptions are made on the boundary values of the solutions. In a recent example, Luis Escauriaza showed that this statement fails if the half-space is replaced by cones with opening angle smaller than 90°. The main result of the chapter is that the result remains true for cones with opening angle larger than 110°.

In Chapter 3 we study the profiles of the singularity of the 1-D complex Burgers equation. The Cauchy problem for the 1-D real-valued viscous Burgers equation $u_t + uu_x = u_{xx}$ is globally well-posed [13]. For complex-valued solutions finite time blow-up is possible from smooth compactly supported initial data, see [23]. It is also proved in [23] that the singularities for the complex-valued solutions are isolated if they are not present in the initial data. We study the singularities in more detail. In particular, we classify the possible blow-up rates and blow-up profiles. It turns out that all singularities are of type II and that the blow-up profiles are regular steady state solutions of the equation.

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Chapter 1

Introduction

1.1 Backward uniqueness of the heat equation

In Chapter 2, we study the problem of backward uniqueness of the heat equation. Consider an open set $\Omega \subset \mathbb{R}^n$. Let u be a bounded solution of the equation

$$u_t - \Delta u + b(x,t) \cdot \nabla u + c(x,t)u = 0 \quad \text{in } \Omega \times (0,T), \tag{1.1}$$

where the coefficients $b = (b_1, \ldots, b_n)$, c are measurable and bounded. We say that Ω has the *backward uniqueness property* if the following statement holds:

(BU) *If a bounded* $u\colon \Omega \times (0,T) \to \mathbb{R}$ *satisfies (1.1) and* $u(\,\cdot\,,T) = 0$, *then* $u \equiv 0$ *in* $\Omega \times (0,T)$.

It is important to emphasize that no assumptions are made about u at the parabolic boundary $(\partial\Omega\times(0,T))\cup(\Omega\times\{0\})$ in the statement (BU). With boundary condition, the backward uniqueness of the heat equation is standard. We will discuss this case later in section 2.1.1.

In fact, we can think about the problem in terms of the control theory: we are given some initial data $u_0\colon \Omega\to\mathbb{R}$, and we wish to find a suitable boundary condition g on the lateral boundary $\partial\Omega\times(0,T)$ so that when we solve equation (1.1) with u_0 as the initial condition and g as the boundary condition, the solution will become exactly zero at time $t=T$. In this interpretation condition (BU) means that we can never achieve the exact boundary control of any non-trivial solution.

By classical results we know that in bounded domains we can achieve exact control, which is discussed in section 2.1.2 and therefore any domain satisfying (BU) has to be unbounded. Classical backward uniqueness results for parabolic equations imply that $\Omega=\mathbb{R}^n$ satisfies (BU). It turns out that the half-space $\Omega=\mathbb{R}^n_+$ also satisfies (BU), although this is harder to prove, see [9]. In general, the smaller the domain, the harder it will be to show that it satisfies (BU). It is immediate that if $\Omega_1\subset\Omega_2$ and Ω_1 satisfies (BU), then also Ω_2 satisfies (BU).

While the control theory for PDEs seems to be the most natural background for (BU), the problem also appeared in regularity theory of parabolic equations, such as the Navier-Stokes equations, harmonic map heat flows, or semi-linear heat equations, see [9, 22, 29]. The specific unbounded domains which arise in this connection are complements of closed balls (for interior regularity), half-spaces (for boundary regularity at C^1 boundaries), or cones (for boundary regularity in Lipschitz domains).

In Chapter 2 we consider the question for cones with opening angle θ. In suitable coordinates

$$\mathcal{O}_\theta = \{x = (x_1, x'), x' \in \mathbb{R}^{n-1}, x_1 > |x| \cos(\theta/2)\}.$$

Luis Escauriaza [5] recently showed that - surprisingly - (BU) fails when $\theta < \pi/2$. We will briefly recall his counterexample in section 2.1.4. Since we also know that (BU) is true for $\theta = \pi$, it is easy to see that there exists a borderline angle $\theta_0 \in [\pi/2, \pi]$ such that (BU) is true for $\theta > \theta_0$, and (BU) fails for $\theta < \theta_0$. The borderline case $\theta = \theta_0$ might perhaps present an extra difficulty.

The main result of Chapter 2 is that the cones $\mathcal{O}_\theta$ satisfy (BU) for

$$\theta > 2 \arccos(1/\sqrt{3}) \sim 109.52^\circ\,.$$

The proof of the main result is in section 2.2. It is tempting to conjecture that $\theta_0 = \pi/2$. This is supported by the fact that $\theta = \pi/2$ is the borderline case for the construction of Escauriaza, as can be seen from the Phragmén-Lindölef principle.

For the classical heat equation, corresponding to the case $b = 0$ and $c = 0$ in (1.1), and $\theta = \pi$ (the half-space), the statement (BU) can be proved by a relatively simple application of Fourier transformation and some classical complex analysis results, see [25]. We were not able to find such simple proof of the case $b = 0$, $c = 0$ when $\theta < \pi$. However, with additional condition that the solution is also 0 at the initial time, we can show the uniqueness using Fourier transformation in a similar way. The results are included in section 2.1.3.

1.2 Isolated singularities of the 1-D complex Burgers equation

A second problem we consider in the thesis in Chapter 3 is about the singularities of the Burgers equation. In a recent paper [23] Poláčik and Šverák study the occurrence of the singularities for the complex 1D viscous Burgers equation. They prove that there exist well-behaved initial data for which the solutions blow up in finite time. (In an earlier paper [17] D. Li and Sinai show that singularities are possible for the complex-valued Navier-Stokes equation. For other results regarding singularities of complex equations, see [3] [18] [19] [26] and [27].) Using the well known Cole-Hopf transformation, Poláčik and Šverák reduce the problem to the study of the zeros of solutions of the 1D complex heat equation and show that the singularities are isolated if they are not presented in the initial data, see Theorem 3.3 of [23]. This chapter can be considered as a continuation of [23] aiming at the description of these isolated singularities and in particular at the study of their profiles.

The Burgers equation [2]

$$u_t + uu_x = u_{xx} \ , \tag{1.2}$$

named after the Dutch physicist Johannes Martinus Burgers (1895-1981), is sometimes used for modeling certain aspects of equations of fluid mechanics. It can be reduced to the standard heat equation by the Cole-Hopf transformation [4] [13]. Let $v(x,t)$ be a positive function satisfying the heat equation $v_t = v_{xx}$, then

$$u = -2\frac{v_x}{v} \ , \tag{1.3}$$

called the Cole-Hopf transformation of v, solves the Burgers equation. Conversely, let u be a (sufficiently regular) solution to the Burgers equation. Let

$$v(x,t) = \exp\left(-\frac{1}{2}\int_0^x u(\xi,t)\ d\,\xi\right).$$

It is easy to check by direct calculation that v satisfies $\left((v_t - v_{xx})/v\right)_x = 0$, and thus there exists a function $C(t)$ such that $v_t - v_{xx} = C(t)v$. Then the function $\tilde{v}(x,t) = v(x,t)\exp\left(\int -C(t)\ dt\right)$ satisfies the heat equation $\tilde{v}_t = \tilde{v}_{xx}$ and $u = -2v_x/v = -2\tilde{v}_x/\tilde{v}$.

Singularities of u are related to zeros of v. One can see that for real and "sufficiently regular" initial data, equation 1.2 has a unique smooth global solution (in some natural classes of functions) given by $u = -2v_x/v$, see [13]. By contrast, for complex-valued Burgers equation blow-up is possible from regular, compactly supported initial data (complex-valued) [23]. In addition, the singularities are isolated if they are not present in the initial data (as quoted in the following theorem).

Theorem 3.3 of [23] Let v be a bounded complex-valued solution of the heat equation in $\mathbb{R} \times (0, \infty)$. Assume v has no zeros in some neighborhood of $\mathbb{R} \times \{0\}$. Then all zeros of v in $\mathbb{R} \times (0, \infty)$ are isolated.

In Chapter 3 we consider the structure of an isolated singularity via the rescaling procedure, which is well-known to be very useful in studying singularities of PDEs. In particular, we classify the possible blow-up rates and blow-up profiles. It turns out that all singularities are of type II and that the blow-up profiles are regular steady state solutions of the equation.

Chapter 2

Backward Uniqueness of the Heat Equation

In this chapter we study the problem of backward uniqueness of the heat equation. Let us recall some settings mentioned in the introduction chapter. Consider an open set $\Omega \subset \mathbb{R}^n$. Let u be a bounded solution of the equation

$$u_t - \Delta u + b(x,t) \cdot \nabla u + c(x,t)u = 0 \quad \text{in } \Omega \times (0,T), \tag{2.1}$$

where the coefficients $b = (b_1, \ldots, b_n)$, c are measurable and bounded. We say that Ω has the *backward uniqueness property* if the following statement holds:

(BU) *If a bounded* $u\colon \Omega \times (0,T) \to \mathbb{R}$ *satisfies (2.1) and* $u(\,\cdot\,, T) = 0$, *then* $u \equiv 0$ *in* $\Omega \times (0,T)$.

We emphasize that no assumptions are made about u at the parabolic boundary $(\partial\Omega \times (0,T)) \cup (\Omega \times \{0\})$ in the statement (BU).

In Section 2.1 we includes some relative results on the topic of backward uniqueness of the heat equation. The main result of this chapter is in Section 2.2

2.1 Background and relative results

In this section we first briefly recall the standard results about backward uniqueness under the assumption of Dirichlet boundary condition and some classical results of boundary controllability in control theory. We then include the elegant proof of [25] on backward uniqueness without boundary condition, in the domain of half space. In other words, (BU) holds with the equation $u_t - \Delta u = 0$ in half space. The same technique is applied to prove a result with a $\pi/2$ cone, with some additional condition. Lastly, we include the surprising example of Escauriaza – that (BU) fails for cones with angle smaller than $90°$.

2.1.1 Backward uniqueness with the Dirichlet boundary condition

Let us first consider the simplest case of the standard heat equation with the Dirichlet boundary condition. Let $\Omega \in \mathbb{R}^n$ be a bounded domain and u is a sufficient regular solution to

$$u_t - \Delta u = 0 \qquad \Omega \times (0,T), \tag{2.2}$$

$$u(x,t) = 0 \qquad \partial\Omega \times (0,T). \tag{2.3}$$

The Laplacian operator $-\Delta$ with the Dirichlet boundary condition (2.3) is self-adjoint in the space $L^2(\Omega)$, with domain $H^2(\Omega) \cap H_0^1(\Omega)$. Let $u_k(x)$ be the orthonormal system of eigenfunctions of $-\Delta$ corresponding to eigenvalues λ_k, where $\lambda_k > 0$. Then

$$u(x,t) = \sum_{k=1}^{\infty} c_k e^{-\lambda_k t} u_k(x).$$

If $u(\cdot,T) = 0$, then all the coefficients c_k are 0, hence $u(x,t) \equiv 0$ in $\Omega \times (0,T)$.

Another way to see the backward uniqueness of the heat equation with the Dirichlet boundary condition is from the concept of "log-convexity". Let $f(t) = ||u(\cdot,t)||_2^2$, and $F(t) = \log f(t)$. One can derive from (2.2)(2.3) that $F(t)$ is a convex function.

$$\begin{aligned}\dot{f}(t) &= \int 2uu_t = \int 2u\Delta u = -2\int |\nabla u|^2,\\ \ddot{f}(t) &= -4\int \nabla u \nabla u_t = 4\int \Delta u u_t = 4\int |\Delta u|^2,\\ \ddot{F}(t) &= \frac{\ddot{f}f-\dot{f}^2}{f^2} = \frac{4\int|\Delta u|^2\int u^2 - 4(\int u\Delta u)^2}{f^2} \geq 0.\end{aligned}$$

We then have

$$F(t) \leq F(0)(T-t)/T + F(T)t/T, \tag{2.4}$$

and in turn,

$$f(t) \leq f(0)^{\frac{T-t}{T}} f(T)^{\frac{t}{T}}. \tag{2.5}$$

If $u(x,T)=0$, that is $f(T)=0$, then it follows that $f(t)=0$, $t\in[0,T]$.

The concept of "log-convexity" was introduced by Agmon and Nirenberg in [1] in 1967. It works for more general equations with variable coefficients. In the rest part of this section, we mostly follow the presentation of the book [15], pp 41-45. For the following equation with bounded coefficients

$$u_t - \Delta u = b(x,t)\cdot\nabla u + c(x,t)u \qquad \Omega\times(0,T), \tag{2.6}$$

$$u(x,t) = 0 \qquad \partial\Omega\times(0,T), \tag{2.7}$$

one can derive a similar differential inequality, provided that $b(x,t)$, $c(x,t)$ have enough regularity.

$$\ddot{F}(t) + C|\dot{F}(t)| + C \geq 0, \tag{2.8}$$

with some positive constant C depending on $b(x,t), c(x,t)$. The differential inequality (2.8) follows from the following lemma and the Schauder estimates for the heat equation.

Let $A = A(t)$ be a linear operator in the Hilbert space H with domain $\mathcal{D}(t)$. By $||\cdot||$, $(,)$ we denote the norm and the scalar product in H. Consider the equation

$$u_t - A(t)u = 0 \quad \text{on } (0,T). \tag{2.9}$$

We assume that $A = A_+ + A_-$ where A_+, A_- are the symmetric and skew-symmetric parts of A. Moreover, they satisfy the following conditions.

$$||A_-u||^2 \leq \alpha(||A_+u||||u|| + ||u||^2) \tag{2.10}$$

and

$$\partial_t(A_+u, u) \geq 2(A_+u, u_t) - \alpha(||A_+u||||u|| + ||u||^2) \tag{2.11}$$

for some constant α.

