Isotopy of nodal symplectic spheres in rational manifolds

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Abstract

In 1985, M. Gromov proved that any symplectic sphere of degree 1 in $\mathbb{CP}^2$ is isotopic to an algebraic line. J. Barraud extended Gromov's work to show that any symplectic sphere of degree d in $\mathbb{CP}^2$ with only positive ordinary double point singularities is symplectically isotopic to an algebraic curve. In this paper, We imitate Barraud's approach and further extend the result to the nodal symplectic spheres in rational manifolds. We prove that if (M, ω) is a rational symplectic 4-manifold, and $A \in H_2(M, \mathbb{Z})$ a homology class with $K_\omega(A) < 0$, then $\mathcal{S}_A^0$, the space of nodal symplectic spheres in the homology class A has only finitely many isotopy classes.

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Chapter 1

Introduction

1.1 Isotopy of symplectic surfaces

The isotopy problem for symplectic submanifolds embedded in a compact symplectic 4-manifold (M, ω) is a very interesting topic in sympectic topology. There are many researches about the isotopy problems for different categories of symplectic submanifolds in different symplectic manifolds.

In 1985, M. Gromov [1] proved that any symplectic sphere of degree 1 in $\mathbb{CP}^2$ is isotopic to an algebraic line, and any symplectic sphere of degree 2 is isotopic to a conic. J. Barraud [2] extended Gromov's work to show that any symplectic sphere of any degree d in $\mathbb{CP}^2$ with only postive ordinary double point singularities is symplectically isotopy to an algebraic curve. B. Siebert amd G. Tian [3] showed that any symplectic surface in $\mathbb{CP}^2$ of degree $d \leq 17$ is symplectically isotopic to an algebraic curve.

There are many different variations on the theme : the embedded symplectic object can vary, and the ambient symplectic 4-manifold can be changed. In some cases, there might be some nodal points or even cusp points involved. For example, E. Opshtein [4] concluded that any 2 symplectic balls $B1, B2$ (meaning that it is the symplectic image of a 4-dimensional euclidean ball) of $(S^2 \times S^2, \omega \oplus \mu\omega)$ are symplectic isotopic. V.M. Kharlamov and V.S. Kulikov [5] pointed out that there are infinitely many ordinary cusidal symplectic plane curves of the same degree with the same number of cusps and the same number of nodes but any one of them is Not symplectically isotopic to another.

1.2 Gromov's work

Definition 1 **(J-holomorphic curve)**

A $W^{k+1,p}$ (in Sobolev sense, and $k \geq 0$, $p>2$) map from an almost complex manifold to another $u : (S, J_s) \to (M, J)$ is ***J-holomorphic*** *(,or* ***pseudo-holomorphic*** *) if and only if it satisfies the "Cauchy-Riemann equation":*

$$du + J(u) \circ du \circ J_s = 0 \tag{1.1}$$

where J is the almost complex structure on M, and J_s is the almost complex structure on S.

In other words, u is a **J-holomorphic curve** if and only if $du : T_pS \to T_{u(p)}(M)$ is a complex linear map for every $p \in S$.

Remark : This condition is also called "Cauchy-Riemann Equation" since it is in some sense a generalization of the "Cauchy-Riemann Equation" in the complex analysis case. We can see that $du + J(u) \circ du \circ J_s = 0$ holds if and only if $J(u) \circ du = du \circ J_s$, and the latter will imply the Cauchy-Riemann Equation in complex analysis if we let S and M both be the complex plane $\mathbb{C}$ with standard complex structure and let u be the function from $\mathbb{C}$ to $\mathbb{C}$.

Remark : Although we define J-holomorphic maps as $W^{k+1,p}$ maps from (S, J_S) to (M, J), it turns out that in fact any J-holomorphic map with regularity $W^{k+1,p}$, $(k+1)p > 2$ is always a C^∞ map between two manifolds if J is a smooth (C^∞) almost complex structure on M.

The story begins with a theorem in Gromov's paper[1], the theorem states:

Theorem 1 (Gromov 1985) *Any J-holomorphic curve $u : \mathbb{CP}^1 \to \mathbb{CP}^2$ of degree 1 is isotopic to an algebraic line*

Since we know that all algebraic line in $\mathbb{CP}^2$ are all symplectic isotopic isotopy to each other, then in fact all J-homorphic curve $u : \mathbb{CP}^1 \to \mathbb{CP}^2$ of degree 1 are all in the same isotopy class.

1.3 Barraud's paper

Barraud [2] learned from Gromov's work and extend it to any degree d curves with only "good singularities". Let ω be the canonical Kähler form on $\mathbb{CP}^2$, we have:

Theorem 2 (Barraud) *Any symplectic sphere in* $(\mathbb{CP}^2, \omega)$ *having only positive ordinary double points is symplectically isotopic (among symplectic spheres with such singularities) to an algebraic curve.*

Let $\mathcal{J}_\omega$ be the space of all ω-tamed almost complex structure in $M = \mathbb{CP}^2$. For any $J \in \mathcal{J}_\omega$, any J-holomorphic sphere with only ordinary double point singularities is always a symplectic submanifold of $\mathbb{CP}^2$. Gromov [1] told us that the reverse is also true : Let C be a symplectic sphere with only positive ordinary double points. Then we can find a ω-tamed almost complex structure along C. In fact, we can even extend this J to the normal bundle of C in $\mathbb{CP}^2$, and then further extend it to the entire $\mathbb{CP}^2$.

Therefore, we can try to find an isotopy among J-holomorphic curves instead of directly finding an isotopy among symplectic spheres. In other words, we can prove Theorem 3, and Theorem 2 will be a direct corollary of Theorem 3.

Theorem 3 (Barraud) *Let J be an ω-tamed almost complex structure on* $\mathbb{CP}^2$ *and C a rational J-curve having only ordinary double point singularities. There exist a path* (J_t, C_t) *where C_t is a J_t-curve with only ordinary doulbe point singularitis joining* (J, C) *to an algebraic curve*

Here, $\mathcal{J}_\omega$ is the space of ω-tamed almost-complex structures on $\mathbb{CP}^2$, and $\mathcal{U}_1$ be the space of degree 1 (with regularity $W^{k+1,p}$, k ≥ 0, p>2) maps from $\mathbb{CP}^1$ to $\mathbb{CP}^2$.

In the space $\mathcal{U}_1 \times \mathcal{J}_\omega$, define

$$\begin{aligned} \mathcal{P} &\equiv \{(u,J) \in \mathcal{U}_1 \times \mathcal{J}_\omega / u \ is \ J-holomorphic\}, \\ \mathcal{M} &\equiv \mathcal{P}/Aut(\mathbb{CP}^1). \end{aligned}$$

In the Gromov's case, he proved the following three properties in order to prove Theorem 1:

- $\mathcal{P}$ is a Banach manifold
- the projection $\pi : \mathcal{P} \to \mathcal{J}_\omega$ is a local submersion
- the projection $\mathcal{M} \to \mathcal{J}_\omega$ is a proper map

In Barraud's case, he needs to deal with similar but a little more complicated situations. For example, let $\mathcal{P}_d$ denote the space of degree d maps from $\mathbb{CP}^1$ to $\mathbb{CP}^2$ (with enough regularity), we can define :

$$\mathcal{P}_d \equiv \{(u,J) \in \mathcal{U}_d \times \mathcal{J}_\omega\} \tag{1.2}$$

where u is a J-holomorphic curve of degree d, and also :

$$\begin{aligned} \mathcal{P}_\frown &\equiv \{(u,J,z_1,...,z_{3d-1}) \in \mathcal{P}_d \times (\mathbb{CP}^1)^{3d-1} \mid u(z_i) = p_i)\} && (1.3) \\ \mathcal{P}_\prec &\equiv \{(u,J,z_1,...,z_{3d-1},z) \in \mathcal{P}_\frown \mid du(z) = 0\} && (1.4) \\ \mathcal{P}_\mathfrak{X} &\equiv \{(u,J,z_1,...,z_{3d-1},z_a,z_b,z_c) \in \mathcal{P}_\frown \times (\mathbb{CP}^1)^3 \setminus \Delta \\ &\qquad \mid u(z_a) = u(z_b) = u(z_c)\} && (1.5) \\ \mathcal{P}_\asymp &\equiv \{(u,J,z_1,...,z_{3d-1},z_a,z_b) \in \mathcal{P}_\frown \times (\mathbb{CP}^1)^2 \setminus \Delta \\ &\qquad \mid u(z_a) = u(z_b), \operatorname{im} du(z_a) = \operatorname{im} du(z_b)\} && (1.6) \end{aligned}$$

and for any integer $d' \leqslant d$, and each subset $\{p_{i_1}, ...p_{i_{3d'}}\} \subset \{p_1, ...p_{3d-1}\}$,

$$\mathcal{P}_\Omega \equiv \{(u, J, z) \in \mathcal{P}_{d'} \times (\mathbb{CP}^1)^{3d'} / u(z_j) = p_{i_j}\} \tag{1.7}$$

Let $\mathcal{M}_d, \mathcal{M}_\frown, \mathcal{M}_\prec, \mathcal{M}_X, \mathcal{M}_\asymp$, and $\mathcal{M}_\Omega$ deontes the quotient of $\mathcal{P}_d, \mathcal{P}_\frown, \mathcal{P}_\prec, \mathcal{P}_\mathfrak{X}, \mathcal{P}_\asymp$, and $\mathcal{P}_\Omega$ by the natural action of $Aut(\mathbb{CP}^1)$.

Let $\pi_d : \mathcal{P}_d \to \mathcal{J}_\omega$, $\pi_\frown : \mathcal{P}_\frown \to \mathcal{J}_\omega$, $\pi_\prec : \mathcal{P}_\prec \to \mathcal{J}_\omega, \pi_\mathfrak{X} : \mathcal{P}_\mathfrak{X} \to \mathcal{J}_\omega$, and $\pi_\Omega : \mathcal{P}_\Omega \to \mathcal{J}_\omega$ be the corresponding projection respectively.

It is not difficult to prove that all these different $\mathcal{P}_c$ are all Banach submanifolds in $\mathcal{U}_d \times \mathcal{J}_\omega$. However, the reason why the projections π_c are local submersions (or proper maps) are not so straightforward and need some efforts to explain.

1.4 Our Generalization

In this paper, we will use Barraud's approach to generalize the result a little further to the symplectic spheres in some other different compact symplectic 4-manifolds. The main theorem is:

Theorem 4 (Main Theorem 1) Let (M, ω) be a rational symplectic 4-manifold and $A \in H_2(M, \mathbb{Z})$ be a homology class with $K_\omega(A) < 0$, J_0, J_1 be two ω-tamed almost complex structures on (M, ω), let u_0 be a J_0-holomorphic sphere, then there exist a path $(u_t, J_t)_{t \in [0,1]}$ where u_t is a nodal J_t-holomorphic spheres.

Here, **rational** means that M can only be $\mathbb{CP}^2$, $S^2 \times S^2$, or blow-ups of them. K_ω is the symplectic cannonical class. **nodal** means immersed submanifold with only positive ordinary double point singularities, J-holomorphic **sphere** means the domain of the J-holomorphic curve is $\mathbb{CP}^1$.

The rational symplectic manifold case is a direct generalization of Barraud's case. We can imitate Barraud's approach and study the isotopies among nodal symplectic spheres in (M, ω) by studying the isotopies among nodal J-holomorphic spheres in (M, ω).

Since every nodal J-holomorphic sphere is always a nodal symplectic sphere in the rational symplectic 4-manifold M and the reverse is also true. The isotopy problem among nodal symplectic spheres is equivalent to a isotopy problem among nodal J-holomorphic spheres. Hence we will naturally have the following result:

Theorem 5 (Main Theorem 2) Let (M, ω) be a rational symplectic 4-manifold, and $A \in H_2(M, \mathbb{Z})$ be a homology class with $K_\omega(A) < 0$, then $\mathcal{S}_A^0$, the space of the nodal symplectic spheres in class A has only finitely many isotopy classes.

This is not a trivial statement. In a 4-dimensional symplectic manifold M, if we consider all symplectic submanifolds in the same homology class $A \in H_2(M, \mathbb{Z})$, we don't always know that there are only finitely many symplectic isotopy classes in the homology class A.

In fact, there are many known examples with infinitely symplectic isotopy classes in a homology class. The followings are some examples and the list goes on.

Theorem 6 (B.D. Park, M. Poddar, S. Vidussi [22]) For any fixed integer $q \geq 1$, there exist simply connected, simplectic 4-manifolds X containing infinitely homologous, pairwise non-isotopic, symplectic surfaces $\{\Xi_{p,q} \mid g.c.d(p,q) = 1\}$ of genus $q + 1$.

Theorem 7 (R. Fintushel, R Stern [23][24]) Let T be a c-embedded symplectic torus in a simply-connected 4-manifold X, then for each $q \geq 2$ there exists an infinite family of mutually non-isotopic symplectic tori representing the homology class $2q[T]$.

Theorem 8 (T. Etgü, B.D. Park [25]) Let T be an essentially embedded symplectic 2-torus in a symplectic 4-manifold X with $b_2^+(X) > 1$. IF $[T] \in H_2(X, \mathbb{Z})$ is primitive, $[T]^2 = 0$, and $H^1(X - vT, \mathbb{Z}) = 0$, then for any $q \geq 3$, there exists an infinite family of mutually non-isotopic tori representing the same homology class $q[T]$.

Theorem 9 (D. Auroux, S.K. Donald, L. Katzarkov [26]) In $\mathbb{CP}^2$, for infinitely many positive integers m, one can choose a pair of integers (ρ_m, d_m) and an infinite of pairwise non-isotopic symplectic curves $S_{m,k} \subset \mathbb{CP}^2$, all of which have degree m, ρ_m cusps and d_m nodes.

Chapter 2

Preliminary Knowledge and Basic Tools

In this chapter, we will introduce some preliminary knowledge and basic tools.

2.1 Symplectic Manifold

Definition 2 (symplectic manifold)

Let M be a smooth manifold of dimension 2n, we say that M is a **symplectic manifold** if M is equipped with a closed nondegenerate differential 2-form ω. We usually denote the symplectic manifold as (M, ω). That is, the smooth manifold M, together with its symplectic structure ω.

The word **closed** here means that $d\omega = 0$. The word **nondegenerate** here means that at every point $p \in M$, if there is a vector $v_p \in T_pM$ has the property that $\omega(v_p, w_p) \equiv 0$ for all $w_p \in T_pM$, then we will have the conclusion that $v_p = 0$.

A direct consequence of the definition is that the n-th exterior power ω^n of the sympelectic form of the 2n-dimensional smooth manifold M will always be a volume form of M. That is, ω^n will be a non-vanishing top form on M. In fact, Darboux [12] tells us that locally any symplectic form on any symplectic manifold of dimension 2n will always be diffeomorphic to the standard symplectic form $\omega_0 = dx_1 \wedge dy_1 + ... + dx_n \wedge dy_n$ on the Euclidean space $\mathbb{R}^{2n}$.

Theorem 10 (Darboux) *Let (M, ω) be a 2n-dimensional symplectic manifold, and $p \in M$, then there is a local coordinate $(\mathcal{U}, x_1, ..., x_n, y_1, ...y_n)$ of M containing a neighborhood $\mathcal{U}$ of the point p such that on $\mathcal{U}$,*

$$\omega = dx_1 \wedge dy_1 + ... + dx_n \wedge dy_n.$$

In this paper, we want to investigate a special kind of maps from a symplectic manifold to another. We are interested in the **J-holomorphic** maps from a symplectic manifold S to another symplectic manifold M. In this paper, we fix the manifold S to be the $\mathbb{CP}^1$. There is a natural Kähler structure on $\mathbb{CP}^1$. A Kähler form is always a symplectic form. Therefore, we can fix the symplectic structure of $S = \mathbb{CP}^1$ to be the standard Kähler form on $\mathbb{CP}^1$ and consider it as a symplectic manifold. Furthermore, a Kähler manifold is also a complex manifold, we also fix the almost complex stucture J_S on $S = \mathbb{CP}^1$ to be the one from the Kähler structure. Since the source manifold S is always $\mathbb{CP}^1$, we call the J-holomorphic curves $u : S \to M$ J-holomorphic **spheres**.

On the other hand, we want (M, ω) to be a 4-dimensional symplectic manifold. We can always define many different almost complex structures on a given symplectic manifold (M, ω). For example, some might be interested in the **ω-compatible** almost complex structures on M. However, in this paper, we are interested in the **ω-tamed** almost complex structures on M.

Definition 3 **(ω-tamed almost complex structure)**

*Let (M,ω) be a symplectic manifold. An almost complex structure $J : TM \to TM$ on M is called ω-**tamed** if $\forall v_p \in T_pM,\ v_p \neq 0$, we always have:*

$$\omega(v_p, Jv_p) > 0$$

Definition 4 **(ω-compatible almost complex structure)**

*Let (M,ω) be a symplectic manifold. A ω-tamed almost complex structure $J : TM \to TM$ on M is called ω-**compatible** if $\forall p \in M,\ \forall v_p, w_p \in T_pM$, we always have :*

$$\omega(Jv_p, Jw_p) = \omega(v_p, w_p)$$

For every ω-tamed almost complex structure J on M, we can define a Riemannian metric $g_J(.,.)$ from J by :

$$g_J(v,w) \equiv \frac{1}{2}\{\omega(v,Jw) + \omega(w,Jv)\}$$

Notice here, the ω-tamed condition of ω gives the positive-definiteness of the Riemannian metric $g_J(.,.)$.

If J is not only ω-tamed, but also ω-compatible, then the $g_J(.,.)$ defined above will have:

$$\begin{aligned} g_J(v,w) &\equiv \frac{1}{2}\{\omega(v,Jw) + \omega(w,Jv)\} = \frac{1}{2}\{\omega(v,Jw) + \omega(Jw,J^2v)\} \\ &= \frac{1}{2}\{\omega(v,Jw) + \omega(Jw,-v)\} = \frac{1}{2}\{\omega(v,Jw) + \omega(v,Jw)\} \\ &= \omega(v,Jw) \end{aligned}$$

For the ω-compatible almost complex structure case, the symplectic form ω, the almost complex structure J, and the induced Riemmanian metric $g_J(.,.)$ is a **compatible triple**, .

