January 1961

EXACT LOWER MOMENTS OF ORDER STATISTICS FOR SMALL SAMPLES FROM THE CHI-POPULATION (1.D.F.)

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Report No. 12

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January 1961

Exact Lower Moments of Order statistics for Small Samples from the Chi-distribution (1 d.f.)[1]

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0. Summary: Numerous contributions have been made to the problem of order statistics in samples from normal and exponential population. For the problem of location with symmetry Fraser [1] derived a locally most powerful rank test against normal alternatives. It is the Wilcoxon test statistic with the ranks replaced by the corresponding expected values of order statistics for a sample of absolute values from the standardized normal-distribution, e.g., the chi-distribution with one degree of freedom. Gupta [3] considered the order statistics from the standardized gamma distribution with the parameter, namely r, defined on the positive integers (that is, from the chi-distribution with even degrees of freedom) and derived expressions for the kth moments of an order statistic and the covariance between two order statistics. He also presented a table of numerical values of the kth moments of an order statistic accurate to six significant digits for $1 \leq k \leq N \leq 15$ and $r = 1\ (1)\ 5$, where N is the sample size. It might be of interest to consider the problem of order statistics in samples from chi-population with odd degrees of freedom. However, this problem seems to be more difficult than the one considered by Gupta [3]. In this paper, the expected values for samples to size four and the mixed and second moments (about the origin) for samples to size five, drawn from the chi-population (1 d.f.) have been evaluated. Numerical values of these to eight decimal places are computed. Section 2 contains general formulae and some definite integrals used in the computation. The results in Section 3 have theoretical interest in showing the relationships between moments of order statistics from chi (1 d.f.) and the standard normal distributions. In Section 4, there is a discussion about the number of integrals required to evaluate the first, second and mixed moments of order statistics for each N, given these moments to (N-1) and the existence of the tables for the normal distribution.

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1. Some of these results were found during the 1958 Summer Statistical Institute sponsored by the National Science Foundation.

2. I. Richard Savage is visiting Harvard University during the year 1960-1961.

1. Notation: Let $X_{1,N} < X_{2,N} < \ldots < X_{N,N}$ be the order statistics in a sample of size N from the chi-population (1 degree of freedom) having the probability density function (pdf)

$$f(x) = \sqrt{\frac{2}{\pi}}\, e^{-x^2/2} \qquad X \geq 0 \tag{1.1}$$
$$= 0 \qquad X < 0 ,$$

and cumulative distribution function (cdf)

$$F(x) = \int_0^x f(t)dt.$$

Let $Z_{1,N} < Z_{2,N} < \ldots < Z_{N,N}$ be the ordered statistics in a sample of size N drawn from the standard normal population with pdf and cdf respectively are

$$\varphi(x) = \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2} , \qquad -\infty < x < \infty \tag{1.2}$$

$$\Phi(x) = \int_{-\infty}^x \varphi(t)dt.$$

Note that when $x \geq 0$,

$$F(x) = 2\,\Phi(x) - 1 \quad \text{and} \quad f(x) = 2\,\varphi(x). \tag{1.3}$$

Let X be a chi-random variable with one degree of freedom (1 d.f.), σ^2 be the variance of X and Z be a normal random variable with mean 0 and variance 1. Let

$$v_{i,N}^{(k)} = E(X_{i,N}^k) = \frac{N!}{(i-1)!(N-i)!} \int_0^\infty x^k f(x)\, F^{i-1}(x)\, [1-F(x)]^{N-i}\, dx$$

$$1 \leq i \leq N \text{ and } k = 1, 2, \ldots , \tag{1.4}$$

$$v_{i,N} = v_{i,N}^{(1)} .$$

$$v_{i,N}^k = (v_{i,N})^k \qquad k = 1, 2, \ldots ; \tag{1.5}$$

$$v_{i,j,N} = E(X_{i,N}\, X_{j,N})$$

$$= \frac{N!}{(i-1)!(j-i-1)!(N-j)!} \iint_{0<x<y<\infty} xyf(x)f(y)F^{i-1}(x)[F(y)-F(x)]^{j-i-1}[1-F(y)]^{N-j}\, dxdy, \tag{1.6}$$

$$1 \leq i < j \leq N.$$

Let $$\nu^{(2)}_{i,N} = \nu_{i,i,N} ,$$

$$\mu^{(k)}_{i,N} = E(Z^{(k)}_{i,N}), \qquad (1.7)$$

for k = 1,2,3 etc. with $\mu_{i,N} = \mu^{(1)}_{i,N}$;

$$\mu_{i,j,N} = E(Z_{i,N}\ Z_{j,N}), \qquad (1.8)$$

for $1 \leq i \leq j \leq N$, and $\mu^{(2)}_{i,N} = \mu_{i,i,N}$, for i = 1,2, . . . , N.