Lemma 2.1.1. *Let $u(t) \in \mathcal{D}(t)$, $u \in C^1([0,T]; H)$ be a solution of the equation (2.9). Let $f(t) = ||u(t)||_2^2$, $F(t) = \log f(t)$. Then $\ddot{F}(t) + C|\dot{F}(t)| + C \geq 0$, on $(0,T)$. The constant C depends on α.*

Proof. By straightforward calculations,

$$\dot{F}(t) = \frac{2(u, u_t)}{||u||^2} = \frac{2(u, A_+u)}{||u||^2}.$$

$$\ddot{F}(t) = \frac{2\partial_t(u, A_+u)}{||u||^2} - \frac{4(u, A+u)^2}{||u||^4}$$
$$\geq \frac{4(A_+u, u_t)}{||u||^2} - \frac{2\alpha(||A_+u||||u|| + ||u||^2)}{||u||^2} - \frac{4(u, A+u)^2}{||u||^4}$$
$$= \frac{4||A_+u||^2}{||u||^2} + \frac{4(A_+u, A_-u)}{||u||^2} - \frac{2\alpha(||A_+u||||u|| + ||u||^2)}{||u||^2} - \frac{4(u, A_+u)^2}{||u||^4}. \quad (2.12)$$

We set $(A_+u, u) = ||A_+u||||u||\theta$ and represent A_+u as $\beta u + u^\perp$, where $u^\perp$ is orthogonal to u and $\beta = (A_+u, u)/||u||^2$. Then $||u^\perp||^2 = ||A_+u||^2(1-\theta^2)$. We thus have that

$$4(A_+u, A_-u) = 4(u^\perp, A_-u) \geq -4||u^\perp||||A_-u||$$
$$\geq -2(||u^\perp||^2 + ||A_-u||^2) \geq -2||A_+u||^2(1-\theta^2) - 2\alpha(||A_+u||||u|| + ||u||^2). \quad (2.13)$$

Combining (2.12) (2.13), it follows that

$$\ddot{F} \geq \frac{2||A_+u||^2(1-\theta^2)}{||u||^2} - \frac{4\alpha(||A_+u||||u|| + ||u||^2)}{||u||^2} = 2\sigma^2(1-\theta^2) - 4\alpha\sigma - 4\alpha, \quad (2.14)$$

where we set $\sigma = ||A_+u||/||u||$.

Observe that $|\dot{F}| = 2\sigma|\theta|$. If $\theta^2 < \frac{1}{4}$, then

$$\ddot{F} \geq \frac{3}{2}\sigma^2 - 4\alpha\sigma - 4\alpha \geq -\frac{3}{4}\alpha^2 - 4\alpha. \quad (2.15)$$

If $\theta^2 \geq \frac{1}{4}$, then $|\dot{F}| \geq \sigma$ and

$$\ddot{F} \geq -4\alpha\sigma - 4\alpha, \quad (2.16)$$

so that $\ddot{F} + 4\alpha|\dot{F}| + 4\alpha \geq 0$. In both cases we have the required inequality with $C = \frac{3}{4}\alpha^2 + 4\alpha$. □

Our operator is $A = \Delta + b(x,t) \cdot \nabla + c(x,t)$ and the space $H = L^2(\Omega)$. The domain of $\mathcal{D}(t)$ is $H^2(\Omega) \cap H_0^1(\Omega)$. $A_+ = \Delta + (-\frac{1}{2}\operatorname{div} b + c)$, $A_- = b(x,t) \cdot \nabla + \frac{1}{2}\operatorname{div} b$. By Schauder estimates, the operator A satisfies the conditions (2.10) (2.11), if $b(x,t)$ has bounded derivatives up to order 2, and $c(x,t)$ has bounded derivatives.

The backward uniqueness of the problem (2.6) (2.7) follows from the next lemma. If $u(x,T) = 0$, Lemma 2.1.2 implies that $u(x,t) \equiv 0$, $\Omega \times [0,T]$.

Lemma 2.1.2. *From the inequality (2.8) we have that*

$$f(t) \leq C_1 f(0)^{1-\lambda(t)} f(T)^{\lambda(t)}, \quad t \in [0,T], \lambda(t) \in [0,1]. \tag{2.17}$$

Proof. Let L be a solution to the differential equality $\ddot{L} + C|\dot{L}| + C = 0$ on $(0,T)$, coincide with F at the end points 0, T. The existence of L can be proven by using a priori estimates on L, $\dot{L}$ that follow from the observation that L satisfies the linear differential equation with the coefficient $C\texttt{sign}\dot{L}$. It follows that $\ddot{F} - \ddot{L} + C|\dot{F} - \dot{L}| \geq 0$. Then $F - L \leq 0$ by maximum principles, and it suffices to bound L. From the equation, $L(t)$ has no local minimum on $(0,T)$, so $\dot{L} > 0$ on $(0,\tau)$ and $\dot{L} < 0$ on (τ, T) for some τ in $(0,T)$. The cases $\tau = 0, T$ are follow along a similar proof. Therefore, L satisfies the linear differential equation $\ddot{L} + C_\tau \dot{L} + C = 0$, where C_τ is C on $(0,\tau)$ and $-C$ on (τ, T).

We separate L into two parts, $L = v + w$, where v is the solution to the homogeneous equation $\ddot{v} + C_\tau \dot{v} = 0$ coinciding with F at the end points, and

$w = L - v$ solves the inhomogeneous equation with zero boundary data.

Since $\dot{v}$ solves a linear first-order homogeneous ODE, it does not change its sign. Let $F(0) \leq F(T)$, then $\dot{v} \geq 0$. Consider the solution V to the ODE: $\ddot{V} + C\dot{V} = 0$ with the same boundary data. We have

$$\ddot{v} - \ddot{V} + C(\dot{v} - \dot{V}) = (C - C_\tau)\dot{v} \geq 0.$$

By maximum principles, $v \leq V$. We solve the equation for V,

$$V(t) = F(0)\frac{e^{-Ct} - e^{-CT}}{1 - e^{-CT}} + F(T)\frac{1 - e^{-Ct}}{1 - e^{-CT}}.$$

When $F(0) \geq F(T)$, we obtain a similar bound with C replaced by $-C$.

To bound w we solve the equations: $\ddot{w} + C\dot{w} + C = 0$ on $(0, \tau)$ and $\ddot{w} - C\dot{w} + C = 0$ on (τ, T) and match the solution at τ. It is easy to see that w is bounded by some constant $C_2(T)$. Replace $F(t)$ by $\log f(t)$ we have the required result, with $\lambda = \frac{1-e^{-Ct}}{1-e^{-CT}}$ in the case $F(0) \leq F(T)$ and $C_1 = e^{C_2 T}$. (We have a similar λ in the case $F(0) > F(T)$.) $\square$

2.1.2 Bounded domains are controllable

By classical results in control theory we know that bounded domains are exact controllable. In other words, for bounded domains, (BU) cannot be satisfied. However, the proof of the controllability is highly non-trivial, even in 1-dim case.

In this section, we include the proof of the null-controllability of the 1-dim heat equation through the boundary. If $u(x,t)$ satisfies the standard heat equation $u_t - \Delta u = 0$ with a reasonable initial condition u_0, then we can find a boundary condition h such that $u(x,T) = 0$. The proof is in two steps. First, we show the equivalence of controllability and observability in the abstract framework of functional analysis. Second, we show the observability by a Carleman inequality.

Let us consider the following boundary control problem.

$$\begin{cases} u_t - \Delta u &= 0 \qquad \Omega \times (0,T), \\ u(0,t) &= 0 \qquad (0,T), \\ u(L,t) &= h(t) \qquad (0,T), \\ u(x,0) &= u_0(x) \qquad (0,L), \end{cases} \tag{I.1}$$

where $u_0 \in L^2(0,L)$. We look for a control $h(t) \in L^2(0,T)$ so that $u(x,T) = 0$.

Controllability and Observability

In this part we follow the definitions in the 1978 survey paper of D. Russell, see [24]. Let X, Y, Z be Hilbert spaces and let $S : X \to Z$ be a bounded linear operator and $C : \mathcal{D}(C) \subseteq Y \to Z$ a closed linear operator with domain $\mathcal{D}(C)$ dense in Y. Then $\{X, Y, Z, S, C\}$ together constitute an abstract control system.

Definition 2.1.3. *The abstract linear control system $\{X, Y, Z, S, C\}$ is controllable if $\mathcal{R}(C) \supseteq \mathcal{R}(S)$. We denote by $\mathcal{R}(C)$, $\mathcal{R}(S)$ the ranges of the operators C*

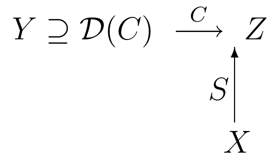

Figure 2.1: Abstract linear control system

and S respectively.

The adjoint operators $S^* : Z \to X$ and $C^* : \mathcal{D}(C^*) \subseteq Z \to Y$ may be defined by

$$(S^*z, x)_X = (z, Sx)_Z, \quad x \in X, z \in Z \tag{2.18}$$

$$(C^*z, y)_Y = (z, Cy)_Z, \quad y \in \mathcal{D}(C), z \in \mathcal{D}(C^*). \tag{2.19}$$

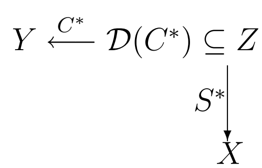

Figure 2.2: Abstract linear observed system

Definition 2.1.4. *The abstract linear observed system* $\{\, X, Y, Z, S^*, C^*\}$ *is observable if there is a positive number K such that*

$$||C^*z||_Y \geq K||S^*z||_X, \quad z \in \mathcal{D}(C^*).$$

The duality of controllability and observability is given by the following theorem.

Theorem 2.1.5. *The abstract linear control system* $\{\ X, Y, Z, S, C\}$ *is controllable if and only if the abstract linear observed system* $\{\ X, Y, Z, S^*, C^*\}$ *is observable.*

Proof. The proof is standard in functional analysis. We include it here for completeness of the thesis.

Controllability $\implies$ observability. Assuming $\mathcal{R}(C) \supseteq \mathcal{R}(S)$, we need to show that there exists $K > 0$ such that $||C^*z||_Y \geq K||S^*z||_X$ for $z \in \mathcal{D}(C^*)$. Otherwise, there is a sequence $z_n \in \mathcal{D}(C^*)$ such that $C^*z_n \to 0$, and $||S^*z_n|| \to \infty$. For any $y \in \mathcal{D}(C)$, $(C^*z_n, y)_Y \to 0$, thus $(z_n, Cy)_Z \to 0$. Because $\mathcal{R}(C) \supseteq \mathcal{R}(S)$, for any $x \in X$, we have $(z_n, Sx)_Z = 0$, thus $(S^*z_n, x)_X = 0$. Uniform Boundedness Theorem implies that S^*z_n is a bounded sequence. This is a contradiction.

Observability $\implies$ Controllability. Define a linear functional for $x \in X$ on $\mathcal{R}(C^*) \subset Y$ as the following.

$$T_x(C^*z) = (x, S^*z)_X.$$

The operator T_x is well defined because that $kerC^* \subseteq kerS^*$. In addition, T_x is also bounded.

$$|T_x(C^*z)| \leq ||x||_X||S^*z||_X \leq \frac{1}{K}||x||_X||C^*z||_Y.$$

Extend T_x to a linear functional on Y. By Riesz representation theorem, there exists some $y \in Y$ such that

$$(y, C^* z)_Y = (x, S^* z)_X = (Sx, z)_Z.$$

This y is in $\mathcal{D}(C^{**})$. As C is a closed operator with domain dense in Y, we have that $\mathcal{D}(C^{**}) = \mathcal{D}(C)$. Thus for each $x \in X$, we found a $y \in Y$ such that $(Cy, z)_Z = (Sx, z)_Z$ for any $z \in Z$. That is, $\mathcal{R}(C) \supseteq \mathcal{R}(S)$. □

Most controllability results for linear systems are proved by a method of establishing the corresponding observability property of the dual system. We put the control problem of the 1-dim heat equation in terms of the abstract framework. $X = L^2(0, L)$, $Y = L^2(0, L)$, $Z = L^2(0, L)$, $Su_0 = u(x, T)$ where u is the solution of the following system with initial data u_0.

$$\text{(I.2)} \qquad \begin{cases} u_t - u_{xx} &= 0 \qquad\qquad (0, L) \times (0, T), \\ u(0, t) &= u(L, t) = 0 \qquad (0, T), \\ u(x, 0) &= u_0(x) \qquad (0, L), \end{cases}$$

The operator C on $L^2(0, T)$ function $h(x)$ is given by solution at time T to the following system.

$$(\texttt{I}.3) \qquad \begin{cases} u_t - u_{xx} = 0 & (0,L)\times(0,T), \\ u(0,t) = 0 & (0,T), \\ u(L,t) = h(t) & (0,T), \\ u(x,0) = 0 & (0,L), \end{cases}$$

We calculate the adjoint operators of S and C respectively. Let $u(x,t)$, $v(x,t)$ be solutions to the system (I.2) with initial data $u(x,0)$ and $v(x,0)$, respectively. We have that

$$\frac{\partial}{\partial t}\int_0^L u(x,t)v(x,T-t)dx = \int_0^T u_t v - u v_t dx = \int \Delta u v - u\Delta v$$
$$= u_x(0,t)v(0,T-t) - u_x(L,t)v(L,T-t) - u(0,t)v_x(0,T-t) + u(L,t)v_x(L,T-t) = 0, \tag{2.20}$$

which implies that

$$\int_0^L u(x,T)v(x,0) = \int_0^L u(x,0)v(x,T).$$

Thus $S^* = S$. Using the same calculation with $u(x,t)$ be a solution to (I.3) and $v(x,t)$ be a solution to (I.2), we have

$$\begin{aligned} &\int_0^L u(x,T)v(x,0)dx - \int_0^L u(x,0)v(x,T)dx \\ &\quad= \int_0^T u(0,t)v_x(0,T-t) - u(L,t)v_x(L,T-t)\,dx = -h(t)v_x(L,T-t). \end{aligned} \tag{2.21}$$

Hence $C^*v_0 = -v_x(L, T-t)$, where $v(x,t)$ is the solution to (I.2) with initial data v_0.

The controllability problem of the 1-dim heat equation is now reduced to the proof of the inequality $||C^*v_0|| \geq K||S^*v_0||$, with some constant K, that is,

$$\int_0^T v_x(L,t)^2 dt \geq K \int_0^L v(x,T)^2 dx, \tag{2.22}$$

with $v(x,t)$ being the solution to the equation (I.2) with initial data $v_0 \in L^2(0,L)$.

proof of the observability

We apply a Carleman type estimate to show the observability (2.22) of the 1-dim heat equation. It is a modification of the estimate in the paper [14] of O. Y. Imanuvilov, (see Lemma 1.1). We prove a simplified version that is enough for our needs.

