Regularity aspects of the Navier–Stokes equations in critical spaces

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[1]Within each category, people are listed in roughly chronological order. Some names which fall into multiple categories have not been repeated.

Abstract

For better or for worse, our current understanding of the Navier–Stokes regularity problem is intimately connected with certain dimensionless quantities known as *critical norms*. In this thesis, we concern ourselves with one of the most basic questions about Navier–Stokes regularity: *How must the critical norms behave at a potential Navier–Stokes singularity?* In Chapter 2, we give a broad overview of the Navier–Stokes theory necessary to answer this question. This chapter is suitable for newcomers to the field. Next, we present two of our published papers [4, 5] which answer this question in the context of *homogeneous Besov spaces*. In Chapter 3, we demonstrate that the critical Besov norms $\|u(\cdot,t)\|_{\dot{B}^{-1+3/p}_{p,q}(\mathbb{R}^3)}$, $p,q \in (3,+\infty)$, must tend to infinity at a potential singularity. Our proof has been streamlined from the published version [4]. In Chapter 4 (joint work with Tobias Barker), we develop a framework of *global weak Besov solutions* with initial data belonging to $\dot{B}^{-1+3/p}_{p,\infty}(\mathbb{R}^3)$, $p \in (3,+\infty)$. To illustrate this framework, we provide applications to blow-up criteria, minimal blow-up initial data, and forward self-similar solutions. This chapter has been reproduced from the published version [5].

Contents

Chapter 1

Introduction

In this thesis, we consider the incompressible Navier–Stokes equations

$$\begin{cases} \partial_t u - \Delta u + u \cdot \nabla u + \nabla p = 0 \\ \operatorname{div} u = 0 \end{cases} \tag{NS}$$

in $Q_T = \mathbb{R}^3 \times (0,T)$ with initial data u_0 belonging to a *critical space* X. The Navier–Stokes equations are endowed with a scaling symmetry

$$\begin{aligned} u &\to \lambda u(\lambda x, \lambda^2 t) \\ p &\to \lambda^2 p(\lambda x, \lambda^2 t) \\ u_0 &\to \lambda u_0(\lambda x), \end{aligned} \tag{1}$$

and a critical space is a Banach space X, continuously embedded into the space of tempered distributions on $\mathbb{R}^3$, whose norm is invariant under translations and the above scaling symmetry. A simple example is $X = L^3$. In particular, the solutions we consider may have infinite energy.

The norm $\|\cdot\|_X$ is considered a 'dimensionless quantity' (for example, in the dimension counting of Caffarelli, Kohn, and Nirenberg [32]), akin to a Reynolds number. When $\|u_0\|_X \ll 1$, one might expect that the solution belongs to a perturbative regime in which the non-linear terms are 'small' and the linear dynamics dominate for all time. In [76], Kato famously validated this heuristic when $X = L^3$. Interestingly, small-data-global-existence was already known to Leray [101], see p. 226–228, in terms of quantities $\|u_0\|_{L^2}\|u_0\|_{L^\infty}$ or $\|u_0\|_{L^2}\|\nabla u_0\|_{L^2}^2$. By now, the situation is well understood, with contributions by many authors (see the survey [56] of Gallagher), culminating in the work [85] of Koch and Tataru in $X = \mathrm{BMO}^{-1}$. Moreover, it was shown by Bourgain and Pavlović [25] that (NS) is ill-posed, in the sense of norm inflation, in the maximal critical space $X = \dot{B}^{-1}_{\infty,\infty}$.

Among critical spaces X in which well-posedness holds, there are essentially two categories. First, we have spaces in which local-in-time well-posedness holds for any divergence-free vector field in X. Typical examples include

$$X = \dot{H}^{\frac{1}{2}}, L^3, \dot{B}^{-1+\frac{3}{p}}_{p,q}, \mathrm{VMO}^{-1} \tag{2}$$

with $p, q \in (3, +\infty)$. The spaces $\dot{B}^{-1+3/p}_{p,q}$ are *homogeneous Besov spaces* of negative regularity, and VMO^{-1} is the closure of Schwartz functions in BMO^{-1}. Second, we have spaces in which only small data results are known, such as

$$\widetilde{X} = L^{3,\infty}, \dot{B}^{-1+\frac{3}{p}}_{p,\infty}, \mathrm{BMO}^{-1} \tag{3}$$

with $p \in (3, +\infty)$. The distinction may also be viewed in terms of the density of Schwartz functions. In the spaces $\widetilde{X}$, the restriction to small data is likely not an artifact of the proof. Our current best understanding of this phenomenon is through *scale-invariant solutions*, that is, solutions invariant under the scaling symmetry (1). The spaces $\widetilde{X}$ contain non-trivial -1-homogeneous initial data whereas the spaces X do not. Let $a \in C^\infty(\mathbb{R}^3 \setminus \{0\})$ be a divergence-free scale-invariant vector field and $u_0 = \sigma a$. When $\sigma \ll 1$, the corresponding solutions are unique in a suitable class of small solutions. It was demonstrated by Jia and Šverák in [72] that, as σ is increased, a curve of smooth solutions persists and conjecturally undergoes certain bifurcations [73]. Numerical evidence of these bifurcations was later found by Guillod and Šverák in [66]. In particular, we expect that when σ increases beyond a certain threshold value, there exist two smooth self-similar solutions emanating from the same initial data. This numerical evidence also has important ramifications for the non-uniqueness of weak Leray–Hopf solutions.

The space $\widetilde{X} = \mathrm{BMO}^{-1}$ is particularly interesting because, among the spaces listed above, it is the only space which contains non-trivial idealized vortex filament initial data. This is discussed in Lemma 2.1.1 and Corollary 2.1.2. Vortex filaments are important ‘coherent structures’ around which three-dimensional fluid flows organize. Vortex filament solutions of the Navier–Stokes equations have been investigated recently by Feng and Šverák in [52], Gallay and Šverák in [59], and Bedrossian, Germain, and Harrop-Griffiths in [21].

It has been a well known open problem since Leray's foundational work [101] to determine whether Navier–Stokes solutions may form *singularities* in finite time. In this context, a singularity is a point in spacetime around which the velocity is no longer essentially bounded. It has been known since the work of Caffarelli, Kohn, and Nirenberg in [32] that finite time blow-up is characterized by singularity formation.

In this thesis, we present two papers [4, 5] motivated by the following question:

(Q) *How must the critical norms* $\|u(\cdot,t)\|_X$ *behave at a potential singularity?*

Let $u_0 \in C_0^\infty(\mathbb{R}^3)$ be a divergence-free vector field and u be the solution of the Navier–Stokes equations with initial data u_0. Let $T^* \in (0,+\infty]$ be its 'maximal time of smoothness'. In [49], Escauriaza, Seregin, and Šverák made the following important contribution: If $T^*(u_0) < +\infty$, then

$$\limsup_{t \to T^*_-} \|u(\cdot,t)\|_{L^3(\mathbb{R}^3)} = +\infty. \tag{4}$$

Since then, the blow-up criterion (4) has been generalized in a number of directions. Notably, Seregin demonstrated in [121] that (4) holds with $\limsup$ replaced by $\lim$. The main idea of (4) and Seregin's theorem is described in the introductions of Chapters 3 and 4. Other generalizations, listed below in roughly chronological order within each category, include

1. treatment of boundaries [120, 110, 16, 7, 46],
2. dimensions $n \geq 4$ [47, 104, 46],
3. $\lim$ rather than $\limsup$ criteria [124, 121, 16, 4, 5], and
4. more general critical spaces [112, 16, 57, 58, 141, 40, 4, 104, 5].

Recently,[1] Tao demonstrated the following *slightly supercritical* blow-up criterion:

$$\limsup_{t \to T^+_-} \frac{\|u(\cdot,t)\|_{L^3(\mathbb{R}^3)}}{(\log\log\log(1/T^* - t))^C} = +\infty. \tag{5}$$

[1] In particular, after the papers [4, 5] comprising this thesis were published.

In contrast to all known proofs of (4), Tao's proof is direct, rather than by contradiction. In [20], Barker and Prange gave an alternative proof which also quantifies the local concentration of L^3 norm at a presumed singularity.[2]

In Chapter 3, we present our first contribution to the story:

Theorem 1 (Blow-up criterion [4])**.** *Let $p,q \in (3,+\infty)$. If $T^* < \infty$, then*

$$\lim_{t \to T^*_-} \|u(\cdot,t)\|_{\dot{B}^{-1+3/p}_{p,q}(\mathbb{R}^3)} = +\infty. \tag{6}$$

In the series of papers [78, 57, 58], Gallagher, Koch, and Planchon generalized (4) to Besov spaces by the 'concentration compactness & rigidity' techniques of Kenig and Merle [79], which feature prominently in dispersive PDE.[3] Theorem 1 sharpens Gallagher et al.'s $\limsup$ criterion. Furthermore, our proof, which some might consider to be more elementary, is based on a simple decomposition of the solution inspired by C. P. Calderón [33],[4] energy estimates, and the backward uniqueness arguments in [49] (which were also exploited by Gallagher et al. in [58]). Theorem 1 additionally sharpens Seregin's $\lim$ criterion in L^3 [121].

The proof we present has been streamlined from the published version in [4]. The (lengthy) appendix of [4] has also been streamlined and incorporated into Chapter 2, where we review the known regularity theory of (NS).

In Chapter 4, we present joint work [5] with Tobias Barker. It is reproduced essentially verbatim from the published version. In [5], we develop a way to analyze **(Q)** and other 'critical problems' through the lens of a special class of solutions, which we term *global weak Besov solutions*. This solution class satisfies three essential properties:

◊ (Existence) *For each divergence-free $u_0 \in \dot{B}^{-1+3/p}_{p,\infty}$, $p \in (3,+\infty)$, there exists a global weak Besov solution with initial data u_0.*

[2]Concentration phenomena had previously been explored in [103, 107, 19].

[3]Questions similar to **(Q)** have also been explored in the dispersive literature. For example, do solutions of the defocusing cubic NLS that belong to $L^\infty_t \dot{H}^{1/2}_x$ in dimension three exist globally and scatter? This was answered in the affirmative in [80] by profile decomposition techniques.

[4]The approach of splitting an infinite energy solution into a part with finite energy and a perturbative part also appears in dispersive PDEs. Key phrases include 'Fourier truncation method', 'almost conservation laws', and the 'I-method', see [23, 24, 41, 42] and many others.

- $\Diamond$ (Weak-strong uniqueness) *These weak solutions are smooth and unique when* u_0 *belongs to the perturbative regime, for example, when* $\|u_0\|_{\dot{B}^{-1+3/p}_{p,\infty}} \ll 1$.

- $\Diamond$ (Weak-$*$ stability) *This class of weak solutions is compact (in a suitable topology) with respect to weak-$*$ convergence of the initial data in* $\dot{B}^{-1+3/p}_{p,\infty}$.

In particular, the weak-$*$ stability property plays a key role in investigations of blow-up criteria [124, 121] and minimal blow-up initial data [118, 71]. To illustrate this, we present the following applications, roughly stated, of our framework to the Navier–Stokes theory:

Theorem 2 (Applications [5])**.** *Let* $p \in (3, +\infty)$.

- $\Diamond$ (Blow-up criterion) *If* u *develops a singularity at time* $T^* < +\infty$ *and* $u(\cdot, T^*)$ *vanishes upon zooming in on the singularity, then* $\|u(\cdot,t)\|_{\dot{B}^{-1+3/p}_{p,\infty}} \to +\infty$ *as* $t \to T^*_-$.

- $\Diamond$ (Minimal blow-up data) *If Navier–Stokes solutions develop singularities, then there exists an initial datum with minimal norm in* $\dot{B}^{-1+3/p}_{p,\infty}$ *and a corresponding solution which develops a singularity. The set of such minimal blow-up initial data is compact modulo symmetries in the weak-$*$ topology.*

- $\Diamond$ (Self-similar solutions) *If* $u_0 \in \dot{B}^{-1+3/p}_{p,\infty}$ *is divergence free and scale invariant, then there exists a corresponding scale-invariant solution.*

Finally, we mention two further papers [6, 7] coauthored with Tobias Barker. In [6], we explore how to generate a Navier–Stokes singularity by 'zooming out' on a smooth ancient solution. In [7], we consider localised blow-up criteria, with boundary, in certain cases where the original proofs rely on global information about the solution. Roughly speaking, we prove that in each spatial neighborhood N of a presumed singularity, we must have $\|u(\cdot,t)\|_{L^3(N)} \to +\infty$ as $t \to T^*_-$. We also describe connections to the Liouville conjecture of Koch, Seregin, Šverák, and Nadirashvili in [82] for mild bounded ancient solutions of (NS).

Unfortunately, we have not succeeded in extending Chapter 4 to the endpoint critical space BMO^{-1}. Therefore, we find it fitting to conclude with the following open problem:

> **(Weak-strong uniqueness in VMO^{-1})** *Let* $u_0 \in L^2 \cap \mathrm{VMO}^{-1}$ *be a divergence-free vector field. Let* u *and* v *be weak Leray–Hopf solutions with initial data* u_0. *Must* $u \equiv v$*?*

Partial progress on this question has been made by Germain in [60, 61], Kukavica and Vicol in [89], and Barker in [15, 18], among others. Notably, in [60], Germain classified the spaces of divergence-free vector fields on which a certain trilinear form that plays a prominent role in weak-strong uniqueness is bounded. Also notably, weak-strong uniqueness in the Besov spaces $\dot{B}^{-1+3/p}_{p,q}$, $3 < p, q < +\infty$ was shown only recently by Barker in [15].[5]

It is tempting to believe that the answer to the above question is 'yes', but for the moment, the proof remains elusive.

[5]One may consider the question also in the class of *suitable* weak Leray–Hopf solutions. In this context, weak-strong uniqueness also holds for small data in the endpoint Morrey space $M^{2,-1}$. See Lemarié-Rieusset [97] and Bradshaw and Tsai [29].

Chapter 2

Review of the Navier–Stokes theory

In this chapter, we review aspects of the mathematical theory of the Navier–Stokes equations. We incorporate material from our published work [4, Appendix].

There are many excellent resources on nearby aspects of the Navier–Stokes theory (in roughly reverse chronological order): Two modern books are by Tsai [140] and Seregin [126]. An introductory modern book focusing on the whole space is by Robinson, Rodrigo, and Sadowski [115]. We recommend the online lectures notes of Šverák [132] and Tao [133]. Two comprehensive books with a harmonic analysis flavor in the whole space are by Lemarié-Rieusset [95, 98]. A review of the mild solution theory, also with a harmonic analysis flavor, is contained in [11, Chapter 5]. An abstract semigroup approach is contained in [129]. A classical book with a dynamical flavor is by Constantin and Foias [43]. Two more classical books are by Temam [135] and Ladyzhenskaya [91]. *We highly recommend the original 1934 paper [101] of Leray*, which was translated into English in [102].

Fefferman's overview of the Clay Millenium Problem on Navier–Stokes regularity is [51]. See also Ladyzhenskaya's response [92].

Further topics in mathematical fluid dynamics, beyond the scope of this thesis, are described in the *Handbook of Mathematical Analysis in Mechanics of Viscous Fluids* [64], which collects specialized survey articles from approximately 2016. Two surveys of particular relevance are [56] by Gallagher, which reviews the Navier–Stokes theory in critical function spaces, and [127] by Seregin and Šverák, which contains a treasure trove of heuristics about Navier–Stokes regularity.

Notational remarks. The constants C are implicitly allowed to depend on the dimension n. Occasionally, we abuse notation for scalar- and vector-valued functions.

Contents

2.1 Function spaces

Good general references for function spaces and interpolation theory include the books of Bergh and Löfstrom [22] and Runst and Sickel [116]. We recommend also the lecture notes of Salo [119]. A classical reference is the book [136] of Triebel.

Besov spaces

Our treatment is based on [11, Chapter 2] by Bahouri, Chemin, and Danchin. See also Lemarié-Riuesset's book [95, Chapters 2–4].

Let $n \geq 1$. There exists a compactly supported smooth function φ on $\mathbb{R}^n$ satisfying the properties

$$\operatorname{supp}(\varphi) \subset \{\xi \in \mathbb{R}^n : 3/4 \leq |\xi| \leq 8/3\}, \tag{2.1.1}$$

$$\sum_{j \in \mathbb{Z}} \varphi(2^{-j}\xi) = 1, \quad \xi \in \mathbb{R}^n \setminus \{0\}. \tag{2.1.2}$$

For each $j \in \mathbb{Z}$, we define the *Littlewood–Paley projection* $\dot{\Delta}_j$ by the Fourier multiplier

$$\dot{\Delta}_j = \varphi(2^{-j}D). \tag{2.1.3}$$

For tempered distributions u_0 on $\mathbb{R}^n$, the convergence of the sum $\sum_{j \leq 0} \dot{\Delta}_j u_0$ typically occurs only in the sense of tempered distributions modulo polynomials.[1] This is inconvenient for PDE purposes. To remove ambiguity, we consider the subspace $\mathcal{S}_h'$ of tempered distributions on $\mathbb{R}^n$ satisfying the 'realization condition'

$$\lim_{\lambda\to+\infty} \|\theta(\lambda D)u_0\|_{L^\infty(\mathbb{R}^n)} = 0 \text{ for all } \theta \in C_0^\infty(\mathbb{R}^n), \tag{2.1.4}$$

that is, the Fourier transform of u_0 'vanishes at the origin' in a suitable sense. In particular, (2.1.4) guarantees

$$\sum_{j\in\mathbb{Z}} \dot{\Delta}_j u_0 = u_0 \tag{2.1.5}$$

unconditionally in the sense of tempered distributions. Importantly, $\mathcal{S}_h'$ is not closed in the standard topology on tempered distributions (for example, consider approximations to the identity).

The *homogeneous Besov seminorms* are defined for all tempered distributions u_0 on $\mathbb{R}^n$:

$$\|u_0\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)} = \left\| 2^{js} \|\dot{\Delta}_j u_0\|_{L^p(\mathbb{R}^n)} \right\|_{\ell^q(\mathbb{Z})}, \quad s \in \mathbb{R},\ p,q \in [1,+\infty]. \tag{2.1.6}$$

The seminorm $\|\cdot\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)}$ restricted to tempered distributions in the class $\mathcal{S}_h'$ becomes a norm, and the *homogeneous Besov spaces* $\dot{B}^s_{p,q}(\mathbb{R}^n)$ are defined by the property that the above norm is finite. As long as the condition

$$s < n/p \text{ or } (s,q) = (n/p, 1) \tag{2.1.7}$$

is satisfied, $\dot{B}^s_{p,q}(\mathbb{R}^3) \cap \dot{B}^{s_1}_{p_1,q_1}(\mathbb{R}^3)$ is a Banach space for $s_1 \in \mathbb{R}$ and $p_1, q_1 \in [1,+\infty]$, and there is no ambiguity modulo polynomials. A different choice of φ in (2.1.1)–(2.1.2) defines an equivalent Besov seminorm.

From Bernstein's inequality

$$\|\dot{\Delta}_j u_0\|_{L^{p_2}(\mathbb{R}^n)} \leq C 2^{nj\left(\frac{1}{p_2}-\frac{1}{p_1}\right)} \|\dot{\Delta}_j u_0\|_{L^{p_1}(\mathbb{R}^n)}, \tag{2.1.8}$$

and the trivial embedding $\ell^{q_1}(\mathbb{Z}) \hookrightarrow \ell^{q_2}(\mathbb{Z})$, $q_1 \leq q_2$, we deduce the Sobolev embedding

[1] This may be characterized as the dual space of Schwartz functions whose Fourier transforms vanish to infinite order at the origin. See p. 28–30 in [95].

theorem in Besov spaces:

$$\dot{B}^{s_1}_{p_1,q_1}(\mathbb{R}^n) \hookrightarrow \dot{B}^{s_2}_{p_2,q_2}(\mathbb{R}^n), \quad s_1 - \frac{n}{p_1} = s_2 - \frac{n}{p_2},\ s_1 \le s_2,\ q_1 \le q_2. \tag{2.1.9}$$

The above choice of indices is sharp due to the scaling property

$$C_s^{-1}\|u_0\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)} \le \lambda^{s-\frac{n}{p}}\|u_0(\cdot/\lambda)\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)} \le C_s\|u_0\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)} \tag{2.1.10}$$

for all $\lambda \in (0,+\infty)$. Using (2.1.10), we may define an equivalent but homogeneous norm $\|\cdot\|_{\widetilde{B}^s_{p,q}(\mathbb{R}^n)}$ by

$$\|u_0\|_{\widetilde{B}^s_{p,q}(\mathbb{R}^n)} = \sup_{\lambda\in(0,+\infty)} \lambda^{s-\frac{n}{p}}\|u_0(\cdot/\lambda)\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)}. \tag{2.1.11}$$

This is discussed in Triebel's book [137].

The Lebesgue spaces $L^p(\mathbb{R}^n)$ are not Besov spaces except when $p = 2$. Rather, $L^p(\mathbb{R}^n)$, $p \in (1,+\infty)$, belongs to the scale of *Triebel–Lizorkin spaces* $F^s_{p,q}(\mathbb{R}^n)$. Besov spaces also arise as real interpolation spaces of Sobolev spaces, see [95, Chapter 3], whereas complex interpolation of Sobolev spaces yields the Bessel potential spaces $H^{s,p}(\mathbb{R}^n)$.

The heat equation in Besov spaces

We now recall the caloric characterization of homogeneous Besov spaces (see [11, Theorem 2.34]). Let $T \in (0,+\infty]$ and $Q_T = \mathbb{R}^n \times (0,T)$. For $u \in L^1_{\mathrm{loc}}(Q_T)$, we define the *Kato norms* [76] by

$$\|u\|_{\mathcal{K}^s_{p,q}(Q_T)} = \left\|t^{-\frac{s}{2}}\|u(\cdot,t)\|_{L^p(\mathbb{R}^n)}\right\|_{L^q((0,T),\,dt/t)}, \quad s\in\mathbb{R},\ p,q\in[1,+\infty]. \tag{2.1.12}$$

The *Kato space* $\mathcal{K}^s_{p,q}(Q_T)$ is defined by the property that the above norm is finite. To simplify notation, we write

$$\mathcal{K}^s_p(Q_T) = \mathcal{K}^s_{p,\infty}(Q_T) \text{ and } \mathcal{K}_p(Q_T) = \mathcal{K}^{s_p}_p(Q_T), \tag{2.1.13}$$

where $s_p = -1 + n/p$. For all $s \in (-\infty, 0)$, there exists a constant $C_s \in (0,+\infty)$ such that

$$C_s^{-1}\|e^{t\Delta}u_0\|_{\mathcal{K}^s_{p,q}(Q_\infty)} \le \|u_0\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)} \le C_s\|e^{t\Delta}u_0\|_{\mathcal{K}^s_{p,q}(Q_\infty)} \tag{2.1.14}$$

for all tempered distributions u_0 on $\mathbb{R}^n$.

In the papers [54, 58], Gallagher et al. employ certain 'time-space homogeneous Besov spaces', see [11, Section 2.6.3]. These are known as *Chemin–Lerner spaces* (see [37, Definition 2.1]). For a 'time-space tempered distribution' u on Q_T, we define

$$\|u\|_{\widetilde{L}^r_T \dot{B}^s_{p,q}} = \left\|2^{js}\|\dot{\Delta}_j u\|_{L^r_t L^p_x(Q_T)}\right\|_{\ell^q(\mathbb{Z})}, \quad s \in \mathbb{R},\, p, q, r \in [1, +\infty]. \tag{2.1.15}$$

Let $\chi \in C_0^\infty(\mathbb{R}^n)$ with $\chi(\xi) = \sum_{j \le 1} \varphi(2^{-j}\xi)$, $\xi \in \mathbb{R}^n \setminus \{0\}$. Define the low-frequency cutoff $\dot{S}_j$ by the Fourier multiplier

$$\dot{S}_j = \chi(2^{-j}D). \tag{2.1.16}$$

The Chemin–Lerner spaces on Q_T are defined by the property that the above norm (2.1.15) is finite and the 'realization condition'

$$\lim_{j \to -\infty} \|\dot{S}_j u\|_{L^1_{t,\mathrm{loc}} L^\infty_x(Q_T)} = 0. \tag{2.1.17}$$

Then $\widetilde{L}^r_T \dot{B}^s_{p,q} \cap \widetilde{L}^{r_1}_T \dot{B}^{s_1}_{p_1,q_1}$ is a Banach space for all $s_1 \in \mathbb{R}$ and $1 \le r_1, p_1, q_1 \le \infty$ when (2.1.7) is satisfied. To estimate solutions in the above spaces, one may exploit the following property. There exist constants $C, c > 0$, such that for all tempered distributions u_0 on $\mathbb{R}^n$,

$$\|e^{t\Delta}\dot{\Delta}_j u_0\|_{L^p(\mathbb{R}^n)} \le Ce^{-ctj^2}\|\dot{\Delta}_j u_0\|_{L^p(\mathbb{R}^n)}, \quad 1 \le p \le \infty, \tag{2.1.18}$$

see [11, Section 2.1.2]. This observation may be combined with Bernstein's inequality (2.1.8), Young's convolution inequality, and paraproducts, see [58, Appendices] and [11, Section 5.6.1].

Other spaces

Let us mention a few further spaces which play a minor role in this thesis.

Lorentz spaces. Let $p \in [1, +\infty)$ and $q \in [1, +\infty]$. The *Lorentz space* $L^{p,q}(\mathbb{R}^n)$ consists of all measurable functions u_0 on $\mathbb{R}^n$ whose *Lorentz quasinorm*[2]

$$\|u_0\|_{L^{p,q}(\mathbb{R}^n)} = \left\|\lambda\,|\{|u_0| > \lambda\}|^{\frac{1}{p}}\right\|_{L^q(\mathbb{R}_+,\, d\lambda/\lambda)} \tag{2.1.19}$$

[2] More generally, one may allow $p, q \in (0, +\infty]$ and a σ-finite measure space X instead of $\mathbb{R}^n$.

is finite. See Bergh and Löfstrom [22, Chapters 1 & 5] and Lemarié-Rieusset [95, p. 18–20]. The Lorentz spaces contain the Lebesgue spaces: $L^{p,p}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$. When $q = +\infty$, the Lorentz spaces are the weak Lebesgue spaces.

When $p > 1$, the Lorentz quasinorm is equivalent to a norm, and $L^{p,q}(\mathbb{R}^n)$ is a Banach space. Lorentz spaces in this range arise as real interpolation spaces of Lebesgue spaces.

We have the trivial embeddings $L^{p,q_1}(\mathbb{R}^n) \hookrightarrow L^{p,q_2}(\mathbb{R}^n)$ when $q_1 \leq q_2$ and $L^{p_2,q}_{\mathrm{loc}}(\mathbb{R}^n) \hookrightarrow L^{p_1,q}_{\mathrm{loc}}(\mathbb{R}^n)$ when $p_2 \geq p_1$.

When $q > 1$, the Lorentz space $L^{1,q}(\mathbb{R}^n)$ contains functions not belonging to $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and which may not be interpretable as distributions. Nevertheless, the space $L^{1,\infty}(\mathbb{R}^n)$ plays a distinguished role in the theory of singular integrals.

Morrey–Campanato spaces. See Adams' book [3] for more information about Morrey–Campanato spaces. The following notation is non-standard.

Let $p \in [1, +\infty]$ and $\alpha \in [-n, 0]$.[3] The *Morrey space* $M^{p,\alpha}(\mathbb{R}^n)$ consists of all locally integrable functions u on $\mathbb{R}^n$ whose *Morrey norm*

$$\|u_0\|_{M^{p,\alpha}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n} \sup_{r\in(0,+\infty)} r^{-\alpha} \left(\fint_{B_r(x)} |u_0|^p \, dy \right)^{\frac{1}{p}} \tag{2.1.20}$$

is finite. When $p = +\infty$, we write $r^{-\alpha}\|u_0\|_{L^\infty(B_r(x))}$ inside the suprema. Notice that $M^{p,0}(\mathbb{R}^n) = L^\infty(\mathbb{R}^n)$. We have the embeddings

$$M^{p_2,\alpha}(\mathbb{R}^n) \hookrightarrow M^{p_1,\alpha}(\mathbb{R}^n) \tag{2.1.21}$$

when $p_1 \leq p_2$, and

$$L^{p_2}(\mathbb{R}^n) \hookrightarrow M^{p_1,\alpha}(\mathbb{R}^n) \tag{2.1.22}$$

when additionally $\alpha = -n/p_2$. Of particular importance in the Navier–Stokes theory is the end-point critical Morrey space $M^{2,-1}(\mathbb{R}^3)$. This is discussed further in Section 2.4 in connection with local energy solutions.