2. Basic formulae: Following is a collection of formulae that are useful in the computation of the desired expected values. Formulae (1)* - (6) and integrals (iv) and (v) are true for an arbitrary distribution. Formulae (1)-(5) can be derived by writing every term on left side of each formula as an integral and summing underneath the integral sign. Formula (6) follows by considering the variance of $(X_{1,N} + \ldots + X_{N,N})/N$.

(1) $i\ \nu^{(k)}_{i+1,N} + (N-i)\ \nu^{(k)}_{i,N} = N\ \nu^{(k)}_{i,N-1}$

for $1 \leq i \leq N-1$ and k = 1,2,3, etc.

(2) $(i-1)\ \nu_{i,j,N} + (j-i)\ \nu_{i-1,j,N} + (N-j+1)\ \nu_{i-1,j-1,N} = N\ \nu_{i-1,j-1,N-1}$,

for $1 < i \leq j \leq N$.

(3) $\sum_{i=1}^{N} \nu_{i,N} = N\ E(X)$.

(4) $\sum_{i=1}^{N} \nu^{(2)}_{i,N} = N\ E(X^2)$.

(5) $\sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \nu_{i,j,N} = 1/2\ N(N-1)\ [E(X)]^2$.

(6) $\sum_{i=1}^{N} \sum_{j=1}^{N} \mathrm{cov}(X_{i,N};\ X_{j,N}) = N\ \sigma^2$.

* for which the corresponding integrals converge.

Integration by parts have been employed wherever is possible. We record some definite integrals related to the specific distribution: limits of integration being $(0,\infty)$ unless otherwise specified.

(i) $\int_b^a x\, f(x)\, dx = f(a) - f(b)$, $a > b > 0$.

(ii) $\int f^2 dx = \frac{1}{\sqrt{\pi}}$

(iii) $\int f^3 dx = \frac{2}{\pi\sqrt{3}}$

(iv) $\int f^m F^n df = \frac{-n}{m+1} \int f^{m+2} F^{n-1}\, dx$,

for all positive integral m and n.

(v) $\int f\, F^n dx = \frac{1}{n+1}$

(vi) $\int F\, f^2 dx = \frac{2}{\pi^{3/2}}$ Arc tan $\frac{1}{\sqrt{2}}$

(vii) $\int F\, f^3 dx = \frac{2}{3\sqrt{3}\,\pi}$

(viii) $\int F^2 f^2 dx = \frac{2}{\pi^{3/2}}$ Arc tan $\frac{1}{\sqrt{8}} = \frac{1}{\sqrt{\pi}} (1 - \frac{4}{\pi}$ Arc tan $\frac{1}{\sqrt{2}}$),

since Arc tan $\frac{1}{\sqrt{8}} + 2$ Arc tan $\frac{1}{\sqrt{2}} = \pi/2$.

(ix) $\int F^2 f^3 dx = \frac{4}{\pi^2\sqrt{3}}$ Arc tan $\frac{1}{\sqrt{15}}$

(x) $\int f^2(y)\, [\int_0^y f^2(x)dx]\, dy = \frac{1}{2\pi}$ and

(xi) $\int f^2(y)\, F(y)\, [\int_0^y f^2(x)dx]\, dy = \frac{2}{\pi^2}$ Arc tan $\frac{1}{\sqrt{5}}$.

When an integral contains a power of F, the integral can be thought of as a multiple integral by using $F(u) = \int_0^u f(t)dt$.

(vi) and (vii) have been evaluated using polar coordinate transformation and integrals (viii) to (xi) have been evaluated using the transformation $u = \rho$, $v = \rho x$, $w = \rho y$ and well-known integrals.

For each N it is sufficient to know one $v_{i,N}^{(k)}$, since the rest of $v_{i,N}^{(k)}$ can be solved for, by using Formula (1) and the $v_{i,N-1}^{(k)}$. It is also sufficient to know $v_{1,j,N}$ for $j = 2,3, \ldots, N$ in order to solve for the rest of $v_{i,j,N}$ by using Formula (2) and the $v_{i,j,N-1}$. Formulae (3), (4), (5) and (6) could be used to check the computations.

3. <u>Certain relationships:</u> In the formulae of this section we strive for some relationships among the moments of order statistics in samples drawn from the chi-population with one degree of freedom. One can also aim for some relationships between the moments of order statistics from the chi (1 d.f.) and the standard normal distribution. These formulae could be used for checking numerical values from existing tables, for computing some in terms of others and for obtaining some values from existing values for the normal distribution. Formulae 3.3, 3.4, 3.5 and 3.6 will be used for the discussion in Section 4. All of the formulae of this Section will be presented in a tabular form and the proofs of these, later.