2.2 Blow Up

In this section, we will explain the meaning of ”blowing up at m distinct points in $\mathbb{CP}^2$, or sometimes written as $\mathbb{CP}^2\#m\overline{\mathbb{CP}^2}$. In mathematics, we can consider ”blow up” submanifolds of a manifold. However, in this paper, we only need finite points blow-ups of $\mathbb{CP}^2$ and $\mathbb{CP}^1 \times \mathbb{CP}^1$.

2.2.1 one point blow up in $\mathbb{C}^2$

Before $\mathbb{CP}^2\#m\overline{\mathbb{CP}^2}$, we can first see the simplest case of blow-up : blow up at one point in the complex plane $\mathbb{C}^2$. There are several approaches to understand this geometric transformation. One way to think about this is : first we begin with a $\mathbb{C}^2$, choose an arbitrarily point p on the plane, remove the point p from the plane and then somehow ”gluing back” a **exceptional curve** $\mathbb{CP}^1$ back to the punctured plane $\mathbb{C}^2 \setminus p$. Roughly speaking, We can think this as replacing the point p by all the lines on $\mathbb{C}$ passing though the point p.

If we let $p = (0,0) \in \mathbb{C}^2$, we can denote the result of this blow-up as $\mathbf{X}_p = \{(q,l) \mid p,q \in l, l \in Gr(1,2)\}$. Here $Gr(1,2)$ is the Grassmannian : all the lines passing though $p = (0,0)$. Apparently, $Gr(1,2)$ can be written as $\mathbb{CP}^1$ by definition. Note that when $q \neq p$, l can only be the unique complex line passing though both p and q. However, if $q = p$, l can be any line passing through p. The collection of all these lines passing through p is a $Gr(1,2)$, or the **exceptional curve** $\mathbb{CP}^1$. $\mathbf{X}_p$ is a subset of $\mathbb{C} \times \mathbb{CP}^1$.

2.2.2 one point blow up in $\mathbb{CP}^2$

Similarly, we can also blow up at the point $p = [0:0:1]$ in $\mathbb{CP}^2$. Since $\mathbb{CP}^2$ can be viewed as the projective completion of the affine plane $\mathbb{C}^2$ by adding the ”line at infinity” L to $\mathbb{C}^2$, and blowing up is a local surgery at the point $p = (0,0)$, there is in fact nothing really different from the previous case. The result of this blowing up can be written as $\mathbf{Y}_p = \{(q, l) \mid p, q \in l\} \subseteq \mathbb{CP}^2 \times \mathbb{CP}^1$. It is trivial that $\mathbf{Y}_p$ is a projective variety since it is a closed subvariety of the product of two projective varieties.

Traditionally, we sometimes call $\mathbf{Y}_p$ as "$\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ ". Notice that when $q \neq p$, l can only be the line passing through p and q. However, if $q = p$, l can be any line passing through p. The collection of all these lines passing through p is also called the **exceptional curve** in $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$

2.2.3 topological description of the blow-up

The historically reason of the name $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ might come from another point of view of the object $\mathbf{Y}_p$, usually known as "topological description" of the blow-up:

Step 1 : We can begin with two identical copies of $\mathbb{CP}^2$, both of them are the projective completion of a $\mathbb{C}^2$ by adding a line at infinity.

Step 2 : Cut out the interior of unit ball $\{(x, y) \in \mathbb{C}^2 \mid \|x\|^2 + \|y\|^2 < 1\}$ from both copies of $\mathbb{CP}^2$.

Step 3 : Now, we have two identical copies of $\mathbb{CP}^2 \setminus \{(x, y) \in \mathbb{C}^2 \mid \|x\|^2 + \|y\|^2 < 1\}$. There are two ways to glue these two copies of $\mathbb{CP}^2 \setminus \{(x, y) \in \mathbb{C}^2 \mid \|x\|^2 + \|y\|^2 < 1\}$ together.

Step 4: If we choose to glue them on the boundaries $\{(x, y) \in \mathbb{C}^2 \mid \|x\|^2 + \|y\|^2 = 1\}$ by using the antipodal map on the boundaries instead of the identical map on the boundaries, then the these two copies of $\mathbb{CP}^2 \setminus \{(x, y) \in \mathbb{C}^2 \mid \|x\|^2 + \|y\|^2 < 1\}$ will have opposite orientations.

Remark : It can be proved that the result of this "gluing" is a manifold holomorphic to the the $\mathbf{Y}_p$ from subsection 2.2.2. The notation $\mathbb{CP}^2\#\overline{\mathbb{CP}^2}$ tells us that this is a connected sum, and the two copies of $\mathbb{CP}^2 \setminus \{(x,y) \in \mathbb{C}^2 \mid \|x\|^2 + \|y\|^2 < 1\}$ have opposite orientations. Notice that, the entire manifold $\mathbb{CP}^2\#\overline{\mathbb{CP}^2}$ is still an orientable manifold. One of the two $\mathbb{CP}^2 \setminus \{(x,y) \in \mathbb{C}^2 \mid \|x\|^2 + \|y\|^2 < 1\}$ needs to change the orientation after the gluing.

2.2.4 multiple points blow up

Finally, We can repeat this one point blow-up operation at m distinct points in $\mathbb{CP}^2$ one by one, and the final result after these m "one point blow-up" is called $\mathbb{CP}^2\#m\overline{\mathbb{CP}^2}$. After we do m "one point blow-up" at m distinct points, we will have m different **exceptional curves** in $\mathbb{CP}^2\#m\overline{\mathbb{CP}^2}$ corresponding to the lines passing through these m points of blow-up.

Furthermore, we can also blow up at m distinct points in other almost complex manifolds such as $\mathbb{CP}^1 \times \mathbb{CP}^1$ in exactly the same way.

The one point blow-up can be performed not only in the topological manifold category but also in the symplectic manifold category. That is, if we have a symplectic manifold to begin with, there is a way to adjust the symplectic form to ensure the result of the blow-up is still a symplectic manifold, and the symplectic form will be the same before and after the blow-up everywhere except a neighborhood U_p of the point of blow-up p. If we have a symplectic manifold M, instead of cutting of one point p , we need to cut off an open neighborhood U_p of p to begin with (, and of course we need to glue the open neighborhood and the exceptional curves back carefully later). This is because we can not arbitrarily extend the original symplectic form ω to the exceptional curve, we need to at least adjust the symplectic form in a small open neighborhood U_p of p, so that the result of the blow-up $\tilde{M}$ will still be a symplectic manifold, and the symplectic form of M and $\tilde{M}$ agree outside U_p. This kind of blow-up are sometimes called **sympletic blow-up**. See [16] for more details.

2.3 Adjunction Formula

In this section, we want to introduce a famous formula about the genus and degree of immersed closed Riemann surfaces (real 2-dimensional) in 2 dimensional almost complex manifolds (real 4-dimensional), known as ”adjunction formula”, or ”adjunction inequality”.

2.3.1 Adjunction Formula and Genus-Degree Formula

Theorem 11 (Adjunction Formula) *Let (M, J) be a 4-(real)-dimensional manifold with an almost complex structure J, and (S, J_s) be a compact Riemann surface (not necessarily connected). and $u : (S, J_s) \to (M, J)$ is a simple J-holomorphic map. Then, we have the adjunction formula:*

$$2\delta(u) \leqslant A \centerdot A - c_1(M) \centerdot A + \chi(S) \tag{2.1}$$

Here $\delta(u)$ is the number of self-intersection of u, $A \in H_2(M, \mathbb{Z})$ is the homology class representing the map u (, that is, $A = [u(S)]$), $c_1(M)$ is the first Chern class of M, and $\chi(S)$ is the Euler number of S. $\chi(S) = 2 - 2g(S)$ where $g(S)$ is the genus of S. The equality holds if all the self-intersections are transveral.

We say a J-holomorphic curve from the compact Riemann surface (S, J) is **simple** if it is not multiply covered. A J-holomorphic curve $u : S \to M$ from compact Riemann surface (S, J_S) is **multiple covered** if there exist a compact Riemann surface $(S', J'_{S'})$, a J-holomorphic curve $u' : S' \to M$, and a holomorhpic branched covering $\phi : S \to S'$ such that $u = u' \circ \phi$, and $deg(\phi) > 1$.

As a special case of the this theorem, if we let M to be $\mathbb{CP}^2$ with the standard Kähler structure J, and (S, J_s) be a connected Riemann surface of genus g. We will have $\chi(S) = 2 - 2g$. If the J-holomorphic map u is of degree d, we have $A = dH$, where $H \in H_2(M, \mathbb{Z})$ is the standard generator in $H_2(M, \mathbb{Z})$ representing a line $\mathbb{CP}^1$ in $M = \mathbb{CP}^2$. Moreover, We know that $c_1(\mathbb{CP}^2) \cdot dH = 3H \cdot dH = 3d$. This gives us the well-known **genus-degree formula** :

Theorem 12 (Genus-Degree Formula) *Let J be the standard Kähler structure on $\mathbb{CP}^2$ and (S, J_s) be a connected Riemann surface. Let $u : (S, J_s) \to (\mathbb{CP}^2, J)$ be a J-holomorphic curve of degree d, then, we have:*

$$\delta(u) + g(S) \leqslant \frac{(d-1)(d-2)}{2} \tag{2.2}$$

The equality holds if all self-intersections are transversal

Remark : A direct consequence of the genus-degree formula is if $g(S) = 0$ and u is an embedding map (hence $\delta(u) = 0$ and the equation holds), then u can only be of degree 1 or degree 2. In other words, an embedded $\mathbb{CP}^1$ in $\mathbb{CP}^2$ can only be of degree 1 or degree 2.

Remark : If we let u be an embedding map from connected Riemann surface S to $\mathbb{CP}^2$ (hence $\delta(u) = 0$ and the equation holds), and $g(S) \neq 0$. then we have a quadratic equation of d :

$$d^2 - 3d + (2 - 2g(S)) = 0$$

There will be at most two integer solutions of d for a given $g(S)$.

Remark : Another important remark here is: for given $g(S)$ and d, if we let all self-intersections of u to be transversal, and the self-intersections are the only singularities of the immersed curve u, then we can see from the equation in the genus-degree formula, the number of self-intersection is a fixed number determined by d and $g(S)$:

$$\delta(u) = \frac{(d-1)(d-2)}{2} - g(S).$$

2.3.2 Some Examples

Example : Disconnected Case

In the statement of the adjunction formula, we have mentioned that S is not necessarily connected.

A easy disconnected example can be found from [9]: Let S be an union of d disjoint $\mathbb{CP}^1$, and u send them into m distinct embedded degree 1 $\mathbb{CP}^1$ in $\mathbb{CP}^2$. Since every 2 different $\mathbb{CP}^1$ in $\mathbb{CP}^2$ intersects exactly once, we have $\delta(u) = \frac{d(d-1)}{2}$. We also know that $A = dH$, $c_1(\mathbb{CP}^2) \cdot A = 3d$, and $\chi(S) = 2d$. Therefore, we still have the equation.

$$2\delta(u) = d(d-1) = d^2 - 3d + (2d) = A \cdot A - c_1(M) \cdot A + \chi(S)$$

Example : $\mathbb{CP}^1 \times \mathbb{CP}^1$

We can also change M to other 4-dimensional manifolds with almost complex structures to find similar results.

For example, let $(M, \omega) = (\mathbb{CP}^1 \times \mathbb{CP}^1, \omega_{FS} \oplus \omega_{FS})$ where ω_{FS} is the standard Fubini-Studi metric on $\mathbb{CP}^1$. Let J be any ω-tame almost complex structure on $M = \mathbb{CP}^1 \times \mathbb{CP}^1$

. Let H_a and H_b be the standard generators of $H_2(M,\mathbb{Z})$. Let S be a connected Riemann surface.

Now, write $A = d_aH_a+d_bH_b$, with $d_a, d_b \in \mathbb{Z}$, we will have $A\centerdot A = 2d_ad_b$, $c_1(M)\centerdot A = 2(d_1 + d_2)$, and $\chi(S) = 2 - 2g(S)$. Therefore, we have the inequality:

$$(\delta(u) + g(S)) \leqslant d_1d_2 - (d_1 + d_2) + 1 = (d_1 - 1)(d_2 - 1)$$

If $u : (S, J_s) \to (\mathbb{CP}^1 \times \mathbb{CP}^1, J)$ is an embedding map, then we have $\delta(u) = 0$, and :

$$g(S) = (d_1 - 1)(d_2 - 1)$$

If $g(S) = 0$, then we have $d_1 = 1$ or $d_2 = 1$.

Example:"blow-up at m points"

We can also let M to be the "blow up at m points" manifolds we mentioned in the last section. We will still have the adjunction formula. For example, if we let M to be $\mathbb{CP}^2\#\overline{\mathbb{CP}^2}$, $S = \mathbb{CP}^1$, and let $u : S \to M$ to be an embedding J-holomorphic map. We will have the equation:

$$2g(S) = A \centerdot A - c_1(M) \centerdot A + 2 \tag{2.3}$$

The homology group $H_2(M,\mathbb{Z})$ can be generated by H and e, where $e \in H_2(M,\mathbb{Z})$ represents the special **exceptional curve** $\mathbb{CP}^1$ in $\mathbb{CP}^2\#\overline{\mathbb{CP}^2}$, and H represents a line on $\mathbb{CP}^2\#\overline{\mathbb{CP}^2}$ that doesn't pass through the blow up point p.

Here, the first Chern class of $M = \mathbb{CP}^2\#\overline{\mathbb{CP}^2}$ is $3H - e$. $c_1(M) \centerdot A = (3H - e) \centerdot A$. If $A = d_0H+d_1e$, then $c_1(M)\centerdot A = 3d_0+d_1$ since $H\centerdot H = 1$, $H\centerdot e = e\centerdot H = 0$, and $e\centerdot e = -1$

(the negative sign comes from the opposite orientation of $\overline{\mathbb{CP}^2}$). $A \centerdot A = d_0^2 - d_1^2$. We can see that the embedding spheres in $M = \mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ can only be in a class $A = d_0 H + d_1 e$ with :

$$d_0^2 - d_1^2 - 3d_0 - d_1 + 2 = 0$$

We can see from here that the choice of the class A is in fact rather limited. Also, in a class A, if we let u to be immersed spheres, with all the self-intersections of u to be transversal and the self-intersections are the only singularities, then the number of self-intersections of u is a fixed number decided by $g(S) = 0$, and $A = d_0 H + d_1 e$.

Similarly, the first Chern class of $\mathbb{CP}^2 \# m\overline{\mathbb{CP}^2}$ is $3H - e_1 - e_2 ...... - e_m$ where e_i represents the ith exceptional curve and H represents a line on $\mathbb{CP}^2 \# m\overline{\mathbb{CP}^2}$ that doesn't pass through the blow up points. If $A = d_0 H + d_1 e_1 + d_2 e_2 ... + d_m e_m$, then $c_1(M) \centerdot A = 3d_0 + d_1 + d_2 + ... + d_m$. $A \centerdot A = d_0^2 - d_1^2 - d_2^2 - ... - e_m^2$.

The first chern class of $\mathbb{CP}^1 \times \mathbb{CP}^1$ is $2H_a + 2H_b$, where H_a and H_b are the standard generators in $H_2(\mathbb{CP}^1 \times \mathbb{CP}^1, \mathbb{Z})$. If $A = d_a H_a + d_b H_b$, then $c_1(\mathbb{CP}^1 \times \mathbb{CP}^1) \centerdot A = 2d_a + 2d_b$. $A \centerdot A = 2d_a d_b$ since $H_a \centerdot H_a = 0$, $H_b \centerdot H_b = 0$, $H_a \centerdot H_b = H_b \centerdot H_a = 1$.

The first chern class of $(\mathbb{CP}^1 \times \mathbb{CP}^1) \# m\overline{\mathbb{CP}^2}$ is $2H_a + 2H_b - e_1 - e_2 - ... - e_n$ where e_i represents the ith exceptional curve. If $A = d_a H_a + d_b H_b + d_1 e_1 + d_2 e_2 + ... + d_m e_m$, then $c_1((\mathbb{CP}^1 \times \mathbb{CP}^1) \# m\overline{\mathbb{CP}^2}) \centerdot A = 2d_a + 2d_b + d_1 + d_2 + ... + d_m$, $A \centerdot A = 2d_a d_b - d_1^2 - d_2^2 - ... - d_m^2$.

For the $M = \mathbb{CP}^2 \# m\overline{\mathbb{CP}^2}$, $M = \mathbb{CP}^1 \times \mathbb{CP}^1$, and $M = (\mathbb{CP}^1 \times \mathbb{CP}^1) \# m\overline{\mathbb{CP}^2}$ cases, we can all have similar discussions just like the $M = \mathbb{CP}^2$, or $M = \mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ cases, and see that the choice of the topological class A for the embedded spheres S in M is in fact rather limited. Moreover, if we let u to be an immersed sphere with all the self-intersections transversal and the self-intersections are the only singularities, then $\delta(u)$, the number of the self-intersection points in u, is in fact a fixed number decided by the class $A = [u(S)] \in H_2(M, \mathbb{Z})$

For the calculation of first Chern classes and intersection numbers, one can easily pick up fundamental knowledge from textbooks like [17][18].