Let $\alpha \in [-n, 1]$. A closely related space is the *Campanato space* $M^{p,\alpha}_{\mathrm{osc}}(\mathbb{R}^n)$, with seminorm

$$\|u_0\|_{M^{p,\alpha}_{\mathrm{osc}}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n} \sup_{r\in(0,+\infty)} r^{-\alpha} \left(\fint_{B_r(x)} |u_0 - (u_0)_{B_r(x)}|^p \, dy \right)^{\frac{1}{p}}, \tag{2.1.23}$$

[3] If $\alpha \in (-\infty, n) \cup (0, +\infty)$, then $M^{p,\alpha}(\mathbb{R}^n) = \{0\}$.

where $(u_0)_{B_r(x)} = \fint_{B_r(x)} u_0 \, dy$. Clearly, $M^{p,\alpha}(\mathbb{R}^n) \hookrightarrow M^{p,\alpha}_{\rm osc}(\mathbb{R}^n)$. When $\alpha = 0$ and $p < +\infty$, the Campanato space coincides with BMO. When $\alpha \in (0,1)$, the Campanato spaces coincide with the homogeneous Hölder spaces $\dot{C}^\alpha(\mathbb{R}^n) = \dot{B}^\alpha_{\infty,\infty}(\mathbb{R}^n)$.[4]

The Koch–Tataru space [85]. The space $\mathrm{BMO}^{-1}(\mathbb{R}^n)$ consists of all tempered distributions u_0 on $\mathbb{R}^n$ that arise as divergences of BMO vector fields. That is,

$$u_0 = \sum_{k=1}^n \partial_k f_k \tag{2.1.24}$$

where $f_1, \ldots, f_n \in \mathrm{BMO}(\mathbb{R}^n)$. The norm in $\mathrm{BMO}^{-1}(\mathbb{R}^n)$ may be defined as the infimum of $\sum_k \|f_k\|_{\mathrm{BMO}(\mathbb{R}^n)}$ over all such representations of u_0. Moreover, we have the following Carleson measure characterization: u_0 belongs to $\mathrm{BMO}^{-1}(\mathbb{R}^n)$ when

$$\|u_0\|_{\mathrm{BMO}^{-1}(\mathbb{R}^n)} := \sup_{x_0 \in \mathbb{R}^n} \sup_{r \in (0,+\infty)} r \left(\fint_0^{r^2} \fint_{B_r(x_0)} |e^{t\Delta} u_0|^2 \, dx \, dt \right)^{\frac{1}{2}} \tag{2.1.25}$$

is finite. See also the paper [10] of Auscher and Frey.

Besov–Morrey spaces. If X is a normed function space on $\mathbb{R}^n$, one may introduce Besov seminorms based on X by replacing $\|\dot{\Delta}_j u_0\|_{L^p(\mathbb{R}^n)}$ with $\|\dot{\Delta}_j u_0\|_X$ in (2.1.6). When X is a Morrey space, this leads to the notion of *Besov–Morrey spaces*, see [87, 109, 99].

Vortex filaments

A velocity field $u_0 \in L^1_{\rm loc}(\mathbb{R}^3)$ corresponds to a *closed vortex filament* if $u_0 \to 0$ as $|x| \to +\infty$ and its vorticity $\omega_0 = \operatorname{curl} u_0$ is a constant multiple of a Dirac mass along a smooth curve. That is, there exists $\alpha \in \mathbb{R}$ and a smooth embedding $\gamma \colon S^1 \to \mathbb{R}^3$ such that, for each vector field $\varphi \in C^\infty_0(\mathbb{R}^3)$,

$$\langle \omega_0, \varphi \rangle = \alpha \int_\gamma \varphi \cdot d\vec{s}. \tag{2.1.26}$$

Notice that ω_0 is divergence free, since for each scalar-valued $\psi \in C^\infty_0(\mathbb{R}^3)$,

$$\langle \operatorname{div} \omega_0, \psi \rangle = -\alpha \int_\gamma \nabla \psi \cdot d\vec{s} = 0. \tag{2.1.27}$$

[4]When $\alpha \in [-n, 0)$, the ‘local Campanato spaces’ on a bounded domain Ω (with a regularity condition), with averages at scale $O(1)$ built into norm, are equivalent to the ‘local Morrey spaces’.

The velocity field u_0 may be recovered from ω_0 by the Biot–Savart law $u_0 = \operatorname{curl}(-\Delta)^{-1}\omega_0$.

Let $\vec{e}, \vec{c} \in \mathbb{R}^3$ with $|\vec{e}| = 1$. If γ is given by $\mathbb{R}e + \vec{c}$, then we may similarly consider the velocity field corresponding to a *straight vortex filament* as long as we replace the decay condition with $u_0 \to 0$ as $|(x - \vec{c}) - (x - \vec{c}) \cdot \vec{e}\,| \to +\infty$.

The following lemma distinguishes $\mathrm{BMO}^{-1}(\mathbb{R}^3)$ from other critical spaces:

Lemma 2.1.1. *The velocity field of a non-trivial straight vortex filament belongs to* $\mathrm{BMO}^{-1}(\mathbb{R}^3)$ *and does not belong to* $\dot{B}^{-1+3/p}_{p,\infty}(\mathbb{R}^3)$ *for any* $p \in (3, +\infty)$ *or* $M^{2,-1}(\mathbb{R}^3)$.

Proof. Let ω_0 be the vorticity of a non-trivial straight vortex filament with $\alpha \neq 0$. After translation and rotation, we may assume that ω_0 is supported on $\mathbb{R}e_z$. Then $\omega_0 = \operatorname{curl} u_0$ and $u_0 = \operatorname{curl} \psi_0$, where

$$\psi_0(x, z) = (-\Delta)^{-1}\omega_0 = -\frac{\alpha}{2\pi} \log |x| \tag{2.1.28}$$

where $(x, z) \in \mathbb{R}^{2+1}$. In particular, $\psi_0 \in \mathrm{BMO}(\mathbb{R}^3)$ and $u_0 \in \mathrm{BMO}^{-1}(\mathbb{R}^3)$. Next, consider $\dot{\Delta}_j u_0$, $j \in \mathbb{Z}$. Since u_0 in non-trivial and not a polynomial, there exists $j_0 \in \mathbb{Z}$ such that $\dot{\Delta}_{j_0} u_0$ is non-trivial. Moreover, since $\dot{\Delta}_{j_0} u_0$ is independent of z, it does not belong to $L^p(\mathbb{R}^3)$ for any $p < +\infty$. In particular, u_0 does not belong to $\dot{B}^{-1+3/p}_{p,\infty}(\mathbb{R}^3)$ for any $p \in (3, +\infty)$.[5] Finally, since u_0 is not locally square integrable, u_0 does not belong to $M^{2,-1}(\mathbb{R}^3)$. $\square$

Corollary 2.1.2. *The velocity field associated to a non-trivial closed vortex filament does not belong to* $\dot{B}^{-1+3/p}_{p,\infty}(\mathbb{R}^3)$ *for any* $p \in (3, \infty)$, $M^{2,-1}(\mathbb{R}^3)$, *or* $\mathrm{VMO}^{-1}(\mathbb{R}^3)$.

Proof. Let u_0 denote the velocity field of a non-trivial closed vortex filament. We zoom in on the vortex filament according to the Navier–Stokes scaling symmetry and pass to a subsequence converging in the sense of distributions to a non-trivial straight vortex filament $\widetilde{u_0}$. Let $X = \dot{B}^{-1+3/p}_{p,\infty}(\mathbb{R}^3)$, $p \in (3, +\infty)$, or $X = M^{2,-1}(\mathbb{R}^3)$. If $u_0 \in X$, then $\|\widetilde{u_0}\|_X \leq \|u_0\|_X$ according to the weak-$*$ convergence properties of X. This contradicts Lemma 2.1.1. Finally, if $u_0 \in \mathrm{VMO}^{-1}(\mathbb{R}^3)$, then $\widetilde{u_0} = 0$. Similar arguments are given in the proof of Theorem 3.1.1. This contradicts that $\widetilde{u_0}$ is non-trivial. $\square$

[5] A slightly different argument is to exploit that, among the homogeneous Besov spaces, the velocity field of a non-trivial point vortex in $\mathbb{R}^2$ only belongs to $\dot{B}^{-1+2/p}_{p,\infty}(\mathbb{R}^2)$, $p \in [1, +\infty]$.

2.2 Linear Stokes theory

We consider the Cauchy problem for the time-dependent Stokes equations:

$$\begin{cases} \partial_t u - \Delta u + \nabla p = f & \text{in } \mathbb{R}^n \times (0,+\infty) \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}^n \times (0,+\infty) \\ u(\cdot,0) = u_0 & \text{in } \mathbb{R}^n \end{cases} \tag{2.2.1}$$

where $\operatorname{div} u_0 = 0$. In the following discussion, we consider u_0 and f 'sufficiently smooth' and 'sufficiently decaying', and we ask that $u \to 0$ as $|x| \to +\infty$.

In the whole space, the situation is particularly simple. When $\operatorname{div} f = 0$, the solution to (2.2.1) is the solution of the heat equation:

$$u(\cdot,t) = e^{t\Delta} u_0 + \int_0^t e^{(t-s)\Delta} f(\cdot,s)\,ds \tag{2.2.2}$$

and $\nabla p = 0$. When $\operatorname{div} f$ is non-zero, we apply the *Leray projection* $\mathbb{P}$ to the forcing term f:

$$u(\cdot,t) = e^{t\Delta} u_0 + \int_0^t e^{(t-s)\Delta} \mathbb{P} f(\cdot,s)\,ds \tag{2.2.3}$$

and $\nabla p = \mathbb{Q} f$. The projection operators $\mathbb{P}$ and $\mathbb{Q}$ onto divergence-free and gradient fields, respectively, are defined by

$$\mathbb{Q} = \nabla(\Delta)^{-1} \operatorname{div} \text{ and } \mathbb{P} = I - \mathbb{Q}. \tag{2.2.4}$$

These projections are Fourier (matrix) multipliers with 0-homogeneous symbols smooth away from the origin. In particular, $\mathbb{P}$ and $\mathbb{Q}$ are bounded operators on $L^p(\mathbb{R}^n)$ for all $p \in (1,+\infty)$. Additionally, $\mathbb{P}$ and $\mathbb{Q}$ are bounded on the homogeneous Besov spaces $\dot{B}^s_{p,q}(\mathbb{R}^n)$ for $s \in \mathbb{R}$ and $p,q \in [1,+\infty]$ (see [11, Proposition 2.30]).[6]

It is often the case that the forcing term f is in divergence form $f = -\operatorname{div} F$, where F is a matrix. Applying div to (2.2.1), we discover that the pressure p satisfies

$$p = (-\Delta)^{-1} \operatorname{div}\operatorname{div} F. \tag{2.2.5}$$

[6] Notice that the endpoints are included!

On the whole space, $\mathbb{P}$ commutes with Δ and the semigroup $e^{t\Delta}$. Applying $\mathbb{P}$ to the heat kernel Γ, we may compute the matrix-valued *Oseen kernel* $K = K(x,t)$ associated to $e^{t\Delta}\mathbb{P}$:

$$K_{ij}(x,t) = \Gamma - (\mathrm{pv}\,\partial_i\partial_j G) * \Gamma, \tag{2.2.6}$$

where G is the fundamental solution of $-\Delta$. See, for example, [95, Chapter 11]. The operator $e^{t\Delta}\mathbb{P}\colon L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ is bounded for all $p \in (1,+\infty)$ by considering it as a composition of operators. The corresponding bound fails when $p \in \{1,+\infty\}$. One justification is as follows. We have

$$K(x,t) = \frac{1}{t^{\frac{n}{2}}} K\left(\frac{x}{t^{\frac{1}{2}}}, 1\right), \tag{2.2.7}$$

where $K(x,1)$ is a smooth matrix-valued function satisfying

$$C^{-1}(1+|x|^n) \leq |K(x,1)| \leq C(1+|x|^n). \tag{2.2.8}$$

Since the kernel is not integrable, we may consider initial data approximating a Dirac mass to show that $e^{t\Delta}\mathbb{P}$ is unbounded on $L^1(\mathbb{R}^n)$. Moreover, if the operator were bounded on $L^\infty(\mathbb{R}^n)$, then a duality argument would imply that it were bounded on $L^1(\mathbb{R}^n)$.

When $f = \operatorname{div} F$, we have further pointwise estimates on the kernel $\widetilde{K}$ associated to $e^{t\Delta}\mathbb{P}\operatorname{div}$ and its derivatives:

$$|\partial_t^k \nabla_x^\ell \widetilde{K}(x,t)| \leq C(k,\ell)(1+|x|)^{-(n+1+k+\ell)} \times t^{-(n+1+2k+\ell)/2}. \tag{2.2.9}$$

In particular, we have

$$\|\partial_t^k \nabla_x^\ell e^{t\Delta}\mathbb{P}\operatorname{div} F\|_{L^{p_2}(\mathbb{R}^n)} \leq C(k,\ell) t^{\frac{1}{2}\left(\frac{n}{p_2} - \frac{n}{p_1} - 1 - 2k - \ell\right)} \|F\|_{L^{p_1}(\mathbb{R}^n)}, \tag{2.2.10}$$

for all $t > 0$ and $1 \leq p_1 \leq p_2 \leq +\infty$.

The semigroup bounds (2.2.10) imply Stokes estimates in the Kato spaces that arise naturally from the caloric characterization (2.1.14) of homogeneous Besov spaces. We summarize them in a single lemma:

Lemma 2.2.1 (Estimates in Kato spaces)**.** *Let* $0 < T \leq +\infty$ *and* $1 \leq p_1 \leq p_2 \leq +\infty$ *such* *that*

$$s_2 - \frac{3}{p_2} = 1 + s_1 - \frac{3}{p_1}. \tag{2.2.11}$$

In addition, assume the conditions

$$s_1 > -2, \qquad \frac{3}{p_1} - \frac{3}{p_2} < 1. \tag{2.2.12}$$

(For instance, if $p_2 = +\infty$, then the latter condition is satisfied when $p_1 > 3$. If $p_1 = 2$, then the latter condition is satisfied when $p_2 < 6$.) Then

$$\left\| \int_0^t e^{(t-\tau)\Delta} \mathbb{P} \operatorname{div} F(\cdot, \tau)\, d\tau \right\|_{\mathcal{K}^{s_2}_{p_2}(Q_T)} \le C(s_1, p_1, p_2) \|F\|_{\mathcal{K}^{s_1}_{p_1}(Q_T)}, \tag{2.2.13}$$

for all $F \in \mathcal{K}^{s_1}_{p_1}(Q_T)$, and the solution u to the corresponding heat equation belongs to $C((0,T]; L^{p_2}(\mathbb{R}^3))$. Let $k, l \ge 0$ be integers. If we further require that

$$t^{\alpha + \frac{|\beta|}{2}} \partial_t^\alpha \nabla^\beta F \in \mathcal{K}^{s_1}_{p_1}(Q_T), \tag{2.2.14}$$

for all integers $0 \le \alpha \le k$ and multi-indices $\beta \in (\mathbb{N}_0)^3$ with $|\beta| \le l$, then we have

$$\begin{aligned} &\left\| t^{k+\frac{l}{2}} \partial_t^k \nabla^l \int_0^t e^{(t-\tau)\Delta} \mathbb{P} \operatorname{div} F(\cdot, \tau)\, d\tau \right\|_{\mathcal{K}^{s_2}_{p_2}(Q_T)} \\ &\quad \le C(k, l, s_1, p_1, p_2) \Big(\sum_{\alpha=0}^{k} \sum_{|\beta| \le l} \| t^{\alpha + \frac{|\beta|}{2}} F \|_{\mathcal{K}^{s_1}_{p_1}(Q_T)} \Big), \end{aligned} \tag{2.2.15}$$

and the spacetime derivatives $\partial_t^k \nabla^l u$ of the solution u belong to $C((0,T]; L^{p_2}(\mathbb{R}^3))$.

Proof. Let us consider the case when α, β are zero. Suppose that s_1, s_2, p_1, p_2, and F obey the hypotheses of the lemma. Then

$$
\begin{aligned}
\Big\|\int_0^t e^{(t-\tau)\Delta}\mathbb{P}\operatorname{div}F(\cdot,\tau)\,d\tau\Big\|_{L^{p_2}(\mathbb{R}^3)} &\le \int_0^t \|e^{(t-\tau)\Delta}\mathbb{P}\operatorname{div}F(\cdot,\tau)\|_{L^{p_2}(\mathbb{R}^3)}\,d\tau \\
&\overset{(2.2.10)}{\le} c\int_0^t (t-\tau)^{\frac{1}{2}(\frac{3}{p_2}-\frac{3}{p_1}-1)}\|F(\cdot,\tau)\|_{L^{p_1}(\mathbb{R}^3)}\,d\tau \\
&\le c\int_0^t (t-\tau)^{\frac{1}{2}(\frac{3}{p_2}-\frac{3}{p_1}-1)}\tau^{\frac{s_1}{2}}\,d\tau \times \sup_{0<\tau<T}\tau^{-\frac{s_1}{2}}\|F(\cdot,\tau)\|_{L^{p_1}(\mathbb{R}^3)} \\
&\overset{(2.2.12)}{\le} c\Big[\big(\frac{s_1}{2}+1\big)^{-1}-2\big(\frac{3}{p_2}-\frac{3}{p_1}+1\big)^{-1}\Big]t^{\frac{1}{2}(\frac{3}{p_2}-\frac{3}{p_1}+s_1+1)}\times\|F\|_{\mathcal{K}_p^{s_1}(Q_T)} \\
&\overset{(2.2.11)}{\le} c\Big[\big(\frac{s_1}{2}+1\big)^{-1}-2\big(\frac{3}{p_2}-\frac{3}{p_1}+1\big)^{-1}\Big]t^{\frac{s_2}{2}}\times\|F\|_{\mathcal{K}_p^{s_1}(Q_T)}.
\end{aligned}
\tag{2.2.16}
$$

This completes the proof of the first estimate. Now let us denote

$$u(\cdot,t) := \int_0^t e^{(t-s)}\mathbb{P}\operatorname{div}F(s)\,ds \tag{2.2.17}$$

for all $0 < t \le T$ and observe the identity

$$u(\cdot,t) = e^{(t-s)\Delta}u(\cdot,\tau) + \int_s^t e^{(t-\tau)\Delta}\mathbb{P}\operatorname{div}F(\cdot,\tau)\,d\tau \tag{2.2.18}$$

for all $0 < s < t$. To prove that $u \in C((0,T];L^{p_2}(\mathbb{R}^3))$, one merely estimates

$$
\begin{aligned}
&\|u(\cdot,t)-u(\cdot,s)\|_{L^{p_2}(\mathbb{R}^3)} \\
&\quad\le \|e^{(t-s)\Delta}u(\cdot,s)-u(\cdot,s)\|_{L^{p_2}(\mathbb{R}^3)} + \int_s^t \|e^{(t-\tau)\Delta}\mathbb{P}\operatorname{div}F(\cdot,\tau)\|_{L^{p_2}(\mathbb{R}^3)}\,d\tau \\
&\quad\le o(1) + c\int_s^t (t-\tau)^{\frac{1}{2}(\frac{3}{p_2}-\frac{3}{p_1}-1)}\tau^{\frac{s_1}{2}}\,d\tau\times\|F\|_{\mathcal{K}_{p_1}^{s_1}(Q_T)} \\
&\quad= o(1) \text{ as } |t-s|\to 0,
\end{aligned}
\tag{2.2.19}
$$

according to the assumption (2.2.12) on the exponents.

Let us now demonstrate how to prove the estimates on spatial derivatives. One estimates the integral in two parts,

$$\begin{aligned}
&\left\| \int_0^t \nabla^l e^{(t-\tau)\Delta} \mathbb{P} \operatorname{div} F(\cdot,\tau)\, d\tau \right\|_{L^{p_2}(\mathbb{R}^3)} \\
&\quad \le c(l) \int_0^{\frac{t}{2}} (t-\tau)^{\frac{1}{2}(\frac{3}{p_2}-\frac{3}{p_1}-1-l)} \|F(\cdot,\tau)\|_{L^{p_1}(\mathbb{R}^3)}\, d\tau \\
&\qquad + c \int_{\frac{t}{2}}^t (t-\tau)^{\frac{1}{2}(\frac{3}{p_2}-\frac{3}{p_1}-1)} \|\nabla^l F(\cdot,\tau)\|_{L^{p_1}(\mathbb{R}^3)}\, d\tau \\
&\quad \le c(l) \int_0^{\frac{t}{2}} (t-\tau)^{\frac{1}{2}(\frac{3}{p_2}-\frac{3}{p_1}-1-l)} \tau^{\frac{s_1}{2}}\, d\tau \times \|F\|_{\mathcal{K}^{s_1}_{p_1}(Q_T)} \\
&\qquad + c \int_{\frac{t}{2}}^t (t-\tau)^{\frac{1}{2}(\frac{3}{p_2}-\frac{3}{p_1}-1)} \tau^{\frac{1}{2}(s_1-l)}\, d\tau \times \|\tau^{\frac{l}{2}} \nabla^l F(\cdot,\tau)\|_{\mathcal{K}^{s_1}_{p_1}(Q_T)} \\
&\quad \le c(l,s_1,p_1,p_2) t^{\frac{1}{2}(s_2-l)} (\|F\|_{\mathcal{K}^{s_1}_{p_1}(Q_T)} + \|\tau^{\frac{l}{2}} \nabla^l F(\cdot,\tau)\|_{\mathcal{K}^{s_1}_{p_1}(Q_T)}).
\end{aligned} \tag{2.2.20}$$

The proof of continuity in $L^{p_2}(\mathbb{R}^3)$ is similar to (2.2.19) except with spatial derivatives in the identity (2.2.18).

The proof of estimates on the temporal derivatives is slightly more cumbersome due to the weighted spaces under consideration and that the temporal derivatives do not preserve the form of the equation. By differentiating the identity (2.2.18) in time, one obtains

$$\partial_t u(\cdot,t) = \partial_t e^{(t-s)\Delta} u(\cdot,s) + e^{(t-s)\Delta} \mathbb{P} \operatorname{div} F(\cdot,s) + \int_s^t e^{(t-\tau)\Delta} \mathbb{P} \operatorname{div} \partial_\tau F(\cdot,\tau)\, d\tau, \tag{2.2.21}$$

and more generally,

$$\begin{aligned}
\partial_t^k u(\cdot,t) &= \partial_t^k e^{(t-s)\Delta} u(\cdot,\tau) + \sum_{\alpha=1}^k \partial_t^{k-\alpha} e^{(t-s)\Delta} \mathbb{P} \operatorname{div} \partial_s^{\alpha-1} F(\cdot,s) \\
&\quad + \int_s^t e^{(t-\tau)\Delta} \mathbb{P} \operatorname{div} \partial_\tau^k F(\cdot,\tau)\, d\tau.
\end{aligned} \tag{2.2.22}$$

(In obtaining the identities, it is beneficial to compare with the differential form of the equation.)

Now set $s := t/2$ and denote the terms by I, II, and III, respectively. We estimate

$$\begin{aligned} \|I\|_{L^{p_2}} &\le c(k)t^{-k}\|u(\cdot,t/2)\|_{L^{p_2}} \\ &\le c(k,p_2)t^{-k+\frac{s_2}{2}}\|u\|_{\mathcal{K}^{s_2}_{p_2}(Q_T)} \\ &\le c(k,s_1,p_1,p_2)t^{-k+\frac{s_2}{2}}\|F\|_{\mathcal{K}^{s_1}_{p_1}(Q_T)}, \end{aligned} \tag{2.2.23}$$

according to our original estimate. Furthermore,

$$\begin{aligned} \|II\|_{L^{p_2}} &\le c(k)\sum_{\alpha=1}^{k} t^{\alpha-k+\frac{1}{2}(\frac{3}{p_2}-\frac{3}{p_1}-1)}\|(\partial_t^{\alpha-1}F)(t/2)\|_{L^{p_1}} \\ &\le c(k,s_1,p_1,p_2)t^{-k+\frac{s_2}{2}}\sum_{\alpha=1}^{k}\|\tau^{\alpha-1}F\|_{\mathcal{K}^{s_1}_{p_1}(Q_T)}, \end{aligned} \tag{2.2.24}$$

and finally,

$$\begin{aligned} \|III\|_{L^{p_2}} &\le c\int_{\frac{t}{2}}^{t}(t-\tau)^{\frac{1}{2}(\frac{3}{p_2}-\frac{3}{p_1}-1)}\|\partial_\tau^k F(\cdot,\tau)\|_{L^{p_1}}\,ds \\ &\le c\int_{\frac{t}{2}}^{t}(t-\tau)^{\frac{1}{2}(\frac{3}{p_2}-\frac{3}{p_1}-1)}\tau^{-k+\frac{s_1}{2}}\,ds \times \|\tau^k\partial_\tau^k F\|_{\mathcal{K}^{s_1}_{p_2}(Q_T)} \\ &\le c(k,s_1,p_1,p_2)t^{-k+\frac{s_2}{2}}\|\tau^k\partial_\tau^k F\|_{\mathcal{K}^{s_1}_{p_2}(Q_T)}. \end{aligned} \tag{2.2.25}$$

This completes the proof of the time derivative estimates. The proof of continuity is similar to (2.2.19) except that one must use the identity (2.2.22).

Regularity in spacetime may be obtained by applying the temporal estimates to the spatial derivatives, since the spatial derivatives preserve the form of the equation. □

What are the key differences between the heat and Stokes equations? We already mentioned that $e^{t\Delta}\mathbb{P}$ is unbounded on $L^1(\mathbb{R}^n)$ and $L^\infty(\mathbb{R}^n)$. Perhaps a more striking difference is the existence of a special class of ‘parasitic solutions’, driven by the pressure, due to Serrin in [128]. Let $\vec{c}(t) \in L^1_{\rm loc}(0,T)$. Then

$$u(x,t) = \vec{c}(t), \quad p(x,t) = \vec{c}\,'(t)\cdot x \tag{2.2.26}$$

is a solution of the time-dependent Stokes and Navier–Stokes equations. More generally, one may allow u to be a potential flow in (2.2.26). Notice that parasitic solutions do not satisfy the solution formula (2.2.3), in terms of integral kernels, except when $\vec{c} \equiv$ const. In particular, the above solutions must be excluded from the solution class in order to restore uniqueness. Since the parasitic solutions are bounded in space but not decaying, it is enough to require the decay condition $u(x,t) \to 0$ as $|x| \to +\infty$ on almost every time slice (or, similarly, $\nabla p(x,t) \to 0$ on almost every time slice). However, often we do wish to discuss solutions u that are 'merely bounded'. In this case, we ask directly that u satisfies the solution formula (2.2.3). For bounded Navier–Stokes solutions, this is equivalent to requiring that $p(\cdot,t) \in \mathrm{BMO}(\mathbb{R}^n)$ on almost every time slice (see [82]). Solutions satisfying (2.2.3) are known as *mild solutions*.

Let us formulate a uniqueness theorem which makes rigorous the above discussion:

Lemma 2.2.2 (Uniqueness)**.** *Let $T \in (0,+\infty)$ and $(u, \nabla p)$ be a tempered distributional solution of the Stokes equations in $\mathbb{R}^n \times (0,T)$ satisfying*

$$u \to 0 \text{ as } |x| \to +\infty \tag{2.2.27}$$

in the following generalized sense: There exists a non-trivial $\theta \in C_0^\infty(\mathbb{R}^n)$ with $\theta \equiv 1$ in a neighborhood of the origin such that

$$\theta(\cdot/\varepsilon)\hat{u} \to 0 \text{ as } \varepsilon \to 0^+ \tag{2.2.28}$$

in the sense of distributions. Assume also that $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n \times (0,T))$ and (there exists a representative such that) $u(\cdot,t) \to 0$ as $t \to 0^+$ in the sense of distributions. Then $u \equiv 0$ and $p \equiv$ const. on $\mathbb{R}^n \times (0,T)$.