Lemma 2.1.6. *Let φ, ψ be functions of the following form.*

$$\varphi(x,t) = \frac{e^{-x}}{t^2(T-t)^2}, \qquad \tilde{\varphi}(x,t) = \frac{e^{-x} - e^L}{t^2(T-t)^2}.$$

Then for any solution u of (I.2) the following estimate holds.

$$\int_0^T \int_0^L s^3\varphi^3 u^2 e^{2s\tilde{\varphi}} dxdt \leq c_0 \int_0^T s\varphi(L,t) u_x(L,t)^2 e^{2s\tilde{\varphi}} dt, \quad ,\forall s \geq s_0 \tag{2.23}$$

Proof. Let u be a solution to (I.2) and $v = e^{s\tilde{\varphi}}u$. Define an operator L as follows.

$$Lv \equiv e^{s\tilde{\varphi}}(\partial_t u - \Delta u) = -\Delta v - s^2|\nabla\tilde{\varphi}|^2 v - s\partial_t\tilde{\varphi}v + \partial_t v + 2s\nabla\tilde{\varphi}\nabla v + s\Delta\tilde{\varphi}v. \quad (2.24)$$

We decompose L into two parts $L = S + A$, where

$$Sv = -\Delta v - s^2|\nabla\tilde{\varphi}|^2 v - s\partial_t\tilde{\varphi}v, \quad Av = \partial_t v + 2s\nabla\tilde{\varphi}\nabla v + s\Delta\tilde{\varphi}v. \quad (2.25)$$

We use the following relation to show the inequality (2.23).

$$\int |Lv|^2 dxdt = \int |Sv|^2 dxdt + \int |Av|^2 dxdt + 2\int (Sv)(Av)dxdt, \quad (2.26)$$

By simple calculations we have that

$$\begin{aligned} 2\int (Sv)(Av)dxdt = &\int 4s\tilde{\varphi}_{,kl}v_{,k}v_{,l}dxdt \\ &+ \int \left(4s^3\tilde{\varphi}_{,kl}\tilde{\varphi}_{,k}\tilde{\varphi}_{,l} - 2s^2\partial_t|\nabla\tilde{\varphi}|^2 - s\Delta^2\tilde{\varphi} + s\partial_t^2\tilde{\varphi}\right)|v|^2 dxdt \\ &+ \int_0^T (-2sv_x^2(0,t)\tilde{\varphi}_x(0,t) + 2sv_x^2(L,t)\tilde{\varphi}_x(L,t))dt. \quad (2.27) \end{aligned}$$

There is some positive constant s_0 such that when $s > s_0$ we have that

$$\begin{aligned} 2\int (Sv)(Av)dxdt &\geq \int s^3\varphi^3 v^2 dxdt + \int_0^T (2s\varphi(0,t)v_x^2(0,t) - 2s\varphi(L,t)v_x^2(L,t))dt \\ &\geq \int s^3\varphi^3 v^2 dxdt - \int_0^T 2s\varphi(L,T)v_x^2(L,t)dt. \quad (2.28) \end{aligned}$$

Since $Lv = 0$ by the heat equation, we finally have that

$$\int_0^T \int_0^L s^3\varphi^3 u^2 e^{2s\tilde{\varphi}} dxdt \leq c_0 \int_0^T s\varphi(L,t)u_x(L,t)^2 e^{2s\tilde{\varphi}} dt. \quad (2.29)$$

□

Now we are ready to proof the observability (2.22). It is easy to see that there exist two positive constant c_1 and c_2 that

$$\varphi(L,t)e^{2s\tilde{\varphi}} \leq c_1, \quad t \in (0,T), \tag{2.30}$$

and

$$\varphi^3(x,t)e^{2s\tilde{\varphi}} \geq c_2, \quad (x,t) \in (0,L) \times (\frac{T}{3}, \frac{2T}{3}). \tag{2.31}$$

The Laplacian Δ generates a strongly continuous semigroup $e^{t\Delta}$ for $t \in (0,\infty)$ on $L^2(0,L)$. There are positive constants M and μ

$$||e^{t\Delta}|| \leq Me^{\mu t}, \quad t \geq 0.$$

Hence, for $t \in (\frac{T}{3}, \frac{2T}{3})$

$$\int_0^L u^2(x,T)dx \leq M^2 e^{2\mu(T-t)} \int_0^L u^2(x,t)dx.$$

By integration over $(\frac{T}{3}, \frac{2T}{3})$,

$$\frac{T}{3}\int_0^L u^2(x,T)dx \leq M^2 e^{4\mu T/3} \int_{T/3}^{2T/3}\int_0^L u^2(x,t)dxdt. \tag{2.32}$$

Combining (2.30) (2.31) (2.32), we have that

$$\int_0^T u_x(L,t)^2 dt \geq K \int_0^L u(x,T)^2 dx.$$

This is the end of the proof. □

We have thus proved the null controllability of the system (I.1). For Neumann boundary control and inner control problems in 1-dim and higher, one can use the same strategy by proving the observability by suitable Carleman inequalities.

2.1.3 Two Backward uniqueness results via Fourier Transform

Theorem 2.1.7 and the proof is taken from [25]. We introduce additional notation:

$$\mathbb{R}^n_+ = \{x \in \mathbb{R}^n || x = (x_1, x_2, ..., x_n), x_n > 0\},$$

$$Q_T = \mathbb{R}^n \times (0, T),$$

where T is a positive fixed number.

Theorem 2.1.7. *Let $u : Q_T \to \mathbb{R}$ be a bounded smooth function satisfying the heat equation $\partial_t u = \Delta u$ in Q_T. Assume that there exists a non-empty open set $\Omega \subset \mathbb{R}^n_+$ such that*

$$\lim_{t \to T-0} \int_\Omega |u(x,t)| dx = 0.$$

Then $u \equiv 0$ in Q_T.

Proof. Using the known regularity theory for the heat equation and the fact that smooth solutions to the heat equation are analytic in spatial variables, we see that one can extend u by zero to the set $Q = \mathbb{R}_n \times (0, \infty)$, and the extension, also denoted by u, is smooth, satisfies the heat equation in Q, and vanishes for $t \geq T$. Also, replacing $u(x,t)$ by $u(x_1, x_2, ..., x_{n-1}, x_n + y_n, t + s)$ for small $y_n > 0$ and $s > 0$, we can assume that all derivatives of u are well-defined, bounded and continuous in the closure $\bar{Q}$ of Q. Making these simplifying assumptions, we will now prove the theorem in several steps.

Step 1. Reduction to the case $n = 1$. The obvious idea here is to use the Fourier transformation along $x' = (x_1, x_2, ..., x_{n-1})$. For each $t > 0$ and $x_n \geq 0$, we define a distribution $\tilde{u}(\cdot, x_n, t)$ on $\mathbb{R}^{n-1}$ by

$$< \tilde{u}(\cdot, x_n, t), \phi(\cdot). = \int_{\mathbb{R}}^{n-1} dx' u(x', x_n, t) \int_{\mathbb{R}^{n-1}} d\xi' \, e^{t|\xi|^2 - ix' \cdot \xi'} \phi(\xi')$$

Here, $\phi \in C_0^\infty(\mathbb{R}^{n-1})$ and $\xi = (\xi_1, \xi_2, ...\xi_{n-1})$. Under suitable assumptions, $\tilde{u}(\cdot, x_n, t)$ is a function and we have

$$\tilde{u}(\xi', x_n, t) = e^{t|\xi'|^2} \int_{\mathbb{R}^{n-1}} u(x', x_n, t) e^{-ix' \cdot \xi'} dx'.$$

A simple calculation shows that, for each fixed $\phi \in C_0^\infty(\mathbb{R}^{n-1})$, the function

$$\tilde{u}(x_n, t) =< \tilde{u}(\cdot, x_n, t), \phi(\cdot) >$$

is bounded in $\mathbb{R}_+ \times (0, +\infty)$, satisfies the heat equation in $\mathbb{R}_+ \times (0, +\infty)$, and vanishes for $t > T$. We now see that it is enough to prove the case $n = 1$.

In what follows, we use the notation $Q = R_+ \times (0, +\infty)$, and (x, t) stands for points of Q.

Step 2. Reduction to the case $|u(x, t)| \leq Ce^{-\alpha x}$. This can be achieved by the following change of variables:

$$u(x, t) = v(x + 2\alpha t, t) e^{\alpha x + \alpha^2 t}, \qquad \alpha > 0.$$

This function v is, of course, defined in a domain different from Q_T , but we can obviously achieve by a suitable shift that the domain of v contains a domain of the form Q in which the theorem is violated, if v does not identically vanish. Moreover, v has the required decay as $x \to \infty$.

Step 3. Proof in the case $n = 1$ *and* $|u(x,t)| < Ce^{-\alpha x}$. We extend u to all $\mathbb{R} \times \mathbb{R}$ by requiring that the extension be an even function of x vanishing for $t \in (-\infty, 0) \cup (T, +\infty)$. The extended function has a discontinuity in t at $t = 0$, but it is smooth in t (for a fixed x) when $t \in (0, \infty)$.

Let $a(x) = u(x, 0)$ and let

$$g(t) = \lim_{x \to 0+0} 2\frac{\partial u}{\partial x}(x,t).$$

Clearly,

$$\frac{\partial u}{\partial t}(x,t) - \frac{\partial^2 u}{\partial t}(x,t) = -\delta(x)g(t) + \delta(t)a(x), \quad x \in \mathbb{R}, \quad ,t \in \mathbb{R}, \tag{2.33}$$

where δ denotes the Dirac distribution.

Denoting by $\hat{g}$ and $\hat{a}$ the Fourier transformations of g and a, respectively, we note that (2.33) implies

$$\hat{g}(i\xi^2) = \hat{a}(\xi), \quad \xi \in \mathbb{R}. \tag{2.34}$$

We now look at the function $\hat{g}$ in more detail. We have

$$g(\tau) = \int_{\mathbb{R}} g(t)e^{-it\tau}dt = \int_0^T g(t)e^{-it\tau}dt \tag{2.35}$$

Hence, $\hat{g}$ is defined for each $\tau \in \mathbb{C}$ and is holomorphic in $\mathbb{C}$. Moreover, standard calculations together with (2.34) and the decay property of a imply that

$$\hat{g}(\tau) = \mathcal{O}(\frac{1}{|\tau|}) \quad \text{as } \tau \to \infty \text{ in } K, \tag{2.36}$$

where $K = \{\tau \in \mathbb{C} \,||\, \operatorname{Re}\tau = 0 \text{ or } \operatorname{Im}\tau = 0\}$, i.e., K is the union of the real and imaginary axes. In fact, (2.36) along the real axis and along the negative part of the imaginary axis follows from (2.35) and integration by parts. (2.36) along the positive part of imaginary axis follows from (2.34) and the fact that

$$\hat{a}(\xi) = \mathcal{O}(\frac{1}{|\xi|^2}) \quad \text{as } \xi \to \infty \text{ and } \xi \in \mathbb{R}.$$

Step 4. The last step in the proof is simple lemma about holomorphic functions.

Lemma 2.1.8. *Let $K \subset \mathbb{C}$ be the union of the real and imaginary axes. Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function satisfying two conditions:*

$$|f(z)| \leq Ae^{a|z|}, \quad z \in \mathbb{C}$$

for some positive constants a and A, and

$$f(\tau) = \mathcal{O}\left(\frac{1}{|\tau|}\right), \quad \text{as } \tau \to \infty \text{ in } K.$$

Then $f \equiv 0$.

Lemma 4.2 can be easily obtained from the Phragmén-Lindelöf theorem for an angle (for details, see Lemma 2.1.11, or [21], Theorem 7.5). □

The next proposition says that if a solution u to the heat equation is 0 on both ends of the time interval $[a, b]$ of the first quadrant, then it must be identically 0 in between.

Theorem 2.1.9. *Let $\mathcal{O}$ be the first quadrant of $\mathbb{R}^2$. The bounded function $u(x, y, t)$ is defined on $\mathcal{O} \times (0, T)$ and satisfies the backward heat equation and the conditions*

$$u_t + \triangle u = 0, \qquad \mathcal{O} \times (0, T) \tag{2.37}$$

$$u(x, y, 0) = u(x, y, T) \equiv 0 \qquad (x, y) \in \mathcal{O}. \tag{2.38}$$

Then $u \equiv 0$ in $\mathcal{O} \times [0, T]$.

Proof. Replacing (x, y, t) by $(x+1, y+1, t)$, we can assume that all derivatives of u are well-defined.

Step 1. Reduction to the case $|u(x, y, t)| < ce^{-c(x^2+y^2)}$. We take a series of change of variables and still denote the function by u. Replacing t by $-t$, u satisfies $u_t + \Delta u = 0$ in $\mathcal{O} \times (-T, 0)$. Shifting the time interval by $T_0 > T$, we have that u satisfies

$$u_t + \triangle u = 0, \qquad \mathcal{O} \times (-T + T_0, T_0)$$

$$u(x, y, -T + T_0) = u(x, y, T_0) \equiv 0 \qquad (x, y) \in \mathcal{O}.$$

We apply the Appell transform of u,

$$v(x,y,t) = \frac{1}{4\pi t} e^{-\frac{x^2+y^2}{4t}} v\left(\frac{x}{t}, \frac{y}{t}, \frac{1}{t}\right). \tag{2.39}$$

The transformed function satisfies the heat equation in $\mathcal{O} \times (-\frac{1}{T_0}, -\frac{1}{-T+T_0})$ and has the desired decay.

Step 2. Reduction to the equation $\Delta v - v = 0$. Now suppose that $u(x,y,t)$ satisfies the heat equation in $\mathcal{O} \times (0,T)$, with the decay

$$|u(x,y,t)| < c\, e^{-c(x^2+y^2)} \tag{2.40}$$

for some $c > 0$. In addition $u(x,y,0) = u(x,y,T) = 0$.