2.4 Banach manifolds and Transversality

Definition 5 *A vector space V is a **Banach space** if it is a complete vector space with a norm $\| \cdot \|$.*

Definition 6 *A manifold M is a **Banach manifold** if it is modeled on Banach spaces. In other words, as a topological space, for each point $p \in M$, there exist an neighborhood U_p homeomorphic to an open set in a Banach space.*

We should first consider about $\mathcal{J}^l$, that is, the space of all C^l almost complex structures J on the manifold M.

$$\mathcal{J}^l \equiv \{J \mid J \in C^l(M, End(TM)); J^2 = -Id \in End(TM)\}$$

When we say J is an C^l almost complex structure in M, we mean: $(i) J \in C^l(M, End(TM))$ and $(ii) J^2 = -Id \in End(TM)$. It is easy to see that $\mathcal{J}^l$ is a Banach manifold from the definition of $\mathcal{J}^l$

Since $\mathcal{J}^l$ is a Banach manifold, we should consider the tangent vectors at a "point" $J \in \mathcal{J}$. By taking derivative to the equation $J^2 = -Id$, we will have $\dot{J}J + J\dot{J} = 0$, and hence :

$$T_J\mathcal{J}^l = \{I \mid I \in C^l(M, End(TM)); IJ + JI = 0\} = C^l(M, \Lambda^{0,1}M \otimes TM)$$

Here, I is (0,1)-form because the fact $IJ = -JI$ gives us $I(Jv) = -JI(v)$ for all vector $v = v_p \in TM$.

For a topological class $A \in H_2(M, \mathbb{Z})$, we can know that $\mathcal{U}_A^{k+1,p} \equiv \{u \mid u \in L^{k+1,p}(S, M); u(S) = A \in H_2(M, \mathbb{Z})\}$, the space of all Sobolev $W^{k+1,p}$ maps in class A is also a Banach manifold.

$\mathcal{U}_A^{k+1,p} \times \mathcal{J}^l$ is the product of two Banach manifolds, and therefore is also a Banach manifold.

Now, we can consider a vector bundle $\mathcal{F}$ over the Banach manifold $\mathcal{U}_A^{k+1} \times \mathcal{J}^l$, the fiber of the vector bundle $\mathcal{F}$ is $L^{k,p}(S, \Lambda^{0,1}S \otimes u^*(TM))$. The total space $\mathcal{F}$ of the vector bundle is also a Banach manifold.

On this Banach bundle $\mathcal{F}$ over $\mathcal{U}_A^{k+1,p} \times \mathcal{J}^l$, we can consider two different section. The first one is :

$$\Phi_A(u, J) \equiv du + J(u) \circ du \circ J_s$$

and the second section is the zero section $Ⓢ_A(u, J) \equiv 0$, which will automatically be identified with the base manifold $\mathcal{U}_A^{k+1,p} \times \mathcal{J}^l$.

It is easy to see that the intersection of these two sections is the subset $\mathcal{P}_A$ of $\mathcal{U}_A^{k+1,p} \times \mathcal{J}^l$ consisting all $L^{k+1,p}$-smooth, J -holomorphic curves with $J \in \mathcal{J}^l$:

$$\mathcal{P}_A \equiv \{(u, J) \mid (u, J) \in \mathcal{U}_A^{k+1,p} \times \mathcal{J}^l; du + J \circ du \circ J_S = 0\}$$

Definition 7 *Two submanifolds L_1 and L_2 of a manifold M are **transversal** to each other, or written as $L_1 \pitchfork L_2$, if $T_pM = T_pL_1 + T_pL_2$ $\forall p \in L_1 \cap L_2$.*

If we can know that these two sections Φ_A and $Ⓢ_A$ are transveral to each other, then we know their intersection $\mathcal{P}_A$ will also be a Banach manifold.

Let us look a little bit closer to $\mathcal{P}_A$, or to be more precise, consider a curve $\gamma(t) = (u_t, J(t))$ in $\mathcal{P}_A$ $\forall t$ with $\gamma(0) = (u_0, J(0)) = (u, J)$.

First, we will automatically have $du_t + J(u_t, t) \circ du_t \circ J_S \equiv 0$ for all t, since $\gamma(t) = (u_t, J(t))$ is a curve in $\mathcal{P}_A$.

Let ∇ be some symmetric connection on TM, If we take the covariant derivative of ∇ with respect to $\frac{\partial}{\partial t}$ on the equation $du_t + J(u_t, t) \circ du_t \circ J_S(t) \equiv 0$, we will get

$$\nabla_{\frac{\partial}{\partial t}}(du_t) + (\nabla_v J)(du_t \circ J_S) + J(u_t, t) \circ \nabla_{\frac{\partial}{\partial t}}(du_t) \circ J_s + \dot{J} \circ du_t \circ J_S = 0 \tag{2.4}$$

where $v = \frac{du_t}{dt}\restriction_{t=0}$, $\dot{J} = \frac{dJ(t)}{dt}\restriction_{t=0}$.

It is not difficult to see that $\nabla_{\frac{\partial}{\partial t}}(du_t)(\xi) = \nabla_\xi v$ $\forall \xi \in TS$, so the condition above will become

$$\nabla v + (\nabla_v J)(du_t \circ J_S) + J \circ \nabla v \circ J_s + \dot{J} \circ du_t \circ J_S = 0 \tag{2.5}$$

Notice that the first three terms of left hand side the equation $\nabla v + (\nabla_v J)(du_t \circ J_S) + J \circ \nabla v \circ J_s$ is exactly $2D_{u,J}(v) \in L^{k,p}(S, \Lambda^{0,1}S \otimes u^*(TM))$ where $D_{u,J}$ is the Gromov's operator. We will study this operator more in the next chapter.

The question whether Φ_A and Ⓢ$_A$ are transversal to each other or not is equivalent to solve a system of equation involving $D_{u,J}$, we will talk about this more in Chapter 4.

2.5 Fredholm map and Sard-Smale Theorem

Definition 8 (Fredholm operator)

A bounded linear transformation $T : X \to Y$ between two Banach spaces X, Y is a Fredholm operator if it has a closed image, finite dimensional kernel and finite dimesntional cokernel.

For a Fredholm operator $T : X \to Y$, we define the **index of** T to be

$$ind(T) \equiv dim(ker(T)) - dim(coker(T))$$

Definition 9 (Fredholm map)

A map (at least C^1) $f : M \to N$ between two Banach manifolds M, N is a Fredholm map if $df_{\restriction x} : T_xM \to T_xN$ is a Fredholm operator at every point $x \in M$.

A Fredholm map $f : M \to N$ is of **index k** if $df_{\restriction x} : T_xM \to T_xN$ is an index k Fredholm operator at every point $x \in M$.

For the problem we want to discuss in this paper, the projections $\widetilde{\pi} : \mathcal{P}_{A,\Omega} \to \mathcal{J}_\omega$, $\pi^{TRI}_{A,\Omega} : \mathcal{P}^{TRI}_{A,\Omega} \to \mathcal{J}_\omega$, $\pi^{\emptyset TP}_{A,\Omega} : \mathcal{P}^{\emptyset TP}_{A,\Omega} \to \mathcal{J}_\omega$, $\pi^{IDTP}_{A,\Omega} : \mathcal{P}^{IDTP}_{A,\Omega} \to \mathcal{J}_\omega$, and $\pi^{COMP}_{A_j,\Omega_j} : \mathcal{P}^{COMP}_{A_j,\Omega_j} \to \mathcal{J}_\omega$ are all Fredholm maps.

For a Fredholm map, we can apply the Sard-Smale's Theorem to it.

Theorem 13 (Sard-Smale's Theorem) *The regular value of an index k Fredholm map $f : M \to N$ is of second category in N provided the Fredholm map is at least C^{k+1}.*

2.6 Evaluation map τ and the operator $\tilde{D}$

Let $D_{u,J}^{N_0}$ be the projection of the Gromov's operator $D_{u,J}$ on the complex vector bundle $N_0 = E/TC = u^*(TM)/\overline{\operatorname{im} du}$.

Let $z_1, ...z_m$ be m distinct marked points on S. We can define the evaluation map τ at each z_i:

$$\mathcal{O}_{D_{u,J}^{N_0}}(N_0) \xrightarrow{\tau} \oplus_{i=1}^m (N_0)_{z_i} \to 0$$

For a section $\alpha \in \mathcal{O}_{D_{u,J}^{N_0}}(S, N_0)$ and the marked points $z_i \in S$, the evaluation map τ takes values at these z_i :

$$\tau(\alpha) = (\alpha_{\restriction z_1}, ..., \alpha_{\restriction z_m})$$

Let $P = \sum_{i=1}^m z_i$, and $\widetilde{N_0} \equiv N_0 \otimes P$ be the complex vector bundle twisted by the divisor P. By lemma 4 in [2], there exist an unique operator $\widetilde{D_{N_0}}$ on $\widetilde{N_0}$ making the following diagram commutative:

$$\begin{array}{ccc} L^{k+1,p}(S, \widetilde{N_0}) & \xrightarrow{\imath} & L^{k+1,p}(S, N_0) \\ \downarrow^{\widetilde{D_{N_0}}} & & \downarrow^{D_{u,J}^{N_0}} \\ L^{k,p}(S, \Lambda^{0,1}S \otimes \widetilde{N_0}) & \to & L^{k,p}(S, \Lambda^{0,1}S \otimes N_0) \end{array} \tag{2.6}$$

It gives us the short exact sequence :

$$0 \to \mathcal{O}_{\widetilde{D_{N_0}}}(S, \widetilde{N_0}) \to \mathcal{O}_{D_{u,J}^{N_0}}(S, N_0) \xrightarrow{\tau} \oplus_{i=1}^m (N_0)_{z_i} \to 0. \tag{2.7}$$

And then the corresponding long exact sequence :

$$0 \to ker\widetilde{D_{N_0}} \to kerD_{u,J}^{N_0} \xrightarrow{\tau} \oplus_{i=1}^m (N_0)_{z_i} \to coker\widetilde{D_{N_0}} \to cokerD_{u,J}^{N_0} \to 0. \tag{2.8}$$

2.7 Automatic genericity

Let S be a Riemann surface of genus g, and L be a complex line bundle over S, let D be a differential operator on L of the form $\overline{\partial} + R$, then by Riemann-Roch Theorem, we know that the index of D, $ind_{\mathbb{R}} D = 2(c_1(L) + 1 - g)$.

Theorem 14 (H. Hofer,V.Lizan) *[11]*

(1) If $c_1(L) < 0$*, then* D *is injective.*

(2) If $c_1(L) \geq 2g - 1$*, the* D *is surjective.*

In the problem we want to discuss in this paper, $S = \mathbb{CP}^1$ so $g = g(S) = 0$. We let L to be the complex line bundle $\widetilde{N_0} = N_0 \otimes P$ on S, and let $\widetilde{D_{N_0}}$ be the operator D on the line bundle $\widetilde{N_0}$.

We can show that $c_1(\widetilde{N_0}) = -1$, therefore the operator $D = \widetilde{D_{N_0}}$ is in fact both injective and surjective.

Chapter 3

Gromov's operator

In this chapter, we will introduce the famous **Gromov's operator** $D_{u,J}$ and give some basic properties. These are all well-known results. In this chapter, we will integrate the notations and logics from [2],[6] and [9], with more details in the calculations and proofs. It is also easy for the readers to find equivalent or similar results from other papers.

We let S always be the Kähler manifold $\mathbb{CP}^1$, and J_S always be the standard complex structure on S. Since the operator is associated with the J-holomorphic curve $u : (S, J_S) \to (M, J)$ and the almost complex structure J on M, we sometimes denote it as $D_{u,J}$.

3.1 Gromov's operator $D_{u,J}$

We can begin with the definition of the operator.

Definition 10 (Gromov's operator $D_{u,J}$)

Let J be an almost complex structure on M, $u : S \to M$ be a J-holomorphic curve in

M, $v \in L^{k+1,p}(S, u^*(TM))$ be a section in $u^*(TM)$, and ∇ be a symmetric connection on TM. We can define $D_{u,J}$ as an operator from $L^{k+1,p}(S, u^*(TM))$ to $L^{k,p}(S, \Lambda^{0,1}S \otimes u^*(TM))$ as:

$$D_{u,J}(v) = \frac{1}{2}(\nabla v + J \circ \nabla v \circ J_s + (\nabla_v J) \circ (du \circ J_s)) \tag{3.1}$$

Here, $L^{k+1,p}(S, M)$ denotes the Sobolev space consists of all the continuous maps $u : S \to M$ which are represented by $W^{k+1,p}$ functions when writing in local coordinates on S and M, and $L^{k,p}(S, \Lambda^{0,1}S \otimes u^*(TM))$ denotes the collection of all $W^{k,p}$ $u^*(TM)$-valued $(0,1)$-forms on S.

Notice that the the choice of the symmetric connection ∇ in the definition is not unique. Therefore, we need to show that $D_{u,J}$ is well-defined. In other words, we need to show that $D_{u,J}$ is in fact independent of the choice of the symmetric connection ∇.

Property 1 *$D_{u,J}$ is independent of the choice of the symmetric connection ∇ on M.*

Proof : By the definition of $D_{u,J}$, for any $\xi \in L^{k+1,p}(S, TS)$, we can write :

$$(D_{u,J}(v))[\xi] = \frac{1}{2}\{\nabla_\xi v + J(\nabla_{J_s\xi} v) + (\nabla_v J) du(J_s \xi)\} \tag{3.2}$$

Let $\widetilde{\nabla}$ be another symmetric connection on TM. Let $\widetilde{D}_{u,J}$ be the operator defined by $\widetilde{\nabla}$. We can prove that $D_{u,J}$ and $\widetilde{D}_{u,J}$ are identical by showing the difference between $D_{u,J}$ and $\widetilde{D}_{u,J}$ is always zero. In other words, for any $v \in L^{k+1,p}(S, u^*(TM))$, and any $\xi \in L^{k+1,p}(S, TS)$, we need to show that:

$$D_{u,J}(v)[\xi] - D_{u,J}(v)[\xi] = 0$$

Let Y, Z be two $W^{k+1,p}$ sections on TM. we can define $Q(Y, Z) = \nabla_Y Z - \widetilde{\nabla}_Y Z$. From Lemma 1 after this property, we know that $Q(Y, Z)$ is a bilinear tensor and $Q(Y, Z) =$

$Q(Z,Y)$. From lemma 2, we know that $(\nabla_Y J)Z - (\widetilde{\nabla}_Y J)Z = Q(Y, JZ) - JQ(Y,Z)$.

From these facts, we can have

$$\begin{aligned}
&2(D_{u,J}v)[\xi] - 2(\widetilde{D}_{u,J}v)[\xi] \\
=&\nabla_\xi v - \widetilde{\nabla}_\xi v + J(\nabla_{J_s\xi}v) - J(\widetilde{\nabla}_{J_s\xi}v) + (\nabla_v J - \widetilde{\nabla}_v J)du(J_s\xi) \\
=&\underline{\nabla_{du(\xi)}v - \widetilde{\nabla}_{du(\xi)}v} + \underline{J(\nabla_{du(J_s\xi)}v) - J(\widetilde{\nabla}_{du(J_s\xi)}v)} + \underline{(\nabla_v J - \widetilde{\nabla}_v J)du(J_s\xi)} \\
=&Q(du(\xi), v) + JQ(du(J_s\xi), v) + Q(v, Jdu(J_s\xi)) - JQ(v, du(J_s\xi)) \\
=&Q(du(\xi), v) + Q(v, Jdu(J_s\xi)) = Q(du(\xi), v) + Q(Jdu(J_s\xi), v) \\
=&Q(du(\xi) + Jdu(J_s\xi), v) = Q((du + Jdu(J_s)\xi), v) = 0 \ \blacksquare
\end{aligned}$$

Remark : Since $D_{u,J}$ is independent of the choice of the symmetric connection ∇, we can choose ∇ to be the ones compatible with J_s to simplify the calculations.

Now, we need only to prove lemma 1 and lemma 2, then we have property 1 :

Lemma 1 *$Q(Y,Z)$ is a bilinear tensor, and $Q(Y,Z) = Q(Z,Y)$*

Proof : Both ∇ and $\widetilde{\nabla}$ are both symmetric connections, we have $\nabla_Y Z - \nabla_Z Y = [Y,Z]$ and $\widetilde{\nabla}_Y Z - \widetilde{\nabla}_Z Y = [Y,Z]$. This us gives $Q(Y,Z) - Q(Z,Y) = [Y,Z] - [Y,Z] = 0$.

$Q(Y,Z)$ is obviously bilinear since all the connections are linear on both arguments. Let f be any real-valued function on S. $Q(Y,Z)$ is a tensor since $Q(fY,Z) = \nabla_{fY}Z - \widetilde{\nabla}_{fY}Z = f\nabla_Y Z - f\widetilde{\nabla}_Y Z = fQ(Y,Z)$ and $Q(Y,fZ) = Q(fZ,Y) = fQ(Z,Y) = fQ(Y,Z)$. $\blacksquare$

Lemma 2 $(\nabla_Z J - \widetilde{\nabla}_Z J)Y = Q(Z, JY) - JQ(Z, Y)$.

Proof : We know that $\nabla_Z(JY) = (\nabla_Z J)Y + J(\nabla_Z Y)$ and $\widetilde{\nabla}_Z(JY) = (\widetilde{\nabla}_Z J)Y + J(\widetilde{\nabla}_Z Y)$. We have $(\nabla_Z J)Y = \nabla_Z(JY) - J(\nabla_Z Y)$ and $(\widetilde{\nabla}_Z J)Y = \widetilde{\nabla}_Z(JY) - J(\widetilde{\nabla}_Z Y)$. Therefore, $(\nabla_Z J - \widetilde{\nabla}_Z J)Y = \nabla_Z(JY) - \widetilde{\nabla}_Z(JY) - J(\nabla_Z Y) + J(\widetilde{\nabla}_Z Y) = Q(Z, JY) - JQ(Z, Y)$. ■

It is obvious that $D_{u,J}(v)[\xi]$ is $\mathbb{R}$**-linear** on both arguments (v and ξ) since all the connections involved here are $\mathbb{R}$-linear on both argument,and both J and J_s are also $\mathbb{R}$-linear.