Proof. For all $\varepsilon > 0$, define T_ε to be the Fourier multiplier with symbol $1 - \theta(\cdot/\varepsilon)$. By the assumption (2.2.28), we have $T_\varepsilon u \to u$ in the sense of tempered distributions as $\varepsilon \to 0^+$. Since T_ε commutes with ∂_t, Δ, and ∇, we have that $(T_\varepsilon u, \nabla T_\varepsilon p)$ is also a solution of the Stokes equations in $\mathbb{R}^n \times (0,T)$. Notice that the operator $\mathbb{P}$ is well defined on the space of tempered distributions whose Fourier transforms are supported away from the spatial origin, so we may apply $\mathbb{P}$ to the Stokes equations for $T_\varepsilon u$. This gives that $T_\varepsilon u$, which belongs to $L^1_{\mathrm{loc}}(\mathbb{R}^n \times (0,T))$, is a solution of the heat equation on $\mathbb{R}^n \times (0,T)$ with $T_\varepsilon u(\cdot,t) \to 0$ as $t \to 0^+$. Hence, $T_\varepsilon u \in C([0,T];\mathcal{S}'(\mathbb{R}^n))$ with $T_\varepsilon u(\cdot,0) = 0$. Standard uniqueness for the heat equation, which may be proven by duality methods, shows that $T_\varepsilon u \equiv 0$. Taking $\varepsilon \to 0^+$, we have $u \equiv 0$. Finally, the Stokes equations give that $\nabla p \equiv 0$. $\square$

Remark 2.2.3 (A condition on the pressure)**.** As mentioned above, one may instead impose the following condition on the pressure: $\nabla p \in L^1_{\mathrm{loc}}(\mathbb{R}^n \times (0,T))$ and

$$\nabla p \to 0 \text{ as } |x| \to +\infty \tag{2.2.29}$$

in the above generalized sense, along with $u(\cdot,t) \to 0$ as $t \to 0^+$. Applying div to the Stokes equations, we have $\Delta p(\cdot,t) = 0$ for a.e. $t \in (0,T)$. Therefore, $p(\cdot,t)$ must be a harmonic polynomial.[7] The decay condition (2.2.29) on ∇p yields that $\nabla p \equiv 0$. Then u satisfies the heat equation, as above. A similar proof is presented in [5, Remark 3.1], which is reproduced in Chapter 4. Alternatively, uniqueness in the whole space can be shown through the vorticity equation. More sophisticated Liouville theorems for ancient solutions of the Stokes equations were presented by Jia, Seregin, and Šverák in [70] in various domains.

The above parasitic solutions also have important ramifications for the local regularity theory. Suppose $u \in L^2_{t,x}(B \times (-1,0))$ is a solution of the heat equation. It is well known that $u \in C^\infty(B_{1/2} \times (-1/4,0))$. What if instead (u,p) satisfies the Stokes equations with, say, $u,p \in L^2_{t,x}(B \times (-1,0))$? Is u smooth on $B_{1/2} \times (-1/4,0)$? No — the parasitic solutions are a counterexample. Rather, what is true is that $u \in L^2_t H^k_x(B_{1/2} \times (-1/4,0))$ for each $k \in \mathbb{N}$ (for example, by (i) the vorticity equations, or (ii) localizing the solutions,[8] correcting the non-zero divergence, and using the formulas on the whole space). This is because, in the parasitic class, *controlling $\partial_t u$ is equivalent to controlling ∇p.*[9] Heuristically speaking, disturbances may propagate strongly and quickly within the fluid (as opposed to heat conduction, in which disturbances propagate instantaneously but only weakly). This can also be seen from the polynomial

[7] This also encompasses the $p(\cdot,t) \in \mathrm{BMO}(\mathbb{R}^n)$ condition mentioned above: An entire harmonic function belonging to $\mathrm{BMO}(\mathbb{R}^n)$ must be constant.

[8] Notice that the pressure appears on the RHS when you localize the equation in space.

[9] The above discussion does not preclude the Caccioppoli-type inequality

$$\|\nabla u\|_{L^2_{t,x}(B_{1/2}\times(-1/4,0))} \leq C\|u\|_{L^2_{t,x}(B\times(-1,0))}, \tag{2.2.30}$$

without pressure p on the RHS, see Bum Ja Jin [74], Wolf [143], and Chen, Strain, Yau, and Tsai [39, 38]. Dong, Kim, and Phan [48] recently showed L^p variants of this inequality by (i) studying the problem with pure-slip (also known as Lions) boundary conditions, and (ii) localizing the solution. Interestingly, the boundary analogue of (2.2.30) with no-slip boundary conditions was recently shown to fail by Chang and Kang in [36]. Even when the pressure p is included on the RHS, the higher boundary regularity is not the same as in the interior case. Counterexamples were given by Kang in [75] and Seregin and Šverák in [122].

(rather than exponential) decay of the Oseen kernel.

Finally, we comment briefly on the Stokes equations in domains with solid boundaries. Here, the situation is significantly more complicated. We recommend the surveys [68] of Hieber and Saal and [131] of Solonnikov.

Let $\Omega \subset \mathbb{R}^n$ be a smooth, bounded domain or $\Omega = \mathbb{R}^n_+$. It is known that

$$L^2(\Omega; \mathbb{R}^n) = L^2_\sigma(\Omega) \oplus \nabla \dot{H}^1(\Omega), \tag{2.2.31}$$

in the sense of orthogonal decomposition, where $L^p_\sigma(\Omega)$, $p \in [1, +\infty]$, is the Banach space of divergence-free vector fields $u \in L^p(\Omega)$ with vanishing normal trace $u \cdot n = 0$ on $\partial\Omega$, and $\dot{H}^1(\Omega)$ is the space of scalar functions $p \in L^1_{\mathrm{loc}}(\Omega)$ with $\nabla p \in L^2(\Omega)$. The operators $\mathbb{P}$ and $\mathbb{Q}$ are defined as the L^2-orthogonal projections onto these subspaces, respectively, and $\mathbb{P} \colon L^p(\Omega) \to L^p_\sigma(\Omega)$ boundedly when $p \in (1, +\infty)$. Since $\mathbb{P}$ enforces a boundary condition $u \cdot n = 0$, it does not commute with the Laplacian Δ. Hence, the linear theory is typically developed for the *Stokes operator* $A = \mathbb{P}\Delta$ on $L^p_\sigma(\Omega)$, $p \in (1, +\infty)$, with no-slip boundary condition built into the domain:

$$D(A) = W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega) \cap L^p_\sigma(\Omega). \tag{2.2.32}$$

It is well known that A generates an analytic semigroup on $L^p_\sigma(\Omega)$. This is known to fail when $p = 1$ and $\Omega = \mathbb{R}^n_+$ [44], and it was only recently shown, by Abe and Giga [1], that the Stokes semigroup generates an analytic semigroup in $L^\infty_\sigma(\Omega)$.[10] Additionally, maximal regularity holds:

$$\|\partial_t u, \nabla^2 u, \nabla p\|_{L^q_t L^p_x(\Omega \times (0,T))} \leq C(\Omega, p, q) \|f\|_{L^q_t L^p_x(\Omega \times (0,T))}, \tag{2.2.33}$$

for all $p, q \in (1, +\infty)$, when u is the solution of the problem

$$\begin{cases} \partial_t u - \Delta u + \nabla p = f \text{ in } \Omega \times (0,T) \\ \operatorname{div} u = 0 \text{ in } \Omega \times (0,T) \\ u(\cdot, t)|_{\partial\Omega} \cdot n = 0 \text{ in } \partial\Omega \times (0,T) \\ u(\cdot, 0) = 0 \text{ in } \Omega. \end{cases} \tag{2.2.34}$$

[10] See [2] for a proof without the compactness argument.

This is due to Solonnikov in [130] with $C = C(\Omega, p, q, T)$ and C independent of T in [63, 65]. Let $f = \operatorname{div} F$ and $\Omega = \mathbb{R}^n_+$. Then we also have the maximal regularity estimate

$$\|\nabla u\|_{L^q_t L^p_x(\mathbb{R}^n_+ \times (0,T))} \leq C(p,q) \|F\|_{L^q_t L^p_x(\mathbb{R}^n_+ \times (0,T))}, \tag{2.2.35}$$

see (1.6) in [62]. However, the above estimate was shown to be false with $\|p\|_{L^q_t L^p_x(\mathbb{R}^n_+ \times (0,T))}$ on the LHS, see the paper [84] of Koch and Solonnikov. This illustrates that estimates for the pressure p (rather than the pressure gradient ∇p) in the presence of boundary may be quite subtle! Finally, the issues with parasitic solutions still exist in the half space, see, for example, [122]. A flexible uniqueness theorem in the half-space was given by Maekawa, Miura, and Prange in [107, Theorem 5].

2.3 Perturbation methods

In this section, we construct smooth solutions of the Navier–Stokes equations by fixed point arguments. The integral formulation of the Navier–Stokes equations with initial data u_0 is

$$u(\cdot, t) = e^{t\Delta} u_0 - \int_0^t e^{(t-s)\Delta} \mathbb{P} \operatorname{div} u \otimes u \, ds, \tag{2.3.1}$$

where the operator $e^{t\Delta} \mathbb{P} \operatorname{div}$ is convolution with the kernel $\widetilde{K}$ as in Section 2.2 (see also [95, Chapter 11]). We will often simply write

$$u(\cdot, t) = e^{t\Delta} u_0 - B(u, u)(\cdot, t), \tag{2.3.2}$$

where B is formally defined by

$$B(v, w)(\cdot, t) = \int_0^t e^{(t-s)\Delta} \mathbb{P} \operatorname{div} v \otimes w \, ds. \tag{2.3.3}$$

A *mild solution* of the Navier–Stokes equations on Q_T with initial data u_0 is a vector field

u on Q_T satisfying (2.3.1) in a suitable function space.[11] Distributional solutions to the Navier–Stokes equations are mild under fairly general hypotheses, as discussed in [95, Chapter 14] and more recently by Lemarié-Rieusset et al. in [100] and Bradshaw and Tsai in [30].

Recall from Chapter 1 that, among critical spaces in which well-posedness holds, there are essentially two types. Consider the chain of embeddings

$$\dot{H}^{\frac{n}{2}-1} \hookrightarrow L^n \hookrightarrow \dot{B}_{p,q}^{-1+\frac{n}{p}} \hookrightarrow VMO^{-1}, \tag{2.3.4}$$

where $p, q \in (n, +\infty)$. The Navier–Stokes equations are locally well-posed for any divergence-free initial data belonging to any of the above spaces. Meanwhile, the Navier–Stokes equations are globally well-posed for any *small* divergence-free initial data belonging to any of the spaces

$$L^{n,\infty} \hookrightarrow \dot{B}_{p,\infty}^{-1+\frac{n}{p}} \hookrightarrow BMO^{-1}, \tag{2.3.5}$$

where $p \in (n, +\infty)$. Similar small data results are valid in the Morrey spaces

$$M^{p,-1} \hookrightarrow M^{2,-1}, \tag{2.3.6}$$

where $p \in (2, +\infty)$.

The result in $\dot{H}^{n/2-1}$ is due to Fujita and Kato in [53]. The result in L^n is due to Kato in [76], with contributions also by Weissler [142], Giga [63], and Giga and Sohr [65]. The results in Besov spaces are due to Cannone [34] and Planchon [113, 114]. The cases BMO^{-1} and VMO^{-1} were treated by Koch and Tataru in [85]. See also the papers [54] and [9] concerning long-time behavior of solutions in Besov spaces and VMO^{-1}, respectively. For the Morrey space results, see Kato [77], Taylor [134], and Lemarié-Rieusset [97]. Finally, we also mention small data well-posedness in the space $\mathcal{X}^{-1}$ of Lei and Lin [94] which is embedded in BMO^{-1} but does not contain $\dot{H}^{1/2}(\mathbb{R}^3)$.

[11] *Remark*. The (slightly informal) notion of mild solution, which simply means 'satisfies the integral formulation' (in a function space strong enough to make sense of the integral formulation), is not particularly meaningful except to exclude the parasitic solutions discussed in Section 2.2. It is not meaningful to speak of 'the mild solution' with a given initial data (for example, the wild weak solutions constructed by convex integration in [31] are mild) without specifying function spaces that guarantee uniqueness. The terminology 'the smooth solution' is also no good, since the parasitic solutions may be smooth. The solutions constructed by the perturbation theory are sometimes informally known as 'strong solutions', though this term is used by Constantin and Foias [43] for certain solutions with H^1 initial data.

Abstract Picard lemma

We require the following two lemmas concerning fixed points of quadratic equations in Banach spaces. See Lemma A.1 and A.2 in [54], Lemma 5 in [9], or Lemma 5.5 in [11].

Lemma 2.3.1 (Abstract Picard lemma)**.** *Let X be a Banach space, $L\colon X \to X$ a bounded linear operator such that $I - L\colon X \to X$ is invertible, and B a continuous bilinear operator on X satisfying*

$$\|B(x,y)\|_X \leq \gamma \|x\|_X \|y\|_X \tag{2.3.7}$$

for some $\gamma > 0$ and all x, $y \in X$. Then for all $a \in X$ satisfying

$$\|(I-L)^{-1}a\|_X < \frac{1}{4\|(I-L)^{-1}\|_X \gamma}, \tag{2.3.8}$$

the Picard iterates $P_k(a)$, defined recursively by

$$P_0(a) = a, \quad P_{k+1}(a) = a + L(P_k) + B(P_k, P_k),\ k \geq 0, \tag{2.3.9}$$

converge in X to the unique solution $x \in X$ of the equation

$$x = a + L(x) + B(x,x) \tag{2.3.10}$$

such that

$$\|x\|_X < \frac{1}{2\|(I-L)^{-1}\|_X \gamma}. \tag{2.3.11}$$

Regarding the hypothesis on L, the operator $I - L\colon X \to X$ is invertible with norm

$$\|(I-L)^{-1}\|_X \leq \frac{1}{1 - \|L\|_X} \tag{2.3.12}$$

whenever $\|L\|_X < 1$.

Lemma 2.3.2 (Propagation of regularity)**.** *We adopt the notation of Lemma 2.3.1. Let $E \hookrightarrow X$ be a Banach space. Suppose that L is bounded on E such that $I - L\colon E \to E$ is invertible and B maps $E \times X \to E$ and $X \times E \to E$ with*

$$\max(\|B(y,z)\|_E, \|B(z,y)\|_E) \leq \eta \|y\|_E \|z\|_X \tag{2.3.13}$$

for some $\eta > 0$ and all $y \in E$, $z \in X$. Finally, suppose that

$$\|(I-L)^{-1}\|_E \eta \le \|(I-L)^{-1}\|_X \gamma. \tag{2.3.14}$$

Then, for all $a \in E$ satisfying (2.3.8)*, the solution x from Lemma 2.3.1 belongs to E and satisfies*

$$\|x\|_E \le 2\|(I-L)^{-1}a\|_E. \tag{2.3.15}$$

This can be helpful, for example, when $u_0 \in \mathrm{VMO}^{-1} \cap \dot{B}^{-1+n/p}_{p,\infty}$, $p \in (n,+\infty)$. The smallness comes from VMO^{-1} rather than $\dot{B}^{-1+n/p}_{p,\infty}$, but the Besov information is nevertheless propagated forward in time.

Review of the strong solution theory

The following proposition is well known, with contributions due to [101, 50, 142, 76, 63, 65], among others.

Proposition 2.3.3 (Subcritical L^p solution theory)**.** *Let $p \in (3,+\infty)$ and $u_0 \in L^p(\mathbb{R}^3)$ be a divergence-free vector field with $\|u_0\|_{L^p(\mathbb{R}^3)} \le M$.*

There exists $T_p = T_p(M) \in (0,+\infty)$ with

$$T_p^{\frac{1-3/p}{2}} = \frac{C_p}{M} \tag{2.3.16}$$

and a mild solution $u \in C([0,T_p];L^p(\mathbb{R}^3))$ satisfying, for all $q \in [p,+\infty]$,

$$\sup_{t\in(0,T_p)} t^{\frac{3}{2}\left(\frac{1}{p}-\frac{1}{q}\right)+k+\frac{l}{2}} \|\partial_t^k \nabla_x^l u(\cdot,t)\|_{L^q(\mathbb{R}^3)} \le C(p,q,k,l)M \tag{2.3.17}$$

for all integers $k,l \ge 0$, and

$$\|u\|_{L^{5p/3}_{t,x}(\mathbb{R}^3\times(0,T_p))} \le C(p)M. \tag{2.3.18}$$

Mild solutions are unique in the class $C([0,T];L^p(\mathbb{R}^3))$.

Let $T^*(u_0) \in (0,+\infty]$ be the maximal time of existence of the mild solution in $C([0,T];L^p(\mathbb{R}^3))$.[12] *If $T^*(u_0) < +\infty$, then $\|u(\cdot,t)\|_{L^q(\mathbb{R}^3)} \to +\infty$ as $t \to T^*(u_0)$ from below for all $q \in [p,+\infty]$.*

[12] That is, $T^*(u_0) = \sup\{T \in (0,+\infty) : u \in C([0,T];L^p(\mathbb{R}^3))\}$.

If also $u_0 \in L^q(\mathbb{R}^3)$ with $q \in (3,+\infty)$, then the two mild solutions guaranteed to exist as above are identical on their (identical) maximal time of existence.

We may also incorporate $p = +\infty$ into the above proposition provided that we write $L^\infty_{t,x}(\mathbb{R}^3 \times (0,T_\infty))$ instead of $C([0,T_\infty]; L^\infty(\mathbb{R}^3))$ (which would require that additionally $u_0 \in \mathrm{BUC}(\mathbb{R}^3)$).

Proof sketch. The existence and uniqueness of mild solutions in $C([0,T]; L^p(\mathbb{R}^3))$, the guaranteed existence time (2.3.16), and the estimate (2.3.17) with $k = l = 0$ are deduced from Lemma 2.3.1 (Abstract Picard lemma) and bilinear estimates which we derive from Lemma 2.2.1 (Estimates in Kato spaces). For higher regularity, we normalize $\|u_0\|_{L^p(\mathbb{R}^3)} = 1$. The estimate (2.3.17) on a short time interval follows from applying Lemma 2.3.1 in Kato spaces with higher derivatives built in.[13] Since *a priori* one may have to shorten the guaranteed existence time (depending on k and l) when applying Lemma 2.3.1, one uses a covering argument to obtain (2.3.17) on the full time interval $(0,T_p)$. The estimate (2.3.18) follows from bilinear estimates in $L^r_{t,x}$ spaces (see [50, 63]) and an application of Lemma 2.3.1 in an intersection space (and possibly shortening T_p). The characterization of the maximal existence time follows from bilinear estimates $B\colon C([0,T]; L^p(\mathbb{R}^3)) \times C([0,T]; L^q(\mathbb{R}^3)) \to C([0,T]; L^q(\mathbb{R}^3))$ (also with the reversed order) and Lemma 2.3.2 (Propagation of regularity). Finally, uniqueness across different Lebesgue spaces follows from applying Lemma 2.3.1 in an intersection space. $\square$

In the critical case $p = 3$, mild solutions are constructed in an auxiliary space, for example, a Kato space or $L^5(Q_T)$, and then shown to belong to $C([0,T]; L^3(\mathbb{R}^3))$ as a byproduct of the construction. Nevertheless, mild solutions are unique in the class $C([0,T]; L^3(\mathbb{R}^3))$. This has been rediscovered by many authors, and we refer to [95, Chapter 27] for a discussion.

Let $s_p = -1 + 3/p$ when $p \in [1,+\infty]$. Recall the Kato spaces defined in Section 2.1.

The following two propositions are also well known, with contributions due to [34, 113, 114] and others (see [54] and the survey [56]):

Proposition 2.3.4 (Subcritical Besov solution theory). *Let $p \in (3,+\infty]$, $\varepsilon \in (0,|s_p|)$, and $s = s_p + \varepsilon$. Let $u_0 \in \dot{B}^s_{p,\infty}(\mathbb{R}^3)$ be a divergence-free vector field with $\|u_0\|_{\dot{B}^s_{p,\infty}(\mathbb{R}^3)} \le M$.*

[13] Here we use the absence of boundaries in an essential way.

There exists $T_\sharp = T_\sharp(M,p,\varepsilon) \in (0,+\infty)$ with

$$T_\sharp^{\frac{\varepsilon}{2}} = \frac{C(p,\varepsilon)}{M} \tag{2.3.19}$$

and satisfying the following property. There exists a mild solution $u \in \mathcal{K}^s_p(Q_{T_\sharp})$ satisfying, for all $q \in [p,+\infty]$,

$$\|t^{\frac{3}{2}(\frac{1}{p}-\frac{1}{q})+k+\frac{l}{2}}\partial_t^k \nabla_x^l u\|_{\mathcal{K}^s_q(Q_{T_\sharp})} \leq C(p,\varepsilon,k,l)M, \tag{2.3.20}$$

for all integers $k,l \geq 0$.

Mild solutions are unique in the class $\mathcal{K}^s_p(Q_T)$.

The local solutions above may be extended to a maximal time of existence $T^*(u_0)$ in $\mathcal{K}^s_p(Q_T)$ according to Proposition 2.3.3 (Subcritical L^p solution theory). Uniqueness results across Kato spaces are also valid, and in particular, when $u_0 \in L^p(\mathbb{R}^3) \cap B^s_{p,\infty}(\mathbb{R}^3)$, the two solutions guaranteed to exist as above coincide, and there is no ambiguity in the definition of $T^*(u_0)$. The proof of Proposition 2.3.4 is similar to that of Proposition 2.3.3, and we omit it.

Let

$$\mathring{\mathcal{K}}^s_p(Q_T) = \left\{ u \in \mathcal{K}^s_p(Q_T) : \|u\|_{\mathcal{K}^s_p(Q_t)} \to 0 \text{ as } t \to 0^+ \right\}. \tag{2.3.21}$$

Then $\mathring{\mathcal{K}}^s_p(Q_T)$ is a closed subspace of $\mathcal{K}^s_p(Q_T)$. Let $\mathring{B}^{s_p}_{p,\infty}(\mathbb{R}^3)$ be the closure of Schwartz functions in $\dot{B}^{s_p}_{p,\infty}(\mathbb{R}^3)$. Recall also the Chemin–Lerner spaces defined in Section 2.1.

Proposition 2.3.5 (Critical Besov solution theory)**.** *Let $p,q \in (3,+\infty)$ and $u_0 \in \dot{B}^{s_p}_{p,q}(\mathbb{R}^3)$ be a divergence-free vector field. There exists $T^*(u_0) \in (0,+\infty]$ and a mild solution u of the Navier–Stokes equations on $Q_{T^*(u_0)}$ belonging to*

$$C([0,T];\dot{B}^{s_p}_{p,q}(\mathbb{R}^3)) \cap \widetilde{L}^1_T \dot{B}^{s_p+2}_{p,q} \cap \widetilde{L}^\infty_T \dot{B}^{s_p}_{p,q} \tag{2.3.22}$$

and

$$C([0,T];\mathring{B}^{s_p}_{p,\infty}(\mathbb{R}^3)) \cap \mathring{\mathcal{K}}_p(Q_T) \cap C((0,T];L^p(\mathbb{R}^3)) \tag{2.3.23}$$

for all $T \in (0,T^*(u_0))$.

Mild solutions are unique in the classes (2.3.22) *and* (2.3.23) *(separately).*

Let $T_1^*(u_0)$ and $T_2^*(u_0)$ be the maximal times of existence of the solution in the classes (2.3.22) *and* (2.3.23)*, respectively. Then $T_1^*(u_0) = T_2^*(u_0)$, and we omit the subscripts. If $T^*(u_0) <$*

$+\infty$, *then* $\|u(\cdot,t)\|_{L^p(\mathbb{R}^3)} \to +\infty$ *and* $u(\cdot,t)$ *becomes discontinuous in* $\dot{B}^{s_p}_{p,\infty}(\mathbb{R}^3)$ *as* $t \to T^*(u_0)$ *from below.*

Again, when Proposition 2.3.3 or Proposition 2.3.4 also apply, the maximal time of existence is the same, and the solutions are identical. Short-time uniqueness is easy to show by a fixed point argument in intersection spaces. The key is to demonstrate that the maximal time of existence is unambiguously defined, that is, u does not exit one solution class before another. This is addressed by the shared characterization of $T^*(u_0)$ in terms of $\|u(\cdot,t)\|_{L^p(\mathbb{R}^3)} \to +\infty$.

A proof of Proposition 2.3.5 was included in [4, Appendix], paraproducts and all. We present here a sketch of a proof. One simplification is in the proof that $T_1^*(u_0) = T_2^*(u_0)$. In [4], we accomplished this by a cumbersome estimate on ∇u in Chemin–Lerner spaces. Below, we instead use a characterization of the blow-up in terms of continuity in $\mathring{B}^{s_p}_{p,\infty}(\mathbb{R}^3)$.

Proof sketch. Let $r \in (2,+\infty)$ such that $s_p + 2/r \in (0,3/p)$. The local existence theory in (2.3.22) is accomplished through Lemma 2.3.1 (Abstract Picard lemma) and bilinear estimates in $\widetilde{L}^r_T \dot{B}^{s_p+2/r}_{p,q}$, which we hinted at in (2.1.18). The local solution in this space is then bootstrapped into the remaining spaces. This is explained in [54, Appendix], in the appendix of our paper [4], and in [11, Section 5.6]. See also [58, Appendix]. The local solution in (2.3.22) may be extended as long as $u \in C([0,T];\dot{B}^{s_p}_{p,q}(\mathbb{R}^3))$. Moreover, by Lemma 2.3.2 (Propagation of regularity) and further bilinear estimates, the solution in (2.3.22) may be extended as long as $u \in C([0,T];\mathring{B}^{s_p}_{p,\infty}(\mathbb{R}^3))$. *This characterizes* $T_1^*(u_0)$.

The local existence theory in (2.3.23) is simple. It follows from Lemma 2.3.1 (Abstract Picard lemma) and bilinear estimates in $\mathcal{K}_p(Q_T)$, which themselves follow from Lemma 2.2.1 (Estimates in Kato spaces). The local solution in $\mathring{\mathcal{K}}_p(Q_T)$ is then bootstrapped into $C([0,T];\mathring{B}^{s_p}_{p,\infty}(\mathbb{R}^3))$ because

$$B\colon \mathring{\mathcal{K}}_p(Q_T) \times \mathring{\mathcal{K}}_p(Q_T) \to C([0,T];\mathring{B}^{s_p}_{p,\infty}(\mathbb{R}^3)). \tag{2.3.24}$$

We delay the proof of (2.3.24) to the next paragraph. Next, the local solution in (2.3.23) is extended to its maximal time of existence $T_2^*(u_0)$ according to the subcritical L^p theory in Proposition 2.3.3. Accordingly, if $T_2^*(u_0) < +\infty$, then $\|u(\cdot,t)\|_{L^p(\mathbb{R}^3)} \to +\infty$ as $t \to T_2^*(u_0)$ from below. Moreover, due to (2.3.24), $T_2^*(u_0)$ *is also characterized by discontinuity in* $\mathring{B}^{s_p}_{p,\infty}$. Since we already established this characterization for $T_1^*(u_0)$, we conclude that $T_1^*(u_0) = T_2^*(u_0)$. By applying Lemma 2.3.1 in an intersection space, the above solutions are identical on $T^*(u_0)$.