Table 3.1

Number of Formula	Result
	When N is even,
3.1[1]	$v_{N,N} = \sum_{i=0}^{N-2} (-1)^i \, 2^{N-1-i} \binom{N}{i} \mu_{N-i,N-i}$
	If N + k is odd,
3.1.1[2]	$v_{N,N}^{(k)} = 1/2 \sum_{i=0}^{N-1} (-1)^i \, 2^{N-i} \binom{N}{i} \mu_{N-i,N-i}^{(k)}$

[1,2] Formulae 3.1 and 3.1.1 were found by Professor Milton Sobel at the 1958 Summer Statistical Institute sponsored by the National Science Foundation.

Table 3.1 continued

3.2 $$\nu_{1,N} = \sum_{i=1}^{N} (-1)^{i-1} \binom{N}{i} \nu_{i,i}$$

3.3 $$\nu^{(2)}_{N,N} = 1 + \nu_{N-1,N,N}$$

3.4 $$\nu^{(2)}_{1,N} = 1 + \nu_{1,2,N} - N(2/\pi)^{1/2}\ \nu_{1,N-1}$$

3.5 When N is even,

$$\nu_{1,N,N} = \sum_{i=1}^{(N-2)/2} (-1)^{i-1} \binom{N}{i} \nu_{i,i}\ \nu_{N-i,N-i} + (-1)^{(N-2)/2}\ 1/2\binom{N}{N/2} \nu^{2}_{N/2,N/2}$$

3.6 When N is odd,

$$\nu_{N-1,N,N} = \sum_{i=0}^{N-2} (-1)^{i}\ 2^{N-1-i} \binom{N}{i}\ \mu_{N-i-1,N-i,N-i}$$

3.7 $$\nu_{1,2,N} = \sum_{i=2}^{N} (-1)^{i-1} \binom{N}{i}\ \nu_{i-1,i,i} + N(2/\pi)^{1/2} \sum_{j=1}^{N-1} (-1)^{j-1} \binom{N}{j} \nu_{j,j}$$

3.8 When N is even,

$$2\,\nu_{1,N,N} = 2^{N}\mu_{1,N,N} + \sum_{i=1}^{N-1} \binom{N}{i} (-1)^{i+1}\ \nu_{1,N-i} \left[\sum_{j=1}^{i} \binom{i}{j} \nu_{j,j} \right]$$

3.9 $$\sum_{j=2}^{N} \nu_{1,j,N} = 1 - \nu^{(2)}_{1,N}$$

Table 3.1 continued

3.10	$\sum_{j=1}^{N} \nu_{j,N,N} = N(2/\pi)^{1/2}\, \nu_{N-1,N-1} + \nu_{N,N}^{(2)} - 1$
3.11	$\sum_{j=i+1}^{N} \nu_{i,j,N} - \sum_{j=1}^{N} \nu_{i-1,j,N} = 1 - \nu_{i,N}^{(2)}$, $1 < i \leq N-1$.

Formula 3.1: When N is even

$$\nu_{N,N} = \sum_{i=0}^{N-2} (-1)^i \, 2^{N-1-i} \binom{N}{i} \mu_{N-i,N-i} \cdot$$

Proof: Consider the expression

$$2^N N \int_0^{\infty} x\, \varphi(x)\, [\Phi\,(x) - 1/2]^{N-1}\, dx \,. \qquad (3.1)$$

If we let $2\,\Phi(x) - 1 = F(x)$, so that $2\varphi(x) = f(x)$ then (3.1) becomes

$$N \int_0^{\infty} x\, f(x)\, [F(x)]^{N-1}\, dx = \nu_{N,N} \cdot$$

On the other hand, if N is even, the integrand in (3.1) is an even function and we can write (3.1) as

$$N\, 2^{N-1} \int_{-\infty}^{+\infty} x\varphi(x)\, [\Phi(x) - 1/2]^{N-1}\, dx =$$

$$N\, 2^{N-1} \sum_{i=0}^{N-1} (-1)^i \binom{N-1}{i} \int_{-\infty}^{+\infty} x\, \varphi(x) \Phi^{N-1-i}\,(x)\, dx$$

which simplifies to

$$\nu_{N,N} = N\, 2^{N-1} \sum_{i=0}^{N-2} (-1)^i \binom{N}{i} 2^{N-i} \mu_{N-i,N-i} \cdot$$

Corollary 3.1.1: If $N + k$ is odd, then

$$\nu_{N,N}^{(k)} = 1/2 \sum_{i=0}^{N-1} (-1)^i \, 2^{N-i} \binom{N}{i} \mu_{N-i,N-i}^{(k)}$$

or symbolically

$$\nu_{N,N}^{(k)} = 1/2 \left(2 \mu_{1,1}^{(k)} - 1\right)^N ,$$

if powers on $\mu_{1,1}^{(k)}$ say $(\mu_{1,1}^{(k)})^i$ are replaced by $\mu_{i,i}^{(k)}$ for $i \geq 1$:

for $i = 0$, we define $\mu_{0,0}^{(k)}$ to be zero.