Property 2 *$D_{u,J}$ is J_s-antilinear on the second argument, that is, $D_{u,J}(v)[J_s\xi] = -J(D_{u,J}(v)[\xi])$ $\forall \xi \in L^{k+1,p}(S, TS)$, $\forall v \in L^{k+1,p}(S, u^*(TM))$.*

Proof : We can easily see this by brute force calculations:

$$\begin{aligned}
2D_{u,J}(v)[J_s\xi] &= \nabla_{J_s\xi} v + J(\nabla_{J_sJ_s\xi} v) + (\nabla_v J)du(J_sJ_s\xi) \\
&= \nabla_{J_s\xi} v - J(\nabla_\xi v) - (\nabla_v J)du(\xi) \\
&= -J(J(\nabla_{J_s\xi} v) + (\nabla_\xi v) - J((\nabla_v J)(du(\xi))) \\
&= -J(J(\nabla_{J_s\xi} v) + (\nabla_\xi v) + (\nabla_v J)(J(du(\xi))) \\
&= -J((\nabla_\xi v) + J(\nabla_{J_s\xi} v) + (\nabla_v J)(J(du(\xi))) \\
&= -J((\nabla_\xi v) + J(\nabla_{J_s\xi} v) + (\nabla_v J)(du(J_s\xi))) \\
&= -2J(D_{u,J}(v)[\xi])
\end{aligned}$$

Here, the first and the last equality are by definition. The second and third are true because $JJ \equiv -Id$ and $J_sJ_s \equiv -Id$. The fourth comes from the fact $J \circ (\nabla_v J) + (\nabla_v J) \circ J = 0$ which comes from $\nabla_v(JJ) = \nabla_v(-Id) \equiv 0$. The sixth equality holds because of the

Cauchy-Riemann equation $J \circ du = du \circ J_S$. ■

3.2 $D_{u,J} = \overline{\partial}_{u,J} + R_{u,J}$

Definition 11 **(** $\overline{\partial}_{u,J}$ **and** $R_{u,J}$ **)**

$D_{u,J}$ can be written in the form $D_{u.J} \equiv \overline{\partial}_{u,J} + R_{u,J}$ with:

$$\overline{\partial}_\xi v \equiv \overline{\partial}_{u.J}(v)[\xi] \equiv \frac{1}{2}[D_\xi v - JD_\xi(Jv)] = \frac{1}{2}\{D_{u,J}(v)[\xi] - J(D_{u,J}(Jv)[\xi])\} \tag{3.3a}$$

$$R_\xi v \equiv R_{u,J}(v)[\xi] \equiv \frac{1}{2}[D_\xi v + JD_\xi(Jv)] = \frac{1}{2}\{D_{u,J}(v)[\xi] + J(D_{u,J}(Jv)[\xi])\} \tag{3.3b}$$

From the definition above, we can have the following properties from easy calculations:

Property 3 *All $D_{u,J}$, $\overline{\partial}_{u,J}$, and $R_{u,J}$ are $\mathbb{R}$-linear on both arguments. $D_{u,J}$ is J_s-linear on the second argument. $\overline{\partial}_{u,J}$ is J-linear, and $R_{u,J}$ is J-anti-linear on the first argument, that is:*

$$D_{u,J}(v)[J_s\xi] = -J(D_{u.J}(v)[\xi]) \tag{3.4a}$$

$$\overline{\partial}_{u.J}(Jv)[\xi] = J\overline{\partial}_{u,J}(v)[\xi] \tag{3.4b}$$

$$R_{u.J}(Jv)[\xi] = -JR_{u,J}(v)[\xi] \tag{3.4c}$$

Proof : It is trivial that $D_{u,J}, \overline{\partial}_{u,J}, and R_{u,J}$ are $\mathbb{R}$-linear on both arguments since all the connections involved here are $\mathbb{R}$-linear on both argument,and both J and J_s are also $\mathbb{R}$-linear.

We have already proved that $D_{u,J}$ is J_s-linear on the second argument in property 2. The other two are also easy to check:

$$\begin{aligned}\overline{\partial}_{u.J}(Jv)[\xi] &= \frac{1}{2}[D_\xi(Jv) - JD_\xi(JJv)] \\ &= \frac{1}{2}[D_\xi(Jv) + JD_\xi(v)] = \frac{1}{2}J[-J(D_\xi(Jv)) + D_\xi(v)] \\ &= J(\overline{\partial}_{u,J}(v)[\xi]). \\ R_{u.J}(Jv)[\xi] &= \frac{1}{2}[D_\xi(Jv) + JD_\xi(JJv)] \\ &= \frac{1}{2}[D_\xi(Jv) - JD_\xi(v)] = \frac{1}{2}(-J)[D_\xi(v) + JD_\xi(Jv)] \\ &= -JR_{u,J}(v)[\xi]. \ \blacksquare\end{aligned}$$

Remark : We sometimes say that $D_{u,J}$ can be split into two part: the J-linear part $\overline{\partial}_{u,J}$, and the J-antilinear part $R_{u,J}$. Also, we can say that $D_{u,J}$, and $\overline{\partial}_{u,J}$ are order one operators on $L^{k+1,p}(S, u^*(TM))$ (they are like "differential operators"), and $R_{u,J}$ is an order zero operator (just a linear map, not a differential operator).

If we let $\overline{\partial}_s$ be the usual $\overline{\partial}$-operator on TS, that is,

$$(\overline{\partial}_s\xi)(\eta) \equiv \frac{1}{2}(\nabla_\eta\xi + J_s\nabla_{J_s\eta}\xi) \tag{3.5}$$

for $\xi, \eta \in L^{k+1,p}(S, TS)$, we can say something more about $D_{u,J}$. Here, again we have different choices for the symmetric connection ∇ on S. We should choose ∇ to be the one compatible with J_s. We will have the following property:

Property 4 *$D_{u,j} \circ du = du \circ \overline{\partial}_s$, that is, $D_{u,J}(du(\xi))[\eta] = du((\overline{\partial}_s\xi)[\eta])$ $\forall \xi,\ \eta \in L^{k+1,p}(S, TS)$.*

Proof

$$
\begin{aligned}
&2(D_{u,J}(du(\xi)))[\eta]\\
=&\nabla_\eta du(\xi) + J\nabla_{J_s\eta} du(\xi) + (\nabla_{du(\xi)}J)(du(J_s\eta))\\
=&(\nabla_\eta du)(\xi) + \underline{du(\nabla_\eta \xi)} + J(\nabla_{J_s\eta} du)(\xi) + \underline{Jdu(\nabla_{J_s\eta}\xi)} + (\nabla_{du(\xi)}J)(du(J_s\eta))\\
=&\underline{du(\nabla_\eta \xi) + Jdu(\nabla_{J_s\eta}\xi)} + (\nabla_\eta du)(\xi) + J(\nabla_{J_s\eta} du)(\xi) + (\nabla_{du(\xi)}J)(du(J_s\eta))\\
=&du(\nabla_\eta \xi + J_s\nabla_{J_s\eta}\xi) + (\nabla_\eta du)(\xi) + J(\nabla_{J_s\eta} du)(\xi) + (\nabla_{du(\xi)}J)(du(J_s\eta))\\
=&2du((\overline{\partial}_s\xi)[\eta]) + [(\nabla_\eta du)(\xi) + J(\nabla_{J_s\eta} du)(\xi) + (\nabla_{du(\xi)}J)(du(J_s\eta))]
\end{aligned}
$$

The first and the last equality are by definition, the second is just taking derivative. In order to prove the property, we need to show that $(\nabla_\eta du)(\xi) + J(\nabla_{J_s\eta} du)(\xi) + (\nabla_{du(\xi)}J)(du(J_s\eta)) = 0$. We can have this from the following calculations :

$$
\begin{aligned}
&(\nabla_\eta du)(\xi) + J(\nabla_{J_s\eta} du)(\xi) + (\nabla_{du(\xi)}J)(du(J_s\eta))\\
=&(\nabla_\eta du)(\xi) + J\underline{(\nabla_\xi du)(J_s\eta)} + (\nabla_{du(\xi)}J)(du(J_s\eta))\\
=&(\nabla_\eta du)(\xi) + J\underline{[J(\nabla_\xi du)(\eta) + (\nabla_\xi J)(du(\eta))]} + (\nabla_{du(\xi)}J)(du(J_s\eta))\\
=&(\nabla_\eta du)(\xi) + J^2(\nabla_\xi du)(\eta) + J(\nabla_\xi J)(du(\eta)) + (\nabla_{du(\xi)}J)(du(J_s\eta))\\
=&(\nabla_\eta du)(\xi) - (\nabla_\xi du)(\eta) + J(\nabla_\xi J)(du(\eta)) + (\nabla_{du(\xi)}J)(du(J_s\eta))\\
=&J(\nabla_\xi J)(du(\eta)) + (\nabla_{du(\xi)}J)(du(J_s\eta)) = J(\nabla_\xi J)(du(\eta)) + (\nabla_{du(\xi)}J)Jdu(\eta)\\
=&(J\circ\nabla_\xi J)(du(\eta)) + ((\nabla_{du(\xi)}J)\circ J)(du(\eta))) = (\nabla_\xi(J^2))(du(\eta)) = 0
\end{aligned}
$$

The first equality comes from lemma 3 after this property, the second comes from lemma 4, and the other steps are rather trivial, hence we have the property. ■

Lemma 3 $(\nabla_\xi du)(\eta) = (\nabla_\eta du)(\xi) \quad \forall \xi,\ \eta \in L^{k+1,p}(S, TS)$.

Proof : We have

$$(\nabla_\xi du)(\eta) = \nabla_\eta(du(\xi)) - du(\nabla_\eta \xi)$$
$$(\nabla_\eta du)(\xi) = \nabla_\xi(du(\eta)) - du(\nabla_\xi \eta).$$

Therfore, subtract the second line from the first, we will have :

$$\begin{aligned}(\nabla_\xi du)(\eta) - (\nabla_\eta du)(\xi) &= \nabla_\eta(du(\xi)) - \nabla_\xi(du(\eta)) - (du(\nabla_\eta \xi) - du(\nabla_\xi \eta)) \\ &= \nabla_\eta(du(\xi)) - \nabla_\xi(du(\eta)) - du(\nabla_\eta \xi - \nabla_\xi \eta) \\ &= [du(\eta), du(\xi)] - du([\eta, \xi]) = 0 \ \blacksquare\end{aligned}$$

Lemma 4 $(\nabla_\xi du) \circ J_s = J \circ (\nabla_\xi du) + \nabla_\xi J \circ du$, *that is,* $(\nabla_\xi du)(J_s(\eta)) = J(\nabla_\xi(du))(\eta) + (\nabla_\xi J)(du(\eta)) \quad \forall \xi,\ \eta \in L^{k+1,p}(S, TS)$.

Proof: We can take covariant derivative to both sides of the Cauchy-Riemann equation $du(J_s \eta) = J(du(\eta))$, and then get:

$$\begin{aligned}\nabla_\xi(du(J_s(\eta))) &= \nabla_\xi(J(du(\eta))) \\ (\nabla_\xi du)(J_s(\eta)) + du(\nabla_\xi(J_s(\eta))) &= (\nabla_\xi J)(du(\eta)) + J(\nabla_\xi(du(\eta))) \\ (\nabla_\xi du)(J_s(\eta)) &= (\nabla_\xi J)(du(\eta)) + J(\nabla_\xi(du(\eta))) - du(\nabla_\xi(J_s(\eta))) \\ (\nabla_\xi du)(J_s(\eta)) &= [J(\nabla_\xi(du(\eta))) - du(\nabla_\xi(J_s(\eta)))] + (\nabla_\xi J)(du(\eta))\end{aligned}$$

In order to complete the proof, we need to show that:

$$J(\nabla_\xi(du(\eta))) - du(\nabla_\xi(J_s(\eta))) = J(\nabla_\xi(du))(\eta)$$

, or equivalently,

$$J(\nabla_\xi(du(\eta))) = J(\nabla_\xi(du))(\eta) + du(\nabla_\xi(J_s(\eta))).$$

Compare this to the fact

$$(\nabla_\xi(du(\eta))) = (\nabla_\xi(du))(\eta) + (du(\nabla_\xi\eta)),$$

we find what we need is :

$$du(\nabla_\xi(J_s(\eta))) = J(du(\nabla_\xi\eta)).$$

However, this is not difficult. Since the symmetric connection ∇ is compatible with J_s, we have :

$$\nabla_\xi\eta + J_s\nabla_\xi(J_s\eta) = 0$$

This implies $du(J_s\nabla_\xi(J_s\eta))+du(\nabla_\xi\eta)) = 0$. Therefore, $J(du(\nabla_\xi(J_s\eta)))+du(\nabla_\xi\eta)) = 0$. Hence, $du(\nabla_\xi(J_s(\eta))) = J(du(\nabla_\xi\eta))$. ■

We can also say something about $R_{u,J}$:

Property 5 *$R_{u,J}$ is an J-antilinear operator with $R_{u,J} \circ du = 0$, that is, $R_{u,J}(du(\xi))[\eta] = 0 \quad \forall\xi,\ \eta \in L^{k+1,p}(S, TS)$.*

Proof :

First, from the calculation above, we know that $R_{u,J}(v)[\eta]$ is J-antilinear at the first argument, that is, $R_{u.J}(Jdu(\xi))[\eta] = -JR_{u,J}(du(\xi))[\eta]$.

From the definition of $R_{u,J}$ and $D_{u,J}$, we have :

$$
\begin{aligned}
&4R_{u.J}(du(\xi))[\eta] \\
\equiv&2D_{u,J}(du(\xi))[\eta] + 2JD_{u,J}(Jdu(\xi))[\eta] \\
\equiv&\nabla_\eta du(\xi) + J\nabla_{J_S\eta} du(\xi) + (\nabla_{du(\xi)}J)\circ du(J_S\eta) \\
+&\underline{J\nabla_\eta(Jdu(\xi))} - \underline{\nabla_{J_S\eta}(Jdu(\xi))} + J(\nabla_{Jdu(\xi)}J)\circ du(J_S\eta) \\
=&\nabla_\eta du(\xi) + J\nabla_{J_S\eta} du(\xi) + (\nabla_{du(\xi)}J)\circ du(J_S\eta) \\
+&\underline{J^2\nabla_\eta(du(\xi)) + J(\nabla_\eta J)(du(\xi))} - \underline{J\nabla_{J_S\eta}(du(\xi)) - (\nabla_{J_S\eta}J)(du(\xi))} + J(\nabla_{Jdu(\xi)}J)\circ(Jdu(\eta)) \\
=&\nabla_\eta du(\xi) + (\nabla_{du(\xi)}J)\circ du(J_S\eta) \\
-&\nabla_\eta(du(\xi)) + J(\nabla_\eta J)(du(\xi)) - (\nabla_{J_S\eta}J)(du(\xi)) + J(\nabla_{Jdu(\xi)}J)\circ(Jdu(\eta)) \\
=&(\nabla_{du(\xi)}J)\circ du(J_S\eta) + J(\nabla_\eta J)(du(\xi)) - (\nabla_{J_S\eta}J)(du(\xi)) + J(\nabla_{Jdu(\xi)}J)\circ(Jdu(\eta)) \\
=&(\nabla_{du(\xi)}J)\circ(Jdu(\eta)) + J(\nabla_{du(\eta)}J)(du(\xi)) - (\nabla_{Jdu(\eta)}J)(du(\xi)) + J(\nabla_{Jdu(\xi)}J)\circ(Jdu(\eta))
\end{aligned}
$$

If we define $N(v,w) \equiv \frac{1}{4}\{(\nabla_v J\circ J)w - (\nabla_w J\circ J)v - (\nabla_{Jw}J)v + (\nabla_{Jv}J)w\}$, it is easy to see from the definition that:

$$R_{u.J}(du(\xi))[\eta] = \frac{1}{4}N(du(\xi), du(\eta)).$$

From the definition of $N(u,v)$, we can see that N is antisymmetric : $N(u,v) = -N(v,u)$, and J-anitlinear on both arguments: $N(Ju,v) = -JN(u,v)$, $N(u,Jv) = -JN(u,v)$ from the antisymmetric of $N(u,v)$ and the J-antilinear of $R_{u,J}$, or directly from the definition of $N(u,v)$. Notice that the antisymmetric of $N(u,v)$ immediately gives us $N(v,v) = 0$ for any v.

If we choose η and ξ such that $J_S\xi = \eta$, we will have:

$$-JR_{u,J}(du(\eta),\xi) = R_{u,J}(du(\eta), J_S\xi) = \frac{1}{4}N(du(\xi), du(\xi)) = 0.$$

It is also trivial that $R_{u,J}(du(\xi),\xi) = \frac{1}{4}N(du(\xi), du(\xi)) = 0$. Therefore, $R_{u,J}(du(a\eta + b\xi), \xi) = 0$ and $R_{u,J}(du(a\eta + b\xi), \eta) = 0$, for any constant a and b.

From the definition of $N(u, v)$, we can also check that $N(u, v)$ is in tensor of type $(1, 2) :\ N(fu, v) = f * N(u, v) = N(u, fv)$, see for example, the **torsion tensor** in [19].

Therefore, we need only to consider $a\eta + b\xi$ for constant a and b instead of $f\eta + g\xi$ for functions f and g in the calculation above.

As a result, we have proved the property $R_{u,J}(du(\xi))[\eta] = 0$ for any $\xi, \eta \in L^{k+1,p}(S, TS)$. ■

Property 6 $\overline{\partial}_{u,J} \circ du = du \circ \overline{\partial}_s$, *that is,* $\overline{\partial}_{u,J}(du(\xi))[\eta] = du((\overline{\partial}_s\xi)[\eta])$ $\forall \xi,\ \eta \in L^{k+1,p}(S, TS)$.

Proof: From property 4, we know that $D_{u,J} \circ du = du \circ \overline{\partial}_s$. From property 5, we have $R_{u,J} \circ du = 0$. We also know that $D_{u,J} = \overline{\partial}_{u,J} + R_{u,J}$ from definition 3, therefore $\overline{\partial}_{u,J} \circ du = D_{u,J} \circ du - R_{u,J} \circ du = du \circ \overline{\partial}_s + 0 = du \circ \overline{\partial}_s$ ■

Property 4 and property 6 are essential for the exact sequences in the next section.