Finally, we prove (2.3.24). We use the caloric characterization of Besov spaces. By extending forward-in-time by zero, we may consider $v, w \in \mathcal{K}_p(Q_\infty)$. We wish to estimate

$$\sup_{\tau \in (0,+\infty)} \tau^{-\frac{s_p}{2}} \|e^{\tau\Delta} B(v,w)(\cdot,t)\|_{L^p(\mathbb{R}^3)} \leq C(p) \|v\|_{\mathcal{K}_p(Q_\infty)} \|w\|_{\mathcal{K}_p(Q_\infty)} \tag{2.3.25}$$

for each $t \in (0,+\infty)$. By rescaling, we may set $\tau = 1$. Then

$$\begin{aligned} &\left\| e^{\Delta} \int_0^t e^{(t-s)\Delta} \mathbb{P} \operatorname{div}(v \otimes w)(\cdot,s)\, ds \right\|_{L^p(\mathbb{R}^3)} \\ &\quad \leq \int_0^t \|e^{(1+t-s)\Delta} \mathbb{P} \operatorname{div}(v \otimes w)(\cdot,s)\|_{L^p(\mathbb{R}^3)}\, ds \\ &\quad \leq C(p) \int_0^t (1+t-s)^{-1-\frac{3}{2p}} s^{-1+\frac{3}{p}}\, ds \times \underbrace{\|v \otimes w\|_{\mathcal{K}_{p/2}^{-2+\frac{6}{p}}(Q_t)}}_{\leq \|v\|_{\mathcal{K}_p(Q_t)} \|w\|_{\mathcal{K}_p(Q_t)}} . \end{aligned} \tag{2.3.26}$$

The factor $(1+t-s)^{-1-\frac{3}{2p}}$ comes from estimating the semigroup $e^{(1+t-s)\Delta}\mathbb{P}\operatorname{div} \colon L^{p/2}(\mathbb{R}^3) \to L^p(\mathbb{R}^3)$. The factor $s^{-1+\frac{3}{p}}$ corresponds to the time weight in the Kato space $\mathcal{K}_{p/2}^{-2+6/p}$ where we estimate $v \otimes w$. When $t \leq 1$, we have

$$\int_0^t (1+t-s)^{-1-\frac{3}{2p}} s^{-1+\frac{3}{p}}\, ds \leq \int_0^1 s^{-1+\frac{3}{p}}\, ds \leq C(p). \tag{2.3.27}$$

When $t > 1$, we have

$$\int_0^t (1+t-s)^{-1-\frac{3}{2p}} s^{-1+\frac{3}{p}}\, ds \leq C(p) t^{-1+\frac{3}{2p}} \leq C(p). \tag{2.3.28}$$

This confirms (2.3.25). If additionally $v, w \in \mathring{\mathcal{K}}_p(Q_T)$, then membership and continuity of $B(v,w)(\cdot,t)$ in $\mathring{B}^{s_p}_{p,\infty}(\mathbb{R}^3)$ may be shown by approximating v, w in the Kato space by $\widetilde{v}, \widetilde{w} \in C([0,T]; L^p(\mathbb{R}^3))$ that vanish in a neighborhood of the origin in time. This completes the proof of (2.3.24). $\square$

2.4 Energy methods and partial regularity

Weak Leray–Hopf solutions

In [101], Leray constructed global-in-time weak solutions of the Navier–Stokes equations with finite energy. Let $u_0 \in L^2(\mathbb{R}^3)$ be a divergence-free vector field and $T \in (0, +\infty]$. A vector field $u \colon Q_T \to \mathbb{R}^3$ is a *weak Leray–Hopf solution* on Q_T with initial data u_0 if the following requirements are satisfied:

1. $u \in L^\infty_t L^2_x \cap L^2_t \dot{H}^1_x(Q_T)$ and $\operatorname{div} u = 0$ in the sense of distributions,

2. u satisfies the Navier–Stokes equations in the following weak sense:

$$\int_{Q_T} -u \cdot \partial_t \varphi + \nabla u : \nabla \varphi - u \otimes u : \nabla \varphi \, dx \, dt = 0 \tag{2.4.1}$$

   for all divergence-free vector fields $\varphi \in C_0^\infty(Q_T)$,

3. u satisfies the global energy inequality

$$\int_{\mathbb{R}^3} |u(x,t)|^2 \, dx + 2 \int_0^t \int_{\mathbb{R}^3} |\nabla u|^2 \, dx \, ds \leq \int_{\mathbb{R}^3} |u_0(x)|^2 \, dx, \tag{2.4.2}$$

   for all $t \in (0, T)$, and

4. $\|u(\cdot, t) - u_0\|_{L^2(\mathbb{R}^3)} \to 0^+$ as $t \to 0^+$.

If $T = +\infty$, then u is a *global weak Leray–Hopf solution*.

The Sobolev embedding $\dot{H}^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$ and interpolation of Lebesgue spaces yield

$$u \in L^l_t L^s_x(Q_T), \quad \frac{2}{l} + \frac{3}{s} = \frac{3}{2}, \; l \in [2, +\infty], \; s \in [3, 6]. \tag{2.4.3}$$

Commonly, $l = s = 10/3$.

It may be shown that weak Leray–Hopf solutions have an *associated pressure*

$$p = (-\Delta)^{-1} \operatorname{div} \operatorname{div} u \otimes u \in L^{5/3}_{t,x}(Q_T) \tag{2.4.4}$$

such that (u, p) satisfies the Navier–Stokes equations in the sense of distributions on Q_T.[14]

[14] The Fourier multiplier argument in Lemma 2.2.2 gives that $\partial_t u - \Delta u = \mathbb{P} f$, where $f = -\operatorname{div} u \otimes u$. Define the associated pressure by (2.4.4). Then we have $\partial_t u - \Delta u + \nabla p = (\mathbb{P} + \mathbb{Q}) f = f$, as desired.

Let $T < +\infty$. The above solutions additionally belong to $C_{\mathrm{wk}}([0,T];L^2(\mathbb{R}^3))$, that is, $u(\cdot,t)$ is continuous on $[0,T]$ as an $L^2(\mathbb{R}^3)$-valued function in the weak topology of $L^2(\mathbb{R}^3)$. Or, more specifically,

$$t \mapsto \int_{\mathbb{R}^3} u(x,t)\cdot\varphi(x)\,dx \tag{2.4.5}$$

belongs to $C([0,T])$ for all vector fields $\varphi \in L^2(\mathbb{R}^3)$. Indeed, under the above assumptions, the Navier–Stokes equations $\partial_t u = \text{RHS}$ imply $\partial_t u \in L^2_t H^{-3/2}_x(Q_T)$. Then, since $u \in L^\infty_t L^2_x(Q_T)$, we may verify the property (2.4.5) when $\varphi \in C_0^\infty(\mathbb{R}^3)$ and argue by density. Weak continuity in time is important because it allows us to define u on every time slice $t \in [0,T]$ rather than almost every time slice. In particular, it allows us to make sense of a potential blow-up profile.

To construct his weak solutions, Leray introduced the *approximate Navier–Stokes equations*

$$\begin{cases} \partial_t u^\varepsilon - \Delta u^\varepsilon + (u^\varepsilon)_\varepsilon \cdot \nabla u^\varepsilon + \nabla p^\varepsilon = 0 \\ \operatorname{div} u^\varepsilon = 0 \end{cases} \tag{NS$_\varepsilon$}$$

where $(f)_\varepsilon$ represents a suitable mollification-in-space of a function f at scale ε. Consider (NS$_\varepsilon$) with initial data $u_0^\varepsilon = (u_0)_\varepsilon$. Then unique smooth solutions u^ε of (NS$_\varepsilon$) satisfying the energy equality exist globally in the energy class.[15] Let $\varepsilon \to 0^+$. With the aid of the Aubins-Lions lemma, we may conclude:

Proposition 2.4.1 (Global weak Leray–Hopf solutions [101])**.** *For each divergence-free vector field $u_0 \in L^2(\mathbb{R}^3)$, there exists a global weak Leray–Hopf solution with initial data u_0.*

If u_0 additionally belongs to the perturbative regimes in Section 2.3, for example, $u_0 \in L^p(\mathbb{R}^3)$, $p \in [3,+\infty]$, then the weak Leray–Hopf solutions and the 'strong solutions' are identical on $\mathbb{R}^3 \times (0, T^*(u_0))$. This is known as *weak-strong uniqueness*. For $u_0 \in \dot{B}^{-1+3/p}_{p,q}(\mathbb{R}^3)$ with $p,q \gg 3$, weak-strong uniqueness was only recently demonstrated by Barker in [15].

The approximation procedure also gives

$$\int_{\mathbb{R}^3} |u(x,t_2)|^2\,dx + 2\iint_{\mathbb{R}^3\times(t_1,t_2)} |\nabla u|^2\,dx\,dt \le \int_{\mathbb{R}^3} |u(x,t_1)|^2\,dx \tag{2.4.6}$$

for almost every $t_1 \in [0,+\infty)$ and every $t_2 \in (t_1,+\infty)$. It is sometimes useful to assume this

[15] One may close a fixed point argument in the norm $C([0,T];L^2(\mathbb{R}^n)) \cap L^2_t \dot{H}^1_x(Q_T)$, for example.

additional property. For example, (2.4.6) appears in estimating the $1/2$-dimensional Hausdorff measure of the set of singular times, showing the temporal decay $\|u(\cdot,t)\|_{L^2(\mathbb{R}^3)} \to 0$ as $t \to +\infty$, and proving 'eventual regularity'.

Suitable weak solutions

Let $z_0 = (x_0, t_0) \in \mathbb{R}^{3+1}$, $R > 0$, and

$$Q(z_0, R) = B_R(x_0) \times (t_0 - R^2, t_0), \quad Q(R) = Q(0, R), \quad Q = Q(1). \tag{2.4.7}$$

We say that (u, p) is a *suitable weak solution* of the Navier–Stokes equations in $Q' = Q(z_0, R)$ if the following requirements are satisfied:

1. $u \in L^\infty_t L^2_x \cap L^2_t H^1_x(Q(z_0, r))$ and $p \in L^{3/2}_{t,x}(Q(z_0, r))$ for all $r \in (0, R)$,
2. (u, p) solves the Navier–Stokes equations in the sense of distributions on Q', and
3. (u, p) satisfies the *local energy inequality*

$$(\partial_t - \Delta)|u|^2 + 2|\nabla u|^2 + \operatorname{div}(|u|^2 + 2p)u \le 0 \tag{2.4.8}$$

in the sense of distributions on Q', that is, allowing only non-negative test functions on Q'.

We say that u is a suitable weak solution in Q' (without reference to the pressure) if there exists $p \in L^{3/2}_{t,x}(Q(z_0, r))$ for all $r \in (0, R)$ such that (u, p) is suitable in Q'. In the literature, occasionally u is asked to belong to $L^\infty_t L^2_x \cap L^2_t H^1_x(Q')$ and p to $L^{3/2}_{t,x}(Q')$.

As before, $u(\cdot, t)$ is continuous on $[t_0 - R^2, t_0]$ with values in $L^2(B_r(x_0))$, endowed with the weak topology, for all $r \in (0, R)$.

If $U \subset \mathbb{R}^{3+1}$ is a domain, we say that (u, p) is a suitable weak solution in U if (u, p) is a suitable weak solution in each parabolic ball $Q' \subset U$. Then, by partition of unity, we may allow $\varphi \in C^\infty_0(U)$ in the local energy inequality.

In the literature (for example, in the book [127]), the local energy inequality is often written in the following (equivalent) form: For all $0 \leq \varphi \in C_0^\infty(Q')$,

$$\begin{aligned}\int_{B_R(x_0)} \varphi |u(x,t)|^2 \, dx + 2 \int_{-\infty}^t \int_{B_R(x_0)} \varphi |\nabla u|^2 \, dx \, ds \\ \leq \int_{-\infty}^t \int_{B_R(x_0)} |u|^2 (\Delta \varphi + \partial_t \varphi) + (|u|^2 + 2p) u \cdot \nabla \varphi \, dx \, ds,\end{aligned} \tag{2.4.9}$$

for every $t \in (t_0 - R^2, t_0)$. Of course, (2.4.8) and (2.4.9) hold with equality when (u,p) is sufficiently smooth.

By more carefully analyzing the convergence as $\varepsilon \to 0^+$ in (NS$_\varepsilon$), we have

Proposition 2.4.2. *The solutions in Proposition 2.4.1 may be taken to be suitable weak solutions.*

The local energy inequality (2.4.8) implies the global energy inequality (2.4.6). This is discussed in Lemarié-Rieusset's book [95] (specifically, Proposition 30.1, p. 318–319).

The following compactness lemma is proven in [105, Theorem 2.2]. The proof relies on the local energy inequality (2.4.8), the embedding $L_t^\infty L_x^2 \cap L_t^2 H_x^1(Q) \hookrightarrow L_{t,x}^{10/3}(Q)$, and the Aubin–Lions lemma [8]:

Lemma 2.4.3 (Compactness)**.** *Let $(u^{(k)}, p^{(k)})_{k \in \mathbb{N}}$ be a sequence of suitable weak solutions on Q satisfying*

$$\sup_{k \in \mathbb{N}} \|u^{(k)}\|_{L^3_{t,x}(Q)} + \|p^{(k)}\|_{L^{3/2}_{t,x}(Q)} < +\infty. \tag{2.4.10}$$

There exists a suitable weak solution (u,p) on Q such that, along a subsequence,

$$\begin{aligned} u^{(k)} &\to u \text{ in } L^3_{\mathrm{loc}}(B \times (-1,0]) \\ p^{(k)} &\rightharpoonup p \text{ in } L^{\frac{3}{2}}_{\mathrm{loc}}(B \times (-1,0]) \end{aligned} \tag{2.4.11}$$

and, for all $R \in (0,1)$, we have

$$u^{(k)} \to u \text{ in } C_{\mathrm{wk}}([-R^2,0]; L^2(B_R)). \tag{2.4.12}$$

We now state an ε-regularity criterion for suitable weak solutions. This version is essentially due to Lin [105]. See also the original paper [32] of Caffarelli, Kohn, and Nirenberg, the paper [93] of Ladyzhenskaya and Seregin, and the survey [49].

Theorem 2.4.4 (ε-regularity)**.** *Let (u,p) be a suitable weak solution on Q satisfying*

$$\|v\|_{L^3_{t,x}(Q)} + \|p\|_{L^{3/2}_{t,x}(Q)} \leq M. \tag{2.4.13}$$

There exists an absolute constant $\varepsilon_0 > 0$ satisfying the following property: If

$$M \leq \varepsilon_0, \tag{2.4.14}$$

then

$$\sup_{Q(3/4)} |v| \leq CM. \tag{2.4.15}$$

Moreover, for each $k \in \mathbb{N}$, $\nabla_x^k v$ is Hölder continuous on $Q(1/2)$ and satisfies

$$\sup_{Q(1/2)} |\nabla_x^k v| \leq C(k,M). \tag{2.4.16}$$

Higher interior regularity (2.4.16) follows from (2.4.15) and the classical paper [128] of Serrin.

Let $u\colon Q(z_0,R) \to \mathbb{R}^3$ be a measurable function. Then z_0 is a *singular point* (or *singularity*) of u if for all $r \in (0,R)$, $u \notin L^\infty_{t,x}(Q(z_0,r))$, and we say that u is *singular* at z_0. Otherwise, we say that z_0 is a *regular point* of u.

The following proposition is contained in Lemma 2.1 and Lemma 2.2 of [118]. For this version, see [6, Proposition 2.3] by Barker and the author.

Proposition 2.4.5 (Persistence of singularities)**.** *Let $(u^{(k)}, p^{(k)})_{k\in\mathbb{N}}$ be a sequence of suitable weak solutions on Q satisfying* (2.4.11)*. If*

$$\limsup_{k\to+\infty} \|u^{(k)}\|_{L^\infty_{t,x}(Q(r))} = +\infty \text{ for all } r \in (0,1), \tag{2.4.17}$$

then

$$u \text{ has a singularity at the space-time origin.} \tag{2.4.18}$$

Local energy solutions

In [95, Chapters 32–33], Lemarié-Rieusset introduced a notion of weak solution with uniformly locally square-integrable initial data. His solutions are called ‘local energy (weak) solutions’ and sometimes ‘local Leray solutions’. This notion was further explored by Kikuchi and Seregin [81], Lemarié-Rieusset [98], and Kwon and Tsai [90], among others. It has been successfully used in investigations of blow-up criteria [124, 121, 19], minimal blow-up initial data [118, 71], and self-similar solutions [72].

Let $p \in [1,+\infty)$ and $L^p_{\mathrm{uloc}}(\mathbb{R}^3)$ denote the set of measurable functions $f\colon \mathbb{R}^3 \to \mathbb{R}$ satisfying

$$\|f\|^p_{L^p_{\mathrm{uloc}}(\mathbb{R}^3)} = \sup_{x_0 \in \mathbb{R}^3} \int_{B_1(x_0)} |f|^p \, dx < +\infty. \tag{2.4.19}$$

Let $\widetilde{L}^p_{\mathrm{uloc}}(\mathbb{R}^3)$ be the closure of test functions in $L^p_{\mathrm{uloc}}(\mathbb{R}^3)$. Then $\widetilde{L}^p_{\mathrm{uloc}}(\mathbb{R}^3)$ is characterized by the property

$$\int_{B_1(x_0)} |f|^p \, dx \to 0 \text{ as } |x_0| \to +\infty. \tag{2.4.20}$$

Locally, L^p_{uloc} functions belong to L^p, but their global behavior is more akin to that of L^∞ functions. The space L^p_{uloc} incorporates L^q for all $q \geq p$ as well as, for example, $L^p + L^q$.

Mild solutions of the Navier–Stokes equations with initial data in $L^p_{\mathrm{uloc}}(\mathbb{R}^3)$, $p \geq 3$, were investigated by Maekawa and Terasawa in [108]. The half-space analogue was investigated in [107] by Maekawa, Miura, and Prange, which successfully employed these solutions to describe concentration phenomena as $t \to T^*_-$. See also Barker and Prange [19].

Let $u_0 \in \widetilde{L}^2_{\mathrm{uloc}}(\mathbb{R}^3)$ be a divergence-free vector field and $T \in (0,+\infty)$. A vector field $u\colon Q_T \to \mathbb{R}^3$ is a *local energy weak solution* on Q_T with initial data u_0 if the following requirements are satisfied:

1. $u \in L^\infty_t(\widetilde{L}^2_{\mathrm{uloc}})_x(Q_T)$ and $\sup_{x_0 \in \mathbb{R}^3} \|\nabla u\|_{L^2_{t,x}(B_1(x_0)\times(0,T))} < +\infty$,
2. there exists $p \in L^{3/2}_{\mathrm{loc}}(\mathbb{R}^3 \times [0,T))$ such that (u,p) is a suitable weak solution on Q_T, and
3. $\|u(\cdot,t) - u_0\|_{L^2(K)} \to 0$ as $t \to 0^+$ for each compact set $K \subset \mathbb{R}^3$.

It is possible to remove the decay conditions on u in exchange for incorporating a pressure decomposition into the above definition. For simplicity, we avoid this extension here.

We borrow the following proposition from [90]:

Proposition 2.4.6 (Local energy estimates)**.** *Let* $T \in (0,+\infty)$ *and* $u_0 \in \widetilde{L}^2_{\mathrm{uloc}}(\mathbb{R}^3)$. *There exists an absolute constant* $\varepsilon_1 > 0$ *satisfying the following property: If*

$$T \leq \frac{\varepsilon_1}{1+\|u_0\|^2_{L^2_{\mathrm{uloc}}(\mathbb{R}^3)}}, \tag{2.4.21}$$

then there exists a local energy weak solution u *on* Q_T *with initial data* u_0 *and satisfying*

$$\sup_{x_0\in\mathbb{R}^3} \|u\|_{L^\infty_t L^2_x(B_1(x_0)\times(0,T))} + \|\nabla u\|_{L^2_{t,x}(B_1(x_0)\times(0,T))} \leq C\|u_0\|_{L^2_{\mathrm{uloc}}(\mathbb{R}^3)}. \tag{2.4.22}$$

Local energy solutions also satisfy a weak-strong uniqueness property, for example, if the initial data belongs to $L^3_{\mathrm{uloc}}(\mathbb{R}^3)$ [95, Theorem 33.2], is small in $L^{3,\infty}(\mathbb{R}^3)$ [69], or is small in $M^{2,-1}(\mathbb{R}^3)$ [97, 29] (see Section 2.1 for Morrey spaces).

Notice that

$$\|u_0\|_{M^{2,-1}(\mathbb{R}^3)} = \sup_{\lambda\in(0,+\infty)} \|\lambda u_0(\lambda\cdot)\|_{L^2_{\mathrm{uloc}}(\mathbb{R}^3)}. \tag{2.4.23}$$

Hence, $M^{2,-1}(\mathbb{R}^3)$ is a natural critical space for the Navier–Stokes equations — it corresponds to requiring a scale-invariant local energy estimate.[16] Local energy weak solutions with initial data in $M^{2,-1}(\mathbb{R}^3)$ were investigated by Lemarié-Rieusset in [97] and Bradshaw and Tsai in [29]. Combining the scaling invariance with the estimate in Proposition 2.4.6, one may control the solution for long times.

Local energy solutions were also investigated in the half-space $\mathbb{R}^3_+$ by Maekawa, Miura, and Prange in [106]. A key difficulty encountered in [106] is to estimate the pressure p (*rather than the pressure gradient* ∇p) arising from the initial data $u_0 \in \widetilde{L}^2_{\mathrm{uloc}}(\mathbb{R}^3)$ in such a way that $up \in L^1_{\mathrm{loc}}([0,T)\times\mathbb{R}^3_+)$. This had been mentioned as an interesting open problem in [16].

2.5 Backward uniqueness and unique continuation

Finally, we recall two theorems concerning backward uniqueness and unique continuation (see [49] by Escauriaza, Seregin, and Šverák and the references therein).

Theorem 2.5.1 (Backward uniqueness)**.** *Let* $Q_+ = \mathbb{R}^3_+ \times (0,1)$. *Suppose* $u \colon Q_+ \to \mathbb{R}^3$ *satisfies the following conditions:*

[16] Whereas $\mathrm{BMO}^{-1}(\mathbb{R}^3)$ essentially corresponds to a scale-invariant local energy estimate at the level of the stream function.

1. $|\partial_t u + \Delta u| \le c(|\nabla u| + |u|)$ *a.e. in* Q_+ *for some* $c > 0$,

2. $u(\cdot, 0) = 0$,

3. $|u(x,t)| \le e^{M|x|^2}$ *in* Q_+, *and*

4. $u, \partial_t u, \nabla^2 u \in L^2_t(L^2_{\mathrm{loc}})_x(Q_+)$.

Then $u \equiv 0$ *on* Q_+.

Theorem 2.5.2 (Unique continuation)**.** *Let* R, $T > 0$ *and* $Q(R,T) = B(R) \times (0,T) \subset \mathbb{R}^{3+1}$. *Suppose* $u\colon Q(R,T) \to \mathbb{R}^3$ *satisfies the following conditions:*

1. $u, \partial_t u, \nabla^2 u \in L^2(Q(R,T))$,

2. $|\partial_t u + \Delta u| \le c(|\nabla u| + |u|)$ *a.e. in* $Q(R,T)$ *for some* $c > 0$, *and*

3. $|u(x,t)| \le C_k(|x| + \sqrt{t})^k$ *in* $Q(R,T)$ *for some* $C_k > 0$ *and all integers* $k \ge 0$.

Then $u(x,0) = 0$ *for all* $x \in B(R)$.

Chapter 3

Blow-up criteria for the Navier–Stokes equations in non-endpoint critical Besov spaces

This chapter contains a streamlined version of the published work [4].

Abstract. We obtain an improved blow-up criterion for solutions of the Navier–Stokes equations in critical Besov spaces. If a strong solution u has maximal existence time $T^* < +\infty$, then the non-endpoint critical Besov norms must become infinite at the blow-up time:

$$\lim_{t \to T^*_-} \|u(\cdot, t)\|_{\dot{B}^{-1+3/p}_{p,q}(\mathbb{R}^3)} = +\infty, \quad p, q \in (3, +\infty).$$

In particular, we introduce *a priori* estimates for the solution based on elementary splittings of initial data in critical Besov spaces and energy methods. These estimates allow us to rescale around a potential singularity and apply backward uniqueness arguments. The proof does not use profile decomposition.

Contents

3.1 Introduction

We are interested in blow-up criteria for solutions of the incompressible Navier–Stokes equations

$$\begin{cases} \partial_t u - \Delta u + u \cdot \nabla u + \nabla p = 0 \\ \operatorname{div} u = 0 \end{cases} \tag{NSE}$$

in $Q_T = \mathbb{R}^3 \times (0, T)$ with divergence-free initial data $u_0 \in C_0^\infty(\mathbb{R}^3)$. It has been known since Leray [101] that a unique smooth solution with sufficient decay at infinity exists locally in time. Furthermore, Leray demonstrated (see p. 226–228) that there exists a constant $c_p > 0$ with the property that if $T^* < \infty$ is the maximal time of existence of the smooth solution, then

$$\|u(\cdot, t)\|_{L^p(\mathbb{R}^3)} \geq c_p \left(\frac{1}{\sqrt{T^* - t}} \right)^{1-3/p} \tag{3.1.1}$$

for all $p \in (3, +\infty]$. Such a characterization exists because the Lebesgue norms in this range are subcritical with respect to the natural scaling symmetry of the Navier–Stokes equations,

$$u(x,t) \to \lambda u(\lambda x, \lambda^2 t), \quad p(x,t) \to \lambda^2 p(\lambda x, \lambda^2 t). \tag{3.1.2}$$

The behavior of the critical L^3 norm near a potential blow-up was unknown until the work of Escauriaza, Seregin, and Šverák [49], who discovered an endpoint local regularity criterion in the spirit of the classical work by Serrin [128]. In particular, they demonstrated that if u is a weak Leray–Hopf solution of the Navier–Stokes equations with initial data $u_0 \in C_0^\infty(\mathbb{R}^3)$ and maximal existence time $T^* < +\infty$, then

$$\limsup_{t \to T^*_-} \|u(\cdot, t)\|_{L^3(\mathbb{R}^3)} = +\infty. \tag{3.1.3}$$

Their proof uses the ε-regularity criterion of Caffarelli, Kohn, and Nirenberg [32] in an essential way. Moreover, it introduced powerful backward uniqueness arguments for studying potential singularities of solutions to the Navier–Stokes equations. The proof is by contradiction: If a solution forms a singularity but remains in the critical space $L_t^\infty L_x^3(Q_{T^*})$, then one may 'zoom in' on the singularity using the scaling symmetry and obtain a weak limit. The limit solution will form a singularity but also vanish identically at the blow-up time. By backwards uniqueness, the limit solution u must be identically zero in space-time, which contradicts that it forms a singularity. This method was adapted by Phuc [112] to cover blow-up criteria in Lorentz spaces.

Later, Seregin [121] improved the blow-up criterion of Escauriaza, Seregin, and Šverák by demonstrating that the L^3 norm must become infinite at a potential blow-up:

$$\lim_{t \to T^*_-} \|u(\cdot,t)\|_{L^3(\mathbb{R}^3)} = +\infty. \tag{3.1.4}$$

The main new difficulty in the proof is that one no longer controls the $L_t^\infty L_x^3$ norm when 'zooming in' on a potential singularity. Seregin addressed this difficulty by relying on certain properties of the *local energy solutions* introduced by Lemarié-Rieusset [95]. However, an analogous theory of local energy solutions was not known in the half space $\mathbb{R}^3_+ = \{x \in \mathbb{R}^3 : x_3 > 0\}$.[1] In order to overcome this obstacle, Barker and Seregin [16] introduced new *a priori* estimates which depend only on the norm of the initial data in the Lorentz spaces $L^{3,q}$, $3 < q < \infty$. This is accomplished by splitting the solution as

$$u = e^{t\Delta} u_0 + w, \tag{3.1.5}$$

where w is a correction in the energy space. The new estimates allowed Barker and Seregin to obtain an analogous blow-up criterion for Lorentz norms in the half space. Later, Seregin and Šverák abstracted this splitting argument into the notion of a *global weak L^3 solution* [123]. We direct the reader to the paper [17] for global weak solutions with initial data in $L^{3,\infty}$.

Recently, there was interest in adapting the 'concentration compactness + rigidity' roadmap of Kenig and Merle [79] to blow-up criteria for the Navier–Stokes equations. This line of thought was initiated by Kenig and G. Koch in [78] and advanced to its current state by Gallagher, Koch, and Planchon in [57, 58]. Gallagher et al. succeeded in extending a version of the blow-up criterion to the negative regularity critical Besov spaces $\dot{B}^{-1+3/p}_{p,q}(\mathbb{R}^3)$, $3 < p, q <$

[1] This theory was recently developed in [106].

$+\infty$. Specifically, it is proved in [58] that if $T^* < +\infty$, then

$$\limsup_{t \to T^*_-} \|u(\cdot, t)\|_{\dot{B}^{-1+3/p}_{p,q}(\mathbb{R}^3)} = +\infty. \tag{3.1.6}$$

Their proof is also by contradiction: If there is a blow-up solution in the space $L^\infty_t(\dot{B}^{-1+3/p}_{p,q})_x$, then one may prove via profile decomposition that there is a blow-up solution in the same space and with minimal norm (made possible by small-data-global-existence results in the spirit of Kato's work [76]). This solution is known as a 'critical element'. Once there exist a critical element, it can be shown that there also exists a critical element vanishing identically at the blow-up time. Hence, one may apply the backward uniqueness arguments of Escauriaza, Seregin, and Šverák to obtain a contradiction. The main difficulty lies in proving the existence of a profile decomposition in Besov spaces, which requires some techniques from the theory of wavelets [83, 13]. A secondary difficulty is to obtain the necessary estimates near the blow-up time in order to apply the ε-regularity criterion. The paper [78] appears to be the first application of Kenig and Merle's roadmap to a parabolic (rather than dispersive) equation. The nonlinear profile decomposition for the Navier–Stokes equations was first proved by Gallagher in [55]. The paper [12] contains further interesting applications of profile decomposition techniques to the Navier–Stokes equations.