Formula 3.2: $\nu_{1,N} = \sum_{i=1}^{N} (-1)^{i-1} \binom{N}{i} \nu_{i,i}$.

Proof: $$\nu_{1,N} = N \int_0^\infty x \, f(x) \, [1-F(x)]^{N-1} \, dx .$$

Expanding $[1-F(x)]^{N-1}$ as a binomial series and integrating termwise, one gets the desired result. Note that Formula 3.2 is true for an arbitrary $F(x)$.

Formula 3.3: $\nu_{N,N}^{(2)} = 1 + \nu_{N-1,N,N}$.

Proof: $$\nu_{N-1,N,N} = N(N-1) \iint_{0<x<y<\infty} x \, y \, f(x) \, f(y) [F(x)]^{N-2} \, dx \, dy .$$

Integrating with respect to y we get

$$\nu_{N-1,N,N} = N(N-1) \int_0^\infty x \, f^2(x) \, [F(x)]^{N-2} \, dx .$$

Writing $f(x)[F(x)]^{N-2} = d\left[\frac{F(x)^{N-1}}{N-1}\right]$ and integrating by parts we get the desired result.

Formula 3.4: $\nu_{1,N}^{(2)} = 1 + \nu_{1,2,N} - N\sqrt{\frac{2}{\pi}} \, \nu_{1,N-1}$

Proof: Use the method of Formula 3.3.

Formula 3.5: When N is even,

$$\nu_{1,N,N} = \sum_{i=1}^{(N-2)/2} (-1)^{i-1}\binom{N}{i} \nu_{i,i}\, \nu_{N-i,N-i} + (-1)^{\frac{N-2}{2}}\, 1/2\binom{N}{N/2} \nu^2_{N/2,N/2} .$$

Proof: $\nu_{1,N,N} = N(N-1) \iint\limits_{0<x<y<\infty} xy\, f(x)\, f(y)\, [F(y)-F(x)]^{N-2}\, dx\, dy$.

Since N is even,

$$\nu_{1,N,N} = N\frac{(N-1)}{2} \int_0^\infty\!\!\int_0^\infty xy\, f(x)\, f(y)\, [F(y)-F(x)]^{N-2}\, dx\, dy .$$

Now, expanding $[F(y) - F(x)]^{N-2}$ and integrating termwise, we get the result.

Formula 3.6: When N is odd,

$$\nu_{N-1,N,N} = \sum_{i=0}^{N-2} (-1)^i\, 2^{N-1-i} \binom{N}{i} \mu_{N-i-1,N-i,N-i}$$

Proof: $\nu_{N-1,N,N} = N(N-1) \int_0^\infty x\, f^2(x)\, F^{N-2}(x)\, dx$ (See the proof of Formula 3.3)

$$= 2^N\, N(N-1) \int_0^\infty x\, \varphi^2(x)\, [\Phi(x) - 1/2]^{N-2}\, dx$$

$$= 2^{N-1}\, N(N-1) \int_{-\infty}^{+\infty} x\, \varphi^2(x)\, [\Phi(x) - 1/2]^{N-2}\, dx .$$

Now expanding $[\Phi(x) - 1/2]^{N-2}$ and integrating term by term, one gets the result.

Formula 3.7: $\nu_{1,2,N} = \sum_{i=2}^{N} (-1)^{i-1}\binom{N}{i} \nu_{i-1,i,i} + N\sqrt{\frac{2}{\pi}} \sum_{j=1}^{N-1} (-1)^{j-1}\binom{N}{j} \nu_{j,j}$.

Proof: $\nu_{1,2,N} = N(N-1) \iint\limits_{0<y<x<\infty} xy\, f(x)\, f(y)\, [1-F(x)]^{N-2}\, dx\, dy$.

Integrating with respect to y one gets

$$\nu_{1,2,N} = -N(N-1)\int_0^\infty x\, f^2(x)[1-F(x)]^{N-2}dx + N(N-1)\sqrt{\frac{2}{\pi}}\int_0^\infty x\, f(x)[1-F(x)]^{N-2}dx.$$

The result follows after expanding $[1-F(x)]^{N-2}$ and integrating termwise.