3.3 Exact Sequences

Let $E = u^*(TM)$, and $p > 2$, from the property $D_{u,J} \circ du = du \circ \overline{\partial}_s$ in the last section, we will have the following commutative diagram with the induced operator $\overline{D}_{u,J}$.

$$\begin{array}{llll} 0 \to L^{k+1,p}(S,TS) & \xrightarrow{du} L^{k+1,p}(S,E) & \longrightarrow L^{k+1,p}(S,E)/\mathrm{im}du \\ \quad \downarrow^{\overline{\partial}_S} & \quad \downarrow^{D_{u,J}} & \quad \downarrow^{\overline{D}_{u,J}} \\ 0 \to L^{k,p}(S,\Lambda^{0,1}S \otimes E) & \xrightarrow{du} L^{k,p}(S,\Lambda^{0,1}S \otimes E) & \longrightarrow L^{k,p}(S,\Lambda^{0,1}S \otimes E)/\mathrm{im}du \end{array} \tag{3.6}$$

Let $\mathcal{O}_{D_{u,J}}(S,E)$ denote the sheaf defined by $kerD_{u,J}$, the kernel of $D_{u,J}$. Since $\overline{\partial} \circ du = du \circ \overline{\partial}_s$, we get the following short exact sequence:

$$0 \longrightarrow \mathcal{O}_{\overline{\partial}_S}(S,TS) \xrightarrow{du} \mathcal{O}_{D_{u,J}}(S,E) \longrightarrow \mathcal{N} \longrightarrow 0 \tag{3.7}$$

where $\mathcal{N} = \mathcal{O}(N_0) \oplus N_1$, and $N_0 = E/TC = E/\overline{\mathrm{im}\, du}$ be the normal bundle of $u(S)$ in M. $N_1 = \overset{p}{\underset{i=1}{\oplus}} \mathbb{C}^{n_i}_{a_i}$ is the skyscraper sheaf having stalk $\mathbb{C}^{n_i}$ at each zero a_i of du. The zeros of du for a J-holomorphic map $u : (S, J_S) \to (M, J)$ are at most finitely many isolated points.

Let $D^{N_0}_{u,J}$ be the projection of $D_{u,J}$ on N_0, we have the exact sequence :

$$0 \longrightarrow \mathcal{O}_{\overline{\partial}_S}(S, TS \otimes A) \xrightarrow{du} \mathcal{O}_{D_{u,J}}(S,E) \longrightarrow \mathcal{O}_{D^{N_0}_{u,J}}(S,N_0) \longrightarrow 0 \tag{3.8}$$

with the divisor $A = \Sigma^p_{i=1}(ni \centerdot ai)$ associated from the sheaf $N_1 = \bigoplus_{i=1}^{p} \mathbb{C}^{n_i}_{a_i}$.

From (3.7) and (3.8), we can have a natural commutative diagram:

$$\begin{array}{lllll} 0 \to \mathcal{O}_{\overline{\partial}_S}(S,TS) & \xrightarrow{du} \mathcal{O}_{D_{u,J}}(S,E) & \longrightarrow \mathcal{N} = \mathcal{O}(N_0) \oplus N_1 & \longrightarrow 0 \\ \quad\downarrow & \quad\downarrow & \quad\downarrow^{\pi} & \\ 0 \to \mathcal{O}_{\overline{\partial}_S}(S,TS\otimes A) & \xrightarrow{du} \mathcal{O}_{D_{u,J}}(S,E) & \longrightarrow \mathcal{O}_{D_{u,J}^{N_0}}(S,N_0) & \longrightarrow 0 \end{array} \tag{3.9}$$

If we let also $H^0_D(E) = kerD_{u,J}$ and $H^1_{D_{u,J}}(E) = cokerD$, from the short exact sequences (3.7) and (3.8), we will have long exact sequences (3.10) and (3.11) respectively :

$$\begin{aligned} 0 \longrightarrow \quad & H^0(S,TS) \xrightarrow{du} \quad H^0_{D_{u,J}}(S,E) \longrightarrow H^0_{D_{u,J}^{N_0}}(S,N_0) \oplus H^0(S,N_1) \longrightarrow \\ \longrightarrow \quad & H^1(S,TS) \longrightarrow \quad H^1_{D_{u,J}}(S,E) \longrightarrow H^1_{D_{u,J}^{N_0}}(S,N_0) \longrightarrow 0 \end{aligned} \tag{3.10}$$

$$\begin{aligned} 0 \longrightarrow \quad & H^0(S,TS\otimes A) \xrightarrow{du} \quad H^0_{D_{u,J}}(S,E) \longrightarrow H^0_{D_{u,J}^{N_0}}(S,N_0) \oplus \longrightarrow \\ \longrightarrow \quad & H^1(S,TS\otimes A) \longrightarrow \quad H^1_{D_{u,J}}(S,E) \longrightarrow H^1_{D_{u,J}^{N_0}}(S,N_0) \longrightarrow 0 \end{aligned} \tag{3.11}$$

From the exact sequences above, S.Ivashikovich and V. Shevchishin tell us that with some conditions, $\mathcal{M}_{A,J}$, the moduli space of nonparametrized J-holomorphic curves in a homology class $A \in H_2(M,\mathbb{Z})$ is a manifold of finite dimension.

Lemma 5 (S.Ivashikovich, V. Shevchishin) *[6]* $\mathcal{N} = \mathcal{O}(N_0) \oplus N_1$ *is isomorphic to* $\mathcal{K}er\overline{D}_{u,J}$ *as a* ***sheaf****, where* $\mathcal{K}er\overline{D}_{u,J} = D^{-1}_{u,J}(du(L^p(S,\Lambda^{0,1}S\otimes E)))/du(L^{1,p}(S,TS))$ *is the quotient sheaf.*

Theorem 15 (S.Ivashikovich, V. Shevchishin) *[6] The space* $\mathcal{K}er\overline{D}_{u,J}$ *is isomorphic to* $H^0_{D_{u,J}^{N_0}}(S,N_0) \oplus H^0(S,N_1)$. $Coker\overline{D_{u,J}}$ *is isomorphic to* $H^1_{D_{u,J}^{N_0}}(S,N_0)$

Theorem 16 (S.Ivashikovich, V. Shevchishin) *[6] If $H^1_{D^{N_0}_{u,J}}(S, N_0) = 0$, then in the neighbourhood of a nonparametrized J-holomorphic curve C, the moduli space $\mathcal{M}_{A,J}$ is a manifold of finite dimension and*

$$T_C\mathcal{M}_{A,J} \cong H^0_{D^{N_0}}(N_0) \oplus H^0(N_1) \tag{3.12}$$

3.4 $D_{u,J}$ as the Vertical Differential

In the last three sections, for a given alsmost complex structure J on M and a given J-holomorphic curve $u : (\mathbb{CP}^1, J_S) \to (M, \omega, J)$, we define $D_{u,J}$ as an operator on S from $L^{k+1,p}(S, u^*TM)$ to $L^{k,p}(S, \Lambda^{0,1}S \times u^*TM)$ and have some instant properties. In this section, we will consider some moduli spaces and vector bundles over these moduli spaces, and view $D_{u,J}$ as a vertical differential on the vector bundle.

To begin with, let (M, ω) be a compact symplectic 2n-(real)-dimension manifold, Let J be a given C^l almost complex stucture on M . Let $S = \mathbb{CP}^1$ be the Kähler manifold with the standard Kähler structure, and corresponding standard almost complex structure J_S on S.

For a given holomogy class $A \in H_2(M, \mathbb{Z})$, and a given J we can first consider the moduli space $\mathcal{M}^*(A, J)$, where

$$\mathcal{M}(A, J) \equiv \{u \mid u \in W^{k+1,p}(S, M), J \circ du = du \circ J_S, [u(S)] = A\}$$

,and

$$\mathcal{M}^*(A; J) \equiv \{u \mid u \in \mathcal{M}(A, J), \text{ u is simple}\}$$

In other words, $\mathcal{M}^*(A, J)$ is the moduli space of all simple J-holomorphic curves from (S, J_S) to (M, J) in the class A. We can have another equivalent definition of the operator $D_{u,J}$ on $\mathcal{M}^*(A, J)$.

First, we can consider the vector bundle $\mathcal{E} \to \mathcal{U}_A^{k+1,p}$, where the base $\mathcal{U}_A^{k+1,p}$ are all

$W^{k+1,p}$ maps from S to M with $[u] = A \in H_2(M, \mathbb{Z})$, and the fiber at $u \in \mathcal{U}_A^{k+1,p}$ is

$$\mathcal{E}_u = L^{k,p}(\Lambda^{0,1} S \otimes u^*TM).$$

This fiber $\mathcal{E}_u$ is apparently of infinite dimensions. We can define a section $\Phi : \mathcal{U}_A^{k+1,p} \to \mathcal{E}$ on the vector bundle by:

$$\Phi(u) \equiv (u, \overline{\partial}_J(u))$$

where

$$\overline{\partial}_J(u) \equiv \frac{1}{2}(du + J \circ du \circ J_S)$$

Now, the zero set of the section Φ is exactly the moduli space $\mathcal{M}(A, J)$, and the subset $\mathcal{M}^*(A, J)$ is the intersection of $\mathcal{M}(A, J)$ with $(\mathcal{U}_A^{k+1,p})^*$, where $(\mathcal{U}_A^{k+1,p})^*$ is the subset of $\mathcal{U}_A^{k+1,p}$ consisting only simple maps.

We now can define the operator $D_{u,J}$. For a given J on M and a $u \in \mathcal{M}^*(A, J)$, consider the differential of the section Φ at the "point" u

$$d\Phi(u) : T_u\mathcal{U}_A^{k+1,p} \to T_{(u,0)}\mathcal{E}.$$

Also, consider the vertical projection that sends the tangent vectors at $(u, 0)$ to its vertical component:

$$\pi_u : T_{(u,0)} = T_u\mathcal{U}_A^{k+1,p} \oplus \mathcal{E}_u \to \mathcal{E}_u$$

We define $D_{u,J}$ to be the composition of these two operators:

$$D_{u,J} \equiv \pi_u \circ d\Phi : T_u\mathcal{U}_A^{k+1,p} \to \mathcal{E}_u$$

Since for a given u, an element in $T_u\mathcal{U}_A^{k+1,p}$ is in fact just a $W^{k+1,p}$ vector field $\xi(z) \in T_{u(z)}M$ along the function u, we can see $T_u\mathcal{U}_A = L^{k+1,p}(S, u^*TM)$. We already know that $\mathcal{E}_u = L^{k,p}(\Lambda^{0,1}S \otimes u^*TM)$. Therefore, for a given u and a given J, we can also view $D_{u,J}$ as an operator from $L^{k+1,p}(S, u^*TM)$ to $L^{k,p}(\Lambda^{0,1}S \otimes u^*TM)$.

For a J-holomorphic map u, if we take derivative and then projection to the equation

$$\overline{\partial}_J(u) \equiv \frac{1}{2}(du + J \circ du \circ J_S),$$

we can find that for a given J and a given u, this new $D_{u,J}$ is indeed the same operator we already defined in previous sections. Therefore, we can in fact view $D_{u,J}$ as the **vertical differential** of the section Φ at u since it is the composition of the vertical projection π_u and the differential $d\Phi$

Notice that now we have only defined $D_{u,J}$ on the simple J-holomorphic maps $u \in \mathcal{M}^*(A, S, J)$ for the new definition at this point while the original definition applies to all $W^{k+1,p}$ maps, not necessarily J-holomorphic. We can extend the new definition furthre to general $W^{k+1,p}$ maps u, J-holomorphic or not.

Now, let $u \in \mathcal{U}_A^{k+1,p}$, given $\xi \in L^{k+1,p}(S, u^*TM)$, we will first consider the parallel transport along the geodesic curves on $M : s \mapsto exp_{u(z)}(s\xi(z))$ which begins from the point $u(z)$ (when $s = 0$) with the tangent vector $\xi(z) \in T_{u(z)}M$. Here, the parallel transport depends on the choice of connection. Therefore, we should begin with the Levi-Civita connection ∇ and then define a complex linear $\tilde{\nabla}$ connection by

$$\tilde{\nabla}_v X \equiv \nabla_v X - \frac{1}{2}J(\nabla_v J)X$$

We should work with this connection $\tilde{\nabla}$ to define the parallel transportation. With the parallel transportation, we will naturally have a vector bundle isomorphism

$$\phi_u(\xi) : u^*TM \to exp_u(\xi)^*TM$$

for a given vector field ξ.

The second step is to define the map

$$\mathcal{F}_{u,J} : L^{k+1,p}(S, u^*TM) \to L^{k,p}(S, \Lambda^{0,1}S \otimes u^*TM)$$

by

$$\mathcal{F}_{u,J}(\xi) \equiv \phi_u(\xi)^{-1}\overline{\partial}_J(exp_u(\xi))$$

Notice that this map $\mathcal{F}_{u,J}$ is sending $\overline{\partial}_J(exp_u(\xi))$ at the "point" $exp_u(\xi)$ back to $\mathcal{E}_u$ via the isomorphism $\phi_u(\xi)^{-1}$. By taking differential and letting ξ going to zero, we can see that the differential $d\mathcal{F}_{u,J}(0)$ of $\mathcal{F}_{u,J}$ at $\xi = 0$ is exactly the vertical differential of the section $\Phi : \mathcal{U}_A^{k+1,p} \to \mathcal{E}$ at u. We should use this $d\mathcal{F}_{u,J}(0)$ to extend our new definition of $D_{u,J}$ since it is also the vertical differential for the section $\Phi(u) = (u, \overline{\partial}_J u)$, weather u is J-holomorphic or not.

Definition 12 (another equivalent definiton of $D_{u,J}$)
For any C^l almost complex structure J on M, and $W^{k+1,p}$ map $u : S \to M$, we can define the operator $D_{u,J} : L^{k+1,p}(S, u^*TM) \to L^{k,p}(S, \Lambda^{0,1}S \otimes u^*TM)$ by

$$D_{u,J}\eta \equiv (d\mathcal{F}_{u,J}(0))\eta$$

for any $\eta \in L^{k+1,p}(S, u^*TM)$.

It is easy to check that for any given $W^{k+1,p}$ u and any C^l almost complex structure J on M this new definition is still equivalent to the old definition in section 3.1 by writing everything down to local coordinates. Notice that the Levi-Civita connection ∇

is torsion free.

Let $\mathcal{J}^l$ be the collection of all C^l almost complex structure on M, for a given u, we can always choose different C^l almost complex structure $J \in \mathcal{J}^l$, and we can now consider the **universal moduli space**

$$\mathcal{M}^*(A, \mathcal{J}^l) \equiv \{(u, J) \mid J \in \mathcal{J}^l, u \in \mathcal{M}^*(A, J)\}$$

An important fact for $\mathcal{M}^*(A, \mathcal{J}^l)$ is that when l is large enough, $\mathcal{M}^*(A, \mathcal{J}^l)$ will be a separable Banach manifold.

Theorem 17 (D. Mcduff, D. Salamon) *[9] For $A \in H^2(M, \mathbb{Z})$, an integer $l \geq 2$, a real number $p > 2$, and an integer $k \in \{0, 1, ..., l-1\}$, the moduli space $\mathcal{M}^*(A, \mathcal{J}^l)$ is a separable $\mathcal{C}^{l-k-1}$ Banach submanifold of $\mathcal{U}_A^{k+1,p} \times \mathcal{J}^l$, where $\mathcal{U}_A^{k+1,p}$ is the collection of all $W^{k+1,p}$ map from S to M.*

We say that an almost complex structure J on M is **regular (for A)** if the vertical differential $\mathcal{D}_{u,J}$ is **surjective** for every $u \in \mathcal{M}^*(A, J)$. If we let $\mathcal{J}_\omega$ be the collection of all ω-tamed almost complex structures on (M, ω), and $\mathcal{J}_{reg}(A) \subset \mathcal{J}_\omega$ be the subset of $\mathcal{J}_\omega$ containing only regular almost complex structures, then $\mathcal{J}_{reg}(A)$ is of second category in $\mathcal{J}_\omega$.

If we choose any $J \in \mathcal{J}_{reg}(A)$, then the moduli space $\mathcal{M}^*(A, J)$ is a manifold of real dimension $2c_1(M) \cdot A + 2$.

Chapter 4

Set UP

Before we begin the proof of the Main theorems, we need to give some background data of the problem.

4.1 The Moduli Spaces.

$$\mathcal{P}_A \equiv \{(u, J) \in \mathcal{U}_A^{k+1,p} \times \mathcal{J}_\omega \mid \ u \text{ is } J\text{-holomorphic } \} \tag{4.1}$$

Here, $\mathcal{U}_A^{k+1,p}$ is the space of all $W^{k+1,p}$ maps from $(\mathbb{CP}^1, J_s)$ to (M, ω, J) with $[u(\mathbb{CP}^1)] = A \in H_2(M, \mathbb{Z})$. $\mathcal{J}_\omega$ is the space of all ω-tamed almost complex structure J on (M, ω). Let

$$\mathcal{P}_{A,\Omega} \equiv \{(u, J, \{z_i\}_{i=1}^{ED(A)})) \mid u \in \mathcal{P}_A, u(z_i) = p_i \in \Omega \in M\} \tag{4.2}$$

Here, $\Omega = \{p_i\}_{i=1}^m$ is an set with $ED(A)$ points in $(M.\omega)$, and $\{z_i\}_{i=1}^{ED(A)}$ are $ED(A)$ points in the domain $\mathbb{CP}^1$.