In this paper, we obtain the following improved blow-up criterion for the Navier–Stokes equations in critical spaces:

Theorem 3.1.1 (Blow-up criterion)**.** *Let $p, q \in (3, +\infty)$ and $u_0 \in \dot{B}^{-1+3/p}_{p,q}(\mathbb{R}^3)$ be a divergence-free vector field. Let u be the solution of the Navier–Stokes equations on $\mathbb{R}^3 \times (0, T^*)$ with initial data u_0 and maximal time of existence T^*. If $T^* < \infty$, then*

$$\lim_{t \to T^*_-} \|u(\cdot, t)\|_{\dot{B}^{-1+3/p}_{p,q}(\mathbb{R}^3)} = +\infty. \tag{3.1.7}$$

The local well-posedness theory in critical Besov spaces is reviewed in Proposition 2.3.5 in Chapter 2, and the solutions of Proposition 2.3.5 are the solutions we consider in Theorem 3.1.1.

Let us discuss the novelty of Theorem 3.1.1. First, it extends Seregin's L^3 criterion (3.1.4) to the scale of Besov spaces and replaces the $\limsup$ condition in Gallagher–Koch–Planchon's criterion (3.1.6). Moreover, our proof does not rely on the profile decomposition techniques in the work of Gallagher et al. [58] and may perhaps be considered to be more elementary. Rather, our methods are based on the rescaling procedure in Seregin's work [121]. Regarding optimality, it is not clear whether Theorem 3.1.1 is valid for the endpoint spaces $\dot{B}^{-1+3/p}_{p,\infty}$ and BMO^{-1}, which contain non-trivial -1-homogeneous functions. Indeed, if the blow-up profile $u(\cdot, T^*)$ is locally a scale-invariant function, then rescaling around the singularity no longer provides useful information.[2] It is likely that this is an essential issue and not merely an artifact of the techniques used here. For instance, one may speculate that if Type I blow-up occurs (in the sense that the solution blows up in L^∞ at the self-similar rate), then the BMO^{-1} norm does not blow-up at the first singular time.

The main difficulty we encounter is in obtaining *a priori* estimates for solutions up to the potential blow-up time. We require that the estimates depend only on the *norm* of the initial data in $\dot{B}^{-1+3/p}_{p,q}$. The low regularity of this space creates a new difficulty because the splitting (3.1.5) does not appear to work in the space $\dot{B}^{-1+3/p}_{p,q}$ when $2/q + 3/p < 1$. One issue is that when obtaining energy estimates for the correction w in (3.1.5), the operator

$$(U, u_0) \mapsto \int_0^T \int_{\mathbb{R}^3} e^{t\Delta} u_0 \cdot \nabla U \cdot U \, dx \, dt \tag{3.1.8}$$

is not known to be bounded for $U \in L^\infty_t L^2_x \cap L^2_t \dot{H}^1_x$ and $u_0 \in \dot{B}^{-1+3/p}_{p,q}$. This is because $e^{t\Delta} u_0$ 'just misses' the critical Lebesgue space $L^r_t L^p_x$ with $2/r + 3/p = 1$. Therefore, to obtain the necessary *a priori* estimates, we rely on a method essentially due to C. P. Calderón [33]. We split the critical initial data $u_0 \in \dot{B}^{-1+3/p}_{p,p}$ into supercritical and subcritical parts:

$$u_0 = U_0 + V_0 \in L^2 + \dot{B}^{s_q+\varepsilon}_{q,q}. \tag{3.1.9}$$

When small, the data V_0 in a subcritical Besov space gives rise to a strong solution V on a prescribed time interval (not necessarily a global strong solution). The supercritical data $U_0 \in L^2$ serves as initial data for a correction U in the energy space. In the published paper [4], we referred to these solutions as *Calderón solutions*. Note that the unboundedness of (3.1.8) is similarly problematic when proving weak-strong uniqueness in Besov spaces. In recent work

[2] Later, Barker [14] obtained the blow-up criterion $\lim_{t \to T^*_-} \|u(\cdot, t)\|_{\dot{B}^{-1+3/p}_{p,\infty}} = +\infty$ using Calderón-type solutions under the additional assumption that $u(\cdot, T^*)$ vanishes in the rescaling limit. This theorem was later incorporated into the paper [5] by Barker and the author, and the preprint [14] was not submitted for publication.

on weak-strong uniqueness, Barker [15] has also dealt with this issue via the splitting (3.1.9). We remark that Calderón's original idea was to construct global weak solutions by splitting L^p initial data for $2 < p < 3$ into small data in L^3 and a correction in L^2. A similar idea was proposed by Lemarié-Rieusset in [96]. This splitting has been further exploited in the papers [54, 9] on the stability in Besov spaces and BMO^{-1}, respectively, of global smooth solutions; in the paper [71] by Jia and Šverák on minimal blow-up data; and elsewhere.

The main difference between this chapter and the published paper [4] is that, here, we do not construct Calderón solutions for abitrarily large times. Rather, we use the estimates afforded by the splitting (3.1.9) only up to the maximal existence time T^*. Hence, we need not go through the construction of weak solutions involving the mollified Navier–Stokes equations. The proof has also streamlined in other, more minor ways.

Let us briefly constrast the Calderón solutions, which are based on the splitting (3.1.9), to the global weak L^3 solutions introduced by Seregin and Šverák in [123], which are based on the splitting (3.1.5). The correction term w in (3.1.5) has zero initial data, which allows one to prove that an appropriate limit of solutions also satisfies the energy inequality up to the initial time. For this reason, the global weak L^3 solutions are compact (in a suitable sense) with respect to weak convergence of initial data in L^3 — importantly, the limit solutions have the desired initial data. In [5] and Chapter 4, this property is called *weak–$*$ stability*. Since the splitting (3.1.9) requires the correction to have non-zero initial condition U_0, the analogous result is not as obvious for Calderón solutions. This is further explained in Remark 3.2.8. We do not seek such a result here, as to avoid burdening the paper technically, but we expect that it is possible by adapting ideas in [125, 15]. Using similar ideas, we expect that one could prove that all Calderón-type solutions agree with the strong solution on a short time interval.

After completion of the paper [4], we learned that Barker obtained a different proof of the blow-up criterion (3.1.7). His proof was based on certain properties of the local energy solutions of Lemarié-Rieusset [95].

3.2 Preliminaries

We often do not distinguish the notation of scalar- and vector-valued functions. We use the notation $\mathrm{NS}(u_0)$ to denote the solution (constructed by the perturbation theory in Chapter 2) of the Navier–Stokes equations with initial data u_0, and its maximal time of existence is denoted $T^*(u_0)$. Recall that $s_p = -1 + 3/p$.

The following lemma allows us to represent critical initial data as the sum of subcritical and supercritical initial data while preserving the divergence free condition. See Proposition 2.8 in [15] or Appendix A in Chapter 4 (reproduced from [5]) for a detailed proof.

Lemma 3.2.1 (Splitting of critical data)**.** *Let* $3 < p < q \leq +\infty$ *and* $\theta \in (0,1)$ *satisfying*

$$\frac{1}{p} = \frac{\theta}{2} + \frac{1-\theta}{q}. \tag{3.2.1}$$

Let $s = s_p/(1-\theta)$*. For all* $\lambda > 0$ *and divergence-free vector fields* $u_0 \in \dot{B}^{s_p}_{p,p}(\mathbb{R}^3)$*, there exist divergence-free vector fields* U_0*,* V_0 *such that* $u_0 = U_0 + V_0$*,*

$$\|U_0\|_{L^2(\mathbb{R}^3)} \leq C\|u_0\|^{p/2}_{\dot{B}^{s_p}_{p,p}(\mathbb{R}^3)} \lambda^{1-p/2}, \tag{3.2.2}$$

$$\|V_0\|_{\dot{B}^{s}_{q,q}(\mathbb{R}^3)} \leq C\|u_0\|^{p/q}_{\dot{B}^{s_p}_{p,p}(\mathbb{R}^3)} \lambda^{1-p/q}. \tag{3.2.3}$$

Additionally,

$$\|U_0\|_{\dot{B}^{s_p}_{p,p}(\mathbb{R}^3)}, \|V_0\|_{\dot{B}^{s_p}_{p,p}(\mathbb{R}^3)} \leq C\|u_0\|_{\dot{B}^{s_p}_{p,p}(\mathbb{R}^3)}. \tag{3.2.4}$$

The proof is by decomposing the Littlewood–Paley components as

$$\dot{\Delta}_j u_0 = \mathbf{1}_{|\dot{\Delta}_j u_0| > \lambda_j} \dot{\Delta}_j u_0 + \mathbf{1}_{|\dot{\Delta}_j u_0| \leq \lambda_j} \dot{\Delta}_j u_0, \tag{3.2.5}$$

where $\lambda_j > 0$, $j \in \mathbb{Z}$, are chosen appropriately. The divergence free condition is kept by applying the Leray projector to the resulting vector fields (it is continuous on homogeneous Besov spaces).

Note that $\dot{B}^s_{q,q}(\mathbb{R}^3)$ is indeed a subcritical space of initial data, since

$$s - \frac{3}{q} = -1 + \frac{\theta}{2(1-\theta)} > -1. \tag{3.2.6}$$

We will occasionally denote $\varepsilon = s - s_q > 0$.

Eventually, we will use energy estimates for the difference of two solutions. Moreover, in our context, *a priori*, one of the solutions may be irregular. For this reason, we must show that

(i) certain energy inequalities are satisfied, and (ii) in the strong solution theory, the difference between the desired solutions actually has finite energy. We do this in Lemmas 3.2.2 and 3.2.3, respectively.

Lemma 3.2.2 (Perturbed energy inequalities)**.** *Let $T \in (0,+\infty)$. Let (u,p) and (V,Q) be suitable weak solutions of the Navier–Stokes equations in Q_T. Let*

$$V \in L_t^l L_x^r(Q_T) \tag{3.2.7}$$

with $l \in [2,+\infty]$, $r \in [3,+\infty]$, and

$$\frac{2}{l} + \frac{3}{r} = 1. \tag{3.2.8}$$

Let

$$u = U + V. \tag{3.2.9}$$

Then U satisfies the perturbed local energy inequality:

$$\partial_t |U|^2 + 2|\nabla U|^2 + \operatorname{div}\left(|U|^2 U + 2PU + |U|^2 V\right) + 2\operatorname{div}(V \otimes U) \cdot U \le \Delta |U|^2. \tag{3.2.10}$$

If also $U \in L_t^\infty L_x^2 \cap L_t^2 \dot{H}_x^1(Q_T)$, then U satisfies the perturbed global energy inequality:

$$\begin{aligned} &\int_{\mathbb{R}^3} |U(x,t_2)|^2\,dx + 2\int_{t_1}^{t_2}\int_{\mathbb{R}^3} |\nabla U(x,t)|^2\,dx\,dt \\ &\quad \le \int_{\mathbb{R}^3} |U(x,t_1)|^2\,dx + 2\int_{t_1}^{t_2}\int_{\mathbb{R}^3} V\otimes U : \nabla U\,dx\,dt \end{aligned} \tag{3.2.11}$$

for almost every $t_1 \in (0,T)$ and for all $t_2 \in (t_1,T]$. Finally, if $U \in C([0,T];L^2(\mathbb{R}^3))$, then (3.2.11) *is satisfied for each $t_1 \in [0,T)$.*

In the above statement, $\operatorname{div}(V\otimes U)\cdot U$ is defined by

$$\langle \operatorname{div}(V\otimes U)\cdot U, \varphi\rangle = -\int_0^T \int_{\mathbb{R}^3} V\otimes U : (U\otimes\nabla\varphi + \varphi\nabla U)\,dx\,dt \tag{3.2.12}$$

for all $\varphi \in C_0^\infty(Q_T)$.

We define the *energy norm*

$$|U|_{2,\mathbb{R}^3\times(S,T)}^2 = \sup_{t\in(S,T)} \int_{\mathbb{R}^3} |U(x,t)|^2\,dx + 2\int_S^T \int_{\mathbb{R}^3} |\nabla U(x,s)|^2\,dx\,ds, \tag{3.2.13}$$

where $0 \le S < T \le +\infty$. If $U \in L_t^\infty L_x^2 \cap L_t^2 \dot{H}_x^1(Q_T)$, then by interpolation between $L_t^\infty L_x^2(Q_T)$ and $L_t^2 L_x^6(Q_T)$, we have

$$\|U\|_{L_t^m L_x^n(Q_T)} \le C|U|_{2,Q_T}, \tag{3.2.14}$$

$$\frac{2}{m} + \frac{3}{n} = \frac{3}{2}, \quad m \in [2,+\infty], n \in [2,6]. \tag{3.2.15}$$

A common choice is $m = n = 10/3$. In particular, when V is as in Lemma 3.2.2, we have

$$\|V_i U_j\|_{L^2_{t,x}(Q_T)} \le C|U|_{2,Q_T} \|V\|_{L_t^l L_x^r(Q_T)} \tag{3.2.16}$$

for all $i, j = 1, 2, 3$. In this case, the pressure P satisfies

$$P = (-\Delta)^{-1} \operatorname{div}\operatorname{div}(U \otimes U + U \otimes V + V \otimes U) \in L^{5/3}_{t,x}(Q_T) + L^2_{t,x}(Q_T). \tag{3.2.17}$$

Proof of Lemma 3.2.2. Let us sketch the proof of the perturbed local energy inequality since the ideas are well known from the proof of weak-strong uniqueness.[3] The main difference is that we argue with local energy inequalities rather than global energy inequalities. Since $U = u - V$, we have the elementary identity

$$\begin{aligned}(\partial_t - \Delta)|U|^2 + 2|\nabla U|^2 &= (\partial_t - \Delta)|u|^2 + 2|\nabla u|^2 + (\partial_t - \Delta)|V|^2 + 2|\nabla V|^2 \\ &\quad - 2(\partial_t - \Delta)u \cdot V - 2(\partial_t - \Delta)V \cdot u.\end{aligned} \tag{3.2.18}$$

The above identity, which is obvious for smooth functions, is used to 'transfer' the energy inequalities from u and V to U, since *a priori* U is not regular enough to prove the energy equality directly. Typically, (3.2.18) is valid in the sense of distributions when $u, V \in (L_t^2 H_x^1)_{\mathrm{loc}}(Q_T)$ and $\partial_t u \in (L_t^2 H_x^{-1})_{\mathrm{loc}}(Q_T)$ (for example, when u and V satisfy the heat equation with RHS f in $L_t^2 H_x^{-1}(Q_T)$). This is not the case in our setting. Rather, we may exploit that $(\partial_t - \Delta)u + \nabla p = -u \cdot \nabla u \in (L_t^{l'} L_x^{r'})_{\mathrm{loc}}(Q_T)$, so $[(\partial_t - \Delta)u + \nabla p] \cdot V$ belongs to $L^1_{\mathrm{loc}}(Q_T)$.

[3] Note that, for weak-strong uniqueness in the class $V \in L_t^\infty L_x^3$, it is also required that V is sufficiently small.

Additionally, $\nabla p \cdot V = \mathrm{div}(pV)$ belongs to $(L^1_t W^{-1,1}_x)_{\mathrm{loc}}(Q_T)$. Together, these facts allow us to make sense of $(\partial_t - \Delta)u \cdot V$. We have similar observations regarding $(\partial_t - \Delta)V \cdot u$. Hence, each term in (3.2.18) makes sense. Furthermore, (3.2.18) may be justified by approximating by smooth functions. Next, we substitute the equations for u and V into (3.2.18), perform elementary manipulations, and exploit the cancellation property of the non-linear term (for example, $(V \cdot \nabla u) \cdot u = \mathrm{div}(|u|^2 V)/2$) that may be justified because V belongs to a critical Lebesgue space. This yields (3.2.10).

Let us mention how to pass from the local energy inequality (3.2.10) to the global energy inequality (3.2.11) (see Lemarié-Rieusset [95], p. 319) when U belongs to the energy space. Let ψ_ε be a suitable mollification-in-time of $\mathbf{1}_{(t_1,t_2)}$ at scale ε. Let $\varphi \in C^\infty_0(B_2)$ with $\varphi \equiv 1$ on B_1. Define

$$\Phi_{\varepsilon,R}(x,t) = \psi_\varepsilon(t)\varphi^2(x/R). \tag{3.2.19}$$

We substitute $\Phi_{\varepsilon,R}$ into the local energy inequality (3.2.10) with $0 < \varepsilon \ll 1$. Since U is in the energy space, $P \in L^{5/3}_{t,x}(Q_T) + L^2_{t,x}(Q_T)$, and $V \in L^l_t L^r_x(Q_T)$, we may justify taking $R \to +\infty$ in each term. This gives

$$2\int_{\mathbb{R}}\int_{\mathbb{R}^3} |\nabla u|^2 \psi_\varepsilon(t)\,dx\,dt \leq \int_{\mathbb{R}}\int_{\mathbb{R}^3} |u|^2 \partial_t \psi_\varepsilon(t)\,dx\,dt + 2\int_{\mathbb{R}}\int_{\mathbb{R}^3} \psi_\varepsilon(t) V \otimes U : \nabla U\,dx\,dt. \tag{3.2.20}$$

If t_1 and t_2 are Lebesgue points of $\|U(\cdot,t)\|_{L^2(\mathbb{R}^3)}$, then letting $\varepsilon \to 0^+$ yields (3.2.11). Finally, since $U \in C_{\mathrm{wk}}([0,T]; L^2(\mathbb{R}^3))$, we may take any $t_2 \in (t_1, T]$ in (3.2.11). $\square$

Lemma 3.2.3 (Perturbations belong to the energy class)**.** *Let $T \in (0,+\infty)$ and $p \in (3,+\infty)$. Let u_0 and V_0 be divergence-free vector fields on $\mathbb{R}^3$ satisfying*

$$u_0 = U_0 + V_0 \tag{3.2.21}$$

with $U_0 \in L^2(\mathbb{R}^3)$. Let

$$Y = L^p(\mathbb{R}^3) \text{ or } \dot{B}^{-1+3/p}_{p,p}(\mathbb{R}^3) \tag{3.2.22}$$

and assume

$$u_0, U_0, V_0 \in Y. \tag{3.2.23}$$

If $Y = \dot{B}^{-1+3/p}_{p,p}(\mathbb{R}^3)$, then additionally assume that $V \in L^2_t L^\infty_x(Q_T)$. Let $u = \mathrm{NS}(u_0)$,

$V = \mathrm{NS}(V_0)$ *and* $U = u - V$. *Assume that* $\min(T^*(u_0), T^*(V_0)) > T$. *Then*

$$U \in C([0,T]; L^2(\mathbb{R}^3)) \cap L^2_t \dot{H}^1_x(Q_T). \tag{3.2.24}$$

Proof. **Step 1. Solvability of perturbed equations**. We study the integral equation

$$\widetilde{U}(\cdot,t) = e^{t\Delta}\widetilde{U}_0 - B(\widetilde{U},\widetilde{U})(\cdot,t) - L(\widetilde{U})(\cdot,t), \tag{3.2.25}$$

where we (formally) define

$$B(f,g)(\cdot,t) = \int_0^t e^{(t-s)\Delta}\mathbb{P}\,\mathrm{div}\, f \otimes g\, ds, \tag{3.2.26}$$

$$L(f)(\cdot,t) = B(f,\widetilde{V})(\cdot,t) + B(\widetilde{V},f)(\cdot,t), \tag{3.2.27}$$

for all vector fields f, g, where $\widetilde{V}$ is a fixed function defined on $Q_{\bar{S}}$ and $\bar{S} > 0$ is fixed. We consider the integral equation in the function space $X_S \cap E_S$, $S \in (0, \bar{S}]$, where $E_S = C([0,S]; L^2(\mathbb{R}^3))$ and X_S is designated below by case. We write $\|\widetilde{U}_0\|_Y \leq M$ and $\|\widetilde{V}\|_{X_{\bar{S}}} \leq N$.

Case A. $Y = L^p(\mathbb{R}^3)$. This is a subcritical scenario. Let $X_S = C([0,S]; L^p(\mathbb{R}^3))$. By the linear Stokes estimates in Lemma 2.2.1 (Estimates in Kato spaces), we have

$$\|B(f,g)\|_{X_S} \leq C(X_S)\|f\|_{X_S}\|g\|_{X_S} \tag{3.2.28}$$

$$\|B(f,g)\|_{E_S}, \|B(g,f)\|_{E_S} \leq C(X_S, E_S)\|f\|_{X_S}\|g\|_{E_S}, \tag{3.2.29}$$

where the above constants are of the form $C_p S^{(1-3/p)/2}$. Hence, by Lemma 2.3.1 (Abstract Picard lemma) and Lemma 2.3.2 (Propagation of regularity), there exists $S = S(M,N,p) \in (0, \bar{S}]$ and a unique mild solution $\widetilde{U} \in X_S \cap E_S$ to the integral equation (3.2.25) on Q_S.[4]

Case B. $Y = \dot{B}^{-1+3/p}_{p,p}(\mathbb{R}^3)$. This is a critical scenario. Let $X_S = \mathring{\mathcal{K}}^{s_p}_p(Q_S)$ (see Section 2.1 for Kato spaces). Again, by the linear Stokes estimates in Lemma 2.2.1, we have the bilinear estimates (3.2.28)–(3.2.29) as above with constants of the form C_p. Hence, there exists $S = S(\widetilde{U_0}, \widetilde{V}, p) \in (0, \bar{S}]$ and a unique mild solution $\widetilde{U} \in X_S \cap E_S$ to the integral equation (3.2.25)

[4] Technically, the uniqueness is only automatic in the class of ‘small solutions’. Smallness is typically guaranteed by shortening the time interval. Then the short-time uniqueness is propagated forward by repeating the argument. This remark also applies to *Case B* below.

on Q_S.

Step 2. Conclusion. *Case A.* $Y = L^p(\mathbb{R}^3)$. We apply the above solvability results on Q_S with $\widetilde{U_0} = U_0$ and $\widetilde{V} = V$. Recall that $S = S(M, N, p)$. By the uniqueness assertions in Proposition 2.3.3 (Subcritical L^p theory), we have $u \equiv \widetilde{U} + V$, so $U \equiv \widetilde{U} \in C([0, S]; L^2(\mathbb{R}^3))$. Then we apply the above solvability results again with $\widetilde{U_0} = U(\cdot, S)$ and $\widetilde{V}(x, t) = V(x, t+S)$. Iterating in this way yields that $U \in C([0, T]; L^2(\mathbb{R}^3))$. Finally, we consider U as a solution of the Stokes equations with RHS $-\operatorname{div} F$, where $F = U \otimes U + U \otimes V + V \otimes U \in L^2(Q_T)$, since also $u, U, V \in L^2_t L^\infty_x(Q_T)$ by the subcritical estimates (2.3.17) in Proposition 2.3.3. This yields $U \in L^2_t \dot{H}^1_x(Q_T)$.

Case B. $Y = \dot{B}^{-1+3/p}_{p,p}(\mathbb{R}^3)$. As before, we apply the above perturbation results on Q_S, where S depends on U_0 and V. This yields that $U \in C([0, S]; L^2(\mathbb{R}^3))$. Since $u, V, W \in C((0, T]; L^p(\mathbb{R}^3))$, we may apply the known $Y = L^p(\mathbb{R}^3)$ case starting from time S to obtain $U \in C([0, T]; L^2(\mathbb{R}^3))$. As in that case, the linear theory gives that $U \in L^2_t \dot{H}^1_x(\mathbb{R}^3 \times (\varepsilon, T])$ for all $0 < \varepsilon \ll 1$. Finally, we use the perturbed global energy inequality (3.2.11) on $\mathbb{R}^3 \times (\varepsilon, T)$ from Lemma 3.2.2 and that $U \in C([0, T]; L^2(\mathbb{R}^3))$ and $V \in L^2_t L^\infty_x(Q_T)$.[5] Specifically, we have

$$2 \int_\varepsilon^T \int_{\mathbb{R}^3} |\nabla U|^2 \, dx \, dt \leq \|U(\cdot, \varepsilon)\|^2_{L^2(\mathbb{R}^3)} + 2 \int_\varepsilon^T \int_{\mathbb{R}^3} |V \otimes U : \nabla U| \, dx \, dt, \tag{3.2.30}$$

which, after Young's inequality, implies

$$\int_\varepsilon^T \int_{\mathbb{R}^3} |\nabla U|^2 \, dx \, dt \leq C \|U(\cdot, \varepsilon)\|^2_{L^2(\mathbb{R}^3)} + C \|V\|^2_{L^2_t L^\infty_x(Q_T)} \|U\|^2_{L^\infty_t L^2_x(Q_T)} \leq \bar{M} \tag{3.2.31}$$

with $\bar{M}$ independent of $\varepsilon \to 0^+$. Hence, $U \in L^2_t \dot{H}^1_x(Q_T)$. □

Since ultimately our scheme relies on zooming in on a singularity, we must characterize the maximal time of existence, if finite, by the formation of a singularity:

Proposition 3.2.4 (Characterization of blow-up). *Let $p \in (3, +\infty)$ and $u_0 \in L^p(\mathbb{R}^3)$ be a divergence-free vector field. If $T^*(u_0) < +\infty$, then $\mathrm{NS}(u_0)$ has a singular point at time $T^*(u_0)$.*

We follow similar arguments in [118, 71].

[5] In this context, the perturbed global energy equality could also be shown directly.

Proof. For all $\varepsilon > 0$, there exist divergence-free vector fields $U_0 \in L^2(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)$ and $V_0 \in L^p(\mathbb{R}^3)$ satisfying

$$u_0 = U_0 + V_0 \tag{3.2.32}$$

$$\|V_0\|_{L^p(\mathbb{R}^3)} \leq \varepsilon. \tag{3.2.33}$$

For example, one may approximate u_0 in $L^p(\mathbb{R}^3)$ by $\widetilde{V}_0 \in C_0^\infty(\mathbb{R}^3)$ and write $V_0 = \mathbb{P}\widetilde{V}_0$. Choose $0 < \varepsilon \ll 1$ so that $T^*(V_0) \geq 2T^*(u_0)$. Let $U = \mathrm{NS}(u_0) - \mathrm{NS}(V_0)$. By Lemmas 3.2.2 and 3.2.3,

$$U \in C([0,T]; L^2(\mathbb{R}^3)) \cap L_t^2 \dot{H}_x^1(Q_T) \text{ for all } T \in (0, T^*(u_0)), \tag{3.2.34}$$

and U satisfies the perturbed global energy inequality (3.2.11). By a Gronwall-type argument in (3.2.11), compare (3.2.72), we have

$$U \in L_t^\infty L_x^2 \cap L_t^2 H_x^1(Q_{T^*(u_0)}). \tag{3.2.35}$$

Hence,

$$u = U + V \in L_{t,x}^{\frac{10}{3}}(Q_{T^*(u_0)}) + L_t^\infty L_x^p(Q_{T^*(u_0)}), \tag{3.2.36}$$

$$p = P + Q \in L_{t,x}^{\frac{5}{3}}(Q_{T^*(u_0)}) + L_{t,x}^2(Q_{T^*(u_0)}) + L_t^\infty L_x^{\frac{p}{2}}(Q_{T^*(u_0)}), \tag{3.2.37}$$

where p, P, Q are the pressures associated to u, U, V, respectively. In particular,

$$\lim_{|x_0| \to \infty} \int_0^{T^*(u_0)} \int_{B_1(x_0)} |u|^3 + |p|^{3/2} \, dx \, dt = 0. \tag{3.2.38}$$

Therefore, by the ε-regularity criterion in Theorem 2.4.4, there exists $R > 0$ and

$$K = \overline{\mathbb{R}^3 \setminus B(R)} \times [T^*(u_0)/2, T^*(u_0)] \tag{3.2.39}$$

satisfying

$$\sup_K |u(x,t)| < +\infty. \tag{3.2.40}$$

Finally, if u has no singular points at time $T^*(u_0)$, then $u \in L^\infty(\mathbb{R}^3 \times (T^*(u_0)/2, T^*(u_0))$, which would contradict the blow-up criterion in Proposition 2.3.3 (Subcritical L^p theory). $\square$

Corollary 3.2.5. *Let $u_0 \in \dot{B}_{p,p}^{-1+3/p}(\mathbb{R}^3)$ be a divergence-free vector field. If $T^*(u_0) < +\infty$,* *then* $\mathrm{NS}(u_0)$ *has a singular point at time* $T^*(u_0)$.