We apply the Laplace transform to u.

$$v(x,y,s) = \int_0^\infty e^{-st} u(x,y,t)dt. \tag{2.41}$$

The function v is analytic in $s \in \mathbb{C}$ because u is compact supported in t. By direct calculation, v satisfies the equation

$$\triangle v - sv = 0, \qquad (x,y) \in \mathcal{O}, \tag{2.42}$$

and the decay $|v| < ce^{-c(x^2+y^2)}$ for all s with Re $s > 0$. Of course c depend on s. If we can prove that $v(x,y,1) = 0$ in $\mathcal{O}$, then $v(x,y,s) = 0$, for all $s \in \mathbb{R}$ and $s > 0$. By analyticity of $v(x,y,s)$ in s, we have that $v(x,y,s) \equiv 0$ in $\mathcal{O} \times \mathbb{C}$. Hence $u(x,y,t) \equiv 0$ in $\mathcal{O} \times (0,T)$.

Step 3. Proof of uniqueness of the equation $\Delta v - v = 0$. Suppose that smooth function $v(x,y)$ satisfies the equation

$$\triangle v - v = 0, \qquad (x,y) \in \mathcal{O}. \tag{2.43}$$

In addition for some constant $c > 0$, $|v| < c\, e^{-c(x^2+y^2)}$. We want to show that $v \equiv 0$.

For simplicity in writing we assume $c = 1$. We extend $v(x,y)$ to the whole plane $\mathbb{R}^2$ by

$$v(x,y) = v(|x|,|y|), \text{ for } (x,y) \in \mathbb{R}^2 \setminus \mathcal{O}. \tag{2.44}$$

Let

$$f(x) = -2 \lim_{y\to 0^+} \frac{\partial v}{\partial y}(x,y) \qquad g(y) = -2 \lim_{x\to 0^+} \frac{\partial v}{\partial x}(x,y) \tag{2.45}$$

The extended function $v(x,y)$ satisfies in distribution sense the equation

$$\triangle v - v = f(x)\delta(y) + g(y)\delta(x), \qquad (x,y) \in \mathcal{O}. \tag{2.46}$$

Apply Fourier transform to the equation we have

$$-(\xi^2 + \eta^2 + 1)\tilde{v}(\xi,\eta) = \tilde{f}(\xi) + \tilde{g}(\eta). \tag{2.47}$$

We now look at the function $\tilde{f}$ in more detail. We have

$$\tilde{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-ix\xi} dx. \tag{2.48}$$

The decay of $f(x)$ implies that

$$|\tilde{f}(\xi)| \leq \int e^{-x^2} e^{x\,\mathrm{Im}\,\xi} dx \leq \int e^{-(x-\mathrm{Im}\,\xi/2)^2+(\mathrm{Im}\,\xi)^2/4} dx < c\, e^{(\mathrm{Im}\,\xi)^2/4} \qquad (2.49)$$

for some absolute constant c. In the same way we see that $\tilde{f}(\xi)$ is analytic in $\xi \in \mathbb{C}$. Moreover, it follows from (2.48) and integration by parts that the function

$$\tilde{f}(\xi) = O(\frac{1}{|\xi|^2}), \text{ as } \xi \to \infty, \text{ for } \xi \in \mathbb{R}.$$

Similarly, the function $\tilde{g}(\eta)$ is analytic and

$$\tilde{g}(\eta) = O(\frac{1}{|\eta|^2}), \text{ as } \eta \to \infty, \text{for } \eta \in \mathbb{R}. \qquad (2.50)$$

From (2.47), we have that

$$\tilde{f}(\xi) = O(\frac{1}{|\xi|^2}), \text{ as } \xi \to \infty, \text{ for } \xi \in K, \qquad (2.51)$$

where $K = \{\xi \in \mathbb{C}, \text{Re } \xi = 0, \text{ or } \text{Im } \xi = 0\}$, and

$$|\tilde{f}(\xi)| < c\, e^{(\mathtt{Im}\,\xi)^2/4}. \qquad (2.52)$$

Step 4. The last step in the proof is a lemma about holomorphic functions.

Lemma 2.1.10. *Let $K \subset \mathbb{C}$ be the union of the real and imaginary axes. Let $h : \mathbb{C} \to \mathbb{C}$ be a holomorphic function satisfying two conditions:*

$$|h(z)| \leq c\, e^{(\mathtt{Im}\,\xi)^2/4}, \quad z \in \mathbb{C}$$

for some positive constants c, and

$$h(\tau) = \mathcal{O}\left(\frac{1}{|\tau|}\right), \qquad \text{as } \tau \to \infty \text{ in } K.$$

Then $h \equiv 0$.

If the rate of increasing of h is bounded by $e^{|z|^\alpha}$ for $\alpha < 2$, then u is bounded directly from Phragmén-Lindelöf theorem. We have $\alpha = 2$. However, the bound in solely in $\operatorname{Im} z$. By a simple trick we can also apply Phragmén-Lindelöf theorem in this case. We notice that for a fixed $\sigma > 0$, if θ greater than and close to 0, the function

$$e^{i\sigma z^2}\tilde{f}(z), \quad z = x + iy$$

is bounded by some C (independent of σ or θ) on the boundary of the angle

$$G_\theta = \{z \in \mathbb{C}, \theta < \arg z < \pi/2\},$$

and is bounded by $C\, e^{C|z|^2}$ in the interior of the angle G_θ. Apply the Phragmén-Lindelöf theorem, see Lemma 2.1.11, we have that $|e^{i\sigma z^2}\tilde{f}(z)| \le C$ on G_θ. Passing $\theta \to 0$ and then $\sigma \to 0$, it follows that $|h(z)| < C$ in the first quadrant. In the same way h is bounded in other quadrants. By Liouville theorem, h is a constant, and this constant has to be zero due to the decay on the axes. □

Lemma 2.1.11. *(Phragmén-Lindelöf theorem for an angle) Let* G *be the interior* *of an angle of* $\alpha\pi$ *radians,* ($0 < \alpha \leq 2$)*, with boundary* Γ*, and let* $f(z)$ *be analytic on* G*, continuous up to the boundary. Suppose* $f(z)$ *satisfies the following conditions:*

1. $f(z) \leq C < \infty$ *on* Γ.

2. $$\liminf_{r\to\infty} \frac{\ln M(r)}{r^{1/\alpha}} \leq 0, \tag{2.53}$$

   where

   $$M(r) = \sup_{|z|=r, z\in G} |f(z)|.$$

   Then

   $$|f(z)| \leq C. \tag{2.54}$$

2.1.4 Counterexample to BU in cones of angle less than $\pi/2$

Recall that $\mathcal{O}_\theta$ is defined as a cone with opening angle θ.

$$\mathcal{O}_\theta = \{x = (x_1, x'), x' \in \mathbb{R}^{n-1}, x_1 > |x|\cos(\theta/2)\}. \tag{2.55}$$

Luis Escauriaza [5] showed that (BU) fails when $\theta < \pi/2$. We briefly recall the counterexample in this section. Let us denote by Γ the standard heat kernel,

i. e. $\Gamma(x,t) = (4\pi t)^{-n/2} \exp(-|x|^2/4t)$, and recall Appell transformation

$$u(x,t) = \Gamma(x,t)v(y,s), \qquad y = \frac{x}{t}, \quad s = \frac{1}{t} \ . \tag{2.56}$$

This transformation takes the solutions $u(x,t)$ of the heat equation into the solutions $v(y,s)$ of the backward heat equation

$$v_s + \Delta v = 0 \ . \tag{2.57}$$

By taking $u(x,t) = h(x)$ for a suitable harmonic function h in $\mathcal{O}_\theta$, we can get a counterexample to the backward heat equation form of (BU) for $\theta < \pi/2$. In dimension 2 we can use the real part of the holomorphic function $z \to \exp(-Az^\alpha)$ (for suitable $A > 0$ and a parameter $\alpha > 2$) to obtain an explicit formula:

$$v(y,s) = \mathrm{Re}\, \frac{1}{s} \exp(-A\frac{(y_1 + iy_2)^\alpha}{s^\alpha} + \frac{|y|^2}{4s}) \ . \tag{2.58}$$

The function $v(y_1, y_2, s)$ is bounded in a sector that $|\arctan \frac{y_2}{y_1}| < \pi/(2\alpha)$, away from the origin. We can shift it to $v(y_1+1, y_2, s)$. The resulted function is bounded in a sector with angle π/α satisfying the backward heat equation $v_s + \triangle v = 0$, and $v(\cdot, \cdot, 0) = 0$. We remark that $\pi/2$ is the borderline of the construction above. The key element of the construction is the existence of a harmonic function in cones with the right decay rates, $e^{-|z|^\alpha}$ for some $\alpha > 2$. For cones with angle smaller than $\pi/2$, one can simply take the real part of $e^{-|z|^\alpha}$ as above. However, for cones with angle greater than or equal to $\pi/2$, such a harmonic function does not exist.

We note that it is enough to construct a counterexample in dimension $n = 2$. The higher-dimensional example can then be constructed by simply considering the two-dimensional function as a function of n variables, independent of $x_3, \ldots, x_n$.

2.2 Backward uniqueness and Carleman inequalities

In this section we prove the main result of this chapter. Let us recall the some settings and the results first. Consider an open set $\Omega \subset \mathbb{R}^n$. Let u be a bounded solution of the equation

$$u_t - \Delta u + b(x,t) \cdot \nabla u + c(x,t)u = 0 \quad \text{in } \Omega \times (0,T), \tag{2.59}$$

where the coefficients $b = (b_1, \ldots, b_n)$, c are measurable and bounded. Recall that Ω has the *backward uniqueness property* if the following statement holds:

(BU) *If a bounded* $u \colon \Omega \times (0,T) \to \mathbb{R}$ *satisfies (2.59) and* $u(\,\cdot\,,T) = 0$, *then* $u \equiv 0$ *in* $\Omega \times (0,T)$.

From the results in [9], (BU) holds for $\theta \geq \pi$. On the other hand, the construction of Escauriaza shows that (BU) fails for $\theta < \pi/2$. It is easy to see that there exists some critical angle θ_0 such that (BU) holds for $\theta > \theta_0$ and (BU) fails for θ_0.

Theorem 2.2.1. *The cones $\mathcal{O}_\theta$ satisfy (BU) for*

$$\theta > 2\arccos(1/\sqrt{3}) \sim 109.52^\circ\,.$$

In other words, the critical angle θ_0 introduced above satisfies

$$\theta_0 \leq 2\arccos(1/\sqrt{3}).$$

Our proof of Theorem (2.2.1) relies on two Carleman-type inequalities, along lines similar to [9]. The first inequality, Proposition 2.2.2, is taken from [9] and is applied in the same way to obtain fast decay rates for the solutions, see Lemma 2.2.3. We note that Carleman inequalities of this type can be found already in [6, 10, 7, 8].

The second inequality, Proposition 2.2.4, is the main new tool used in our proof. The heuristics behind this inequality is somewhat similar to the heuristics behind the second Carleman-type inequality in [9] (Proposition 6.2). However, the proof of Proposition 2.2.4 requires a new idea, as for $\theta < \pi$ certain critical terms appearing in the proofs of the inequalities lose convexity.

In addition to determining the critical angle, another interesting open problem is to optimize the assumptions on the coefficient $b(x,t)$ and $c(x,t)$, in the spirit of [16]. For example, it is conceivable that the result remains true for $b \in L^{n+2}_{x,t}$ and $c \in L^{(n+2)/2}_{x,t}$, but it might be a difficult problem to decide whether this is the case.

In what follows we will work with the inequality

$$|u_t - \Delta u| \le c_1(|\nabla u| + |u|) \tag{2.60}$$

rather than (2.59). It is not hard to see that when assuming the boundedness of b and c, the two formulations are equivalent.

2.2.1 Backward uniqueness

Without loss of generality we assume $T = 1$ and work with the backward form of (2.60). Recall that

$$\mathcal{O}_\theta = \{x = (x_1, x'), x' \in \mathbb{R}^{n-1}, x_1 > |x| \cos(\theta/2)\}. \tag{2.61}$$

Suppose that $u(x,t)$ is a solution to the backward heat equation for some $\theta > 2 \arccos(1/\sqrt{3})$.

$$|u_t + \Delta u| \le c_1(|\nabla u| + |u|) \qquad \text{in } \mathcal{O}_\theta \times (0,1), \tag{2.62}$$

$$u(\cdot, 0) = 0 \qquad \text{in } \mathcal{O}_\theta\,. \tag{2.63}$$

In addition,

$$|u| < M \qquad \text{in } \mathcal{O}_\theta \times (0,1). \tag{2.64}$$

Then $u \equiv 0$.

To prove the above statement we firstly need the following Carleman inequality from [9] (Proposition 6.1), by which we obtain a decay result for solutions of backward heat equation.

Proposition 2.2.2 ([9])**.** *For any function $u \in C_0^\infty(\mathbb{R}^n \times (0,2); \mathbb{R}^n)$ and any positive number a,*

$$\int_{\mathbb{R}^n \times (0,2)} h^{-2a}(t) e^{-\frac{|x|^2}{4t}} \left(\frac{a}{t}|u|^2 + |\nabla u|^2 \right) dxdt \\ \leq c_0 \int_{\mathbb{R}^n \times (0,2)} h^{-2a}(t) e^{-\frac{|x|^2}{4t}} |\partial_t u + \Delta u|^2 dxdt, \tag{2.65}$$

where c_0 is a positive absolute constant and $h(t) = te^{\frac{1-t}{3}}$.

Lemma 2.2.3 below immediately implies exponential decay of the solution u satisfying (2.62)-(2.64). The proof is by using the Carleman inequality in Proposition 2.2.2. The decay of u enables us to apply the Carleman inequality in Proposition 2.2.4 and reach the conclusion in Theorem 2.2.1.

Lemma 2.2.3. *Let B_R denote the ball with radius R in $\mathbb{R}^n$. Assume that $R > 2$. Consider a function u satisfying the following conditions, with some positive* *constants* c_1 *and* M*.*

$$|u_t + \triangle u| \le c_1(|\nabla u| + |u|) \qquad \text{in } B_R \times (0,T), \tag{2.66}$$

$$u(x,0) = 0 \qquad \text{in } B_R, \tag{2.67}$$

$$|u| < M \qquad \text{in } B_R \times (0,T). \tag{2.68}$$

Then there exists some constants β*,* γ*, such that for* $t \in (0,\gamma)$*,*

$$u(0,t) \le \frac{c_2}{\min\{1,T\}} M e^{-\beta \frac{R^2}{t}}, \tag{2.69}$$

where β *is a small enough absolute constant,* c_2 *depends on* c_1*,* γ *depends on* c_1 *and* T*.*

Notice that the constants β, γ and c_2 do not depend on R. It follows immediately the exponential decay of the bounded solution to the equation (2.60) with respect to the distance to the lateral boundary. We will give the proof of the lemma in the next section.