4.1.1 The expected dimension ED(A) of the moduli space $\mathcal{P}_A$

$ED(A)$ is the expected (complex) dimension of the moduli space $\mathcal{P}_A$. We can compute the expected dimension explicitly by the equation: $ED(A) = -K_\omega(A) - 1 = c_1(M) \centerdot A - 1$, where K_ω is the symplectic canonical class, and $c_1(M)$ is the first chern class of M. We can see some easy examples:

Example : $M = \mathbb{CP}^2$**, Barraud's case.**

Let H be the generator of $H_2(\mathbb{CP}^2, \mathbb{Z}) = \mathbb{Z}$. Let $A = dH \in H_2(\mathbb{CP}^2, \mathbb{Z})$. Note that $c_1(\mathbb{CP}^2) = 3H$. Then we have :

$$ED(A) = c_1(M) \centerdot A - 1 = 3H \centerdot dH - 1 = 3dH \centerdot H - 1 = 3d - 1$$

This is why we have 3d-1 marked points in the Barraud's case.

Example : $M = \mathbb{CP}^2 \# m\overline{\mathbb{CP}^2}$**, the blow-ups of** $\mathbb{CP}^2$**.**

First, we know that $H_2(\mathbb{CP}^2 \# m\overline{\mathbb{CP}^2}, \mathbb{Z}) = \mathbb{Z}^{m+1}$. Let $H \in H_2(\mathbb{CP}^2 \# m\overline{\mathbb{CP}^2}, \mathbb{Z})$ be the generator that represents a line in $\mathbb{CP}^2 \# m\overline{\mathbb{CP}^2}$ that doesn't intersect with any of the exceptional curves, and $e_1, e_2, ..., e_m \in H_2(\mathbb{CP}^2 \# m\overline{\mathbb{CP}^2}, \mathbb{Z})$ are the generator that represent the exceptional curves. Let $A = d_0 H + d_1 e_1 + d_2 e_2 + ... + d_m e_m$, note that $c_1(\mathbb{CP}^2 \# m\overline{\mathbb{CP}^2}) = 3H - e_1 - e_2 - e_m$, and $H \centerdot H = 1$, $e_i \centerdot e_i = -1$, $H \centerdot e_i = e_i \centerdot H = 0$, and $e_i \centerdot e_j = 0$, when $i \neq j$. Then we have :

$$ED(A) = c_1(M) \centerdot A - 1 = 3d_0 + d_1 + d_2 + ...d_m - 1$$

Example : $M = \mathbb{CP}^1 \times \mathbb{CP}^1$**.**

Let H_a, H_b be the standard generators of $H_2(\mathbb{CP}^1 \times \mathbb{CP}^1, \mathbb{Z}) = \mathbb{Z}^2$. Let $A = d_a H_a + d_b H_b \in H_2(\mathbb{CP}^1 \times \mathbb{CP}^1, \mathbb{Z})$. Note that $c_1(\mathbb{CP}^1 \times \mathbb{CP}^1) = 2H_a + 2H_b$, $H_a \cdot H_a = H_b \cdot H_b = 0$, $H_a \cdot H_b = H_b \cdot H_a = 1$. Therefore, we have :

$$ED(A) = c_1(M) \cdot A - 1 = 2d_a + 2d_b - 1$$

Example : $M =(\mathbb{CP}^1 \times \mathbb{CP}^1)\#m\overline{\mathbb{CP}^2}$**, the blow-ups of** $\mathbb{CP}^1 \times \mathbb{CP}^1$**.**

Let H_a, H_b be the standard generators of $H_2((\mathbb{CP}^1 \times \mathbb{CP}^1)\#m\overline{\mathbb{CP}^2}, \mathbb{Z}) = \mathbb{Z}^{m+2}$ represent the curves in $\mathbb{CP}^1 \times \mathbb{CP}^1$ that don't intersect with any of the exceptional curves, and $e_1, e_2, ..., e_m \in H_2((\mathbb{CP}^1 \times \mathbb{CP}^1)\#m\overline{\mathbb{CP}^2}, \mathbb{Z})$ are the generators represent the exceptional curves. Let $A = d_a H_a + d_b H_b + d_1 e_1 + d_2 e_2 + ...d_m e_m \in H_2((\mathbb{CP}^1 \times \mathbb{CP}^1)\#m\overline{\mathbb{CP}^2}, \mathbb{Z})$. Note that $c_1((\mathbb{CP}^1 \times \mathbb{CP}^1)\#m\overline{\mathbb{CP}^2}) = 2H_a + 2H_b - e_1 - e_2 - ... - e_m$, and $H_a \cdot H_a = H_b \cdot H_b = 0$, $e_i \cdot e_i = -1$, $H_a \cdot H_b = H_b \cdot H_a = 1$, $H_a \cdot e_i = e_i \cdot H_a = H_b \cdot e_i = e_i \cdot H_b = 0$, $e_i \cdot e_j = 0$, when $i \neq j$. Therefore we have :

$$ED(A) = c_1(M) \cdot A - 1 = 2d_a + 2d_b + d_1 + d_2 + ...d_m - 1$$

4.1.2 $\overline{\mathcal{P}_{A,\Omega}}$

As usual, we don't deal with moduli space $\mathcal{P}_A$ directly, but are interested in the one with marked points: $\mathcal{P}_{A,\Omega}$. To be more precise, according to the Gromov compactification theorem, the object we should really study is the compactification $\overline{\mathcal{P}_{A,\Omega}}$ of $\mathcal{P}_{A,\Omega}$ since we also need to consider the "boundary" of $\mathcal{P}_{A,\Omega}$.

The statement of Main Theorem 1 only involves nodal J-holomorphic maps from $(\mathbb{CP}^1, J_s)$ to (M, ω, J), that is, curves **with only ordinary double point singularities**. It is not the entire $\mathcal{P}_{A,\Omega}$, let alone $\overline{\mathcal{P}_{A,\Omega}}$. If we call the collection of "nodal J-holomorphic spheres" to be the "good part" of the moduli space $\overline{\mathcal{P}_{A,\Omega}}$. Then the spirit of the proof of the theorem is to find a way to show that the "good part" is path-connected as a submanifold in $\overline{\mathcal{P}_{A,\Omega}}$.

Now, let us give a list of the "bad parts" in $\overline{\mathcal{P}_{A,\Omega}}$. Although what we want are curves with only ordinary double point singularities, $\overline{\mathcal{P}_{A,\Omega}}$ includes some much worse singularities.

First, an element in the "good part" can have self-intersections as a J-holomorphic curve from $(\mathbb{CP}^1, J_s)$ to (M, ω, J). However, for any point p in M, there can only be at most 2 (maybe 1, maybe 0) preimages in $\mathbb{CP}^1$. Therefore, a curve with any point having 3 or more preimages belongs to the "bad parts". We should think them as one of the bad parts:

$$\begin{aligned}\mathcal{P}^{TRI}_{A,\Omega} \equiv &\{(u, J, \{z_i\}_{i=1}^{ED(A)}, z_a, z_b, z_c) \mid (u, J, \{z_i\}_{i=1}^{ED(A)}) \in \mathcal{P}_{A,\Omega};\\ &z_a, z_b, z_c \in \mathbb{CP}^1; z_a \neq z_b, z_b \neq z_c, z_c \neq z_a; u(z_a) = u(z_b) = u(z_c)\}\end{aligned} \tag{4.3}$$

Second, an element in the "good part" is a J-holomorphic curve from $(\mathbb{CP}^1, J_s)$ to (M, ω, J), so we have the "Cauchy-Rieman Equation". Therefore, $du(J_s v) = J \circ du(v)$ for any $v \in T_z\mathbb{CP}^1$, $z \in \mathbb{CP}^1$. This means for any point $z \in \mathbb{CP}^1$, $du(T_z\mathbb{CP}^1)$ can only be of dimension 2 or 0. We only want the curves with ordinary double points. We should think the curves having $du(T_z\mathbb{CP}^1) = 0$ for some point z in $\mathbb{CP}^1$ as one of the bad parts

:

$$\mathcal{P}_{A,\Omega}^{\emptyset TP} \equiv \{(u, J, \{z_i\}_{i=1}^{ED(A)}, z) \mid (u, J, \{z_i\}_{i=1}^{ED(A)}) \in \mathcal{P}_{A,\Omega}; z \in \mathbb{CP}^1; du(T_z\mathbb{CP}^1) = 0\} \tag{4.4}$$

Third, an element in the "good part" can have ordinary double points. For an ordinary double point $p \in M$, there exists $z_1, z_2 \in \mathbb{CP}^1, z_1 \neq z_2$ such that $u(z_1) = u(z_2)$, but $du(T_{z_1}\mathbb{CP}^1) \neq du(T_{z_2}\mathbb{CP}^1)$. Clearly, we want both $du(T_{z_1}\mathbb{CP}^1)$ and $du(T_{z_2}\mathbb{CP}^1)$ to be 2-dimensional subspace of T_pM from the last paragraph.

From the Cauchy-Riemann Equation, we know that $du(T_{z_1}\mathbb{CP}^1) \cap du(T_{z_2}\mathbb{CP}^1)$ can only be of dimension 2 or 0. This is because if there are some $v \in T_{z_1}\mathbb{CP}^1$, $w \in T_{z_2}\mathbb{CP}^1$ with $du(v) = du(w) \neq 0$, then we have $du(J_s v) = J \circ du(v) = J \circ du(w) = du(J_s w) \neq 0$. Therefore, an element in the "good part" can only have double points with $du(T_{z_1}\mathbb{CP}^1) \cap du(T_{z_2}\mathbb{CP}^1) = \{0\}$. We should think the curves with $du(T_{z_1}\mathbb{CP}^1) = du(T_{z_2}\mathbb{CP}^1) \neq \{0\}$ as one of the bad parts:

$$\mathcal{P}_{A,\Omega}^{IDTP} \equiv \{(u, J, \{z_i\}_{i=1}^{ED(A)}, z_1, z_2) \mid (u, J, \{z_i\}_{i=1}^{ED(A)}) \in \mathcal{P}_{A,\Omega}; z_1, z_2 \in \mathbb{CP}^1; z_1 \neq z_2; u(z_1) = u(z_2); du(T_{z_1}\mathbb{CP}^1) = du(T_{z_2}\mathbb{CP}^1)\} \tag{4.5}$$

The last kind of the bad parts in the "boundary" of the moduli space $\overline{\mathcal{P}_{A,\Omega}}$ are reducible curves.

Let $C_k = (u_k, J_k) \in \mathcal{P}_A$ be a sequence of irreducible J-holomorphic curves from $(\mathbb{CP}^1, J)$ to (M, ω, J) with $[u(\mathbb{CP}^1)] = A$. It is possible that they converges to the union of several J-holomorphic curves: $\lim_{k\to\infty} C_k = C_\infty = \sum_{j=1}^{N} Z_j$ where $Z_j = (u_j, J_\infty) \in$

$\mathcal{P}_{A_j}$ with the property $\sum_{j=1}^{N} A_j = A \in H_2(M, \mathbb{Z})$.

We know that for each C_k, we assign $ED(A)$ marked points to them. However, as C_k converges to $C_\infty = \sum_{j=1}^{N} Z_j$, we don't know which component Z_j these marked points end up with. It is possible that some of the components have many marked points while some components have only few or even none.

For each component $Z_j \in \mathcal{P}_{A_j}$, we have $ED(A_j) = c_1(M) \centerdot A - 1$. We can see that $\sum_{j=1}^{n} ED(A_j) = c_1(M) \centerdot A_j - n = ED(A) - (n-1) < ED(A)$. This tell us that at least one of the components have more than enough (more than the number $ED(A_j)$) marked points. We can use this component Z_j as a representative of the entire C_∞. Therefore, we should also let these sets belong to the bad parts.

For $C_\infty = \Sigma_{j=1}^{N} Z_j$, with $\lim_{k\to\infty} C_k = C_\infty$. For $\mathbb{CP}^2$ and its blow-ups $\mathbb{CP}^2 \# m\overline{\mathbb{CP}^2}$, $[C_k] = A = d_0 H + d_1 e_1 + ... + e_m \ \forall k$.

Let $[Z_j] = A_j = d_{j,0} H + d_{j,1} e_1 + ... + d_{j,m} e_m$, then we must have $d_l = \Sigma_{j=1}^{N} d_{j,l}$ for $l = 0, 1, ..., m$, so that $A = \Sigma_{j=1}^{N} A_j$. Notice that A can not split into all random combinations $\Sigma_{j=1}^{N} A_j$ of A. For example, we can not have A_j such that expected dimension $ED(A_j)$ is negative or higher than $ED(A)$.

For $\mathbb{CP}^1 \times \mathbb{CP}^1$ and its blow-ups $(\mathbb{CP}^1 \times \mathbb{CP}^1) \# m\overline{\mathbb{CP}^2}$, $[C_k] = A = d_a H_a + d_b H_b + d_1 e_1 + ... + d_m e_m \ \forall k$, we can have similar discussions.

For $\Sigma_{j=1}^{N} A_j = A$, and subset $\{p_{j_1}, ..., p_{j_{ED(A_j)+1}}\} \subset \{p_1, ..., p_{ED(A)}\}$, the following set is one of the bad parts:

$$\mathcal{P}^{COMP}_{A_j,\Omega_j} \equiv \{(u, J, \{z_i\}_{i=1}^{ED(A_j)+1}) \mid (u, J) \in \mathcal{P}_{A_j}; u(z_i) = p_{j_i} \ \forall i\} \tag{4.6}$$

For all these $\mathcal{P}_c$, we should define the projection to $\mathcal{J}_\omega$.

$$\pi : \mathcal{P}_A \to \mathcal{J}_\omega$$

$$\widetilde{\pi} : \mathcal{P}_{A,\Omega} \to \mathcal{J}_\omega$$

$$\pi^{TRI}_{A,\Omega} : \mathcal{P}^{TRI}_{A,\Omega} \to \mathcal{J}_\omega$$

$$\pi^{\emptyset TP}_{A,\Omega} : \mathcal{P}^{\emptyset TP}_{A,\Omega} \to \mathcal{J}_\omega$$

$$\pi^{IDTP}_{A,\Omega} : \mathcal{P}^{IDTP}_{A,\Omega} \to \mathcal{J}_\omega$$

$$\pi^{COMP}_{A_j,\Omega_j} : \mathcal{P}^{COMP}_{A_j,\Omega_j} \to \mathcal{J}_\omega$$

4.2 Evaluation maps

We are now dealing with J-holomorphic maps from $(\mathbb{CP}^1, J_S)$ to (M, ω, J) with a fixed number ($ED(A)$) of marked points on each curve. to study the moduli space, the next step is to define the evaluation maps.

We can define different kinds of evaluation maps at given points. For example, we can define $\tau_0(z) \equiv u(z)$, or $\tau_1(z) \equiv [du(z)]^{1,0}$. Here, $[du(z)]^{1,0} = du(z)$ for all J-holomorphic u.

Also, For $N_0 = E/TC = u^*(TM)/\overline{\mathrm{im}\, du}$ to be the complex vector bundle F on $S = \mathbb{CP}^1$. For a section $\alpha \in \mathcal{O}_{D_{u,J}^{N_0}}(\mathbb{CP}^1, N_0)$ and the marked points $z_i \in \mathbb{CP}^1$, we can also consider a natural evaluation map that takes values on the marked points :

$$\tau(\alpha) = (\alpha_{\restriction_{z_1}}, ..., \alpha_{\restriction_{z_m}})$$

Let $P = \Sigma_{i=1}^m z_i$ be a divisor on $\mathbb{CP}^1$ where z_i are m distinct marked points on $\mathbb{CP}^1$. We let $D_{u,J}^{N_0}$ defined in section 3.3 to be the operator on N_0, and then derived the unique operator $\widetilde{D_{N_0}}$ on $\widetilde{N_0} = N_0 \otimes P$ that makes the following diagram commutative:

$$\begin{array}{ccc} L^{k+1,p}(\mathbb{CP}^1, \widetilde{N_0}) & \xrightarrow{\imath} & L^{k+1,p}(\mathbb{CP}^1, N_0) \\ \downarrow^{\widetilde{D_{N_0}}} & & \downarrow^{D_{u,J}^{N_0}} \\ L^{k,p}(\mathbb{CP}^1, \Lambda^{0,1}\mathbb{CP}^1 \otimes \widetilde{N_0}) & \rightarrow & L^{k,p}(\mathbb{CP}^1, \Lambda^{0,1}\mathbb{CP}^1 \otimes N_0) \end{array}$$

4.3 The Banach Bundles

Now, we come back to the transversality problem. We will consider different categories including the good part and all the "bad parts". In a category c, we want to let $\mathcal{P}_c$ to be the intersection of two Banach manifolds Φ_c and $Ⓢ_c$. We should set up some vector bundles $\mathcal{F}_c$'s over some base manifold $\mathcal{B}_c$'s before considering two different sections Φ_c and $Ⓢ_c$ in the bundles and their intersection $\mathcal{P}_c$. The plan is that for all the "good part" and "bad parts" $\mathcal{P}_c$ mentioned in section 4.1, we would like to find them as the intersection of two sections Φ_c and $Ⓢ_c$ in some vector bundle $\mathcal{F}_c$ over some base manifold $\mathcal{B}_c$

4.3.1 $\mathcal{F}_A$ over $\mathcal{B}_A$

We can begin with the easiest case: $\mathcal{P}_A$. Recall that (M, ω) is a compact 2n-(real)-dimensional symplectic manifold with the symplectic form ω, and $\mathcal{U}_A^{k+1,p}$ is the space of all $W^{k+1,p}$ curves from $(\mathbb{CP}^1, J_s)$ to (M, ω, J) with $[u(\mathbb{CP}^1)] = A \in H_2(M, \mathbb{Z})$. $\mathcal{J}_\omega$ is the space of all ω-tamed almost complex structure J on (M, ω). Let $\mathcal{B}_A \equiv \mathcal{U}_A^{k+1,p} \times \mathcal{J}_\omega$ be the base manifold. $\mathcal{B}_A$ is a Banach manifold since both $\mathcal{U}_A^{k+1}$ and $\mathcal{J}_\omega$ are.