This follows from Propositions 2.3.5 and 3.2.4.

The next lemma will be required to apply Theorem 2.5.2 (Unique continuation):

Lemma 3.2.6 (Epochs of regularity)**.** *Let* $u = U + V$ *be as in Lemma 3.2.2 with also* $U \in L^\infty_t L^2_x \cap L^2_t \dot{H}^1_x(Q_T)$. *Let* $q \in (3, +\infty)$ *and* $V \in C((0,T]; L^q(\mathbb{R}^3))$. *Then there exists a dense open set* $G \subset (0,T)$ *such that* $u \in C^\infty(\mathbb{R}^3 \times G)$.

Probably one may also prove that the $1/2$-dimensional Hausdorff measure of the set of singular times is zero.

Proof. Let Π be the set of times $t_1 \in (0,T)$ such that $U(\cdot,t_1) \in H^1(\mathbb{R}^3)$ and U satisfies the perturbed global energy inequality (3.2.11) for all $t_2 \in (t_1, T]$. This ensures

$$\lim_{t \to t_1^+} \|U(\cdot,t) - U(\cdot,t_1)\|_{L^2(\mathbb{R}^3)} = 0. \tag{3.2.41}$$

The set Π has full measure in $(0,T)$ and, in particular, is dense in $(0,T)$.

Let $t_1 \in \Pi$ be fixed. Using the perturbation theory in $L^2(\mathbb{R}^3) \cap L^6(\mathbb{R}^3)$ in Lemma 3.2.3 (Perturbations belong to the energy class), Step 1, Case A, we have the following: For each $t_1 \in \Pi$, there exists $\varepsilon = \varepsilon(t_1) > 0$, a vector field $\widetilde{U} \in C([t_1, t_1+\varepsilon]; L^2(\mathbb{R}^3) \cap L^6(\mathbb{R}^3))$ with $\widetilde{U}(\cdot,t_1) = U(\cdot,t_1)$, and a pressure $\widetilde{P} \in L^\infty_t L^3_x(\mathbb{R}^3 \times (t_1, t_1+\varepsilon))$ satisfying

$$\begin{aligned} \partial_t \widetilde{U} - \Delta \widetilde{U} + \operatorname{div} \widetilde{U} \otimes \widetilde{U} + \operatorname{div} V \otimes \widetilde{U} + \operatorname{div} \widetilde{U} \otimes V &= -\nabla \widetilde{P} \\ \operatorname{div} \widetilde{U} &= 0 \end{aligned} \tag{3.2.42}$$

in the sense of distributions on $\mathbb{R}^3 \times (t_1, t_1+\varepsilon)$. Recall that $V \in L^\infty_{t,x}(\mathbb{R}^3 \times (t_1, t_1+\varepsilon))$ by Proposition 2.3.3 (Subcritical L^p theory). By energy estimates for the Stokes equations with RHS $-\operatorname{div} F$, $F = \widetilde{U} \otimes \widetilde{U} + V \otimes \widetilde{U} + \widetilde{U} \otimes V \in L^2_{t,x}(\mathbb{R}^3 \times (t_1, t_1+\varepsilon))$, we have $\widetilde{U} \in L^2_t \dot{H}^1_x(\mathbb{R}^3 \times (t_1, t_1+\varepsilon))$.

We wish to conclude that $\widetilde{U} \equiv U$ on $\mathbb{R}^3 \times (t_1, t_1+\varepsilon)$. This is done by weak-strong uniqueness. Specifically, integration by parts yields that $\widetilde{U}$ satisfies the perturbed global energy equality (3.2.11) starting from every $s_1 \in [t_1, t_1+\varepsilon)$. We write $D = U - \widetilde{U}$. Then D obeys the energy inequality

$$
\begin{aligned}
&\int_{\mathbb{R}^3} |D(x,s_2)|^2\,dx + 2\int_{s_1}^{s_2}\int_{\mathbb{R}^3} |\nabla D|^2\,dx\,dt \\
&\quad \le \int_{\mathbb{R}^3} |D(x,s_1)|^2\,dx + 2\int_{s_1}^{s_2}\int_{\mathbb{R}^3} (\widetilde{U}+V)\otimes D : \nabla D\,dx\,dt
\end{aligned}
\tag{3.2.43}
$$

for almost every $s_1 \in [t_1, t_1+\varepsilon)$ and every $s_2 \in (s_1, t_1+\varepsilon]$. The proof of (3.2.43) is similar to that of Lemma 3.2.2 (Perturbed energy inequalities) except at the global level: We expand $|D|^2 = |U|^2 + |\widetilde{U}|^2 - 2U\cdot\widetilde{U}$ and $|\nabla D|^2 = |\nabla U|^2 + |\nabla\widetilde{U}|^2 - 2\nabla U : \nabla\widetilde{U}$ and use the following elementary identity to estimate the cross-terms:

$$
\begin{aligned}
&\int_{\mathbb{R}^3} U(x,s_2)\cdot\widetilde{U}(x,s_2)\,dx + 2\int_{s_1}^{s_2}\int_{\mathbb{R}^3} \nabla U : \nabla\widetilde{U}\,dx\,dt - \int_{\mathbb{R}^3} U(x,s_1)\cdot\widetilde{U}(x,s_1)\,dx \\
&\quad = \int_{s_1}^{s_2}\int_{\mathbb{R}^3} (\partial_t U - \Delta U)\cdot\widetilde{U} + U\cdot(\partial_t\widetilde{U} - \Delta\widetilde{U})\,dx\,dt.
\end{aligned}
\tag{3.2.44}
$$

The identity (3.2.44) is valid for compactly supported smooth vector fields and is applicable to our situation by approximation. The energy inequality (3.2.43) yields

$$
\begin{aligned}
&|D|^2_{2,\mathbb{R}^3\times(s_1,s_2)} \le C\times[\|\widetilde{U}\|_{L^m_t L^6_x(\mathbb{R}^3\times(s_1,s_2))} + \|V\|_{L^n_t L^q_x(\mathbb{R}^3\times(s_1,s_2))}]|D|^2_{2,\mathbb{R}^3\times(s_1,s_2)} \\
&\quad + \int_{\mathbb{R}^3} |D(x,s_1)|^2\,dx,
\end{aligned}
\tag{3.2.45}
$$

where $m,n \in [2,+\infty)$ and $2/m + 1/2 = 2/n + 3/q = 1$. Note that the above norms of $\widetilde{U}$ and V are finite when $s_1 = t_1$ and $s_2 = t_1+\varepsilon$. Taking $s_1 = t_1$ and $|s_2 - s_1| \ll 1$, we may absorb the energy norm on the RHS into the LHS. Since $D(\cdot,t_1)\equiv 0$, we have $D\equiv 0$ on (s_1,s_2). We repeat this Gronwall-type argument finitely many times until $U\equiv\widetilde{U}$ on $\mathbb{R}^3\times(t_1,t_1+\varepsilon)$.

Now $u = U+V \in C([t_1,t_1+\varepsilon];L^6(\mathbb{R}^3)) + C([t_1,t_1+\varepsilon];L^q(\mathbb{R}^3))$. From here, it is not difficult to bootstrap via Duhamel's formula and the linear estimates in Lemma 2.2.1 (Estimates in Kato spaces) to $C((t_1,t_1+\varepsilon];L^\infty(\mathbb{R}^3))$. Then Proposition 2.3.3 (Subcritical L^p theory) yields that u is smooth on $\mathbb{R}^3\times(t_1,t_1+\varepsilon]$.

Finally, $G\subset(0,T)$ is defined as follows:

$$
G := \bigcup_{t_1\in\Pi} (t_1, t_1+\varepsilon(t_1)). \tag{3.2.46}
$$

Clearly, G is open, and

$$\overline{G} \supseteq \overline{\Pi} = [0,T]. \tag{3.2.47}$$

$\square$

Here is the compactness result that we will use in Theorem 3.1.1. For simplicity, we normalize the time scale.

Proposition 3.2.7 (Compactness)**.** *Let $(u_0^{(n)})_{n\in\mathbb{N}}$ be a sequence of divergence-free vector fields satisfying*

$$\|u_0^{(n)}\|_{\dot{B}^{-1+3/p}_{p,p}(\mathbb{R}^3)} \leq M \tag{3.2.48}$$

and $T^*(u_0^{(n)}) \geq 1$ for all $n \in \mathbb{N}$. Let $u^{(n)} = \mathrm{NS}(u_0^{(n)})$ and $p^{(n)}$ be the associated pressure. There exists a suitable weak solution (u,p) on $\mathbb{R}^3 \times (0,1)$ and a subsequence (which we do not reindex) satisfying

$$u^{(n)} \to u \text{ in } L^3_{\mathrm{loc}}(\mathbb{R}^3 \times (0,1]), \tag{3.2.49}$$

$$p^{(n)} \rightharpoonup p \text{ in } L^{\frac{3}{2}}_{\mathrm{loc}}(\mathbb{R}^3 \times (0,1]), \tag{3.2.50}$$

and, for all $x_0 \in \mathbb{R}^3$,

$$u^{(n)} \to u \text{ in } C_{\mathrm{wk}}([1/2,1]; L^2(B_{1/2}(x_0))). \tag{3.2.51}$$

Moreover,

$$u = U + V, \tag{3.2.52}$$

$$U \in L^\infty_t L^2_x \cap L^2_t \dot{H}^1_x(\mathbb{R}^3 \times (0,1)), \tag{3.2.53}$$

and

$$V \in C((0,1]; L^q(\mathbb{R}^3)) \tag{3.2.54}$$

is a mild solution of the Navier–Stokes equations on $\mathbb{R}^3 \times (0,1)$ with $q \in (p, +\infty)$.

Remark 3.2.8 (On weak–$*$ stability)**.** Upon passing to a subsequence, we also have

$$u_0^{(n)} \overset{*}{\rightharpoonup} u_0 \text{ in } \dot{B}^{-1+3/p}_{p,p}(\mathbb{R}^3). \tag{3.2.55}$$

With additional effort, one may prove

$$u = \mathrm{NS}(u_0) \text{ on } \mathbb{R}^3 \times (0, \min(T^*(u_0), 1)). \tag{3.2.56}$$

This was done, for example, in joint work with T. Barker in [5] (see Theorem 4.1.2 in Chapter 4), which was partially inspired by the present work. One more-or-less immediate consequence is an alternative proof of the (conditional) existence of minimal blow-up initial data in $\dot{B}^{-1+3/p}_{p,q}(\mathbb{R}^3)$ where $p,q \in (3,+\infty)$, see Corollary 4.1.11 in Chapter 4. The original result is due to Gallagher–Koch–Planchon in [58].

The main issue in showing (3.2.56) is that it is not obvious that the perturbed global energy inequality is satisfied with $t_1 = 0$ (equivalently, that $\|U(\cdot,t) - U_0\|_{L^2(\mathbb{R}^3)} \to 0$ as $t \to 0^+$). This issue is already present at the level of weak Leray–Hopf solutions: Let $(u_0^{(n)})_{n\in\mathbb{N}} \subset C_0^\infty(\mathbb{R}^3)$ be a sequence of divergence-free vector fields with $\|u_0^{(n)}\|_{L^2(\mathbb{R}^3)} = 1$. Let $(u^{(n)})_{n\in\mathbb{N}}$ be a sequence of associated weak Leray–Hopf solutions. Assume that $u_0^{(n)} \rightharpoonup 0$ in $L^2(\mathbb{R}^3)$. There exists a subsequence satisfying $u^{(n)} \overset{*}{\rightharpoonup} u$ in $L^\infty_t L^2_x \cap L^2_t \dot{H}^1_x(\mathbb{R}^3\times(0,+\infty))$. *Is $u \equiv 0$?* When $u_0^{(n)} \overset{*}{\rightharpoonup} u_0$ in $L^3(\mathbb{R}^3)$, the issue may be dealt with by showing that $\|u(\cdot,t) - e^{t\Delta}u_0\|_{L^2} \leq C(\|u_0\|_{L^3(\mathbb{R}^3)})t^{\frac{1}{4}}$, as in Seregin and Šverák [123], since this *quantitative* rate of decay near the initial time persists under weak limits. The same issue was encountered in the paper [71] of Jia and Šverák on minimal blow-up initial data in $L^3(\mathbb{R}^3)$.

Proof of Proposition 3.2.7. Assume the above hypotheses. Let $q \in (p,+\infty)$.

Step 1. Splitting arguments. According to Lemma 3.2.1, there exists $\varepsilon = \varepsilon(p,q) \in (0,|s_q|)$ such that, for all $\Lambda > 0$, we may decompose $u_0^{(n)}$ into divergence-free vector fields

$$u_0^{(n)} = U_0^{(n)} + V_0^{(n)} \tag{3.2.57}$$

satisfying

$$\|U_0^{(n)}\|_{L^2(\mathbb{R}^3)} \leq C(M,\Lambda,p,q), \tag{3.2.58}$$

$$\|V_0^{(n)}\|_{\dot{B}^{s_q+\varepsilon}_{q,q}(\mathbb{R}^3)} \leq \Lambda. \tag{3.2.59}$$

By Proposition 2.3.4 (Subcritical Besov theory), there exists a constant $\gamma = \gamma(p,q) \in (0,1)$ satisfying the following property: If

$$\|V_0^{(n)}\|_{\dot{B}^{s_q+\varepsilon}_{q,q}(\mathbb{R}^3)} \leq \gamma, \tag{3.2.60}$$

then there exists a unique mild solution $V^{(n)} \in \mathcal{K}^{s_q+\varepsilon}_q(\mathbb{R}^3\times(0,1))$ with initial data $V_0^{(n)}$ and satisfying

$$\|t^{k+\frac{l}{2}}\partial_t^k \nabla_x^l V^{(n)}\|_{\mathcal{K}_q^{s_q+\varepsilon}(\mathbb{R}^3\times(0,1))} \leq C(q,\varepsilon,k,l)\|V_0^{(n)}\|_{\dot{B}_{q,q}^{s_q+\varepsilon}(\mathbb{R}^3)} \tag{3.2.61}$$

for all integers $k, l \geq 0$. Let $\Lambda \leq \gamma$ in the decomposition (3.2.57). The corresponding mild solutions $V^{(n)}$ exist on $\mathbb{R}^3 \times (0,1)$ and satisfy (3.2.61). In particular,

$$\|V^{(n)}\|_{L_t^l L_x^q(\mathbb{R}^3\times(0,1))} \leq C(p,q)\Lambda \tag{3.2.62}$$

with

$$\frac{2}{l} + \frac{3}{q} = 1. \tag{3.2.63}$$

Let

$$u^{(n)} = U^{(n)} + V^{(n)}. \tag{3.2.64}$$

According to Lemma 3.2.3,

$$U^{(n)} \in C([0,T]; L^2(\mathbb{R}^3)) \cap L_t^2 \dot{H}_x^1(Q_T) \text{ for all } T \in (0,1). \tag{3.2.65}$$

Step 2. Convergence of $V^{(n)}$ and $Q^{(n)}$. Due to (3.2.61) and the Ascoli-Arzela theorem, we may pass to a subsequence, still denoted by n, such that

$$V^{(n)} \overset{*}{\rightharpoonup} V \text{ in } \mathcal{K}_q^{s_q+\varepsilon}(\mathbb{R}^3 \times (0,1)) \tag{3.2.66}$$

$$\partial_t^k \nabla_x^l V^{(n)} \to \partial_t^k \nabla_x^l V \text{ in } C(K), \tag{3.2.67}$$

for all compact $K \subset \mathbb{R}^3 \times (0,1]$ and integers $k, l \geq 0$.

By the Calderón–Zygmund estimates and (3.2.61), the associated pressures satisfy

$$Q^{(n)} = (-\Delta)^{-1} \operatorname{div}\operatorname{div} V^{(n)} \otimes V^{(n)} \overset{*}{\rightharpoonup} Q \text{ in } L_{t,\mathrm{loc}}^\infty L_x^m(\mathbb{R}^3 \times (0,1]) \tag{3.2.68}$$

for all $m \in [q/2, +\infty)$. In particular, the convergence occurs weakly in $L_{\mathrm{loc}}^{3/2}(\mathbb{R}^3 \times (0,1])$. Hence, (V, Q) is a suitable weak solution on $\mathbb{R}^3 \times (0,1)$, as shown in Lemma 2.4.3 (Compactness). Since $V \in \mathcal{K}_q^{s_q+\varepsilon}(Q_T)$, we conclude that $V \in C((0,T]; L^q(\mathbb{R}^3))$ is a mild solution of the Navier–Stokes equations on Q_T.

Step 3. Convergence of $U^{(n)}$ and $P^{(n)}$. Lemma 3.2.2 implies the perturbed global energy inequality

$$\begin{aligned} &\int_{\mathbb{R}^3} |U^{(n)}(x,t_2)|^2 \, dx + 2 \int_{t_1}^{t_2} \int_{\mathbb{R}^3} |\nabla U^{(n)}(x,t)|^2 \, dx \, dt \\ &\quad \leq \int_{\mathbb{R}^3} |U^{(n)}(x,t_1)|^2 \, dx + 2 \int_{t_1}^{t_2} \int_{\mathbb{R}^3} V^{(n)} \otimes U^{(n)} : \nabla U^{(n)} \, dx \, dt \end{aligned} \tag{3.2.69}$$

for all $t_1 \in [0,1)$ and $t_2 \in (t_1, 1]$. In particular,

$$|U^{(n)}|^2_{2,\mathbb{R}^3\times(t_1,t_2)} \leq \|U^{(n)}(\cdot,t_1)\|^2_{L^2(\mathbb{R}^3)} + C\|V^{(n)}\|_{L^l_t L^q_x(\mathbb{R}^3\times(t_1,t_2))} |U^{(n)}|^2_{2,\mathbb{R}^3\times(t_1,t_2)}. \tag{3.2.70}$$

When $C\|V^{(n)}\|_{L^l_t L^q_x(\mathbb{R}^3\times(t_1,t_2))} \leq 1/2$, we may absorb the second term on the RHS into the LHS. Starting with $t_1 = 0$ and repeating the argument $O(\Lambda)$ times[6] yields

$$|U^{(n)}|^2_{2,\mathbb{R}^3\times(0,1)} \leq \|U_0^{(n)}\|^2_{L^2(\mathbb{R}^3)} e^{C\Lambda} \leq C(M,\Lambda,p,q), \tag{3.2.71}$$

since the energy norm is allowed to double with each repetition. Along a subsequence, we have

$$U^{(n)} \overset{*}{\rightharpoonup} U \text{ in } L^\infty_t L^2_x \cap L^2_t \dot{H}^1_x(\mathbb{R}^3 \times (0,1)). \tag{3.2.72}$$

By the Calderón–Zygmund estimates, we have that $P^{(n)} = P_1^{(n)} + P_2^{(n)}$ satisfies

$$P_1^{(n)} \rightharpoonup P_1 \text{ in } L^{\frac{5}{3}}_{t,x}(Q_T) \tag{3.2.73}$$

$$P_2^{(n)} \rightharpoonup P_2 \text{ in } L^2_{t,x}(Q_T) \tag{3.2.74}$$

along a subsequence. Finally, Lemma 2.4.3 (Compactness) implies that (u,p) is a suitable weak solution on $\mathbb{R}^3 \times (0,1)$. $\square$

3.3 Proof of Theorem 3.1.1

We are ready to prove Theorem 3.1.1. The proof follows the scheme set forth in [121] except that we use the Calderón-type splitting to control the sequence of solutions.

[6]Alternatively, when $\Lambda \ll 1$, we need not repeat the argument.

Proof of Theorem 3.1.1. By Sobolev embedding in Besov spaces, we have

$$u_0 \in \dot{B}^{s_p}_{p,q}(\mathbb{R}^3) \hookrightarrow \dot{B}^{s_m}_{m,m}(\mathbb{R}^3) \text{ where } m = \max(p,q). \tag{3.3.1}$$

Without loss of generality, we may assume $p = q = m$.

Step 1. Rescaling. Let $u = \mathrm{NS}(u_0)$ be the solution of the Navier–Stokes equations with initial data $u_0 \in \dot{B}^{s_p}_{p,p}(\mathbb{R}^3)$ and $T^*(u_0) < \infty$ as in the statement of Theorem 3.1.1. In Corollary 3.2.5, we proved that u must form a singularity at time $T^*(u_0)$. By the translation and scaling symmetries of the Navier–Stokes equations, we may assume that the singularity occurs at the spatial origin and time $T^*(u_0) = 1$.

Suppose for contradiction that there exists a sequence $t_n \uparrow 1$ and constant $M > 0$ such that

$$\|u(\cdot,t_n)\|_{\dot{B}^{s_p}_{p,p}(\mathbb{R}^3)} \leq M. \tag{3.3.2}$$

The solution $u(\cdot,t)$ is continuous on $[0,1]$ with values in the tempered distributions, due to the splitting in Proposition 3.2.7 (Compactness). By lower semi-continuity, we must have

$$\|u(\cdot,1)\|_{\dot{B}^{s_p}_{p,p}(\mathbb{R}^3)} \leq M. \tag{3.3.3}$$

Let us zoom in around the singularity. For each $n \in \mathbb{N}$, we define

$$u^{(n)}(x,t) := \lambda_n u(\lambda_n x, t_n + \lambda_n^2 t), \quad (x,t) \in Q_1, \tag{3.3.4}$$

where $\lambda_n := (1-t_n)^{1/2}$. Then $u^{(n)} = \mathrm{NS}(u_0^{(n)})$ is the solution of the Navier–Stokes equations on Q_1 with initial data $u_0^{(n)} = \lambda_n u(\lambda_n x, t_n)$, and

$$\|u_0^{(n)}\|_{\dot{B}^{s_p}_{p,p}(\mathbb{R}^3)} \leq M. \tag{3.3.5}$$

Step 2. Limiting procedure. We now apply Proposition 3.2.7 (Compactness) to the sequence $(u_0^{(n)})_{n\in\mathbb{N}}$. Upon passing to a subsequence, we have

$$u^{(n)} \to v \text{ in } L^3_{\mathrm{loc}}(\mathbb{R}^3 \times (0,1]) \tag{3.3.6}$$

$$p^{(n)} \rightharpoonup q \text{ in } L^{\frac{3}{2}}_{\mathrm{loc}}(\mathbb{R}^3 \times (0,1]) \tag{3.3.7}$$

and, in particular,

$$u^{(n)}(\cdot,1) \to v(\cdot,1) \text{ in the sense of tempered distributions,} \tag{3.3.8}$$

where (v,q) is a suitable weak solution on $\mathbb{R}^3 \times (0,1)$ and satisfies the properties in Proposition 3.2.7. According to Proposition 2.4.5 (Persistence of singularities), v has a singularity at $(x,t) = (0,1)$.

Next, we observe that the solution v vanishes identically at time $t = 1$:

$$v(\cdot,1) = 0. \tag{3.3.9}$$

Indeed, since $u(\cdot,1) \in \dot{B}^{s_p}_{p,p}(\mathbb{R}^3)$, we have

$$\langle u^{(n)}(\cdot,1), \varphi \rangle = \langle u(\cdot,1), \lambda_n^{-2}\varphi(\cdot/\lambda_n)\rangle \to 0 \tag{3.3.10}$$

for all Schwartz functions φ. The property (3.3.10) is a consequence of the density of Schwartz functions in $\dot{B}^{s_p}_{p,p}(\mathbb{R}^3)$. Indeed, (3.3.10) is valid with $u(\cdot,1)$ replaced by a Schwartz function ψ, and therefore,

$$\begin{aligned} |\langle u(\cdot,1), \lambda_n^{-2}\varphi(\cdot/\lambda_n)\rangle| &\leq |\langle \psi, \lambda_n^{-2}\varphi(\cdot/\lambda_n)\rangle| + |\langle u(\cdot,1) - \psi, \lambda_n^{-2}\varphi(\cdot/\lambda_n)| \\ &\leq o(1) + C\|u(\cdot,1) - \psi\|_{\dot{B}^{s_p}_{p,p}(\mathbb{R}^3)} \|\varphi\|_{\dot{B}^{-s_p}_{p',p'}(\mathbb{R}^3)} \text{ as } n \to \infty, \end{aligned} \tag{3.3.11}$$

where p' is the Hölder conjugate of p, for all Schwartz functions φ.

Step 3. Backward uniqueness. We will demonstrate

$$\omega = \operatorname{curl} v \equiv 0 \text{ on } \mathbb{R}^3 \times (1/2,1). \tag{3.3.12}$$

If (3.3.12) is satisfied, then the vector identity

$$\Delta v = \nabla \operatorname{div} v - \operatorname{curl}\operatorname{curl} v \tag{3.3.13}$$

implies

$$\Delta v = 0 \text{ on } \mathbb{R}^3 \times (1/2,1). \tag{3.3.14}$$

The harmonic function v will belong to

$$v = U + V \in L^{\frac{10}{3}}_{t,x}(\mathbb{R}^3 \times (1/2,1)) + L^\infty_t L^q_x(\mathbb{R}^3 \times (1/2,1)), \tag{3.3.15}$$

and hence $v \equiv 0$ on $\mathbb{R}^3 \times (1/2,1)$. This contradicts that v has a singularity at time $t = 1$.

We now prove (3.3.12). Based on (3.3.15) and

$$q = P + Q \in L^{\frac{5}{3}}_{t,x}(\mathbb{R}^3 \times (1/2,1)) + L^2_{t,x}(\mathbb{R}^3 \times (1/2,1)) + L^\infty_t L^{\frac{q}{2}}_x(\mathbb{R}^3 \times (1/2,1)), \tag{3.3.16}$$

we have

$$\int_{1/4}^1 \int_{B_1(x_0)} |v|^3 + |q|^{\frac{3}{2}}\, dx\, dt \to 0 \text{ as } |x_0| \to +\infty. \tag{3.3.17}$$

As in Proposition 3.2.4 (Characterization of blow-up), the ε-regularity criterion in Theorem 2.4.4 implies that there exist $R > 0$ and

$$K = \overline{\mathbb{R}^3 \setminus B_R} \times [1/2, 1] \tag{3.3.18}$$

satisfying

$$\sup_K |v| + |\nabla v| + |\nabla^2 v| < +\infty. \tag{3.3.19}$$

Since $\partial_t \omega - \Delta \omega = -\operatorname{curl}(v \cdot \nabla v)$, we have

$$|\partial_t \omega - \Delta \omega| \le C(|\nabla \omega| + |\omega|) \text{ in } K. \tag{3.3.20}$$

Additionally, $w(\cdot, 1) = 0$ due to (3.3.9). Now, according to Theorem 2.5.1 (Backward uniqueness), $\omega \equiv 0$ in K.

It remains to demonstrate that $\omega \equiv 0$ in $\overline{B(R)} \times (1/2, 1)$. Let $G \subset (0,1)$ be a dense open set such that v is smooth on $\Omega = \mathbb{R}^3 \times G$ as guaranteed by Lemma 3.2.6 (Epochs of regularity). Let $z_0 = (x_0, t_0) \in \Omega \cap K$ such that $|x_0| = 2R$. In particular, $\omega \equiv 0$ in a neighborhood of z_0. In addition, by the smoothness of v, there exist $0 < \varepsilon \ll 1$ and C depending on z_0 such that

$$|\partial_t \omega - \Delta \omega| \le C(|\nabla \omega| + |\omega|) \text{ in } \Omega' = B_{4R}(x_0) \times (t_0 - \varepsilon, t_0 + \varepsilon) \subset \Omega. \tag{3.3.21}$$

Hence, the assumptions of Theorem 2.5.2 (Unique continuation) are satisfied in Ω', and $\omega \equiv 0$ in Ω'. This implies that $\omega \equiv 0$ in $\mathbb{R}^3 \times (t_0 - \varepsilon, t_0 + \varepsilon)$. Since $t_0 \in G \cap (1/2, 1)$ was arbitrary, we obtain that $\omega \equiv 0$ in Ω. By density, $\omega \equiv 0$ on $\mathbb{R}^3 \times (1/2, 1)$. The proof is complete. $\square$

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Chapter 4

Global weak Besov solutions of the Navier–Stokes equations and applications

This chapter reproduces the published version of the paper [5] (joint work with Tobias Barker).

Abstract. We introduce a notion of global weak solution to the Navier–Stokes equations in three dimensions with initial values in the critical homogeneous Besov spaces $\dot{B}^{-1+\frac{3}{p}}_{p,\infty}$, $p > 3$. These solutions satisfy a certain stability property with respect to the weak-$*$ convergence of initial conditions. To illustrate this property, we provide applications to blow-up criteria, minimal blow-up initial data, and forward self-similar solutions. Our proof relies on a new splitting result in homogeneous Besov spaces that may be of independent interest.