The following Carleman inequality in Proposition 2.2.2 is a key tool used in our proof of the backward uniqueness in cones. We define the set

$$Q_\theta = (\mathcal{O}_\theta \cap \{x_1 > 1\}) \times (0,1).$$

The purpose of "cutting off the corner" is to avoid singularities at the origin.

Proposition 2.2.4. *Let* $\phi(x,t) = a\Lambda(t)\varphi(x) + t^2$*, where* $\Lambda(t) = \dfrac{1-t}{t^{\alpha/2}}$*, and* $\varphi(x) = x_1^\alpha - \varepsilon^\alpha r^\alpha$*, where* $r = |x|$*, and* $\varepsilon = \cos(\theta/2)$*. For any* $\varepsilon \in (0, 1/\sqrt{3})$*, that is,*

$\theta \in (2\arccos(1/\sqrt{3}), \pi)$, *there exists some* $\alpha = \alpha(\varepsilon) \in (1, 2)$ *such that the following inequality holds for* $u \in C_0^\infty(Q_\theta)$ *and* $a > a_0$ *for some constant* a_0.

$$\int_{Q_\theta} e^{2\phi(x,t)} \left[a\left(\Lambda(t) + \varphi(x)\right) u^2 + |\nabla u|^2\right] dxdt \\ \leq 4 \int_{Q_\theta} e^{2\phi(x,t)} |\partial_t u + \Delta u|^2 dxdt. \tag{2.70}$$

We apply this Carleman inequality to prove the main result of the paper in the remaining part of this section. The proof of Proposition 2.2.4 is postponed to Section 2.4.

For $x \in \mathcal{O}_\theta$ we denote by $d_\theta(x)$ the distance between x and the boundary of $\mathcal{O}_\theta$, explicitly given by

$$d_\theta(x) = x_1 \sin(\theta/2) - |x'| \cos(\theta/2). \tag{2.71}$$

Let ${\mathcal{O}_\theta}^{+2} = \{x \in \mathcal{O}_\theta \mid d_\theta(x) > 2\}$. With any other number c, the set ${\mathcal{O}_\theta}^{+c}$ is defined in the same way.

The next lemma is a consequence of the decay result from Lemma 2.2.3 and Proposition 2.2.4. It implies Theorem 2.2.1 immediately.

Lemma 2.2.5. *Assume that for some* $\theta \in (2\arccos(1/\sqrt{3}), \pi)$ *a function* u *satisfies (2.62) – (2.64), then there is a number* $\gamma_1(c_1)$ *such that*

$$u(x,t) \equiv 0 \tag{2.72}$$

in $\mathcal{O}_\theta \times (0, \gamma_1)$.

Proof. Lemma 2.2.3 implies that

$$|u(x,t)| \leq c_2 M e^{-\beta \frac{d_\theta^2(x)}{t}} \tag{2.73}$$

for all $(x,t) \in {\mathcal{O}_\theta}^{+2} \times (0,\gamma)$. By local gradient estimates for the heat equation [20] we can assume that

$$|u(x,t)| + |\nabla u(x,t)| \leq c_3 M e^{-\frac{\beta d_\theta^2(x)}{2t}} \tag{2.74}$$

for all $(x,t) \in {\mathcal{O}_\theta}^{+3} \times (0,\gamma/2]$.

By scaling we define a function v by

$$v(y,s) = u(\lambda y, \lambda s^2 - \gamma_1) \tag{2.75}$$

for $(y,s) \in \mathcal{O}_\theta \times (0,1)$ with $\lambda = \sqrt{2\gamma_1}$. This function satisfies the relations

$$|\partial_s v + \Delta v| \leq c_1 \lambda (|\nabla v| + |v|) \qquad \text{in } \mathcal{O}_\theta \times (0,1), \tag{2.76}$$

$$v(y,s) = 0 \qquad \text{in } \mathcal{O}_\theta \times (0,1/2), \tag{2.77}$$

and

$$|v(y,s)| + |\nabla v(y,s)| \leq c_3 M e^{-\frac{\beta \lambda^2 d_\theta^2(y)}{2(\lambda^2 s - \gamma_1)}} \leq c_3 M e^{-\beta \frac{d_\theta^2(y)}{2s}} \tag{2.78}$$

for $1/2 < s < 1$ and $y \in {\mathcal{O}_\theta}^{+3/\lambda} = \{y \in \mathcal{O}_\theta \mid d_\theta(y) > 3/\lambda\}$.

To apply Proposition 2.2.4, we need certain decay of $|v(y,s)|$ when $|y|$ is large. Notice that the preferred decay can be obtained by considering a cone with slightly small opening. Proposition 2.2.4 holds for angles in the interval $(2\arccos(1/\sqrt{3}), \pi)$. We thus consider the median of θ and $2\arccos(1/\sqrt{3})$,

$$\delta = \frac{\theta + 2\arccos(1/\sqrt{3})}{2}.$$

In the smaller cone $\mathcal{O}_\delta = \{x \in \mathbb{R}^n, x_1 > |x|\cos(\delta/2)\}$ we have the estimate $d_\theta(y) \geq |y| \sin\left(\frac{\theta-\delta}{2}\right)$. It follows (2.78) that

$$|v(y,s)| + |\nabla v(y,s)| \leq c_3 M e^{-\beta' \frac{|y|^2}{s}} \tag{2.79}$$

for $1/2 < s < 1$ and $y \in \mathcal{O}_\delta \cap {\mathcal{O}_\theta}^{+3/\lambda}$, with the constant $\beta' = \beta \sin^2(\frac{\theta-\delta}{2})/2$. We can further have

$$|v(y,s)| + |\nabla v(y,s)| \leq c_3' M e^{-\beta' \frac{|y|^2}{s}} \tag{2.80}$$

for $1/2 < s < 1$ and $y \in \mathcal{O}_\delta \cap \{y_1 > 3/\lambda\}$ for some other constant c_3'.

Next we work on the smaller cone $\mathcal{O}_\delta$ with opening δ, where we have exponential decay (2.80) and the following properties inherited from $\mathcal{O}_\theta$.

$$|\partial_s v + \Delta v| \leq c_1 \lambda (|\nabla v| + |v|) \qquad \text{in } \mathcal{O}_\delta \times (0,1), \tag{2.81}$$

$$v(y,s) = 0 \qquad \text{in } \mathcal{O}_\delta \times (0,1/2). \tag{2.82}$$

Proposition 2.2.4 requires support condition for the Carleman inequality. For that purpose, let us fix two smooth cut-off functions such that

$$\psi_1(y_1) = \begin{cases} 0, & y_1 < 3/\lambda + 2, \\ 1, & y_1 > 3/\lambda + 3, \end{cases}$$

$$\psi_2(\tau) = \begin{cases} 0, & \tau < -3/4, \\ 1, & \tau > -1/2. \end{cases}$$

We set (for the definition of ϕ, see Proposition 2.2.4)

$$\phi_B(y,s) = \frac{1}{a}\phi(y,s) - B = (1-s)\frac{y_1^\alpha - \varepsilon^\alpha |y|^\alpha}{s^{\alpha/2}} + \frac{s^2}{a} - B,$$

where $\varepsilon = \cos(\delta/2)$, $B = \frac{2}{a}\phi(y_\lambda, \frac{1}{2})$, with $y_\lambda = (3/\lambda + 3, 0, \cdots, 0)$ and

$$\eta(y,s) = \psi_1(y_1)\psi_2(\frac{\phi_B}{B}), \qquad w(y,s) = \eta(y,s)v(y,s).$$

The function w is not compact supported in $Q_\delta = (\mathcal{O}_\delta \cap \{x_1 > 1\}) \times (0,1)$. However, it follows from (2.78) and the special structure of the weight function ϕ in Proposition 2.2.4 that, with w replacing u in (2.70), integrals on both sides converge. If we multiply w by an additional cut-off function ξ such that

$$\xi(x) = \begin{cases} 1, & \tau < R \\ 0, & \tau > 2R \end{cases}$$

and $|\nabla\xi| < c/R$, $|\nabla^2\xi| < c/R^2$, then apply Proposition 2.2.4 to the compact supported function $w\xi$, and let $R \to \infty$, we finally obtain

$$\int_{Q_\delta} e^{2a\phi_B}\left[a\left(\Lambda(s) + \varphi\right)w^2 + |\nabla w|^2\right] dyds \le 4\int_{Q_\delta} e^{2a\phi_B}|\partial_s w + \Delta w|^2 dyds. \quad (2.83)$$

From (2.81) we have

$$\begin{aligned} |\partial_s w + \triangle w| &\leq c_1\lambda(|\nabla w| + |w|) \\ &\quad + c_4(|\nabla v| + |v|)(|\partial_s \eta + |\nabla \eta| + |\Delta \eta|). \qquad (2.84) \end{aligned}$$

We can see that for a large enough $a(\Lambda(s) + \varphi(y)) > 1$ in the support of w, where $\phi_B/B \geq -3/4$ because of the definition of ψ_2. In addition, we take $\gamma_1(c_1)$ small enough such that $16c_1^2\lambda^2 < 1/2$. We then have

$$\begin{aligned} I &\equiv \int_{Q_\delta} e^{2a\phi_B}\left(w^2 + |\nabla w|^2\right) dyds \qquad (2.85) \\ &\leq 32c_4^2 \int_{Q_\delta} e^{2a\phi_B}(|v|^2 + |\nabla v|^2)(|\partial_s \eta| + |\nabla \eta| + |\Delta \eta|)^2 dyds. \qquad (2.86) \end{aligned}$$

To estimate the right hand side, we need to look into details of derivatives of η. In view of the definitions of ψ_1 and ψ_2, the support of derivatives of $\eta(y,s)$ is the closure of the set

$$\{y_1 > 3/\lambda + 2,\ -3B/4 < \phi_B < -B/2\}$$
$$\cup \{3/\lambda + 2 < y_1 < 3/\lambda + 3,\ \phi_B > -B/2\}.$$

However, the second set has empty intersection with $\mathcal{O}_\delta \times (1/2, 1)$, where the function v is nonzero. Hence the support of the term $(|\nabla v| + |v|)(|\partial_s \eta + |\nabla \eta| + |\Delta \eta|)$ is closure of the set

$$\{y_1 > 3/\lambda + 2, 1/2 < s < 1, -3B/4 < \phi_B(y,s) < -B/2\}.$$

of which we denote by $\chi(y,s)$ the characteristic function.

Next we estimate the derivatives of $\eta(y,s)$ in the support of the term $(|\nabla v| + |v|)(|\partial_s \eta + |\nabla\eta| + |\Delta\eta|)$. Recall that $\phi(y,s) = a\Lambda(s)\varphi(y) + s^2$, where $\Lambda(s) = \dfrac{1-s}{s^{\alpha/2}}$ and $\varphi(y) = y_1^\alpha - \varepsilon^\alpha |y|^\alpha$. The function $\Lambda(s)$ and the derivative $\Lambda'(s)$ are bounded for $s \in (1/2, 1)$. The function φ and its derivatives up to the second order are bounded by (const. $|y|^\alpha$). The cut-off functions ψ_1 and ψ_2 and derivatives up to the second order are bounded by some absolute constant. The value of B is bounded from below regardless of the value of the parameter a. Thus

$$(|\partial_s \eta| + |\nabla\eta| + |\Delta\eta|)^2 < c_5 |y|^{2\alpha}. \tag{2.87}$$

We now estimate (2.86) by using (2.79) and (2.87). We see that

$$I \leq c_6 M e^{-Ba} \int_{Q_\delta} |y|^{2\alpha} e^{-2\beta' \frac{|y|^2}{s}} \chi(y,s) dy ds$$

for some constant c_6. The last integral is bounded. Passing to the limit as $a \to \infty$ we see that $v(y,s) = 0$ where $\phi_B(y,s) > 0$ and $1/2 < s < 1$. Using the property of unique continuation across the spatial boundaries (see Theorem 4.1 in [9]), we show that $v(y,s) = 0$ if $y \in \mathcal{O}_\theta$ and $0 < s < 1$. This proves the lemma. $\square$

2.3 Proof of Lemma 2.2

The proof of Lemma 2.2.3 is based on the Carleman inequality in [9], which we quoted in Proposition 2.2.2.

Proof of Lemma 2.2.3. The proof is similar to the one in [9]. We still include the proof here for the convenience of the reader.

In what follows, we always assume that the function u is extended by zero to negative values of t.

The assumption that $R > 2$ results in no loss, since the conclusion is only useful when R is large. According to the local gradient estimates of the heat equation [20], in the smaller cylinder $(x,t) \in B_{R-1} \times (0, T/2)$, we can assume that

$$|u| + |\nabla u(x,t)| \leq \frac{c_7}{\min\{1,T\}} M \tag{2.88}$$

with some absolute constant c_7.

We fix $t \in (0, \min\{1,T\}/12)$ and introduce a new function v by the usual parabolic scaling:

$$v(y,s) = u(\lambda y, \lambda^2 s - t/2).$$

The function v is well defined on the set $Q_\rho = B_\rho \times (0,2)$, where $\rho = (R-1)/\lambda$ and $\lambda = \sqrt{3t} \in (0, \min\{1, \sqrt{T}\}/2)$. We have the following relations for v.

$$|\partial_s v + \Delta v| \leq c_1 \lambda (|\nabla v| + |v|), \tag{2.89}$$

$$|v(y,s)| + |\nabla v(y,s)| < \frac{c_7}{\min\{1,T\}} M \tag{2.90}$$

for all $(y,s) \in Q_\rho$,

$$v(y,s) = 0 \tag{2.91}$$

for $(y,s) \in B_\rho \times (0, 1/6]$.