Consider the vector bundle $\mathcal{F}_A$ over the base manifold $\mathcal{B}_A$ with fiber $L^{k,p}(\Lambda^{0,1}\mathcal{CP}^1 \otimes u_*TM)$ at each point $(u, J) \in \mathcal{B}_A$. The total space $\mathcal{F}_A$ is also a Banach manifold. $\mathcal{F}_A$ is a Banach bundle over $\mathcal{B}_A$.

Define the section Φ_A in the vector bundle $\mathcal{F}_A$ over $\mathcal{B}_{\mathcal{A}}$ by :

$$\Phi_A(u, J) \equiv du + J(u) \circ du \circ J_S$$

Define the zero section $Ⓢ_A(u, J)$ in the vector bundle $\mathcal{F}_A$ over $\mathcal{B}_\mathcal{A}$ by :

$$Ⓢ_A(u, J) \equiv 0$$

Let $\mathcal{P}_A$ denote the intersection of these two sections : $\mathcal{P}_A \equiv \Phi_A \cap Ⓢ_A$. In other words, $\mathcal{P}_A = \Phi_A^{-1}(0)$.

4.3.2 $\mathcal{F}_{A,\Omega}$ over $\mathcal{B}_{A,\Omega}$

Now, we should add some marked points. To be precise, we should add $ED(A)$ marked points M for the $\mathcal{P}_A$ case. Let $\Omega = (p_1, ...p_{ED(A)})$ be a set of chosen marked points on M.

First, we let the based manifold to be $\mathcal{B}_{A,\Omega} \equiv \mathcal{B}_A \times (\mathbb{CP}^1)^{ED(A)}$ and let $\mathcal{F}_{A,\Omega} \equiv \mathcal{F}_A \oplus (M)^{ED(A)}$ be the vector bundle over $\mathcal{B}_{A,\Omega}$ with fiber $L^{k,p}(\Lambda^{0,1}\mathcal{CP}^1 \otimes u_* TM) \oplus (M)^{ED(A)}$ at $(u, J, z_1, ..., z_{ED(A)})$. $\mathcal{F}_{A,\Omega}$ is a Banach bundle over the Banach manifold $\mathcal{B}_{A,\Omega}$.

Define the section $\Phi_{A,\Omega}$ in the vector bundle $\mathcal{F}_{A,\Omega}$ over $\mathcal{B}_{A,\Omega}$ by :

$$\Phi_{A,\Omega}(u, J, z_1, ...z_{ED(A)}) \equiv (du + J(u) \circ du J_S, u(z_1), ..., u(z_{ED(A)}))$$

Define the zero section $Ⓢ_{A,\Omega}$ in the vector bundle $\mathcal{F}_{A,\Omega}$ over $\mathcal{B}_{A,\Omega}$ by :

$$Ⓢ_{A,\Omega}(u, J, z_1, ..., z_{ED(A)}) \equiv (0, p_1, ..., p_{ED(A)})$$

Let $\mathcal{P}_{A,\Omega}$ to be the intersection of $\Phi_{A,\Omega}$ and $Ⓢ_{A,\Omega}$. In other words, $\mathcal{P}_{A,\Omega} = \Phi_{A,\Omega}^{-1}(0 \times S_{A,\Omega})$ where $S_{A,\Omega} = \Omega = (p_1, ..., p_{ED(A)})$ are a set of chosen marked point on M.

4.3.3 $\mathcal{F}_{A,\Omega}^{TRI}$ over $\mathcal{B}_{A,\Omega}^{TRI}$

For the space $\mathcal{P}_{A,\Omega}^{TRI}$, we should let the base manifold to be $\mathcal{B}_{A,\Omega}^{TRI} \equiv \mathcal{B}_{A,\Omega} \times (\mathbb{CP}^1)^3$, and let $\mathcal{F}_{A,\Omega}^{TRI} \equiv \mathcal{F}_{A,\Omega} \oplus (M)^3$ to be the Banach bundle over $\mathcal{B}_{A,\Omega}^{TRI}$.

Define the section $\Phi^{TRI}_{A,\Omega}$ in the vector bundle $\mathcal{F}^{TRI}_{A,\Omega}$ over $\mathcal{B}^{TRI}_{A,\Omega}$ by :

$$\Phi^{TRI}_{A,\Omega}(u, J, z_1, ..., z_{ED(A)}, z_a, z_b, z_c) \equiv (du+J(u)\circ duJ_S, u(z_1), ..., u(z_{ED(A)}), u(z_a), u(z_b), u(z_c))$$

Define the zero section $Ⓢ^{TRI}_{A,\Omega}$ in the vector bundle $\mathcal{F}^{TRI}_{A,\Omega}$ over $\mathcal{B}^{TRI}_{A,\Omega}$ by :

$$Ⓢ^{TRI}_{A,\Omega}(u, J, z_1, ..., z_{ED(A)}, z_a, z_b, z_c) \equiv (0, p_1, ..., p_{ED(A)}, p, p, p)$$

Let $\mathcal{P}^{TRI}_{A,\Omega}$ be the intersection of $\Phi^{TRI}_{A,\Omega}$ and $Ⓢ^{TRI}_{A,\Omega}$. In other words, $\mathcal{P}^{TRI}_{A,\Omega} = \Phi^{TRI}_{A,\Omega}{}^{-1}(0\times S^{TRI}_{A,\Omega})$ where $S^{TRI}_{A,\Omega} = S_{A,\Omega} \times (p, p, p)$ for some $p \in M$.

4.3.4 $\mathcal{F}^{\emptyset TP}_{A,\Omega}$ over $\mathcal{B}^{\emptyset TP}_{A,\Omega}$

For the space $\mathcal{P}^{\emptyset TP}_{A,\Omega}$, we should let the base manifold $\mathcal{B}^{\emptyset TP}_{A,\Omega} \equiv \mathcal{B}_{A,\Omega} \times (\mathbb{CP}^1)$, and let $\mathcal{F}^{\emptyset TP}_{A,\Omega} \equiv \mathcal{F}_{A,\Omega} \oplus \mathcal{T}^1_J$ where $\mathcal{T}^1_J$ has the fiber $\Lambda^{1,0}\mathbb{CP}^1 \otimes u^*TM$ at (u, J, z).

Define the section $\Phi^{\emptyset TP}_{A,\Omega}$ in the vector bundle $\mathcal{F}^{\emptyset TP}_{A,\Omega}$ over $\mathcal{B}^{\emptyset TP}_{A,\Omega}$ by :

$$\Phi^{\emptyset TP}_{A,\Omega}(u, J, z_1, ..., z_{ED(A)}, z) \equiv (du + J(u) \circ duJ_S, u(z_1), ..., u(z_{ED(A)}), du_{\restriction z})$$

Define the zero section $Ⓢ^{\emptyset TP}_{A,\Omega}$ in the vector bundle $\mathcal{F}^{\emptyset TP}_{A,\Omega}$ over $\mathcal{B}^{\emptyset TP}_{A,\Omega}$ by :

$$Ⓢ^{\emptyset TP}_{A,\Omega}(u, J, z_1, ..., z_{ED(A)}, z) \equiv (0, p_1, ..., p_{ED(A)}, 0)$$

Let $\mathcal{P}^{\emptyset TP}_{A,\Omega}$ be the intersection of $\Phi^{\emptyset TP}_{A,\Omega}$ and $Ⓢ^{\emptyset TP}_{A,\Omega}$. In other words, $\mathcal{P}^{\emptyset TP}_{A,\Omega} = \Phi^{\emptyset TP}_{A,\Omega}{}^{-1}(0\times S^{\emptyset TP}_{A,\Omega})$ where $S^{\emptyset TP}_{A,\Omega} = S_{A,\Omega} \times 0$.

4.3.5 $\mathcal{F}^{IDTP}_{A,\Omega}$ over $\mathcal{B}^{IDTP}_{A,\Omega}$

For the space $\mathcal{P}^{IDTP}_{A,\Omega}$, we should let the base manifold $\mathcal{B}^{IDTP}_{A,\Omega} \equiv \mathcal{B}_{A,\Omega} \times (\mathbb{CP}^1)^2$, and let $\mathcal{F}^{IDTP}_{A,\Omega} \equiv \mathcal{F}_{A,\Omega} \oplus (\mathcal{T}^1_J)^2$ where $\mathcal{T}^1_J$ has the fiber $\Lambda^{1,0}\mathbb{CP}^1 \otimes u^*TM$ at (u, J, z).

Define the section $\Phi_{A,\Omega}^{IDTP}$ in the vector bundle $\mathcal{F}_{A,\Omega}^{IDTP}$ over $\mathcal{B}_{A,\Omega}^{IDTP}$ by :

$$\Phi_{A,\Omega}^{IDTP}(u, J, z_1, ..., z_{ED(A)}, z_a, z_b) \equiv (du + J(u) \circ duJ_S, u(z_1), ..., u(z_{ED(A)}), du_{\restriction z_a}, du_{\restriction z_b})$$

Define the zero section $Ⓢ_{A,\Omega}^{IDTP}$ in the $\mathcal{F}_{A,\Omega}^{IDTP}$ over $\mathcal{B}_{A,\Omega}^{IDTP}$ by :

$$Ⓢ_{A,\Omega}^{IDTP}(u, J, z_1, ..., z_{ED(A)}, z_a, z_b) \equiv (0, p_1, ..., p_{ED(A)}, du_{\restriction z_a}, du_{\restriction z_b})$$
$$\text{with } u(z_a) = u(z_b), \text{and } \operatorname{im} du_{\restriction z_a} = \operatorname{im} du_{\restriction z_b}.$$

Let $\mathcal{P}_{A,\Omega}^{IDTP}$ be the intersection of $\Phi_{A,\Omega}^{IDTP}$ and $Ⓢ_{A,\Omega}^{IDTP}$. In other words, $\mathcal{P}_{A,\Omega}^{IDTP} = \Phi_{A,\Omega}^{IDTP^{-1}}(0 \times S_{A,\Omega}^{IDTP})$ where $S_{A,\Omega}^{IDTP} = S_{A,\Omega} \times (du_{\restriction z_a}, du_{\restriction z_b})$ with $u(z_a) = u(z_b)$, and $\operatorname{im} du_{\restriction z_a} = \operatorname{im} du_{\restriction z_b}$.

4.3.6 $\mathcal{F}_{A_j,\Omega_j}^{COMP}$ over $\mathcal{B}_{A_j,\Omega_j}^{COMP}$

For the space $\mathcal{P}_{A_j,\Omega_j}^{COMP}$, we should let the base manifold $\mathcal{B}_{A_j,\Omega_j}^{COMP} \equiv \mathcal{B}_{A_j} \times (\mathbb{CP}^1)^{ED(A_j)+1}$, and let $\mathcal{F}_{A_j,\Omega_j}^{COMP} \equiv \mathcal{F}_{A_j} \oplus (M)^{ED(A_j)+1}$.

Define the section $\Phi_{A_j,\Omega_j}^{COMP}$ in the vector bundle $\mathcal{F}_{A_j,\Omega_j}^{COMP}$ over $\mathcal{B}_{A_j,\Omega_j}^{COMP}$ by :

$$\Phi_{A_j,\Omega_j}^{COMP}(u, J, z_1, ..., z_{ED(A_j)+1}) \equiv (du + J(u) \circ du \circ J_S, u(z_1), ..., u(z_{ED(A_j)+1}))$$

Define the zero section $Ⓢ_{A_j,\Omega_j}^{COMP}$ in the $\mathcal{F}_{A_j,\Omega_j}^{COMP}$ over $\mathcal{B}_{A_j,\Omega_j}^{COMP}$ by :

$$Ⓢ_{A_j,\Omega_j}^{COMP}(u, J, z_1, ..., z_{ED(A)+1}) \equiv (0, p_{i_1}, ..., p_{i_{ED(A)+1}}) \tag{4.7}$$

Let $\mathcal{P}_{A_j,\Omega_j}^{COMP}$ be the intersection of $\Phi_{A_j,\Omega_j}^{COMP}$ and $Ⓢ_{A_j,\Omega_j}^{COMP}$. In other words, $\mathcal{P}_{A_j}^{COMP} = \Phi_{A_j,\Omega_j}^{COMP^{-1}}(0 \times S_{A_j,\Omega_j}^{COMP})$ where $S_{A_j,\Omega_j}^{COMP} = \{(p_{i_1}, ..., p_{i_{ED(A_j)+1}})\}$.

4.4 Transversality

In section 4.3, we have set up some Banach bundles $\mathcal{F}_c$ over some Banach manifolds $\mathcal{B}_c$, and then we define two different sections Φ_c and $\circledS_c$ in the vector bundle $\mathcal{F}_c$. By taking linearization of the section Φ_c, we find that the question of the transversality between this two sections is equivalent to another problem of solving a system of differential equations.

Let ∇ be any symmetric connection on TM and take covariant derivative to $\Phi_A(u, J) = du + J(u) \circ du \circ J_S$. Let $(u_t, J(t))$ be a curve in $\mathcal{P}_A$ with $(v, \delta J)$ tangent to $\mathcal{P}_A$ at $(u, J) = (u_0, J_0)$. Here, v is a continuous vector field on M along $u(S)$, and $\delta J \in C^1(M, \Lambda^{0,1}M \otimes TM)$, which means $J(\delta J) + (\delta J)J = 0$ at every point along $u(S)$.

$$\begin{aligned}
&\nabla_{\frac{\partial}{\partial t}} \Phi(u, J) \\
&= \underline{\nabla_{(-)}v + J \circ \nabla_{J_S(-)}v + (\nabla_v J) \circ du(J_S-)} + (\delta J) \circ du \circ (J_S-)
\end{aligned}$$

Recall the Gromov's operator $D_{u,J} : L^{k+1,p}(S, u^*(TM)) \to L^{k,p}(S, \Lambda^{0,1}S \otimes u^*(TM))$ is defined by $D_{u,J}(v)(-) = \nabla_{(-)}v + J \circ \nabla_{J_S(-)}v + (\nabla_v J) \circ du(J_S-)$ which is just exactly the first three terms above. The question of transversality between Φ_A and $\circledS_A$ is equivalent to:

For $\alpha \in L^{k,p}(S, \Lambda^{0,1}S \otimes u^*(TM))$, can we find $(v, \delta J)$ tangent to Φ_A at (u, J) for the equation ??

$$D_{u,J} + (\delta J) \circ du \circ J_S = \alpha$$

Similarly, for the case of $\mathcal{P}_{A,\Omega}$

$$\begin{array}{l} \mathcal{F}_{A,\Omega} \\ \downarrow \Phi_{A,\Omega}, \circledS_{A,\Omega} \\ \mathcal{B}_{A,\Omega} \end{array}$$

For $\alpha \in L^{k,p}(S, \Lambda^{0,1}S \otimes u^*(TM))$, and $v_i \in T_{u(z_i)}M$, can we find $(v, \delta J, \xi_i)$ (here $\xi_i \in T_{z_i}S$) tangent to $\mathcal{P}_{A,\Omega}$ at (u, J, z_i) such that :

$$\begin{cases} D_{u,J}(v) + (\delta J) \circ du \circ J_S = \alpha & (1) \\ v(z_i) + d_{z_i}u(\xi_i) = v_i & (2) \end{cases}$$

First, we should notice that (2) is the behavior of u at a single point z_i.

If we take $\delta J = 0$, with $D_{u,J}$ as a order 1 operator, we can in fact solve the system (1)+(2) to find v in a neighborhood U_i of each marked point z_i, and a neighborhood U_j of each singularity z_j of U. We can always find local solution by some version of uniqueness and existence theorem of differential equation.

Now, we have found ξ_i only involves one marked point z_i, and $\delta J = 0$ inside these U_i, and we have a local solution of v inside these U_i and U_j. Therefore, $(v_{\restriction_{U_i \cup U_j}}, 0, \xi_i)$ is in fact a local solution of the system (1)+(2) in these neighborhoods U_i and U_j in $S = \mathbb{CP}^1$.

We can easily extend $v_{\restriction_{U_i \cup U_j}}$ to be a continuous vector v field on M along $C = u(S)$. Outside $U_i \cup U_j$, we don't always have $D_{u,J}(v) = \alpha$, but we can always find $(\delta J)_{\restriction_{TC}}$ to satisfy $(\delta J) \circ du \circ J_S = \alpha - D_{u,J}(v)$ along C since $du \neq 0$ outside U_j.

Now, we have $(\delta J)_{\restriction_{TC}}$ as an anti-complex map from TC to $TM_{\restriction_C}$ along $C = u(S)$. We can extend it to $(\delta J)_{\restriction_{TM_{\restriction_C}}}$, an anti-complex map from $TM_{\restriction_C}$ to $TM_{\restriction_C}$, and then further extend it to an anti-complex map (δJ) on the entire TM to TM.

This $(v, \delta J, \xi_i)$ is a global solution of the system (1)+(2). This tell us that $\Phi_{A,\Omega}$ and $Ⓢ_{A,\Omega}$ intersect transversally and $\mathcal{P}_{A,\Omega} = \Phi_{A,\Omega} \cap Ⓢ_{A,\Omega}$ is hence a Banach manifold.

The $\mathcal{P}^{COMP}_{A_j,\Omega_j}$ case is essentially the same, we can always find a global solution and that means $\Phi^{COMP}_{A_j,\Omega_j}$ intersects $Ⓢ^{COMP}_{A_j,\Omega_j}$ transversally, and hence $\mathcal{P}^{COMP}_{A_j,\Omega_j} = \Phi^{COMP}_{A_j,\Omega_j} \cap Ⓢ^{COMP}_{A_j,\Omega_j}$ is a Banach space.

For the $\mathcal{P}^{\emptyset TP}_{A,\Omega}$, $\mathcal{P}^{IDTP}_{A,\Omega}$, and $\mathcal{P}^{TRI}_{A,\Omega}$ cases, we have some more linear equations in the system about du_{z_j} at singular points. Since each of these new linear equations only involves one singular point z_j, we can still find local solutions in the neighborhood $U_i \cup U_j$ and then extend to a global solution for the system. Therefore, we have the transversality and $\mathcal{P}^{\emptyset TP}_{A,\Omega}$, $\mathcal{P}^{IDTP}_{A,\Omega}$, and $\mathcal{P}^{TRI}_{A,\Omega}$ are all Banach manifolds.