Contents

4.1 Introduction

In this paper, we investigate certain classes of global-in-time weak solutions of the incompressible Navier–Stokes equations in three dimensions:

$$\left.\begin{aligned} \partial_t v - \Delta v + v \cdot \nabla v = -\nabla q \\ \operatorname{div} v = 0 \end{aligned}\right\} \text{ in } \mathbb{R}^3 \times \mathbb{R}_+. \tag{NSE}$$

In the recent paper [123], G. Seregin and V. Šverák introduced a notion of *global weak L^3 solution* to the Navier–Stokes equations which enjoys the following property. Given a sequence of global weak L^3 solutions with initial data $u_0^{(n)} \rightharpoonup u_0$ in L^3, there exists a subsequence converging in the sense of distributions to a global weak L^3 solution with initial data u_0. This property, known as *weak-$*$ stability*, plays a distinguished role in the regularity theory of the Navier–Stokes equations. For example, such sequences of solutions arise naturally when zooming in on a potential singularity of the Navier–Stokes equations, as in the papers [121, 124][1] by Seregin and [49] by Escauriaza, Seregin, and Šverák.

The main idea in [123] is to decompose a solution of the Navier–Stokes equations as

$$v = V + u, \tag{4.1.1}$$

where V is the linear evolution of the initial data $u_0 \in L^3$,

$$V(x,t) := \int_{\mathbb{R}^3} \Gamma(x-y,t) u_0(y)\, dy, \tag{4.1.2}$$

and u is a perturbation belonging to the global energy space

$$u \in L_t^\infty L_x^2 \cap L_t^2 \dot{H}_x^1(Q_T) \text{ for all } T > 0. \tag{4.1.3}$$

Here, $\Gamma \colon \mathbb{R}^3 \times \mathbb{R}_+ \to \mathbb{R}$ denotes the heat kernel in three dimensions, and

$$Q_T := \mathbb{R}^3 \times]0,T[, \quad 0 < T \leq \infty, \tag{4.1.4}$$

denotes a parabolic cylinder. It is reasonable to expect that solutions of the form $v = V + u$ enjoy weak-$*$ stability, since the linear evolution V is continuous in many nice topologies with respect to weak convergence of initial data, while the correction term u is "merely a perturbation".

In the paper [17], Barker *et al.* created a notion of *global weak $L^{3,\infty}$ solution* that contains the solutions in [123] as well as the scale-invariant solutions investigated by Jia and Šverák in [72] as a special case. These solutions exhibit some interesting phenomena. For instance, global weak $L^{3,\infty}$ solutions exist even when a local-in-time mild solution is not known to exist[2] (unlike in the L^3 case). It appears that such solutions may be non-unique even from the initial time, see the examples of the forward self-similar solutions computed by Guillod and Šverák in [66]. On the other hand, weak-$*$ stability continues to hold in spite of the conjectured non-uniqueness. The authors of [17] also showed that global weak $L^{3,\infty}$ solutions provide a natural

[1] In these papers, Seregin also investigated weak-$*$ stability in the context of local Leray solutions, which were discovered by Lemarié-Rieusset [95].

[2] We mention that for divergence-free initial data in $L^{3,\infty}$, there exists an associated global-in-time Lemarié-Rieusset local energy solution of the Navier–Stokes equations [95].

class in which to investigate *minimal blow-up initial data*. Šverák also mentioned the possibility of investigating the *radius of smoothness* (resp. *uniqueness*) associated to each initial data $u_0 \in L^{3,\infty}$. This is the maximal time such that each global weak L^3 solution with prescribed initial data u_0 is smooth (resp. unique).

Recently, the second author proposed in the paper [14] to investigate notions of solution in critical spaces X that generalize the solutions described above. Namely, one desires a notion of *global X solution* that satisfies a weak-$*$ stability property when, for example,

$$X = \dot{H}^{\frac{1}{2}}, L^3, L^{3,\infty}, \dot{B}^{-1+\frac{3}{p}}_{p,\infty}, \mathrm{BMO}^{-1}, \dot{B}^{-1}_{\infty,\infty}, \quad 3 < p < \infty. \tag{4.1.5}$$

The second author established the existence of *global $\dot{B}^{-\frac{1}{4}}_{4,\infty}$ solutions* with the decomposition $v = V + u$ utilized in previous works. Moreover, he proved that under natural hypotheses, $\dot{B}^{-\frac{1}{4}}_{4,\infty}$ is the largest critical space in which such a decomposition is viable. Therefore, a notion of global X solution for the critical homogeneous Besov spaces $X = \dot{B}^{-1+\frac{3}{p}}_{p,\infty}$ with $4 < p < \infty$ must be based on a new structure.[3]

In this paper, we develop a notion of global weak Besov solution *of the Navier–Stokes equations associated to initial data in the critical homogeneous Besov spaces $\dot{B}^{-1+\frac{3}{p}}_{p,\infty}(\mathbb{R}^3)$, $3 < p < \infty$.*

In Section 4.3, we prove the following results. Let $3 < q \leq p < \infty$, and $0 < T \leq \infty$. We include forcing terms of the form $\operatorname{div} F$ with $F \in \mathcal{F}_q(Q_T)$, defined as the space of locally integrable functions $F \colon Q_T \to \mathbb{R}^{3\times 3}$ such that

$$\|F\|_{\mathcal{F}_q(Q_T)} := \sup_{t \in]0,T[} t^{1-\frac{3}{2q}} \|F(\cdot,t)\|_{L^q(\mathbb{R}^3)} < \infty. \tag{4.1.6}$$

Theorem 4.1.1 (Existence)**.** *Let $u_0 \in \dot{B}^{-1+\frac{3}{p}}_{p,\infty}(\mathbb{R}^3)$ be a divergence-free vector field and $F \in \mathcal{F}_q(Q_\infty)$. There exists a global weak Besov solution v with initial data u_0 and forcing term* $\operatorname{div} F$.

Theorem 4.1.2 (Weak–$*$ stability)**.** *Suppose that $(v^{(n)})_{n\in\mathbb{N}}$ is a sequence of global weak Besov solutions with initial data $u_0^{(n)}$ and forcing terms* $\operatorname{div} F^{(n)}$*, respectively. Furthermore, suppose*

[3] While critical spaces are not strictly necessary for weak-$*$ stability (see p. 5 of the second author's paper [15], for example), they are convenient for the applications we have in mind.

that

$$u_0^{(n)} \overset{*}{\rightharpoonup} u_0 \text{ in } \dot{B}^{-1+\frac{3}{p}}_{p,\infty}(\mathbb{R}^3), \quad F^{(n)} \overset{*}{\rightharpoonup} F \text{ in } \mathcal{F}_q(Q_\infty). \tag{4.1.7}$$

Then there exists a subsequence converging strongly in $L^3_{\mathrm{loc}}(Q_\infty)$ to a global weak Besov solution v with initial data u_0 and forcing term $\operatorname{div} F$.

Theorem 4.1.3 (Weak-strong uniqueness)**.** *There exists a constant $\varepsilon_0 := \varepsilon_0(p,q) > 0$ such that for all $u_0 \in \dot{B}^{-1+\frac{3}{p}}_{p,\infty}(\mathbb{R}^3)$ divergence-free and $F \in \mathcal{F}_q(Q_T)$ satisfying*

$$\|u_0\|_{\dot{B}^{-1+\frac{3}{p}}_{p,\infty}(\mathbb{R}^3)} + \|F\|_{\mathcal{F}_q(Q_T)} \leq \varepsilon_0, \tag{4.1.8}$$

there exists a unique weak Besov solution on Q_T with initial data u_0 and forcing term $\operatorname{div} F$.[4] *This solution belongs to $L^\infty_{\mathrm{loc}}(\mathbb{R}^3 \times \mathbb{R}_+)$.*

The second half of this paper is dedicated to applications of global weak Besov solutions. Namely, we provide applications to certain critical problems concerning blow-up criteria, minimal blow-up initial data, and forward self-similar solutions. We present these results at the end of the introduction. The reader interested only in applications is invited to skip to Section 4.1.1.

To motivate our notion of solution, it is instructive to write the perturbed Navier–Stokes system satisfied by the correction term in the decomposition $v = V + u$ used in the previous works [123, 17, 14]:

$$\left.\begin{array}{c} \partial_t u - \Delta u + (u+V)\cdot\nabla u + u\cdot\nabla V = -\nabla q - \operatorname{div} V\otimes V \\ \operatorname{div} u = 0 \end{array}\right\} \text{ in } \mathbb{R}^3\times\mathbb{R}_+ \tag{4.1.9}$$

with zero initial condition. The associated global energy inequality is

$$\|u(\cdot,t)\|^2_{L^2} + 2\int_0^t\int_{\mathbb{R}^3} |\nabla u|^2\,dx\,dt' \leq 2\int_0^t\int_{\mathbb{R}^3} (V\otimes u + V\otimes V) : \nabla u\,dx\,dt'. \tag{4.1.10}$$

In order for the RHS of (4.1.10) to make sense, we require that

$$V \in L^4_{t,\mathrm{loc}} L^4_x(\mathbb{R}^3\times\mathbb{R}_+). \tag{4.1.11}$$

[4] In particular, the solution agrees with the unique small mild solution on Q_T with initial data u_0 and forcing term F, and $u(\cdot,t) \overset{*}{\rightharpoonup} u_0$ in $\dot{B}^{-1+\frac{3}{p}}_{p,\infty}$ as $t \to 0^+$.

As demonstrated by the second author in [14], the quantitative scale-invariant version of (4.1.11) is $\|u_0\|_{\dot{B}^{-\frac{1}{4}}_{4,\infty}} \leq M$, due to the caloric characterization of Besov spaces. Roughly speaking, the forcing term should belong to an L^2-based space, whereas $V \otimes V$ may only belong to spaces with integrability $\geq \frac{p}{2}$ for initial data $u_0 \in \dot{B}^{-1+\frac{3}{p}}_{p,\infty}$. When $p \gg 1$, the obstacle is sometimes interpreted as "slow decay at spatial infinity."

The notion of global weak Besov solution developed in this paper is based on the decomposition

$$v = P_k(u_0) + u, \tag{4.1.12}$$

where $P_k(u_0)$ is the *kth Picard iterate*, $k \geq 0$, defined by

$$P_0(u_0)(\cdot, t) := S(t)u_0, \tag{4.1.13}$$

$$P_{k+1}(u_0)(\cdot, t) := S(t)u_0 - B(P_k, P_k), \quad k \geq 0, \tag{4.1.14}$$

and B is the bilinear term in the integral formulation of the Navier–Stokes equations (see (4.2.17) for the precise definition):

$$B(u, v)(\cdot, t) := \int_0^t S(t-s)\mathbb{P} \operatorname{div} u \otimes v(\cdot, s)\, ds\, dx. \tag{4.1.15}$$

The papers [123, 17, 14] utilized the decomposition (4.1.12) with $k = 0$. Observe that if v solves (NSE), then $u = v - P_k$ solves

$$\left.\begin{array}{r} \partial_t u - \Delta u + P_k \cdot \nabla u + u \cdot \nabla P_k + u \cdot \nabla u = -\nabla p - \operatorname{div} F_k \\ \operatorname{div} u = 0 \end{array}\right\} \text{ in } \mathbb{R}^3 \times \mathbb{R}_+ \tag{4.1.16}$$

with initial condition $u(\cdot, 0) = 0$, where the forcing term $F_k(u_0)$, $k \geq 0$, is defined by

$$F_k(u_0) := P_k \otimes P_k - P_{k-1} \otimes P_{k-1}, \tag{4.1.17}$$

and we use the convention that $P_{-1}(u_0) = 0$. One expects the correction u to belong to the energy class if F_k belongs to $L^2(\mathbb{R}^3 \times]0, T[)$.

Here is our key observation:

Lemma 4.1.4 (Finite energy forcing)**.** *Let $p \in]3,\infty[$ and $u_0 \in \dot{B}^{-1+\frac{3}{p}}_{p,\infty}(\mathbb{R}^3)$ be a divergence-free vector field with $\|u_0\|_{\dot{B}^{-1+\frac{3}{p}}_{p,\infty}(\mathbb{R}^3)} \leq M$. Then for all integers $k \geq k(p) := \lceil \frac{p}{2} \rceil - 2$, the forcing term $F_k(u_0)$ satisfies*

$$\|F_k(u_0)\|_{L^2(\mathbb{R}^3\times]0,T[)} \leq T^{\frac{1}{4}} C(k,M,p). \tag{4.1.18}$$

The proof of Lemma 4.1.4 is based on a self-improvement property of the bilinear term B. Heuristically, if a vector field V belongs to an L^p-based space, then $B(V,V)$ belongs to an $L^{\frac{p}{2}}$-based space (as well as the original space). For instance, let $u_0 \in \dot{B}^{-\frac{1}{2}}_{6,\infty}(\mathbb{R}^3)$. Then $V := Su_0$ belongs to an L^6-based space, and $F_1(u_0)$ satisfies

$$F_1 = -B(V,V) \otimes V - V \otimes B(V,V) + B(V,V) \otimes B(V,V). \tag{4.1.19}$$

Since $B(V,V)$ belongs to an L^3-based space (and an L^6-based space), an application of Hölder's inequality implies that F_1 belongs to an L^2-based space. The same reasoning applies *mutatis mutandis* with the inclusion of a forcing term $\operatorname{div} F$ with F belonging to $\mathcal{F}_q(Q_T)$ with $q \in]3,p]$.[5] The self-improvement property of B was already exploited in the papers [54, 58]. The phenomenon that F_1 is a higher order term is already present in the Picard iterates for the ODE $\dot{x} = ax^2$, $x(0) = x_0$, where $a, x_0 \in \mathbb{R}$.

Here is our main definition:

Definition 4.1.5 (Weak Besov solution)**.** *Let $T > 0$, $u_0 \in \mathrm{BMO}^{-1}(\mathbb{R}^3)$ be a divergence-free vector field, and $F \in \mathcal{F}(Q_T)$.*[6]

We say that a distributional vector field v on $Q_T := \mathbb{R}^3\times]0,T[$ is a weak Besov solution *to the Navier–Stokes equations on Q_T with initial data u_0 and forcing term $\operatorname{div} F$ if there exists an integer $k \geq 0$ such that the following requirements are satisfied.*

First, there exists a pressure $q \in L^{\frac{3}{2}}_{\mathrm{loc}}(Q_T)$ such that v satisfies the Navier–Stokes equations on Q_T in the sense of distributions:

$$\partial_t v - \Delta v + v \cdot \nabla v = -\nabla q + \operatorname{div} F, \qquad \operatorname{div} v = 0. \tag{4.1.20}$$

[5] See (4.1.6) or (4.2.18)-(4.2.19) for the relevant definition.

[6] The requirements $u_0 \in \mathrm{BMO}^{-1}(\mathbb{R}^3)$ and $F \in \mathcal{F}(Q_T)$ ensure that the Picard iterates $P_k(u_0, F)$ are well-defined. We refer the reader to (4.2.10), (4.2.21), and (4.2.29)-(4.2.30) for the respective definitions.

Second, v may be decomposed as

$$v = u + P_k(u_0, F), \tag{4.1.21}$$

where $u \in L_t^\infty L_x^2 \cap L_t^2 \dot{H}_x^1(Q_T)$ and $u(\cdot, t)$ is weakly L^2-continuous on $[0, T]$. Furthermore, we require that $\lim_{t\downarrow 0}\|u(\cdot,t)\|_{L^2(\mathbb{R}^3)} = 0$ and $F_\ell \in L^2(Q_T)$ for all integers $\ell \geq k$. Finally, we require that (v, q) satisfies the following local energy inequality for every $t \in]0, T[$ and all non-negative test functions $\varphi \in C_0^\infty(Q_\infty)$:

$$\begin{aligned} &\int_{\mathbb{R}^3} \varphi(x,t)|v(x,t)|^2 \, dx + 2\int_0^t \int_{\mathbb{R}^3} \varphi |\nabla v|^2 \, dx \, dt' \\ &\quad \leq \int_0^t \int_{\mathbb{R}^3} |v|^2(\partial_t \varphi + \Delta\varphi) + v(|v|^2 + 2q)\cdot\nabla\varphi - 2F : \nabla(\varphi v) \, dx \, dt'. \end{aligned} \tag{4.1.22}$$

We say that the weak Besov solution is based on the kth Picard iterate *if the above properties are satisfied for a given integer $k \geq 0$.*

We say that v is a global weak Besov solution *(or weak Besov solution on $Q_\infty := \mathbb{R}^3 \times \mathbb{R}_+$) with initial data u_0 and forcing term* $\operatorname{div} F$ *if there exists an integer $k \geq 0$ such that for all $T > 0$, v is a weak Besov solution on Q_T based on the kth Picard iterate with initial data u_0 and forcing term* $\operatorname{div} F$.

Let us explain the requirement that $F_\ell \in L^2(Q_T)$ for all $\ell \geq k$. Its role is to ensure that v is also a weak Besov solution based on the ℓth Picard iterate for all $\ell \geq k$, see Proposition 4.3.4. In other words, one may always raise the order of the Picard iterate. Similarly, one may lower the Picard iterate depending on the regularity of the initial data. Hence, our notion of weak Besov solution in Definition 4.1.5 is not overly sensitive to the order of the Picard iterate.

Before turning to applications, we present another key ingredient in our arguments: a decay property for the correction term near the initial time.

Proposition 4.1.6 (Decay property)**.** *Let $p \in]3, \infty[$, $q \in]3, p]$, and $k \geq k(p) := \lceil \frac{p}{2} \rceil - 2$. Assume that v is a global weak Besov solution based on the kth Picard iterate with initial data u_0 and forcing term* $\operatorname{div} F$*. Let $\|u_0\|_{\dot{B}_{p,\infty}^{-1+\frac{3}{p}}(\mathbb{R}^3)} + \|F\|_{\mathcal{F}_q(Q_\infty)} \leq M$. Then*

$$\|u(\cdot,t)\|_{L^2(\mathbb{R}^3)} \leq C(k, M, p, q) t^{\frac{1}{4}}. \tag{4.1.23}$$

The proof is given in Section 4.3.2. Notably, (4.1.23) is used to obtain the global energy inequality for the correction term starting from the initial time, see Corollary 4.3.9. A similar issue is already present in Seregin's paper [125] for sequences of weak Leray-Hopf solutions with initial data converging to zero weakly in L^2.

To illustrate the main issue in Proposition 4.1.6, we consider the special case $u_0 \in L^{3,\infty}$ with zero forcing term and decompose the solution as $v = V + u$ as in the paper [17]. That is, $k = 0$ in Definition 4.1.5. Heuristically, the energy of the correction term should originate entirely in the forcing term $-\operatorname{div} V \otimes V$. Let $\|u_0\|_{L^{3,\infty}} \leq M$. Since

$$\int_{\mathbb{R}^3} \int_0^t |V \otimes V|^2 \, dx \, dt \leq C t^{\frac{1}{2}} M^2 \text{ for all } t > 0, \tag{4.1.24}$$

we expect the following *a priori* estimate:

$$\|u(\cdot,t)\|_{L^2}^2 + 2 \int\limits_0^t \int\limits_{\mathbb{R}^3} |\nabla u|^2 \, dx \, dt' \leq t^{\frac{1}{2}} C(M). \tag{4.1.25}$$

When $u_0 \in L^3$, the proof of (4.1.25) is via a Gronwall-type argument that does not extend to the more general case. For instance, consider the following estimate for the lower order term in the global energy inequality:

$$\left| \int_0^T \int_{\mathbb{R}^3} V \otimes u : \nabla u \, dx \, dt' \right| \leq C \|V\|_{L_t^\infty L_x^{3,\infty}(Q_T)} \|u\|_{L_t^2 L_x^{6,2}(Q_T)} \|\nabla u\|_{L^2(Q_T)}, \tag{4.1.26}$$

where the quantity $\|V\|_{L_t^\infty L_x^{3,\infty}(Q_T)}$ is not "locally small" unless $M \ll 1$.

In the paper [17], this issue is overcome using splitting arguments[7] inspired by C. P. Calderón [33] and a decomposition of initial data in Lorentz spaces. In the present work, we require the following new decomposition of initial data in Besov spaces.[8]

Lemma 4.1.7 (Splitting of initial data)**.** *Let $p \in]d, \infty[$. There exist $p_2 \in]p, \infty[$, $\delta_2 \in]0, -s_{p_2}[$, $\gamma_1, \gamma_2 > 0$, and $C > 0$, each depending only on p, such that for each divergence-free vector field $g \in \dot{B}^{s_p}_{p,\infty}(\mathbb{R}^d)$ and $N > 0$, there exist divergence-free vector fields $\bar{g}^N \in \dot{B}^{s_{p_2}+\delta_2}_{p_2,p_2}(\mathbb{R}^d) \cap$*

[7] Related splitting arguments have also previously been used by Jia and Šverák in [71], in order to show estimates near the initial time for Lemarié-Rieusset local energy solutions of the Navier–Stokes equations with L^3 initial data.

[8] This lemma was obtained by the second author in [14, Proposition 1.5], which will not be submitted for journal publication.

$\dot{B}^{s_p}_{p,\infty}(\mathbb{R}^d)$ *and* $\widetilde{g}^N \in L^2(\mathbb{R}^d) \cap \dot{B}^{s_p}_{p,\infty}(\mathbb{R}^d)$ *with the following properties:*

$$g = \widetilde{g}^N + \bar{g}^N, \tag{4.1.27}$$

$$\|\widetilde{g}^N\|_{L^2(\mathbb{R}^d)} \leq CN^{-\gamma_2}\|g\|_{\dot{B}^{s_p}_{p,\infty}(\mathbb{R}^d)}, \tag{4.1.28}$$

$$\|\bar{g}^N\|_{\dot{B}^{s_{p_2}+\delta_2}_{p_2,p_2}(\mathbb{R}^d)} \leq CN^{\gamma_1}\|g\|_{\dot{B}^{s_p}_{p,\infty}(\mathbb{R}^d)}. \tag{4.1.29}$$

Furthermore,

$$\|\widetilde{g}^N\|_{\dot{B}^{s_p}_{p,\infty}(\mathbb{R}^d)}, \|\bar{g}^N\|_{\dot{B}^{s_p}_{p,\infty}(\mathbb{R}^d)} \leq C\|g\|_{\dot{B}^{s_p}_{p,\infty}(\mathbb{R}^d)}. \tag{4.1.30}$$

The most notable feature of Lemma 4.1.7 is that the summability index q is reduced from ∞ in both terms of the decomposition. Therefore, the splitting is not a simple “diagonal splitting” that could be obtained via complex interpolation of Besov spaces. Moreover, it does not appear to obviously follow from the abstract real interpolation theory, since Besov spaces are not real interpolation spaces of Besov spaces except in special cases (such as when the integrability index p is kept constant), see [88, Section 4] for an example. Further discussion and the proof of a general splitting result, Proposition 4.5.6, are contained in Section 4.5.

4.1.1 Applications

The second part of the paper is focused on applications of global weak Besov solutions to three problems concerning the three-dimensional Navier–Stokes equations.

Blow-up criteria

Our first application is an improved blow-up criterion for the Navier–Stokes equations in critical spaces:

Corollary 4.1.9 ($\dot{B}^{-1+\frac{3}{p}}_{p,\infty}$ blow-up criterion)**.** *Let* $T^* > 0$*,* $u_0 \in L^\infty(\mathbb{R}^3)$ *be a divergence-free vector field, and* $F \in L^\infty_t L^q_x(\mathbb{R}^3\times]0,T^*[)$ *for some* $q \in]3,\infty[$*. Suppose that* $v \in L^\infty(\mathbb{R}^3\times]0,T[)$ *is a mild solution of the Navier–Stokes equations on* $\mathbb{R}^3\times]0,T[$ *with initial data* u_0 *and forcing term* $\operatorname{div} F$ *for all* $T \in]0,T^*[$*. Suppose that there exists a sequence of times* $t_n \uparrow T^*$ *such that*

$$\sup_{n\in\mathbb{N}}\|v(\cdot,t_n)\|_{\dot{B}^{-1+\frac{3}{p}}_{p,\infty}(\mathbb{R}^3)} < \infty \tag{4.1.31}$$

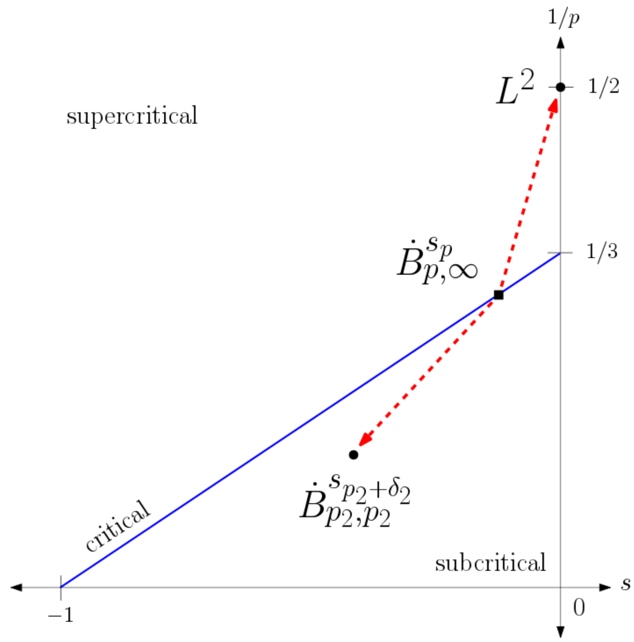

Figure 4.8: Illustration of Lemma 4.1.7 with $d = 3$. The initial data $g \in \dot{B}^{s_p}_{p,\infty}$ is split along the dashed red lines into $\widetilde{g} \in L^2$ and $\bar{g} \in \dot{B}^{s_{p_2}+\delta_2}_{p_2,p_2}$.

for some $p \in]3, \infty[$*. Finally, assume that* $v(\cdot, T^*)$ *satisfies*

$$\sqrt{T^* - t_n}\, v(\sqrt{T^* - t_n}(\cdot - x^*), T^*) \overset{*}{\rightharpoonup} 0 \textit{ in } \mathcal{D}'(\mathbb{R}^3), \tag{4.1.32}$$

for some $x^* \in \mathbb{R}^3$*. Then* v *is regular at* (x^*, T^*)*.*[9] *If* (4.1.32) *is verified for all* $x^* \in \mathbb{R}^3$*, then* $v \in L^\infty(\mathbb{R}^3 \times]0, T^*[)$*.*

Corollary 4.1.9 is a special case of Theorem 4.4.1, which may be regarded as a quantitative version of the corollary. For comparison, the following weaker criterion was obtained without forcing term by the first author in [4]:

$$\lim_{t \uparrow T^*} \|v(\cdot, t)\|_{\dot{B}^{-1+\frac{3}{p}}_{p,q}(\mathbb{R}^3)} = \infty \text{ with } p, q \in]3, \infty[. \tag{4.1.33}$$

We also mention the preceding works [49, 124, 121, 112, 16, 78, 57, 58, 141] in this direction.[10]

[9] We say that v is *regular* at $(x^*, T^*) \in \mathbb{R}^{3+1}$ if $v \in L^\infty(B(x^*, R) \times]T^* - R^2, T^*[)$ for some $R > 0$. Otherwise, it is *singular* at (x^*, T^*).

[10] Corollary 4.1.9 without forcing term appeared in the recent preprint [14] of the second author which is not

Specifically, our methods are based on the rescaling procedure and backward uniqueness arguments introduced by Escauriaza, Seregin, and Šverák in [49] and further developed by Seregin in [124, 121].

The requirement (4.1.32) states that the blow-up profile $v(\cdot, T^*)$ vanishes in the limit of the rescaling procedure. This assumption excludes, for example, the situation that $v(\cdot, T^*)$ is scale-invariant, in which case zooming on the singularity would provide no new information. If $v(\cdot, T^*)$ belongs to the closure of Schwartz functions in $\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3)$, then (4.1.32) is automatically satisfied. See Section 4.4.1 for further remarks.

The reason for the restriction $q > 3$ on the forcing term is to ensure that the maximal time of existence is indicated by the formation of a singularity. Note, for instance, that solutions of the equation $\Delta u = \operatorname{div} F$ in $\mathbb{R}^d$ with F belonging to $L^d(\mathbb{R}^d)$ may not be locally bounded when $d \geq 2$.