By the assumption that $R > 2$ and $\lambda < 1/2$, we have $\rho > 2$. In order to apply Proposition 2.2.2, we take two smooth cut-off functions in Q_ρ:

$$\psi_\rho(y) = \begin{cases} 0, & |y| > \rho - 1/2, \\ 1, & |y| < \rho - 1, \end{cases}$$

$$\psi_t(s) = \begin{cases} 0, & 7/4 < s < 2, \\ 1, & 0 < s < 3/2. \end{cases}$$

By assumption, these functions take values in $[0,1]$ and are such that $|\nabla^k \psi_\rho| < C_k$, $k = 1, 2$, and $|\partial_s \psi_t| < C_0$. We set $\eta(y,s) = \psi_\rho(y)\psi_t(s)$ and

$$w(y,s) = \eta(y,s)v(y,s). \tag{2.92}$$

It follows from (2.89) that

$$|\partial_s w + \Delta w| \leq c_1 \lambda (|\nabla w| + |w|) + c_8 \chi (|\nabla v| + |v|), \tag{2.93}$$

where c_8 is a positive constant depending only on c_1 and C_k, $k = 0, 1, 2$, $\chi(y,s) = 1$ for $(y,s) \in \omega = \{\rho - 1 < |y| < \rho, 0 < s < 2\} \cup \{|y| < \rho - 1, 3/2 < s < 2\}$, and

$\chi(y,s) = 0$ for $(y,s) \notin \omega$. The set ω is where the cut-off function η is not constantly 1 in Q_ρ. Obviously, the function w is compactly supported on $\mathbb{R}^2 \times (0,2)$. The inequality (2.65) also holds for scale valued functions. Therefore we may apply Proposition 2.2.2 and obtain

$$\int_{Q_\rho} h^{-2a}(s)e^{-\frac{|y|^2}{4s}} \left(\frac{a}{s}|w|^2 + |\nabla w|^2\right) dyds \\ \leq c_0 \int_{Q_\rho} h^{-2a}(s)e^{-\frac{|y|^2}{4s}} |\partial_s w + \Delta w|^2 dyds. \quad (2.94)$$

Taking $a > 2$, and applying (2.93) we finally obtain that

$$I \equiv \int_{Q_\rho} h^{-2a}(s)e^{-\frac{|y|^2}{4s}} (|w|^2 + |\nabla w|^2) dyds \leq 4c_0(c_1^2\lambda^2 I + c_8^2 I_1), \quad (2.95)$$

where

$$I_1 = \int_{Q_\rho} \chi(y,s)h^{-2a}(s)e^{-\frac{|y|^2}{4s}} (|\nabla v|^2 + |v|^2) dyds.$$

Taking a sufficiently small value for $\gamma = \gamma(c_1)$ such that in the range $\lambda \in (0,\gamma)$, we can assume that the inequality $4c_0c_1^2\lambda^2 \leq 1/2$ holds, and therefore (2.95) implies that

$$I \leq 8c_0c_8^2 I_1. \quad (2.96)$$

Notice that near the origin $\{y = 0, s = 0\}$, where the parametric function $h^{-2a}(s)e^{-\frac{|y|^2}{4s}}$ is not integrable, the characteristic function χ is 0. By (2.90) we have

$$\begin{aligned} I_1 &\le \frac{c_7^2 M^2}{\min\{1,T^2\}}\left\{\int_{3/2}^2 \int_{|y|<\rho-1} h^{-2a}(s)e^{-\frac{|y|^2}{4s}}dyds \right.\\ &\left. + \int_0^2 \int_{\rho-1<|y|<\rho} h^{-2a}(s)e^{-\frac{|y|^2}{4s}}dyds\right\} \qquad (2.97)\\ &\le \frac{c_9 M^2}{\min\{1,T^2\}}\left[h^{-2a}(3/2)+\int_0^2 h^{-2a}(s)e^{-\frac{(\rho-1)^2}{4s}}ds\right], \qquad (2.98)\end{aligned}$$

where c_9 is an absolute constant.

Using (2.98) we obtain the estimate

$$\begin{aligned} D &\equiv \int_{B_1}\int_{1/2}^1 |w|^2 dyds = \int_{B_1}\int_{1/2}^1 |v|^2 dyds \\ &\le c_{10}\int_{Q_\rho} h^{-2a}(s)e^{-\frac{|y|^2}{4s}}(|w|^2+|\nabla w|^2)dyds \\ &\le \frac{c_{11}M^2}{\min\{1,T^2\}}\left[h^{-2a}(3/2)+\int_0^2 h^{-2a}(s)e^{-\frac{\rho^2}{16s}}ds\right] \\ &= \frac{c_{11}M^2}{\min\{1,T^2\}}e^{-\beta\rho^2}\left[h^{-2a}(3/2)e^{\beta\rho^2}+\int_0^2 h^{-2a}(s)e^{\beta\rho^2-\frac{\rho^2}{16s}}ds\right].\end{aligned}$$

We take $\beta < 1/64$ and then let

$$a = \beta\rho^2/(2\log h(3/2)).$$

This choice of a leads to the estimate

$$D \le c_{11}e^{-\beta\rho^2}\left[1+\int_0^2 g(s)ds\right],$$

where $g(s) = h^{-2a}(s)e^{-\frac{\rho^2}{32s}}$. By simple calculation we have that

$$g'(s) = h^{-2a}(s)e^{-\frac{\rho^2}{32s}}\left[-\frac{\beta\rho^2}{\log h(3/2)}\left(\frac{1}{s}-\frac{1}{3}\right)+\frac{\rho^2}{32s^2}\right].$$

One can readily verify that $g(2) < 1$ and $g'(s) \geq 0$ for any $s \in (0, 2)$ if $\beta < \frac{1}{64} \log h(3/2)$. Therefore,

$$D \leq 3 \frac{c_{11} M^2}{\min\{1, T^2\}} e^{-\beta \rho^2} = 3 \frac{c_{11} M^2}{\min\{1, T^2\}} e^{-\beta \frac{R^2}{12t}}.$$

On the other hand, the regularity theory implies that

$$|u(0,t)|^2 = |v(0, 1/2)|^2 \leq c_{12} D.$$

Finally we obtain that

$$|u(0,t)| \leq \frac{c_2}{\min\{1, T\}} M e^{-\beta \frac{R^2}{24t}}.$$

□

2.4 Proof of the Carleman inequality

One of the difficulties in the proof of the Carleman inequality in Proposition 2.2.4 is that – by comparison with the case $\theta = \pi$ – some loss of the convexity of the weight φ cannot be avoided. Therefore we have to investigate in more detail some of the terms which can be neglected when $\theta \geq \pi$.

Proof of Proposition 2.2.4. Let u be an arbitrary function in $C_0^\infty(Q_\theta)$ and $v = e^\phi u$. Then

$$Lv \equiv e^\phi (\partial_t u + \Delta u) = \Delta v + |\nabla \phi|^2 v - \partial_t \phi v + \partial_t v - 2 \nabla \phi \nabla v - \Delta \phi v. \qquad (2.99)$$

We decompose L into symmetric and skew-symmetric parts

$$L = S + A,$$

where

$$Sv = \Delta v + |\nabla\phi|^2 v - \partial_t \phi v \tag{2.100}$$

and

$$Av = \partial_t v - 2\nabla\phi\nabla v - \Delta\phi v. \tag{2.101}$$

The right hand side of the inequality (2.70) is

$$\int |Lv|^2 dxdt = \int |Sv|^2 dxdt + \int |Av|^2 dxdt + \int ([S,A]v)v dxdt, \tag{2.102}$$

where $[S,A] = SA - AS$ is the commutator of S and A. By simple calculations we have that

$$([S,A]v, v) = \int 4\phi_{,kl} v_{,k} v_{,l} dxdt \tag{2.103}$$

$$+ \int \left(2\nabla\phi\nabla|\nabla\phi|^2 - \Delta^2\phi + \partial_t^2\phi - 2\partial_t|\nabla\phi|^2\right)|v|^2 dxdt. \tag{2.104}$$

The Hessian of the function $\phi = a\Lambda(t)(x_1^\alpha - \varepsilon^\alpha r^\alpha)$ is not positive-definite. To compensate the term $\int 4\phi_{,kl} v_{,k} v_{,l} dxdt$ in (2.103) we introduce a function $F(x,t)$ to be determined.

$$\begin{aligned}(Sv, Fv) &= \int \Delta v F v + (|\nabla\phi|^2 - \partial_t\phi)Fv^2 dx \\ &= \int -F|\nabla v|^2 + (\frac{1}{2}\triangle F + |\nabla\phi|^2 F - \partial_t\phi F)v^2 dxdt.\end{aligned}$$

Cauchy-Schwartz inequality implies that

$$\begin{aligned}(Sv, Sv) &\geq -(Sv, Fv) - \frac{1}{4}\int F^2 v^2 dxdt \\ &\geq \int F|\nabla v|^2 - (\frac{1}{2}\triangle F + |\nabla\phi|^2 F - \partial_t \phi F + \frac{1}{4}F^2)v^2 dxdt. \quad (2.105)\end{aligned}$$

Combining (2.103) and (2.105) we have

$$([S, A]v, v) + (Sv, Sv) \geq \int 4\phi_{,kl} v_{,k} v_{,l} + F|\nabla v|^2 dxdt \quad (2.106)$$

$$+\int \left(2\nabla\phi\nabla|\nabla\phi|^2 - \Delta^2\phi + \partial_t^2\phi - 2\partial_t|\nabla\phi|^2\right) v^2 dxdt \quad (2.107)$$

$$+\int -(\frac{1}{2}\triangle F + |\nabla\phi|^2 F - \partial_t\phi F + \frac{1}{4}F^2)v^2 dxdt. \quad (2.108)$$

By calculation the Hessian of φ is

$$D^2\varphi(x) = \alpha(\alpha-1)\begin{pmatrix} x_1^{\alpha-2} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} - \alpha\,\varepsilon^\alpha\, r^{\alpha-2} E_n \quad (2.109)$$

$$+\alpha(2-\alpha)\,\varepsilon^\alpha\, r^{\alpha-4} x^T x, \quad (2.110)$$

where E_n reprensents the n dimensional identity matrix, $x = (x_1, \ldots, x_n)$ is the row vector and x^T denotes the transpose of x. It is easy to see that

$$D^2\varphi(x) + \alpha\,\varepsilon^\alpha\, r^{\alpha-2} E_n \geq 0.$$

We thus let $f(x) = \alpha\,\varepsilon^\alpha\, r^{\alpha-2}$ and

$$F(x,t) = 4a\Lambda(t)f(x) + 1. \quad (2.111)$$

With this choice of $F(x,t)$, the right hand side of line (2.106) is positive and

$$\int 4\phi_{,kl}v_{,k}v_{,l} + F|\nabla v|^2 dxdt \geq \int |\nabla v|^2 dxdt.$$

Grouping the remaining terms according to the orders of the parameter a, with A_3 denoting the terms with a^3 and etc., we have that

$$\begin{aligned} ([S,A]v,v) + (Sv,Sv) \ \geq\ & \int |\nabla v|^2 dxdt \\ & + \int (A_3 + A_2 + A_1 + A_0) v^2 dxdt, \end{aligned} \tag{2.112}$$

where

$$A_3 = 2\nabla\phi\nabla|\nabla\phi|^2 - 4a^3\Lambda^3 f|\nabla\varphi|^2, \tag{2.113}$$

$$A_2 = -4a^2\Lambda\Lambda'|\nabla\varphi|^2 + 4a^2\Lambda\Lambda'\varphi f - 4a^2\Lambda^2(t)f^2 - a^2\Lambda^2|\nabla\varphi|^2, \tag{2.114}$$

$$A_1 = -a\Lambda\Delta^2\varphi + a\Lambda''\varphi - 2a\Lambda(t)\Delta f + a\Lambda'\varphi - 2a\Lambda(t)f + 8at\Lambda f, \tag{2.115}$$

$$A_0 = 7/4 + 2t. \tag{2.116}$$

We analyze A_3 first. With a^3 as a coefficient A_3 must be non-negative in the set Q_θ. By letting $x_1/r \to \varepsilon$, we see easily that $\varepsilon < 1/\sqrt{3}$ is a necessary condition for $A_3 \geq 0$. Next we show that under the condition $\varepsilon < 1/\sqrt{3}$, we indeed have that $A_3 \geq 0$. Denoting by $\nabla\varphi^T$ the transpose of the row vector $\nabla\varphi$, we notice that

$$A_3 = 4a^3\Lambda^3(t)\nabla\varphi\left(D^2\varphi(x) - \alpha\,\varepsilon^\alpha\, r^{\alpha-2}\right)\nabla\varphi^T. \tag{2.117}$$

It is easy to see that

$$\begin{aligned} & D^2\varphi(x) - \alpha\,\varepsilon^\alpha\, r^{\alpha-2} \\ \geq \; & \alpha(\alpha-1)\begin{pmatrix} x_1^{\alpha-2} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} - 2\alpha\,\varepsilon^\alpha\, r^{\alpha-2} E_n. \end{aligned} \tag{2.118}$$

By the fact that $x_1^{\alpha-2} > r^{\alpha-2}$, we have

$$D^2\varphi(x) - \alpha\,\varepsilon^\alpha\, r^{\alpha-2} \geq \alpha r^{\alpha-2} \begin{pmatrix} \alpha - 1 - 2\,\varepsilon^\alpha & 0 \\ 0 & -2\,\varepsilon^\alpha\, E_{n-1} \end{pmatrix}, \tag{2.119}$$

where E_{n-1} is the $n-1$ dimensional identity matrix.