Chapter 5

The Proof of the Main Theorem

Theorem 18 (Main Theorem 1) Let (M, ω) be a rational symplectic 4-manifold and $A \in H_2(M, \mathbb{Z})$ be a homology class with $K_\omega(A) < 0$, J_0, J_1 be two ω-tamed almost complex structures on (M, ω), let u_0 be a J_0-holomorphic sphere, then there exist a path $(u_t, J_t)_{t \in [0,1]}$ where u_t is a nodal J_t-holomorphic sphere.

Proof :

Recall that a **rational symplectic manifold** can only be $\mathbb{CP}^2$, $\mathbb{CP}^1 \times \mathbb{CP}^1$, or m points blow-ups of them.

First , we can decide that $ED(A)$, the expected dimension of the moduli space P_A, is :

$$ED(A) = -K_\omega(A) - 1 = c_1(M) \cdot A - 1 \tag{5.1}$$

The number $ED(A)$ is the number of marked points we want to add on the domain $\mathbb{CP}^1$. In Barraud's paper, he choose $3d - 1$ ordered marked points on $\mathbb{CP}^1$. (We can see that it is in fact a special case of our theorem since for degree d curves in his case, $K_\omega(dH) = -3d < 0$).

Barraud gave the marked points an order. However, the key point in the proof is about the existence of a path (u_t, J_t), but not counting how many paths are there, we can also let the marked points to be just a collection of unlabeled points without a given order.

Let $J \in J_\omega$ be an ω-tamed almost complex structure on (M, ω), and C be the J-holomprphic curve $u : (\mathbb{CP}^1, J_s) \to (M, \omega, J)$ with the standard complex structure J_s on $\mathbb{CP}^1$, and $[C] = [u(\mathbb{CP}^1)] = A \in H_2(M, \mathbb{Z})$

Let $P = -\sum_{p_i \in \Omega} p_i$ be a divisor on C. Here, Ω is a set of marked points on C. The number of the marked points, $\mid \Omega \mid$, is decided by the equation (5.1). Let $N_0 = u^*(T\mathbb{CP}^2)/TC$ be the normal bundle of C in $\mathbb{CP}^2$. Let $\widetilde{N_0} = N_0 \otimes P$ be the complex bundle twisted by the divisor P, $\widetilde{N_0}$ is still a complex vector bundle on C. Let $D_{u,J}^{N_0}$ be the projection of Gomov's operator $D_{u,J}$ to N_0. According to the dicuss in section 2.6, there is an unique induced operator $\widetilde{D_{N_0}} = \widetilde{\partial_{N_0}} + \widetilde{R_{N_0}}$ on $\widetilde{N_0}$:

$$\widetilde{D_{N_0}} : L^{k+1,p}(S, \widetilde{N_0}) \to L^{k,p}(S, \Lambda^{0,1}S \otimes \widetilde{N_0})$$

Furthermore, we know that the sequence

$$0 \longrightarrow \mathcal{O}_{\widetilde{D_{N_0}}}(S, \widetilde{N_0}) \longrightarrow \mathcal{O}_{D_{u,J}^{N_0}}(S, N_0) \xrightarrow{\tau} N_0 \mid_\Omega \longrightarrow 0 \tag{5.2}$$

is a short exact sequence and hence we can have the corresponding long exact sequence :

$$0 \to ker\widetilde{D_{N_0}} \to kerD^{N_0}_{u,J} \xrightarrow{\tau} N_0 \mid_\Omega \to coker\widetilde{D_{N_0}} \to cokerD^{N_0}_{u,J} \to 0. \tag{5.3}$$

Here, for the J-holomorphic curve C, τ is the evaluation map of N_0 on the set of marked points Ω. Let $s \in L^{k+1,p}(S, \widetilde{N_0})$ be a section of the normal bundle of C in $\mathbb{CP}^2$ and $p \in \Omega$ is one of the marked point, then $\tau(p) = (p, s(p)) \in N_0 \mid_p$.

We are interested in the opertor $\widetilde{D_{N_0}}$, especially its surjectivity and injecitivity. This is because the index of $\widetilde{D_{N_0}}$ will be the same as the index of the linearization $d\widetilde{\pi}$ of the projection map $\widetilde{\pi} : \mathcal{P}_{A,\Omega} \to \mathcal{J}_\omega$ where

$$\widetilde{\pi}(u, J, \{z_i\}_{i=1}^{ED(A)}) = J$$

In fact, We know that (from Proposition 7.12 in [27], for example,) $d\widetilde{\pi}$ is an Fredholm operator and $\widetilde{\pi}$ is an Fredholm map, and

$$ker(d\widetilde{\pi}) = ker(\widetilde{D_{N_0}}) \quad \text{and} \quad coker(d\widetilde{\pi}) = coker(\widetilde{D_{N_0}}) \tag{5.4}$$

From the automatic genericity in section 2.7, we can see that $\widetilde{D_{N_0}}$ is both surjective and injective by computing the first Chern class of the line bundles.

$$c_1(\widetilde{N_0}) = c_1(N_0) - ED(A) = A \bullet A - ED(A) = -1 \tag{5.5}$$

From the theorem in section 2.7, we know that if $c_1(\widetilde{N_0}) \geq 2g-1$, then $coker\widetilde{D_{N_0}} = 0$, $\widetilde{D_{N_0}}$ is surjective; and if $c_1(\widetilde{N_0}) \leq -1$, then $ker\widetilde{D_{N_0}} = 0$, so $\widetilde{D_{N_0}}$ is injective. In our case, since $g(\mathbb{CP}^1) = 0$, and $c_1(\widetilde{N_0}) = -1$, we know that the operator $\widetilde{D_{N_0}}$ is both injective and surjetive.

The surjectivity of $\widetilde{D_{N_0}}$ implies the surjectivity of $d\widetilde{\pi}$, and therefore $\widetilde{\pi}$ is a local submersion. The injectivity of $\widetilde{D_{N_0}}$ implies the injectivity of $d\widetilde{\pi}$, and therefore $\widetilde{\pi}$ is in fact a local diffeomorphism if we let the domain of the projection to be the unparametrized curves in $\mathcal{M}_{A,\Omega} \equiv \mathcal{P}_{A,\Omega}/Aut(\mathbb{CP}^1)$. Therefore, in our setting, we let $\mathcal{P}_{A,\Omega}$ to be parametrized curves, the complex index will be 3.

Notice here, the condition $K_\omega(A) < 0$ in the statement of the main theorem is essential since we need $-K_\omega(A) = c_1(N_0)$ to be positive enough to guarantee the surjectivity of D_{N_0} and $\widetilde{D_{N_0}}$.

In Barraud's case, the main object are J-holomorphic curves of degree d from $\mathbb{CP}^1$ to $\mathbb{CP}^2$. The space we are interested in is $\mathcal{P}_d \equiv \{(u, J) \in \mathcal{U}_d \times \mathcal{J}_\omega \mid \ u \text{ is J-holomorhic}\}$ where $\mathcal{U}_d$ is the collection of degree d curves from $\mathbb{CP}^1$ to $\mathbb{CP}^2$, and $\mathcal{J}_\omega$ is the collection of all ω-tamed almost complex structures on (M, ω).

Here, we need to deal with similar but a little more complicated situations. Let $\mathcal{P}_A$ denote the space of J-holomorphic maps from $\mathbb{CP}^1$ (with standard Kähler structure J_S) to the symplectic 4-manifold (M, ω). The main objects are J-holomorphic curves from $(\mathbb{CP}^1, J)$ to the symplectic 4-manifold (M, ω) with $[u(\mathbb{CP}^1)] = A \in H_2(M, \mathbb{Z})$. Let

$$\mathcal{P}_A \equiv \{(u, J) \in \mathcal{U}_A^{k+1,p} \times \mathcal{J}_\omega \mid u \text{ is } J\text{-holomorphic } \} \tag{5.6}$$

where $\mathcal{U}_A^{k+1,p}$ is the space of all $W^{k+1,p}$ maps from $(\mathbb{CP}^1, J_s)$ to (M, ω, J) with $[u(\mathbb{CP}^1)] = A \in H_2(M, \mathbb{Z})$, and $\mathcal{J}_\omega$ denotes the space of all ω-tamed almost complex structures on (M,ω). Let

$$\mathcal{P}_{A,\Omega} \equiv \{(u, J, \{z_i\}_{i=1}^{ED(A)})) \mid u \in \mathcal{P}_A, u(z_i) = p_i \in \Omega \in M\} \tag{5.7}$$

where $\{z_i\}_{i=1}^{ED(A)}$ are the $ED(A)$ distinct points in $\mathbb{CP}^1$, and $\Omega = \{p_i\}_{i=1}^{ED(A)}$ are $ED(A)$ points in M. This is analogues to the $\mathcal{P}_{\frown}$ space in Barraud's case.

From the argument in Chapter 4, we can find the $\mathcal{P}_{A,\Omega}$ is a Banach manifold since it is the intersection of two transveral sections ($\Phi_{A,\Omega}$ and $Ⓢ_{A,\Omega}$) of the vector bundle $\mathcal{F}_{A,\Omega}$ over $\mathcal{B}_{A,\Omega} = \mathcal{B}_A \times (\mathbb{CP}^1)^{ED(A)}$. For a fixed J, the complex dimension of the preimage of the projection $\widetilde{\pi} : \mathcal{P}_{A,\Omega} \to \mathcal{J}_\omega$ (which means the complex dimension of the J-holomorphic curves with the given almost complex structure J passing through these given points $\{p_i\}_{i=1}^{ED(A)}$ in the given homological class A) will equal to the index of $\widetilde{\pi} : \mathcal{P}_{A,\Omega} \to \mathcal{J}_\omega$. From the surjectivity and injectivity of $\widetilde{D_{N_0}}$, we know that it equals to 3 (complex index).

Furthremore, $\mathcal{M}_{A,\Omega} \equiv \mathcal{P}_{A,\Omega}/\mathrm{Aut}(\mathbb{CP}^1)$ is also an Banach manifold. The complex dimension of the Banach manifold $\mathcal{M}_{A,\Omega}$ equals to the the dimension of $\mathcal{P}_{A,\Omega}$ subtracts three. $\mathcal{M}_{A,\Omega}$, the set of unparametized curves with the given almost complex structure J passing through the points $\Omega = \{p_i\}_{i=1}^{ED(A)}$ $\mathcal{M}_{A,\Omega}$, is of dimension zero.

Likewise, we also need to consider other types of $\mathcal{P}_c$ for all different possible singular type c (analogues to $\mathcal{P}_{\prec}, \mathcal{P}_X, \mathcal{P}_{\asymp}$, and $\mathcal{P}_\Omega$ in Barraud's case), and they are all Banach manifolds for similar reasons. Furthremore, $\mathcal{M}_c = \mathcal{P}_c/\mathrm{Aut}(\mathbb{CP}^1)$ are also Banach manifolds after quotient by the automorphism groups on $\mathbb{CP}^1$.

Consider the projection $\widetilde{\pi} :\ \mathcal{P}_{A,\Omega} \to \mathcal{J}_\omega$, and its linearization $d\widetilde{\pi} : T\mathcal{P}_{A,\Omega} \to T\mathcal{J}_\omega$.

We know that $d\widetilde{\pi}$ is an Fredholm operator and $\widetilde{\pi}$ is an Fredholm map, the proof see Proposition 7.12 in [27].

Likewise, π_c and $d\pi_c$ for all different kinds of "bad parts" are also Fredholm maps and Fredholm operators.

We can calculate the index of these projections, it turns out that $\text{ind}_{\mathbb{C}}\, \widetilde{\pi} = 3$, and $\text{ind}_{\mathbb{C}}\, \pi_c = 2$, for all kinds of "bad parts" c. This is not surprising since for a given J, the "bad parts" are of lower dimension than the "good part". This also tells us that an almost complex structure J is a regular value of one of these π_c if and only if it is not reached by this π_c.

We say a ω-tamed almost complex structure $J \in \mathcal{J}_\omega$ on (M, ω) is a **regular value** of the projection $\widetilde{\pi} : \mathcal{P}_{A,\Omega} \to \mathcal{J}_\omega$ if and only if $\widetilde{\pi}$ is a local submersion at J.

We say J is a **regular value** of $\pi_c : \mathcal{P}_c \to \mathcal{J}_\omega$ for the "bad parts" if and only of it is NOT in the image of the projection π_c. This definition is not surprising at all. Suppose J is in the image of a projection π_c. This means that there is a J-holomorphic curve of type c in the "bad parts". By letting $Aut(\mathbb{CP}^1)$ acts on the said J-holomorphic curve, we get a family of J-holomorphic curves of the same type, this family of curves are of real dimension at least 6, which contradicts with the fact that π_c is of (read) index 4 and the cokernel of π_c is of (real) dimension at least 2.

We say a ω-tamed almost complex structure $J \in \mathcal{J}_\omega$ on (M, ω) is **generic for** $\{A, \Omega\}$ if and only if J is a regular value of $\widetilde{\pi}$ and all π_c for all different possible "bad parts".

Sard-Smale theorem tells us that the regular valus of an index k Fredholm map is of second category provided the Fredholm map is C^{k+1}. Here, these Fredholm maps are projections therefore will be always smooth enough. Therefore, for $\{A, \Omega\}$, the set of almost complex structure generic for $\{A, \Omega\}$ is of second category in $\mathcal{J}_\omega$.

In fact, we can even know that for $\{A, \Omega\}$, the generic almost complex structure is path-connected in $\mathcal{J}_\omega$. To show that, we should begin with two generic J_0 and J_1, they are therefore both regular value for $\widetilde{\pi}$ by definition, there will always be a regular path $(J_t)_{t\in[0,1]}$ in $\mathcal{J}_\omega$ connecting them (See for example Theorem 3.1.7 in [1]).. However, along the path $(J_t)_{t\in[0,1]}$, we don't know if every J_t is always a regular value of $\widetilde{\pi}$ or not, but we do know it can only be either a regular value of $\widetilde{\pi}$ or $\dim_{\mathbb{R}} \mathit{coker}\, \widetilde{\pi} = 1$.

However, the automatic genericity tells us that for any $J \in J_\omega$, $\widetilde{\pi}$ is always a local submersion. This tells us that we can always find $(J_t)_{i\in[0,1]}$ to be a path of regular values of $\widetilde{\pi}$ for the given J_0 and J_1. Finally, since the choice of J_0 and J_1 are arbitrary, this means that the regular values of the projection $\widetilde{\pi}$ is a path-connected subset of $\mathcal{J}_\omega$.

Now, the regular value of π_c (the projection of a certain tye c "bad parts") are the almost complex structures that are not in the image of the projection π_c. The codimension of these projections are at least complex dimensional one (real dimensional 2). We know that if we take away a real codimensional 2 subset from a path connected set, the remaining set will still be a path-connected set. The regular values of $\widetilde{\pi}$ is a path-connected set and the image of the other π_c are all of real codimensional 2, we know that the generic almost complex structures is a path connected subset of $\mathcal{J}_\omega$ for any given pair (A, Ω).

Next, for generic J_0 and J_1, we should consider the restriction of the moduli space on the path of generic structures $(J_t)_{t\in[0,1]}$.

Let $\widetilde{\mathcal{P}_{A,\Omega}} = (\mathcal{P}_{A,\Omega})_{\restriction_{(J_t)_{t\in[0,1]}}}$, and $\widetilde{\mathcal{M}_{A,\Omega}} = (\widetilde{\mathcal{P}_{A,\Omega}})_{\restriction_{(J_t)_{t\in[0,1]}}}/\mathrm{Aut}(\mathbb{CP}^1)$. We can show that $\widetilde{\mathcal{M}_{A,\Omega}}$ is a compact set.

In general, we don't know if $(\mathcal{P}_{A,\Omega})_{\restriction_{(J_t)}}$ is compact or not, we need its compactification. However, all these (J_t) here are generic almost complex structures. This means that they are not in the image of any of the π_c. Therefore, the preimage of the projection $\widetilde{\pi}$ of these (J_t) does not include the curves in the "bad parts", the preimage of the projection $\widetilde{\pi}$ consists of only good nodal pseudo-holomorphic curves. Therefore, the compactification of the set $\widetilde{\mathcal{P}_{A,\Omega}}$ is the set itself. $\widetilde{\mathcal{P}_{A,\Omega}}$ (and hence $\widetilde{\mathcal{M}_{A,\Omega}}$) is already a compact set.

This means that for arbitrary (ω-tamed) generic J_0, J_1, and u_0 a nodal J_0-holomorphic sphere in the class A, we can always find a path $(u_t, J_t)_{t\in[0,1]}$ such that u_t is a nodal J_t-holomorphic sphere in the class A.

The last minor problem is that the almost complex structures J_0 and J_1 may not always be generic for the given $\{A, \Omega\}$, and the isotopy we have above is from a generic J_0 to another generic J_1. It is possible that we need to add some local diffemorphisms to change the J a little bit or change the marked points on M to another set of marked points on M in the two ends of the isotopy in some cases. ■

Like we mentioned in the first chapter, the isotopy problems of J-holomorphic curves can be translated into isotopy problems of symplectic submanifolds, we can have the following theorem. The fact that $\widetilde{\mathcal{M}_{A,\Omega}}$ is a compact set and therefore has only finitely many components implies that there are only finitely many isotopy classes among nodal symplectic spheres in a homology class A.

Theorem 19 (Main Theorem 2) Let (M, ω) be a rational symplectic 4-manifold, and $A \in H_2(M, \mathbb{Z})$ be a homology class with $K_\omega(A) < 0$, then $\mathcal{S}^0_A$, the space of the nodal symplectic spheres in class A has only finitely many isotopy classes.

Chapter 6

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