Formula 3.8: When N is even

$$2\,\nu_{1,N,N} = 2^N \mu_{1,N,N} + \sum_{i=1}^{N-1}\binom{N}{i}(-1)^{i+1}\nu_{1,N-i}\left[\sum_{j=1}^{i}\binom{i}{j}\nu_{j,j}\right].$$

Proof: $\nu_{1,N,N} = 2^N N(N-1)\iint\limits_{0<x<y<\infty} xy\,\varphi(x)\,\varphi(y)\,[\Phi(y)-\Phi(x)]^{N-2}\,dx\,dy.$

The integrand is symmetrical with respect to origin and in x and y. Consider

$$\mu_{1,N,N} = N(N-1)\iint\limits_{-\infty<x<y<\infty} xy\,\varphi(x)\,\varphi(y)\,[\Phi(y)-\Phi(x)]^{N-2}\,dx\,dy$$

$$= \frac{2}{2^N}\,\nu_{1,N,N} + N(N-1)\int_{x=-\infty}^{0}\int_{y=0}^{\infty} xy\,\varphi(x)\,\varphi(y)\,[\Phi(y)-\Phi(x)]^{N-2}\,dx\,dy.$$

Expanding $[\Phi(y)-\Phi(x)]^{N-2}$ and changing x to $-z$, one gets

$$\mu_{1,N,N} = \frac{2}{2^N}\,\nu_{1,N,N} + \sum_{i=0}^{N-2}(-1)^{i+1}N(N-1)\binom{N-2}{i}.$$

$$\left[\int_0^\infty z\,\varphi(z)\,[1-\Phi(z)]^{N-2-i}\,dz\right]\left[\int_0^\infty y\,\varphi(y)\,[\Phi(y)]^{i}\,dy\right]$$

$$= \frac{2}{2^N}\,\nu_{1,N,N} + \sum_{i=0}^{N-2}\binom{N-2}{i}\frac{N(N-1)}{2^N}(-1)^{i+1}\left[\int_0^\infty z\,f(z)[1-F(z)]^{N-2-i}\,dz\right]\cdot$$

$$\left[\int_0^\infty y\,f(y)\,[1+F(y)]^{i}\,dy\right].$$

Now expanding $[1 \pm F]^{\alpha}$ in powers of F and integrating termwise we obtain the result.

Formula 3.9: $\sum_{j=2}^{N}\nu_{1,j,N} = 1-\nu_{1,N}^{(2)}$

<u>Proof</u>: $$\text{L.H.S.} = \sum_{j=2}^{N} \frac{N!}{(j-2)!(N-j)!} \iint\limits_{0<x<y<\infty} xy\, f(x)\, f(y)[F(y)-F(x)]^{j-2}[1-F(y)]^{N-j} dx\, dy.$$

Taking the summation underneath the integral sign, we get

$$\text{L.H.S.} = N(N-1) \iint\limits_{0<x<y<\infty} xy\, f(x)\, f(y)[1-F(x)]^{N-2}\, dx\, dy$$

$$= N(N-1) \int_0^\infty x\, f^2(x)[1-F(x)]^{N-2}\, dx\ .$$

Now the result follows after one integration by parts.

<u>Formula 3.10</u>: $$\sum_{j=1}^{N} \nu_{j,N,N} = N \sqrt{\frac{2}{\pi}}\, \nu_{N-1,N-1} + \nu_{N,N}^{(2)} - 1\ .$$

<u>Proof</u>: Following the same method as in Formula 3.9, we get

$$\text{L.H.S.} = N(N-1) \iint\limits_{0<x<y<\infty} xy\, f(x)\, f(y)\, [F(y)]^{N-2}\, dx\, dy\ .$$

Integrating with respect to x, we obtain the result.

<u>Formula 3.11</u>: For $1 < i \leq N-1$

$$\sum_{j=i+1}^{N} \nu_{i,j,N} - \sum_{j=i}^{N} \nu_{i-1,j,N} = 1 - \nu_{i,N}^{(2)}\ .$$

<u>Proof</u>: Following the method of Formula 3.9, we have

$$\sum_{j=i+1}^{N} \nu_{i,j,N} = \frac{N(N-1)\ldots(N-i)}{(i-1)!} \iint\limits_{0<x<y<\infty} xy\, f(x)\, f(y) F^{i-1}(x)[1-F(x)]^{N-i-1} dx\, dy$$

$$= \frac{N!}{(N-i-1)!(i-1)!} \int_0^\infty x\, f^2(x) F^{i-1}(x)[1-F(x)]^{N-i-1}\, dx\ .$$

Write $(N-i)\, f(x)[1-F(x)]^{N-i-1} = d\, [1-F(x)]^{N-i}$ and integrate by parts. The result follows.

4. Minimum number of integrals to be evaluated: It will be useful, especially when we are interested in exact lower moments, to find out the number of integrals required, to evaluate the first, second and mixed moments of order statistics in a sample of size N given the first, second and mixed moments of order statistics in samples to size (N-1) and the existence of the tables for the normal distribution. Hence the following theorem.