The first derivatives of ϕ are as follows.

$$\varphi_{,1}(x,t) = \alpha x_1^{\alpha-1} - \alpha\,\varepsilon^\alpha\, r^{\alpha-2} x_1, \quad \varphi_{,k}(x,t) = -\alpha\,\varepsilon^\alpha\, r^{\alpha-2} x_k, \quad k = 2, \ldots, n.$$

We notice that $\varphi_{,1} \geq \alpha(1-\varepsilon^\alpha) x_1^{\alpha-1}$. Thus

$$A_3 \geq 4a^3\Lambda^3(t)\alpha^3 r^{\alpha-2}\left[(\alpha - 1 - 2\,\varepsilon^\alpha)(1-\varepsilon^\alpha)^2 x_1^{2\alpha-2} - 2\,\varepsilon^{3\alpha}\, r^{2\alpha-4}|x'|^2\right].$$

Taking into account that $x_1/r > \varepsilon$ and $|x'|^2/r^2 < 1 - \varepsilon^2$,

$$A_3 \geq 4a^3\Lambda^3(t)\alpha^3 r^{3\alpha-4}\,\varepsilon^{2\alpha-2}\left[(\alpha - 1 - 2\,\varepsilon^\alpha)(1-\varepsilon^\alpha)^2 - 2\,\varepsilon^{\alpha+2}(1-\varepsilon^2)\right].$$

Let us denote the quantity in the bracket above by $m(\alpha, \varepsilon)$.

$$m(\alpha,\varepsilon) = (\alpha - 1 - 2\,\varepsilon^\alpha)(1-\varepsilon^\alpha)^2 - 2\,\varepsilon^{\alpha+2}(1-\varepsilon^2).$$

A_3 is non-negative if $m(\alpha,\varepsilon)$ is. In the set $(\alpha,\varepsilon)\in(1,2)\times(0,1/\sqrt{3})$, the function $m(\alpha,\varepsilon)$ is monotone increasing with respect to α and is monotone decreasing with respect to ε. Notice that

$$m(2,1/\sqrt{3})=0.$$

Hence for any ε smaller than and close to $1/\sqrt{3}$, we can choose a corresponding $\alpha<2$ such that $m(\alpha,\varepsilon)\geq 0$, and in turn $A_3\geq 0$. We fix this α in the rest of the proof.

For the estimate of A_2 we notice that $\nabla v=(\nabla\phi)v+e^{\phi}\nabla u$, as $v=e^{\phi}u$. To bound $|e^{\phi}\nabla u|$, we want to apply the inequality $|e^{\phi}\nabla u|^2/2\leq|\nabla v|^2+|\nabla\phi|^2v^2$. We thus look at the inequality (2.112) in the following way.

$$\begin{aligned}([S,A]v,v)+(Sv,Sv) \;\geq\; & \int(|\nabla v|^2+|\nabla\phi|^2v^2)dxdt\\ &+\int[A_3+(A_2-|\nabla\phi|^2)+A_1+A_0]v^2dxdt.\end{aligned}$$

Next we estimate $A_2-|\nabla\phi|^2$.

$$A_2-|\nabla\phi|^2=-4a^2\Lambda(t)\Lambda'(t)\left(\left(1+\frac{\Lambda(t)}{2\Lambda'(t)}\right)|\nabla\varphi|^2-\varphi f+\frac{\Lambda(t)}{\Lambda'(t)}f^2\right). \quad (2.120)$$

$\Lambda(t)=\frac{1-t}{t^{\alpha/2}}$ and $\Lambda'(t)=-\frac{\alpha/2+(1-\alpha/2)t}{t^{\alpha/2+1}}$. $|\Lambda(t)/\Lambda'(t)|<\frac{1}{2\alpha}$.

$$A_2-|\nabla\phi|^2\geq-4a^2\Lambda(t)\Lambda'(t)\left(\left(1-\frac{1}{4\alpha}\right)|\nabla\varphi|^2-\varphi f-\frac{1}{2\alpha}f^2\right). \quad (2.121)$$

Notice that $|\nabla\varphi|^2\geq|\varphi_{,1}|^2\geq\alpha^2(1-\varepsilon^\alpha)^2x_1^{2\alpha-2}$ and $\varphi f\leq\alpha\,\varepsilon^\alpha(1-\varepsilon^\alpha)x_1^{2\alpha-2}$. For the choice of α we made above (see the expression of $m(\alpha,\varepsilon)$), $\varepsilon^\alpha\leq(\alpha-1)/2$.

Taking into account that $r \geq x_1 > 1$,

$$A_2 - |\nabla \phi|^2 \geq -4a^2 \Lambda(t)\Lambda'(t) \left(\frac{3}{8} x_1^{2\alpha-2} - \frac{1}{4} r^{2\alpha-4} \right) \geq -\frac{a^2}{2} \Lambda(t)\Lambda'(t) x_1^{2\alpha-2}. \quad (2.122)$$

Finally, we estimate A_1. Recall that

$$A_1 = -a\Lambda\Delta^2\varphi + a\Lambda''\varphi - 2a\Lambda(t)\Delta f + a\Lambda'\varphi - 2a\Lambda(t) f + 8at\Lambda f.$$

A simple observation is that

$$A_1 \geq a(\Lambda''\varphi + a\Lambda')\varphi - a\Lambda(\Delta^2\varphi + 2\Delta f + 2f), \quad (2.123)$$

and $\Lambda''(t) + \Lambda'(t) > \alpha - 1 > 0$. The terms in the second parenthesis are of homogeneity less than or equal to $\alpha - 2$, thus under the control of A_2. Consequently, there exists some constant a_0 depending on φ such that for $a > a_0$,

$$A_2 + A_1 \geq -\frac{a^2}{4} \Lambda(t)\Lambda'(t) x_1^{2\alpha-2} + a(\alpha - 1)\varphi. \quad (2.124)$$

In addition $|\Lambda'(t)| \geq 1$ and $x_1 > 1$. We thus have

$$\begin{aligned} ([S, A]v, v) + (Sv, Sv) \quad &\geq \quad \int (|\nabla v|^2 + |\nabla \phi|^2 v^2) dxdt \\ &\quad + \int \frac{a^2}{4} \Lambda(t) v^2 dxdt + \int a(\alpha - 1)\varphi v^2 dxdt \end{aligned}$$

In turn

$$\begin{aligned} \int_{Q_\theta} e^{2a\Lambda(t)\varphi(x) + 2t^2} \left[\left(\frac{a^2}{4} \Lambda(t) + a(\alpha - 1)\varphi \right) u^2 + \frac{1}{2} |\nabla u|^2 \right] dxdt \\ \leq \int_{Q_\theta} e^{2a\Lambda(t)\varphi(x) + 2t^2} |\partial_t u + \Delta u|^2 dxdt. \quad (2.125) \end{aligned}$$

To simplify, we can assume $\alpha > 3/2$, and $a > 2$, it follows that

$$\int_{Q_\theta} e^{2a\Lambda(t)\varphi(x)+2t^2} \left[a\left(\Lambda+\varphi\right)u^2 + |\nabla u|^2\right] dxdt$$
$$\leq 4\int_{Q_\theta} e^{2a\Lambda(t)\varphi(x)+2t^2}|\partial_t u + \Delta u|^2 dxdt. \qquad (2.126)$$

$\square$

Chapter 3

Isolated singularities of the 1D complex viscous Burgers equation

3.1 Structure of the isolated singularities

In this chapter, we consider the structure of the isolated singularities of the Burgers equation

$$u_t + uu_x = u_{xx} \ , \tag{3.1}$$

via the re-scaling procedure, which is well-known to be very useful in studying singularities of PDEs. We will present the settings and the results first and give the proofs in the sections following.

With initial data $u_0 \in L^1(\mathbb{R})$, we define the solution u by means of the Cole-Hopf transformation, that is, $u = -2v_x/v$, where v is the bounded solution to the heat equation with initial data $v_0(x) = \exp\left(-\frac{1}{2}\int_0^x u_0(\xi)\,d\xi\right)$. The function v is analytic in $\mathbb{R}\times(0,\infty)$ and the zero set is isolated. Thus the function u is analytic and solves the Burgers equation in $\mathbb{R}\times(0,\infty)$ excluding the zero set of v. By shifting the origin and re-scaling the function we can assume that $(0,0)$ is an isolated singularity in a neighborhood $Q = (-1,1)\times(-1,1)$ and

$$G(t) = \sup_{|x|<1} |u(x,t)|$$

is attained at some x in $(-1,1)$ for any fixed t in the interval $(-1,0)$. Let

$$M(t) = \sup_{|x|<1, s<t} |u(x,s)|.$$

There exists a sequence $t_j \to 0^-$ (that is, $t_j < 0$, $t_j \to 0$) such that $G(t_j) = M(t_j)$. Let x_j be such that $M(t_j) = |u(x_j,t_j)|$. Of course x_j need not be unique. The Burgers equation has a natural scaling invariance

$$u(x,t) \to \lambda u(\lambda x, \lambda^2 t).$$

We consider the re-scaled functions

$$w_j(y,s) = \frac{1}{M(t_j)} u(x_j + \frac{y}{M(t_j)}, t_j + \frac{s}{M(t_j)^2}).$$

They are equilibria of equation (1) in the (y,s) variables defined in the cylinders $Q_j = \{(y,s)\,|\,(x_j + \frac{y}{M(t_j)}, t_j + \frac{s}{M(t_j)^2}) \in Q\}$, which contain any given bounded subset of $\mathbb{R} \times (-\infty, 0]$ if j is large enough. We are interested in the limiting process as $t_j \to 0^-$. What is the rate of blow-up? Does the limit of w_j exist? If so, in which sense? What is the limit function? We answer these questions in the following theorem.

Theorem 3.1.1. *With the notations introduced above, we have*

(i) there exists a constant $C_0 > 0$ such that $|M(t)| > C_0/|t|$ for $t \in (-1, 0)$,

(ii) there is a $T_0 > 0$ such that when $-T_0 < t < 0$, $M(t) = G(t)$,

(iii) for a suitable choice of $x(t)$ (it may not be unique) such that $M(t) = |u(x(t), t)|$, the re-scaled functions

$$w^{(t)}(y, s) := \frac{1}{M(t)} u \left(x(t) + \frac{y}{M(t)}, t + \frac{s}{M^2(t)} \right)$$

converge as $t \to 0^-$ uniformly in any bounded subset of $\mathbb{R} \times (-\infty, 0]$. Moreover

$$\lim_{t \to 0^-} w^{(t)}(y, s) = \frac{-2}{y \pm 2i}, \tag{3.2}$$

which are steady state solutions of the Burgers equation.

Singularities for which some scale-invariant quantity is bounded are often called type I singularities. The most common definition of type I singularity for equations with the scaling invariance $u(x, t) \to \lambda u(\lambda x, \lambda^2 t)$ uses the quantity $\sup \sqrt{-t}\, |u(x, t)|$. Singularities which are not of type I are called type II. In this terminology the blow-up of the solution u to the Burgers equation is of type II.

The proof of Theorem 3.1.1 uses analyticity and explicit calculation. Namely, the function v in the Cole-Hopf transformation $u = -2v_x/v$ is analytic and in turn has a representation in caloric polynomial series.

3.2 Representation in caloric polynomials and outline of the proof

3.2.1 Caloric polynomials

Let us first recall the definition of caloric polynomials. A function $f(x,t)$ is said to be parabolically m-homogeneous if $f(\lambda x, \lambda^2 t) = \lambda^m f(x,t)$. The m-th caloric polynomial $P_m(x,t)$ is a parabolically m-homogeneous polynomial satisfying the heat equation. It is unique, modulo a multiplicative factor, and can be given by:

$$P_m(x,t) = \sum_{k=0}^{[m/2]} \frac{m!}{k!(m-2k)!} x^{m-2k} t^k.$$

For m even, $m = 2k$, the polynomial P_m is of the form

$$P_m(x,t) = (x^2 + a_1 t) \cdots (x^2 + a_k t), \quad 0 < a_1 < \cdots < a_k.$$

For m odd, $m = 2k+1$, P_m is of the form

$$P_m(x,t) = x(x^2 + a_1 t) \cdots (x^2 + a_k t), \quad 0 < a_1 < \cdots < a_k.$$

The a_j's might be different. The caloric polynomials are simply the heat extension of the polynomials x^m.

Poisson transform, see [30]:

$$\mathcal{P}[\phi] = \int_{-\infty}^{\infty} \Gamma(x-y,t)\phi(y)dy. \tag{3.3}$$

The Poisson transform of x^m is the heat polynomial of degree m.

$$\mathcal{P}[x^m] = P_m(x,t) = m! \sum_{k=0}^{[m/2]} \frac{x^{m-2k}}{(m-2k)!} \frac{t^k}{k!}. \tag{3.4}$$

To verify, change variable and using integration by parts as follows.

$$\begin{aligned}
\mathcal{P}[x^m] &= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4t} y^m dy && (3.5)\\
&= \frac{1}{\sqrt{4\pi t}} (\sqrt{4t})^{m+1} \int_{-\infty}^{\infty} e^{-z^2} (z + \frac{x}{\sqrt{4t}})^m dz && (3.6)\\
&= \frac{1}{\sqrt{\pi}} \sum_{k=0}^{[m/2]} \binom{m}{2k} 4^k x^{m-2k} t^k \int_{-\infty}^{\infty} e^{-z^2} z^{2k} dz && (3.7)\\
&= P_m(x,t). && (3.8)
\end{aligned}$$

In the last equation, we use integration by parts to get that

$$\int_{-\infty}^{\infty} e^{-z^2} z^{2k} dz = \frac{2k-1}{2} \int_{-\infty}^{\infty} e^{-z^2} z^{2k-2} dz, \tag{3.9}$$

and the fact that

$$\int_{-\infty}^{\infty} e^{-z^2} dz = \sqrt{\pi}.$$

A solution $v(x,t)$ to the heat equation can be written as an absolutely convergent caloric polynomial series

$$v(x,t) = \sum_{k=0}^{\infty} \alpha_k P_k(x - x_0, t - t_0)$$

in a strip $|t - t_0| < \sigma$ (with the coefficients $\alpha_k = \partial_x^k v(x_0, t_0)/k!$) if and only if it has the property

$$v(x,t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi(t-s)}} e^{-\frac{|x-y|}{4(t-s)}} v(y,s)dy$$

for every pair t, s such that $t_0 - \sigma < s < t < t_0 + \sigma$, see Definition 8.1 and Theorem 11.1 in [28].