Theorem 4.1: In order to obtain the first, second and mixed moments of order statistics in a sample of size N, it is sufficient to evaluate (N-3) integrals when N is even, for example evaluate $v_{1,j,N}$ for $j = 1,3, \ldots, (N-2)$; and (N-2) integrals when N is odd, for example evaluate $v_{1,N}$ and $v_{1,j,N}$ for $j = 3,4, \ldots, (N-1)$. The number of integrals is also necessary in the sense that we do not have any more constraints among the moments of order statistics independent of those given in Sections 2 and 3.

Proof: If N is even, $v_{N,N}$ can be computed from the normal tables (See Theorem 3.1) and using Formula (1) the rest of $v_{i,N}$ can be found. By Formula (2), the rest of $v_{i,j,N}$ can be solved for in terms of $v_{1,j,N}$, $j = 1,2, \ldots, N$. It can be easily shown that each of Formulae 3.3, 3.4 and 3.5 gives an independent constraint among the $v_{1,j,N}$, $j = 1,2, \ldots, N$. Hence there are at most (N-3) independent $v_{i,j,N}$. Thus it is sufficient to evaluate $v_{1,j,N}$ for $j = 1,3,4, \ldots, (N-2)$. It is clear that the above number of integrals is also necessary if there are no constraints among the moments independent of those given in Sections 2 and 3.

If N is odd, evaluate one $v_{i,N}$, for example evaluate $v_{1,N}$. Using Formula (1) we can solve for the rest of $v_{1,N}$. Also from Corollary 3.1.1, we can obtain $v^{(2)}_{N,N}$ in terms of second moments of order statistics from the standard normal distribution. Applying Formula (2) we can solve for the rest of $v^{(2)}_{i,N}$. It can be easily shown that each of Formulae (3.3) (or 3.6) and 3.4 gives an independent constraint among the $v_{1,j,N}$, $j = 2, \ldots, N$. Hence it is enough if we can evaluate (N-3) of them, for example evaluate $v_{1,j,N}$ for $j = 3,4, \ldots,$ (N-1). The above number of integrals to be evaluated will also be necessary if there are no more constraints among the moments independent of those obtained in Sections 2 and 3. This completes the proof of the theorem.

Remark 4.1: Formulae (3), (4) and (5) do not constitute constraints independent of those given by Formulae (1) and (2). Formulae (3) and (4) can be obtained by summing Formula (1) on $i = 1,2, \ldots, N-1$, with $k = 1$ and 2 respectively. Similarly Formula (5) can be obtained by summing Formula (2) on $i = 1,2, \ldots, N-1$ and $j = i+1, i+2, \ldots, N$.

Let us demonstrate how one obtains the moments of order statistics in samples of sizes three and four. Assume that the table of these moments for $N = 2$ and the table of these for the standard normal distribution are available.

$N = 3$.

Evaluate $\nu_{1,3}$. Solve for $\nu_{2,3}$ and $\nu_{3,3}$ from the following equations obtained from Formula (1).

$$\nu_{2,3} + 2\nu_{1,3} = 3\nu_{1,2} \quad \text{and}$$
$$2\nu_{3,3} + \nu_{2,3} = 3\nu_{2,2}\,.$$

From Corollary 3.1.1, we can compute $\nu_{3,3}^{(2)}$ which is given by the equation

$\nu_{3,3}^{(2)} = 4\,\mu_{3,3}^{(2)} - 6\,\mu_{2,2}^{(2)} + 6$. Using Formula (1) with $k = 2$, we can solve for $\nu_{1,3}^{(2)}$ and $\nu_{2,3}^{(2)}$. Also

Formula 3.3 gives $\quad \nu_{2,3,3} = \nu_{3,3}^{(2)} - 1$

" 3.4 " $\quad \nu_{1,2,3} = \nu_{1,3}^{(2)} + 3\sqrt{\frac{2}{\pi}}\;\nu_{1,2} - 1\,.$

Formula (2) with $i = 2$, $j = 3$, and $N = 3$ gives

$\nu_{2,3,3} + \nu_{1,3,3} + \nu_{1,2,3} = 3\,\nu_{1,2,3}$, from which we can solve for $\nu_{1,3,3}$.

$N = 4$.

From Formula 3.1 we have

$$\nu_{4,4} = 4(\,2\,\mu_{4,4} - 4\,\mu_{3,3} + 3\,\mu_{2,2})\,.$$

We can solve for $\nu_{1,4}$, $\nu_{2,4}$ and $\nu_{3,4}$ from the following equations obtained from Formula (1).

$$v_{2,4} + 3\, v_{1,4} = 4\, v_{1,3}$$

$$2\, v_{3,4} + 2\, v_{2,4} = 4\, v_{2,3}$$ and

$$3\, v_{4,4} + v_{3,4} = 4\, v_{3,3} \, .$$

Now evaluate $v_{1,4}^{(2)}$ and solve for $v_{2,4}^{(2)}$, $v_{3,4}^{(2)}$ and $v_{4,4}^{(2)}$, using Formula (1) with $k = 2$. Also

Formula 3.3 gives $v_{3,4,4} = v_{4,4}^{(2)} - 1$

" 3.4 " $v_{1,2,4} = v_{1,4}^{(2)} + 4\sqrt{\frac{2}{\pi}}\; v_{1,3} - 1$

" 3.5 " $v_{1,4,4} = -\left(4\sqrt{\frac{2}{\pi}}\; v_{3,3} + 3\, v_{2,2}^{2} \right)$.

Formula (2) with

$i = 2$, $j = 3$ gives $v_{2,3,4} + v_{1,3,4} + 2\, v_{1,2,4} = 4\, v_{1,2,3}$

$i = 2$, $j = 4$ " $v_{2,4,4} + 2\, v_{1,4,4} + v_{1,3,4} = 4\, v_{1,3,3}$

$i = 3$, $j = 4$ " $2\, v_{3,4,4} + v_{2,4,4} + v_{2,3,4} = 4\, v_{2,3,3}$

from which we can solve for $v_{1,3,4}$; $v_{2,3,4}$ and $v_{2,4,4}$.

Acknowledgement: We thank Dr. John S. White of General Motors Corporation and Professor Ingram Olkin of the University of Minnesota for a critical reading of the manuscript.

TABLE I

Expected Values of Order Statistics from the χ distribution (1 d.f.)

E(.) \ N	1	2	3	4	
X_1	$\sqrt{\frac{2}{\pi}}$	$\frac{2}{\sqrt{\pi}}(\sqrt{2}-1)$	$\frac{3}{\sqrt{\pi}}(\sqrt{2}-2+\frac{4}{\pi}\alpha)$	$\frac{4}{\sqrt{\pi}}(\sqrt{2}-6+\frac{24\alpha}{\pi})$	where α = Arc tan $\frac{1}{\sqrt{2}}$, β = Arc tan $1/\sqrt{5}$ and δ = Arc tan $1/\sqrt{15}$
X_2		$\frac{2}{\sqrt{\pi}}$	$\frac{6}{\sqrt{\pi}}(1-\frac{4}{\pi}\alpha)$	$\frac{48}{\sqrt{\pi}}(1-\frac{5\alpha}{\pi})$	
X_3			$\frac{12}{\pi^{3/2}}\alpha$	$\frac{12}{\sqrt{\pi}}(\frac{16\alpha}{\pi}-3)$	
X_4				$\frac{12}{\sqrt{\pi}}(1-\frac{4\alpha}{\pi})$	
X_1^2	1	$1-\frac{2}{\pi}$	$1-\frac{2}{\pi}(3-\sqrt{3})$	$1-\frac{4}{\pi}(3-\frac{4}{\sqrt{3}})$	$1-\frac{20}{3\pi}(3-\sqrt{3}-\frac{6\sqrt{3}\,\beta}{\pi})$
X_2^2		$1+\frac{2}{\pi}$	$1-\frac{2}{\pi}(2\sqrt{3}-3)$	$1-\frac{4}{\pi}(2\sqrt{3}-3)$	$1-\frac{20}{\pi}(\frac{8\sqrt{3}\,\delta}{\pi}-1)$
X_3^2			$1+\frac{2\sqrt{3}}{\pi}$	1	$1-\frac{20\sqrt{3}}{\pi}(1-\frac{12\delta}{\pi})$
X_4^2				$1+\frac{8}{\pi\sqrt{3}}$	$1+\frac{40}{\pi\sqrt{3}}(1-\frac{12\delta}{\pi})$

TABLE I (continued)