3.2.2 Proof of Theorem 3.1.1

In our setting, the bounded solution v of the heat equation satisfies the above conditions and has an isolated zero at $(0,0)$. We thus have the following series starting from a positive integer m.

$$v(x,t) = \sum_{k=m}^{\infty} \alpha_k P_k(x,t), \qquad \alpha_k \in \mathbb{C}.$$

The series converges absolutely in a strip $\mathbb{R} \times (-\sigma, \sigma)$, where σ is the length from the initial time to $t = 0$ (recall that we have shifted the origin). We can assume $\alpha_m = 1$. If not, we replace $v(x,t)$ by $v(x,t)/\alpha_k$, which does not affect the Cole-Hopf transformation. The derivative $v_x(x,t)$ is also analytic in $\mathbb{R} \times$

$(-\sigma, \sigma)$ and has the absolutely convergent caloric polynomial expansion $v_x(x,t) = \sum_{k=m}^{\infty} \alpha_k \partial_x P_k(x,t)$. We notice that $\partial_x P_k(x,t)$ also satisfies the heat equation. By uniqueness of P_k

$$\partial_x P_{k+1}(x,t) = (k+1)P_k(x,t). \tag{3.10}$$

In turn we have the following representation.

$$u(x,t) = \frac{-2v_x(x,t)}{v(x,t)} = -2\frac{mP_{m-1}(x,t) + \sum_{k=m+1}^{\infty} \alpha_k k P_{k-1}(x,t)}{P_m(x,t) + \sum_{k=m+1}^{\infty} \alpha_k P_k(x,t)}. \tag{3.11}$$

To simplify the representation we notice that when $t = -1$, the caloric polynomial $P_k(x,-1)$ becomes the Hermite polynomial $H_k(x)$. From (3.10) we have $H'_{k+1}(x) = (k+1)H_k(x)$. In addition, Hermite polynomials have only simple roots. Hence $H_{k+1}(x)$ and $H_k(x)$ have no common zeros. (Actually their zeros are interlacing.) We will often refer to the zero sets of Hermite polynomials and therefore we introduce the notation

$$V_k := \{x \in \mathbb{R}, H_k(x) = 0\}.$$

The set V_k consists of k real numbers and $V_{k+1} \cap V_k = \emptyset$.

We let $t = -\tau^2$ and $\xi = x/\tau$. Then we have

$$\begin{aligned} u(x,t) = u(\tau\xi, -\tau^2) &= -\frac{2v_x(\tau\xi,-\tau^2)}{v(\tau\xi,-\tau^2)} \\ &= -\left(\frac{2}{\tau}\right)\frac{H'_m(\xi) + \sum_{k=m+1}^{\infty} \alpha_k \tau^{k-m} H'_k(\xi)}{H_m(\xi) + \sum_{k=m+1}^{\infty} \alpha_k \tau^{k-m} H_k(\xi)}. \end{aligned} \tag{3.12}$$

The proof of Theorem 3.1.1 , which consists of the following three lemmas, is based on analysis of the representation (3.12).

Lemma 3.2.1. *Let u be the solution of the Burgers equation defined by the Cole-Hopf transformation $u = -2v_x/v$ with an isolated singularity at the origin. Given the representation (3.12), for each $b \in V_m$, we have an analytic function $\xi(\tau)$ with $\xi(0) = b$ such that $|u(x,t)|$ attains a local maximum on $x(t) = \tau\xi(\tau)$. The m analytic curves yield all the local maxima of $|u(x,t)|$ for $t < 0$ in a neighborhood of* 0.

For each time $t < 0$, the maximum $G(t) = \sup_{|x|<1} |u(x,t)|$ is attained on one of the m curves. Actually, it is attained on a fixed curve for all time t because of analyticity.

Lemma 3.2.2. *There exists one curve $x(t) = \tau\xi(\tau)$ and a time $T_0 > 0$ such that for $t \in (-T_0, 0)$*

$$G(t) = |u(x(t),t)|$$

and $G(t)$ is monotone increasing.

Recall that $M(t) = \sup_{|x|<1,s<t} |u(x,s)|$. The second statement in Lemma 3.2.2 implies that $G(t) = M(t)$, which means that we can take limit along the continuous time parameter t instead of t_j. The last ingredient of the proof is the study of the blow-up profile.

Lemma 3.2.3. *Let $w^{(t)}(y,s)$ be the re-scaled function along a fixed curve $x = x(t)$ where $M(t)$ is attained.*

$$w^{(t)}(y,s) := \frac{1}{M(t)} u\left(x(t) + \frac{y}{M(t)}, t + \frac{s}{M^2(t)}\right).$$

Then as $t \to 0^-$, $w^{(t)}(y,s)$ converges uniformly in any bounded subset of $\mathbb{R} \times (-\infty, 0]$ and

$$\lim_{t\to 0^-} w^{(t)}(y,s) = -2/(y \pm 2i),$$

which are regular steady state solutions of the Burgers equation.

3.3 Proof of the lemmas

Proof of Lemma 3.2.1

We apply the same method as in Lemma 3.1 in [23]. As u is a complex-valued function, let us look at

$$|u(\tau\xi, -\tau^2)|^2 = \frac{4}{\tau^2}\left|\frac{H'_m(\xi) + \sum_{k=m+1}^{\infty} \alpha_k \tau^{k-m} H'_k(\xi)}{H_m(\xi) + \sum_{k=m+1}^{\infty} \alpha_k \tau^{k-m} H_k(\xi)}\right|^2. \tag{3.13}$$

We differentiate the right hand side with respect to ξ. By the isolated singularity assumption, the denominator of the derivative is nonzero when $\tau \neq 0$ in a neighborhood of 0. Therefore the equation

$$\partial_\xi |u(\tau\xi, -\tau^2)|^2 = 0 \tag{3.14}$$

is equivalent to the numerator of the derivative being zero, or to the equation

$$\Phi(\xi,\tau) := 2H_m H'_m[H_m H''_m - (H'_m)^2] + \tau R(\xi,\tau) = 0, \tag{3.15}$$

where $R(\xi,\tau)$ is an analytic function of (ξ,τ) when ξ is bounded and τ is in a neighborhood of 0. $R(\xi,\tau)$ has vanishing derivatives in ξ of order 1, 2, ..., $4m-2$

When $m = 1$, $H_m H''_m - (H'_m)^2$ is a negative constant. Let us write explicitly $V_m = \{b_1^{(m)}, b_2^{(m)}, ..., b_m^{(m)}\}$, $V_{m-1} = \{b_1^{(m-1)}, b_2^{(m-1)}, ..., b_{m-1}^{(m-1)}\}$. When $m > 1$, we have the following inequality .

$$\begin{aligned} H_m(\xi)H''_m(\xi) - (H'_m(\xi))^2 &= H_m^2(\xi)\left(\frac{H'_m(\xi)}{H_m(\xi)}\right)' \\ &= -H_m^2(\xi)\sum_{k=1}^{m}\frac{1}{(\xi - b_k^{(m)})^2} = -\sum_{k=1}^{m}\prod_{j\neq k}(\xi - b_j^{(m)})^2 < 0. \end{aligned} \tag{3.16}$$

The inequality is strict because $b_k^{(m)}$ are all different. We now have

$$\Phi(\xi,\tau) = 2m\prod_{l=1}^{m}(\xi - b_l^{(m)})\prod_{n=1}^{m-1}(\xi - b_n^{(m-1)})\left(-\sum_{k=1}^{m}\prod_{j\neq k}(\xi - b_j^{(m)})^2\right) + \tau R(\xi,\tau).$$

We look for analytic curves $\xi = \xi(\tau)$ in the form of $\xi(\tau) = b + \tau\phi(\tau)$ with $b \in V_m \cup V_{m-1}$ defined for small τ on which $\Phi(\xi,\tau)$ vanishes. Substituting the expression $\xi(\tau) = b + \tau\phi(\tau)$ into (3.15), it is easy to see that we get an equation of the form

$$\phi(\tau) = F_b(\tau,\phi(\tau)) \tag{3.17}$$

where $F_b = F_b(\tau,\phi)$ is analytic in ϕ and depends on ϕ only through $\tau\phi$. Therefore $\phi_0 := F_b(0,\phi)$ is independent of ϕ and $\frac{\partial F_b}{\partial\phi}(0,\phi) = 0$. By the standard implicit function theorem, equation (3.17) has an analytic solution ϕ defined on a neighborhood of 0 with $\phi(0) = \phi_0$. Thus we obtain $2m-1$ distinct analytic curves on which the equation (3.15) holds.

By the Weierstrass preparation theorem [12], the $2m-1$ analytic curves found above give all the real solutions to (3.15) in a neighborhood of $(0,0)$. That is, for a fixed $\tau \neq 0$ they yield all the critical points for (3.13).

Recall that local maximums of $|u|$ are attained when ξ is near V_m, thus those curves with $\xi(0) \in V_m$ give all the local maxima since $V_m \cap V_{m-1} = \emptyset$. $\square$

Remark 3.3.1. *From the proof of Lemma 3.2.1 we can see that the blow-up rate of $u(x,t)$ at the origin must be $|t|^{-k/2}$, where $k \in \mathbb{N}$ and $k \geq 2$. That is, the blow-up is of type II. On the other hand, it is easy to construct a function $u(x,t)$ with blow-up rate $|t|^{-k/2}$ for all integers $k \geq 2$ by the Cole-Hopf transformation from caloric polynomial series.*

Proof of Lemma 3.2.2

Let us denote $x_k(t) = \tau\xi_k(\tau) = b_k\tau + \tau^2\phi_k(\tau)$, where $b_k \in V_m$ the zero set of H_m and $\phi_k(\tau)$ is the analytic function we found for b_k, see Lemma 3.2.1. In a neighborhood of $(0,0)$, $G(t) := \sup_{|x|<1} |u(x,t)|$ is given by

$$\max_{k=1,\ldots,m} |u(x_k(t), -\tau^2)|.$$

Because $x_k(-\tau^2)$ is analytic in τ and $x_k(0) = 0$, there exists a neighborhood of 0 such that the functions $v_x(x_k(-\tau^2), -\tau^2)$ and $v(x_k(-\tau^2), -\tau^2)$ are analytic in τ. Since $b_k \notin V_{m-1}$,

$$v_x(x^k(-\tau^2), -\tau^2) = mH_{m-1}(b_k)\tau^{m-1} + O(\tau^m).$$

For some integer $n_k > m$ and $\gamma_{n_k} \in \mathbb{C}$ we have

$$v(x^k(-\tau^2), -\tau^2) = \gamma_{n_k}\tau^{n_k} + O(\tau^{n_k+1}).$$

Thus $|u(x^k(-\tau^2), -\tau^2)|^2$ are meromorphic functions of τ and dominated by the lowest order terms.

$$|u(x^k(-\tau^2), -\tau^2)|^2 = \theta_0^k\tau^{2m-2-2n_k} + \theta_1^k\tau^{2m-1-2n_k} + \theta_2^k\tau^{2m-n_k} + \cdots, \quad \theta_0^k > 0.$$

By comparing one by one n_k, θ_0^k, θ_1^k, θ_2^k,...., it is clear that there exist some $k \in \{1, ..., m\}$ and $T_0 > 0$ such that when $0 < \tau < \sqrt{T_0}$, $G(t) = |u(x^k(t), t)|$.

Because $G^2(t)$ is dominated by $\theta_0^k\tau^{2m-2-2n_k}$, it is obvious that $G(t)$ is monotone for $t \in (-T_0, 0)$ by reducing T_0 if necessary. We point out lastly that the curve $x(t)$ where $G(t)$ is attained need not be unique. For example, it is possible that $v(x, t)$ is even in x. $\square$

Proof of Lemma 3.2.3

Recall from the proof of Lemma 3.2.2 that on the curve $x(t) = b\tau + \tau^2\phi(\tau)$ where the maxima $M(t)$ is attained we have

$$v_x(x(t),t) = mH_{m-1}(b)\tau^{m-1} + O(\tau^m);$$

$$v(x(t),t) = \gamma_n\tau^n + O(\tau^{n+1}), \qquad \text{for a fixed } n > m;$$

$$M(t) = 2\left(\frac{v_x\bar{v}_x}{v\bar{v}}\right)^{1/2} = 2\tau^{-(n-m+1)}\sqrt{|mH_{m-1}(b)/\gamma_n|^2 + O(\tau)}. \tag{3.18}$$

Taking into account that $v_{xx}(x(t),t) = O(\tau^{\max\{m-2,\ 0\}})$, for (y,s) in a bounded subset of $R \times (-\infty, 0]$ we have

$$v_x\left(x(t) + \frac{y}{M(t)}, t + \frac{s}{M^2(t)}\right) = v_x\big(x(t),t\big) + O(\tau^{n-1}), \tag{3.19}$$

$$v\left(x(t) + \frac{y}{M(t)}, t + \frac{s}{M^2(t)}\right) = v(x(t),t) + v_x\big(x(t),t\big)\frac{y}{M(t)} + O(\tau^{2n-m}). \tag{3.20}$$

Putting together (3.19), (3.20),

$$\begin{aligned} w^{(t)}(y,s) &= \frac{-2v_x\big(x(t),t\big) + O(\tau^{n-1})}{v_x\big(x(t),t\big)y + M(t)v\big(x(t),t\big) + M(t)O(\tau^{2n-m})} \\ &= \frac{-2 + O(\tau^{n-m})}{y - 2M(t)/u\big(x(t),t\big) + O(\tau^{n-m})}. \end{aligned} \tag{3.21}$$

To finish the proof it only remains to show that $M(t)/u\big(x(t),t\big)$ converges to either i or $-i$. It is obvious that $|M(t)/u\big(x(t),t\big)| = 1$. From the fact that $x(t)$ is a critical point of $|u|^2$ we can derive $\lim_{t\to 0^-} \mathrm{Re}\ M(t)/u(x(t),t) = 0$. Actually,

$$\partial_x(u\bar{u}) = 0 \iff u_x\bar{u} + u\bar{u}_x = 0 \iff \mathrm{Re}\,(u_x/u) = 0 \iff \mathrm{Re}\,v_x/v = \mathrm{Re}\,v_{xx}/v_x.$$

In the last equation, the right hand side is $O(\tau^{-1})$ because $H_{m-1}(b)$ is away from 0 when $b \in V_m$, so is $\mathrm{Re}\,v_x/v$. Thus $\mathrm{Re}\,u(x,t) = O(\tau^{-1})$ on the curve. While $M(t) = O(\tau^{-(n-m+1)})$ and $n-m+1 \geq 2$, we conclude that limit of $M(t)/u\big(x(t),t\big)$ exits and the limit is either i or $-i$. $\square$

Remark 3.3.2. *Both $\pm i$ are possible. It is easy to construct an example where $v(x,t)$ is even in x. Then $v_x(x,t)$ is odd in x. If $M(t)$ is attained at $x(t)$, it is also attained at $-x(t)$. The two curves $x = x(t)$ and $x = -x(t)$ will give opposite signs.*

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