N	1	2	3	4	5
x_5^2					$1 + \frac{40\sqrt{3}\,\delta}{\pi^2}$
x_1x_2		$\frac{2}{\pi}$	$\frac{2}{\pi}(3 + \sqrt{3} - 3\sqrt{2})$	$\frac{12}{\pi}(1 - 2\sqrt{2} + \frac{4}{3\sqrt{3}} + \frac{4\sqrt{2}\,\alpha}{\pi})$	$\frac{20}{\pi}(1 + \frac{1}{\sqrt{3}} - 6\sqrt{2} + \frac{24\sqrt{2}\,\alpha}{\pi} + \frac{2\sqrt{3}\,\delta}{\pi})$
x_1x_3			$\frac{2}{\pi}(3\sqrt{2} - 2\sqrt{3})$	$\frac{4}{\pi}(3 + 6\sqrt{2} - \frac{8}{\sqrt{3}} - \frac{24\sqrt{2}\,\alpha}{\pi})$	$\frac{20}{\pi}(3 + 12\sqrt{2} - \frac{2}{\sqrt{3}} - \frac{60\sqrt{2}\,\alpha}{\pi} - \frac{12\beta}{\pi} - \frac{4\sqrt{3}\,\delta}{\pi})$
x_2x_3			$\frac{2\sqrt{3}}{\pi}$	$\frac{4}{\pi}(2\sqrt{3} - 3)$	$\frac{20}{\pi}(\sqrt{3} - 3 + \frac{12\beta}{\pi} - \frac{2\sqrt{3}\,\delta}{\pi})$
x_1x_4				$\frac{12}{\pi}(\frac{4\sqrt{2}\,\alpha}{\pi} - 1)$	$\frac{20}{\pi}[\frac{12}{\pi}(4\sqrt{2} - 1) - 3 - 9\sqrt{2} + \frac{36\beta}{\pi}]$
x_2x_4				$\frac{4}{\pi}(3 - \frac{4}{\sqrt{3}})$	$\frac{20}{\pi}(3 - \frac{4}{\sqrt{3}} + \frac{24\alpha}{\pi} - \frac{48\beta}{\pi} + \frac{8\sqrt{3}\,\delta}{\pi})$
x_3x_4				$\frac{8}{\pi\sqrt{3}}$	$\frac{40}{\pi}(\frac{1}{\sqrt{3}} - \frac{6\alpha}{\pi} + \frac{6\beta}{\pi} - \frac{\sqrt{3}\,\delta}{\pi})$

TABLE I (continued)

N	1	2	3	4	5
X_1X_5					$\frac{120}{\pi^2}\left[\frac{\pi}{\sqrt{2}} - 2(\sqrt{2} - 1)\alpha - 4\beta\right]$
X_2X_5					$\frac{240}{\pi^2}(3\beta - 2\alpha)$
X_3X_5					$\frac{40}{\pi}\left(\frac{6\alpha}{\pi} - \frac{6\beta}{\pi} - \frac{2\sqrt{3}\,\delta}{\pi}\right)$
X_4X_5					$\frac{40\sqrt{3}\,\delta}{\pi^2}$

TABLE II

Numerical Values of Expected values of Order Statistics from the χ distribution (1 d.f.)

$E(\cdot)$ \ N	1	2	3	4	5
x_1	0.79788456	0.46738996	0.33490293	0.26208228	
x_2		1.12837917	0.73236399	0.55336491	
x_3			1.32638676	0.91136308	
x_4				1.46472798	
x_1^2	1.00000000	0.36338023	0.19279847	0.12070214	0.08307731
x_2^2		1.63661977	0.70454374	0.40908747	0.27120146
x_3^2			2.10265779	1.00000000	0.61591648
x_4^2				2.47021039	1.25605568
x_5^2					2.77374906
x_1x_2		0.63661977	0.31156816	0.18955768	0.12863436
x_1x_3			0.49563337	0.27624476	0.18148084

N E (·)	1	2	3	4	5
X_2X_3			1.10265779	0.59091253	0.38040446
X_1X_4				0.41349542	0.24905402
X_2X_4				0.87929786	0.52015408
X_3X_4				1.47021039	0.83679982
X_1X_5					0.35775347
X_2X_5					0.74516270
X_3X_5					1.19300492
X_4X_5					1.77374907

The Arc tangents have been computed using tables of Arc tan x and the interpolation formula $\text{Arc tan } x = \text{Arc tan } x_o + \frac{Ph}{1+xx_o} - \frac{1}{3}\left(\frac{Ph}{1+xx_o}\right)^3$ where $x = x_o + Ph$. [See (2), 1953] we record

$$\alpha = \text{Arc tan } 1/\sqrt{2} = 0.61547\,97085$$

$$\beta = \text{Arc tan } 1/\sqrt{5} = 0.42053\,43352 \text{ and}$$

$$\delta = \text{Arc tan } 1/\sqrt{15} = 0.25268\,02551 .$$

TABLE III

Numerical Values: Variances and Covariances of Order Statistics from the χ distribution (1 d.f.)

N	1	2	3	4
x_1	0.36338023	0.14432686	0.08063850	0.05201502
x_2		0.36338023	0.16818672	0.10287475
x_3			0.34335597	0.16941735
x_4				0.32478233
x_1x_2		0.10922668	0.06629731	0.04453054
x_1x_3			0.05142255	0.03739265
x_2x_3			0.13125989	0.08659619
x_1x_4				0.02961618
x_2x_4				0.06876880
x_3x_4				0.13531139

TABLE IV

Expected values of the largest order statistic for even N, Computed using Formula 3.1 and Teichroew's Tables. (See (4)-1956.)

N	$\nu_{N,N}$	N	$\nu_{N,N}$
4	1.46472798	14	2.02252221
6	1.65399631	16	2.07705370
8	1.78336710	18	2.12406668
10	1.88071558	20	2.17252108
12	1.95829680		

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