Finally, let us mention that the "concentration+rigidity" roadmap of Kenig and Merle [79] was utilised by Koch and Kenig in [78], and subsequently by Koch, Gallagher and Planchon in [57]-[58] to show the following. Namely, if the maximal time of existence $T_{\max}(u_0)$ is finite, then

$$\limsup_{t \uparrow T_{\max}(u_0)} \|v(\cdot, t)\|_X = \infty \tag{4.1.34}$$

for $X = \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ [78], $L^3(\mathbb{R}^3)$[57][11], and $\dot{B}^{-1+\frac{3}{p}}_{p,q}(\mathbb{R}^3)$ $(3 < p, q < \infty)$ [58]. The approach of the aforementioned papers relies on profile decompositions of sequences of elements bounded in the above spaces X. In [13, Remark 3.1], it is conjectured that profile decompositions fail for the continuous embedding $\dot{B}^{-1+\frac{3}{p}}_{p,\infty}(\mathbb{R}^3) \hookrightarrow \dot{B}^{-1+\frac{3}{p'}}_{p',\infty}(\mathbb{R}^3)$ $(3 < p \leq p')$. Thus, it seems challenging to use the approach in [78] and [58]-[57] to obtain Corollary 4.1.9.

Minimal blow-up problems

Our second application is to minimal blow-up questions in the context of global weak Besov solutions. The existence of minimal blow-up initial data was first proven by Rusin and Šverák in [118] in the class of mild solutions with initial data belonging to $\dot{H}^{\frac{1}{2}}$ (provided that such solutions may form singularities in finite time). Analogous results were established for L^3 by Jia and Šverák in [71] and for $\dot{B}^{-1+\frac{3}{p}}_{p,q}$, $p, q \in]3, \infty[$ by Gallagher, G. Koch, and Planchon

intended to be submitted for publication.

[11] The result for $L_3(\mathbb{R}^3)$ and $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ was first obtained in [49].

in [58].

While local-in-time mild solutions are not known to exist for each solenoidal initial data in $L^{3,\infty}$ or $\dot{B}^{-1+\frac{3}{p}}_{p,\infty}$, the minimal blow-up data problem in these spaces may be reformulated for certain classes of weak solutions. This was originally observed by the second author, Seregin, and Šverák in [17] for global weak $L^{3,\infty}$ solutions. We now formulate the problem for global weak Besov solutions:

Definition 4.1.10 (Critical space)**.** *Let $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ be a Banach space consisting of divergence-free distributional vector fields on $\mathbb{R}^3$. We say that $\mathcal{X}$ is a* critical space *provided that*

(i) $\mathcal{X}$ is continuously embedded in $\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3)$,

(ii) $\mathcal{X}$ and $\|\cdot\|_{\mathcal{X}}$ are invariant under spatial translation and the scaling symmetry of the Navier–Stokes equations, and

(iii) $\bar{B}^{\mathcal{X}} := \{u_0 \in \mathcal{X} : \|u_0\|_{\mathcal{X}} \leq 1\}$ is sequentially compact in the topology of distributions.

Let $\mathcal{X}$ be a critical space which is embedded into $\dot{B}^{-1+\frac{3}{p}}_{p,\infty}(\mathbb{R}^3)$ for some $p \in]3,\infty[$. By Theorem 4.1.3, there exists $\rho > 0$ satisfying

> *(small data implies smooth)* $u_0 \in \mathcal{X}$ and $\|u_0\|_{\mathcal{X}} < \rho$ implies that any global weak Besov solution with initial data u_0 has no singular points

Then the following quantity is well defined:

> $\rho_{\mathcal{X}} := \sup(\{\rho > 0 :$ for all $u_0 \in \mathcal{X}$ satisfying $\|u_0\|_{\mathcal{X}} \leq \rho$, any global weak Besov solution with initial condition u_0 has no singular points$\})$.

Under the assumption that $\rho_{\mathcal{X}} < \infty$, one may ask whether the above supremum is attained: Does there exist a global weak Besov solution $\widetilde{v}$ with initial data $\widetilde{u}_0 \in \mathcal{X}$ such that $\widetilde{v}$ has a singular point and $\|\widetilde{u}_0\|_{\mathcal{X}} = \rho_{\mathcal{X}}$? Such $\widetilde{u_0}$ is referred to as *minimal blow-up initial data*. We answer this question in the affirmative below:

Corollary 4.1.11 (Minimal blow-up data)**.** *Let $\mathcal{X}$ be a critical space continuously embedded into $\dot{B}^{-1+\frac{3}{p}}_{p,\infty}(\mathbb{R}^3)$ for some $p \in]3,\infty[$. Suppose that $\rho_{\mathcal{X}} < \infty$. Then there exists a solenoidal vector field $\widetilde{u_0} \in \mathcal{X}$ with $\|\widetilde{u_0}\|_{\mathcal{X}} = \rho_{\mathcal{X}}$ such that $\widetilde{u_0}$ is initial data for a singular global weak Besov solution. The set of such $\widetilde{u_0}$ is sequentially compact (modulo spatial translation and the scaling symmetry of the Navier–Stokes equations) in the topology of distributions.*

Our more general result is Theorem 4.4.7, which treats the problem of minimal blow-up perturbations of global solutions, thus generalizing the work [117] of Rusin for $\dot{H}^{\frac{1}{2}}$ initial data. On the other hand, Corollary 4.1.11 already contains the previously known results for $\mathcal{X} = \dot{H}^{\frac{1}{2}}, L^3$, and $\dot{B}^{-1+\frac{3}{p}}_{p,q}$ with $p, q \in]3, \infty[$ due to weak-strong uniqueness for global weak Besov solutions. While Corollary 4.1.11 only asserts the sequential compactness in the topology of distributions, we may upgrade to convergence in norm if the critical space is uniformly convex, as in the examples above. For $\mathcal{X} = \dot{B}^{-1+\frac{3}{p}}_{p,\infty}$, compactness is in the weak-$*$ topology. A minor point is that our approach also accounts for any possible changes in the set of minimal blow-up initial data under renormings of the critical space.

Self-similar solutions

Our final application concerns forward-in-time self-similar solutions of the Navier–Stokes equations. A locally integrable vector field $v \colon \mathbb{R}^3 \times \mathbb{R}_+ \to \mathbb{R}^3$ is *discretely self-similar* with scaling factor $\lambda > 1$ (λ-DSS) provided that

$$v(x,t) = \lambda v(\lambda x, \lambda^2 t) \text{ for a.e. } (x,t) \in \mathbb{R}^3 \times \mathbb{R}_+. \tag{4.1.35}$$

The vector field is *self-similar* (scale-invariant) if the relation (4.1.35) is verified for all $\lambda > 0$. We consider also the analogous definition for vector fields on $\mathbb{R}^3$ and for distributional vector fields.

Although self-similar solutions of the Navier–Stokes equations have a rich history going back to Leray [101][12], the existence of large-data forward self-similar solutions was settled only recently by Jia and Šverák in [72]. These solutions have important consequences for the potential non-uniqueness of weak Leray-Hopf solutions, as investigated in [73, 66]. While the solutions in [72] correspond to scale-invariant data in $C^\alpha_{\text{loc}}(\mathbb{R}^3 \setminus \{0\})$, there is also an abundant literature on the existence of (discretely) self-similar solutions evolving from rough initial data [139, 86, 26, 98, 27, 28, 35]. In particular, we are interested in the paper [28] of Bradshaw and Tsai, which established the existence of discretely self-similar solutions associated to initial data $u_0 \in \dot{B}^{-1+\frac{3}{p}}_{p,\infty}(\mathbb{R}^3)$, $p \in]3, 6[$. Our final application is the following extension of their work:

[12] In [101], Leray posed the question of whether backward self-similar solutions of the Navier–Stokes equations exist. These were subsequently ruled out in [111] and [138].

Theorem 4.1.12 (Existence of (discretely) self-similar solutions)**.** *Suppose* $u_0 \in \dot{B}^{-1+\frac{3}{p}}_{p,\infty}(\mathbb{R}^3)$ *is a divergence-free vector field for some* $p \in]3,\infty[$*. If* u_0 *is* λ*–DSS for some scaling factor* $\lambda > 1$*, then there exists a* λ*–DSS global weak Besov solution with initial data* u_0*. If* u_0 *is scale-invariant, then there exists a scale-invariant global weak Besov solution with initial data* u_0.

To prove Theorem 4.1.12, we approximate $u_0 \in \dot{B}^{-1+\frac{3}{p}}_{p,\infty}(\mathbb{R}^3)$ by a sequence of (discretely) self-similar initial data belonging to the Lorentz space $L^{3,\infty}(\mathbb{R}^3)$. The proof is completed by applying the weak-$*$ stability property to an associated sequence of (discretely) self-similar solutions whose existence was established in [26].

4.2 Preliminaries

4.2.1 Function spaces

Let $d, m \in \mathbb{N}$. We begin by recalling the definition of the *homogeneous Besov spaces* $\dot{B}^s_{p,q}(\mathbb{R}^d;\mathbb{R}^m)$. Our treatment follows [11, Chapter 2]. There exists a non-negative radial function $\varphi \in C^\infty(\mathbb{R}^d)$ supported on the annulus $\{\xi \in \mathbb{R}^d : 3/4 \leq |\xi| \leq 8/3\}$ such that

$$\sum_{j\in\mathbb{Z}} \varphi(2^{-j}\xi) = 1, \quad \xi \in \mathbb{R}^3 \setminus \{0\}. \tag{4.2.1}$$

The homogeneous Littlewood-Paley projectors $\dot{\Delta}_j$ are defined by

$$\dot{\Delta}_j f = \varphi(2^{-j}D)f, \quad j \in \mathbb{Z}, \tag{4.2.2}$$

for all tempered distributions f on $\mathbb{R}^d$ with values in $\mathbb{R}^m$. The notation $\varphi(2^{-j}D)f$ denotes convolution with the inverse Fourier transform of $\varphi(2^{-j}\cdot)$ with f.

Let $p, q \in [1,\infty]$ and $s \in]-\infty, d/p[$.[13] The homogeneous Besov space $\dot{B}^s_{p,q}(\mathbb{R}^d;\mathbb{R}^m)$ consists of all tempered distributions f on $\mathbb{R}^d$ with values in $\mathbb{R}^m$ satisfying

$$\|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^d;\mathbb{R}^m)} := \left\| \left(2^{js}\|\dot{\Delta}_j f\|_{L^p}\right)_{j\in\mathbb{Z}} \right\|_{\ell^q} < \infty \tag{4.2.3}$$

and such that $\sum_{j\in\mathbb{Z}} \dot{\Delta}_j f$ converges to f in the sense of tempered distributions on $\mathbb{R}^d$ with values

[13] The choice $s = d/p$, $q = 1$ is also valid.

in $\mathbb{R}^m$. In this range of indices, $\dot{B}^s_{p,q}(\mathbb{R}^d;\mathbb{R}^m)$ is a Banach space. When $s \geq 3/p$ and $q > 1$, the spaces must be considered *modulo polynomials*, see Section 4.5. Note that other reasonable choices of the function φ defining $\dot{\Delta}_j$ lead to equivalent norms. In general, Besov spaces may also be characterized as real interpolation spaces of Bessel potential spaces, see [22, 95]. For now, we only consider $d = m = 3$.

We now recall a particularly useful property of Besov spaces, i.e., their characterization in terms of the heat kernel. For all $s \in]-\infty, 0[$, there exists a constant $c := c(s) > 0$ such that for all tempered distributions f on $\mathbb{R}^3$,

$$c^{-1} \sup_{t>0} t^{-\frac{s}{2}} \|S(t)f\|_{L^p(\mathbb{R}^3)} \leq \|f\|_{\dot{B}^s_{p,\infty}(\mathbb{R}^3)} \leq c \sup_{t>0} t^{-\frac{s}{2}} \|S(t)f\|_{L^p(\mathbb{R}^3)}, \tag{4.2.4}$$

where we use the notation

$$(Sf)(\cdot,t) = S(t)f = \Gamma(\cdot,t) * f, \quad t > 0, \tag{4.2.5}$$

and $\Gamma\colon \mathbb{R}^3 \times \mathbb{R}_+ \to \mathbb{R}$ is the heat kernel in three dimensions. This motivates the definition of the *Kato spaces* $\mathcal{K}^s_p(Q_T)$ with parameters $s \in \mathbb{R}$, $p \in [1,\infty]$, and $0 < T \leq \infty$. The Kato spaces consist of all locally integrable vector fields $u\colon Q_T \to \mathbb{R}^3$ satisfying

$$\|u\|_{\mathcal{K}^s_p(Q_T)} := \operatorname*{ess\,sup}_{t\in]0,T[} t^{-\frac{s}{2}} \|u(\cdot,t)\|_{L^p(\mathbb{R}^3)} < \infty. \tag{4.2.6}$$

We abbreviate

$$\mathcal{K}_p(Q_T) := \mathcal{K}^{s_p}_p(Q_T), \quad s_p := -1 + 3/p. \tag{4.2.7}$$

Therefore, for all $p \in]3,\infty]$, there exists a constant $c := c(p) > 0$ such that

$$c^{-1}\|Su_0\|_{\mathcal{K}_p(Q_\infty)} \leq \|u_0\|_{\dot{B}^{s_p}_{p,\infty}(\mathbb{R}^3)} \leq c\|Su_0\|_{\mathcal{K}_p(Q_\infty)}, \tag{4.2.8}$$

for all vector fields $u_0 \in \dot{B}^{s_p}_{p,\infty}(\mathbb{R}^3)$. As demonstrated in [34] and [113], the characterization (4.2.8) is particularly well suited for constructing mild solutions of the Navier–Stokes equations.

Next, for all $0 < T \leq \infty$, consider the space $\mathcal{X}_T$ consisting of all locally integrable functions u on Q_T such that

$$\|u\|_{\mathcal{X}_T} := \operatorname*{ess\,sup}_{t\in]0,T[} t^{\frac{1}{2}}\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^3)} + \sup_{x\in\mathbb{R}^3}\sup_{R\in]0,\sqrt{T}[} R^{-\frac{3}{2}}\|u\|_{L^2(B(x,R)\times]0,R^2[)} < \infty. \quad (4.2.9)$$

This is the largest space on which the bilinear operator B is known to be bounded, see the paper [85] of H. Koch and D. Tataru. We use the following Carleson measure characterization of $\|\cdot\|_{\mathrm{BMO}^{-1}(\mathbb{R}^3)}$. Namely, for all tempered distributions f on $\mathbb{R}^3$, we define

$$\|f\|_{\mathrm{BMO}^{-1}(\mathbb{R}^3)} := \|Sf\|_{\mathcal{X}_\infty}. \quad (4.2.10)$$

The space $\mathrm{BMO}^{-1}(\mathbb{R}^3)$ consists of all tempered distributions on $\mathbb{R}^3$ with finite BMO^{-1} norm.

Let us clarify the relationships between various function spaces of initial data. The Lorentz space $L^{3,\infty}(\mathbb{R}^3)$ is continuously embedded into $\dot{B}^{s_p}_{p,\infty}(\mathbb{R}^3)$ for all $p \in]3,\infty]$. This may be proven using (4.2.4) and Young's convolution inequality for Lorentz spaces. Next, the Bernstein inequalities for frequency-localized functions imply an analogue of the Sobolev embedding theorem for homogeneous Besov spaces. Finally, regarding $\mathrm{BMO}^{-1}(\mathbb{R}^3)$, Hölder's inequality gives

$$\sup_{x\in\mathbb{R}^3}\sup_{R>0} R^{-\frac{3}{2}}\|S(t)f\|_{L^2(B(x,R)\times]0,R^2[)} \leq c_p \sup_{t>0} t^{-\frac{s_p}{2}}\|S(t)f\|_{L^p(\mathbb{R}^3)}, \quad (4.2.11)$$

when $p \in]2,\infty[$ and f is tempered distribution on $\mathbb{R}^3$. These relationships are summarized below:

$$L^{3,\infty}(\mathbb{R}^3) \hookrightarrow \dot{B}^{s_{p_1}}_{p_1,\infty}(\mathbb{R}^3) \hookrightarrow \dot{B}^{s_{p_2}}_{p_2,\infty}(\mathbb{R}^3) \hookrightarrow \mathrm{BMO}^{-1}(\mathbb{R}^3) \hookrightarrow \dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3), \quad (4.2.12)$$

where $3 < p_1 < p_2 < \infty$. The above continuous embeddings are strict.

We now present a useful interpolation inequality for Kato spaces. Let $0 < T \leq \infty$ and $u\colon Q_T \to \mathbb{R}^3$ be a locally integrable vector field. The interpolation inequality is

$$\|u\|_{\mathcal{K}^s_p(Q_T)} \leq \|u\|^{\theta}_{\mathcal{K}^{s_1}_{p_1}(Q_T)}\|u\|^{1-\theta}_{\mathcal{K}^{s_2}_{p_2}(Q_T)}, \quad (4.2.13)$$

provided that $s_1, s_2 \in \mathbb{R}$, $p_1, p_2 \in [1, \infty]$, $\theta \in]0, 1[$, and

$$s := \theta s_1 + (1-\theta)s_2, \quad \frac{1}{p} := \frac{\theta}{p_1} + \frac{1-\theta}{p_2}. \tag{4.2.14}$$

A common scenario is $u \in L_t^\infty L_x^2(Q_T) \cap \mathcal{K}_{p_2}(Q_T)$ with $p_2 \in]4, \infty[$. Observe that $L_t^\infty L_x^2(Q_T) = \mathcal{K}_2^0(Q_T)$, so (4.2.13) implies $u \in \mathcal{K}_4^s(Q_T)$ with $s = -1/2 + 1/(2(p_2-2))$. Hence, we obtain

$$\|u\|_{L^4(Q_T)} \leq c(p_2) T^{\frac{1}{4(p_2-2)}} \|u\|_{L_t^\infty L_x^2(Q_T)}^{\theta} \|u\|_{\mathcal{K}_{p_2}(Q_T)}^{1-\theta}. \tag{4.2.15}$$

This embedding fails when $p_2 = \infty$.

4.2.2 Linear estimates

Our next goal is to present certain estimates for the time-dependent Stokes system in Kato spaces. The Leray projector $\mathbb{P}$ onto divergence-free vector fields is the Fourier multiplier with matrix-valued symbol $I - \xi \otimes \xi / |\xi|^2$. The operators $\{S(t)\mathbb{P}\operatorname{div}\}_{t \geq 0}$ are convolution operators with the gradient of the Oseen kernel, see [95, Chapter 11]. Specifically, there exists a smooth function $G\colon \mathbb{R}^3 \to \mathbb{R}^{3\times 3}$ satisfying

$$|\nabla^\ell G(x)| \leq \frac{c(\ell)}{(1+|x|^2)^{\frac{3+\ell}{2}}}, \quad x \in \mathbb{R}^3,\ 0 \leq \ell \in \mathbb{Z}, \tag{4.2.16}$$

$$(S(t)\mathbb{P}\operatorname{div} F)_i := \sum_{j,k=1}^{3} \frac{1}{t^2} \frac{\partial G_{ij}}{\partial x_k}\Big(\frac{\cdot}{\sqrt{t}}\Big) * F_{jk}, \quad t > 0,\ 1 \leq i \leq 3, \tag{4.2.17}$$

for matrix-valued functions $F\colon \mathbb{R}^3 \to \mathbb{R}^{3\times 3}$. Let us define a space of forcing terms in analogy with the Kato spaces. For all $s \in \mathbb{R}$, $p \in [1, \infty]$, and $0 < T \leq \infty$, the space $\mathcal{F}_p^s(Q_T)$ consists of all locally integrable matrix-valued functions $F\colon Q_T \to \mathbb{R}^{3\times 3}$ such that

$$\|F\|_{\mathcal{F}_p^s(Q_T)} := \operatorname*{ess\,sup}_{t \in]0,T[} t^{-\frac{s}{2}} \|F(\cdot, t)\|_{L^p(\mathbb{R}^3)} < \infty. \tag{4.2.18}$$

We often abbreviate

$$\mathcal{F}_p(Q_T) := \mathcal{F}_p^{s_p'}(Q_T), \quad s_p' := -2 + 3/p. \tag{4.2.19}$$

In analogy with $\mathcal{X}_T$, we also define the space $\mathcal{Y}_T$ consisting of all locally integrable $F\colon Q_T \to \mathbb{R}^{3\times 3}$ such that

$$\|F\|_{\mathcal{Y}_T} := \operatorname*{ess\,sup}_{t\in]0,T[} t\|F(\cdot,t)\|_{L^\infty(\mathbb{R}^3)} + \sup_{x\in\mathbb{R}^3} \sup_{R\in]0,\sqrt{T}[} R^{-3}\|F\|_{L^1(B(x,R)\times]0,R^2[)} < \infty. \quad (4.2.20)$$

Finally, our admissible class of forcing terms in Definition 4.1.5 is

$$\mathcal{F}(Q_T) := \mathcal{Y}_T \cup \left[\cup_{q\in]3,\infty[}\mathcal{F}_q(Q_T)\right]. \quad (4.2.21)$$

The following estimates were proven by the first author in [4, Lemma 4.1]:

Lemma 4.2.1 (Estimates in Kato spaces)**.** *Let $0 < T \le \infty$, $1 \le p_1 \le p_2 \le \infty$, such that*

$$s_2 - \frac{3}{p_2} = 1 + s_1 - \frac{3}{p_1}. \quad (4.2.22)$$

In addition, assume the conditions

$$s_1 > -2, \qquad \frac{3}{p_1} - \frac{3}{p_2} < 1. \quad (4.2.23)$$

(For instance, if $p_2 = \infty$, then the latter condition is satisfied when $p_1 > 3$. If $p_1 = 2$, then the latter condition is satisfied when $p_2 < 6$.) Then

$$\left\|\int_0^t S(t-\tau)\mathbb{P}\operatorname{div} F(\cdot,\tau)\,d\tau\right\|_{\mathcal{K}^{s_2}_{p_2}(Q_T)} \le C(s_1,p_1,p_2)\|F\|_{\mathcal{F}^{s_1}_{p_1}(Q_T)}, \quad (4.2.24)$$

for all distributions $F \in \mathcal{F}^{s_1}_{p_1}(Q_T)$, and the solution u to the corresponding heat equation belongs to $C(]0,T];L^{p_2}(\mathbb{R}^3))$. Let $k,l \ge 0$ be integers. If we further require that

$$t^{\alpha+\frac{|\beta|}{2}}\partial_t^\alpha\nabla^\beta F \in \mathcal{F}^{s_1}_{p_1}(Q_T), \quad (4.2.25)$$

for all integers $0 \le \alpha \le k$ and multi-indices $\beta \in (\mathbb{N}_0)^3$ with $|\beta| \le l$, then we have

$$\begin{aligned}&\left\|t^{k+\frac{l}{2}}\partial_t^k\nabla^l\int_0^t S(t-\tau)\mathbb{P}\operatorname{div} F(\cdot,\tau)\,d\tau\right\|_{\mathcal{K}^{s_2}_{p_2}(Q_T)}\\ &\quad\le C(k,l,s_1,p_1,p_2)\Big(\sum_{\alpha=0}^{k}\sum_{|\beta|\le l}\|t^{\alpha+\frac{|\beta|}{2}}F\|_{\mathcal{F}^{s_1}_{p_1}(Q_T)}\Big),\end{aligned} \quad (4.2.26)$$

and the spacetime derivatives $\partial_t^k \nabla^l u$ of the solution u belong to $C(]0,T];L^{p_2}(\mathbb{R}^3))$.

Let us define operators L, B by

$$L(F)(\cdot,t) := \int_0^t S(t-s)\mathbb{P}\operatorname{div} F(\cdot,s)\,ds, \quad B(U,V) := L(U\otimes V), \tag{4.2.27}$$

for certain matrix-valued functions F and vector fields U,V on Q_T. Lemma 4.2.1 (see also p. 526 of [34], for example) asserts that for all $p \in]3,\infty[$,

$$L\colon \mathcal{F}_p(Q_T) \to \mathcal{K}_p(Q_T), \quad B\colon \mathcal{K}_p(Q_T)\times\mathcal{K}_p(Q_T) \to \mathcal{K}_p(Q_T), \tag{4.2.28}$$

boundedly and with operator norms independent of $0 < T \leq \infty$. As was demonstrated in [85], L and B are also bounded as operators $L\colon \mathcal{Y}_T \to \mathcal{X}_T$ and $B\colon \mathcal{X}_T\times\mathcal{X}_T \to \mathcal{X}_T$ with norms independent of $T \in]0,\infty]$. This leads to the following important consequence. If $u_0 \in \dot{B}^{s_p}_{p,\infty}(\mathbb{R}^3)$ is a divergence-free vector field and $F \in \mathcal{K}_q(Q_T)$ with $p \in]3,\infty[$ and $q \in]3,p]$, then the Picard iterates $P_k(u_0,F)$ are well defined for all $k \geq 0$:

$$P_0(u_0,F)(\cdot,t) := S(t)u_0 + L(F)(\cdot,t), \tag{4.2.29}$$

$$P_{k+1}(u_0,F) := P_0(u_0,F) - B(P_k,P_k), \quad k \geq 0, \tag{4.2.30}$$

and $P_{-1}(u_0,F) := 0$. For simplicity, we often omit the dependence on u_0 and F. Here, P_0 belongs to $\mathcal{K}_p(Q_T)\cap\mathcal{X}_T$, and

$$\|P_0\|_{\mathcal{K}_p(Q_T)} + \|P_0\|_{\mathcal{X}_T} \leq C(p,q)(\|u_0\|_{\dot{B}^{s_p}_{p,\infty}(\mathbb{R}^3)} + \|F\|_{\mathcal{F}_q(Q_T)}). \tag{4.2.31}$$

Supposing now that $\|P_0(u_0,F)\|_{\mathcal{K}_p(Q_T)} + \|P_0(u_0,F)\|_{\mathcal{X}_T} \leq M$, we obtain

$$\|P_k\|_{\mathcal{K}_p(Q_T)} + \|P_k\|_{\mathcal{X}_T} \leq C(k,M,p,q), \tag{4.2.32}$$

where the constant is a polynomial in M with no constant term and degree depending only on k. Lemma 4.2.1 also has the following consequence regarding vector fields with finite kinetic energy. Namely, for all $p \in]3, \infty]$,

$$\|B(U,V)\|_{L_t^\infty L_x^2(Q_T)} \leq c(p) \min(\|U\|_{L_t^\infty L_x^2(Q_T)} \|V\|_{\mathcal{K}_p(Q_T)}, \|V\|_{L_t^\infty L_x^2(Q_T)} \|U\|_{\mathcal{K}_p(Q_T)}). \tag{4.2.33}$$

This is useful in our construction of strong solutions in Section 4.3.5.

We now exploit the self-improvement property of the bilinear term described in the introduction to prove a version of Lemma 4.1.4. Throughout the paper, we define for $k \geq 0$:

$$F_k(u_0, F) := P_k \otimes P_k - P_{k-1} \otimes P_{k-1}. \tag{4.2.34}$$

Lemma 4.2.2 (Finite energy forcing)**.** *Let $T > 0$, $u_0 \in \mathrm{BMO}^{-1}(\mathbb{R}^3)$ be a divergence-free vector field, and $F \in \mathcal{F}(Q_T)$. Suppose that*

$$\|P_0(u_0, F)\|_{\mathcal{K}_p(Q_T)} + \|P_0(u_0, F)\|_{\mathcal{X}_T} \leq M \tag{4.2.35}$$

for some $p \in]3, \infty[$. Then for all $k \geq k(p) := \lceil \frac{p}{2} \rceil - 2$, we have

$$F_k(u_0, F) \in L^2(Q_T). \tag{4.2.36}$$

$$P_{k+1}(u_0, F) - P_k(u_0, F) \in C([0,T]; L^2(\mathbb{R}^3)) \cap L_t^2 \dot{H}_x^1(Q_T), \tag{4.2.37}$$

In addition, the following bounds are satisfied:

$$\|F_k\|_{\mathcal{F}_2(Q_T)} + \|P_{k+1} - P_k\|_{\mathcal{K}_2(Q_T)} \leq C(k, M, p), \tag{4.2.38}$$

$$\|F_k\|_{L^2(Q_T)} + \|P_{k+1} - P_k\|_{L_t^\infty L_x^2(Q_T)} \leq T^{\frac{1}{4}} C(k, M, p). \tag{4.2.39}$$

Proof of Lemma 4.2.2. Define $p(k) := 2k+4$. By interpolation, we may assume without loss of generality that $p = p(k)$ in the statement. That is, $\|P_0(u_0, F)\|_{\mathcal{K}_{p(k)}(Q_T)} \leq M$. For all integers $0 \leq \ell \leq k$, we define $q(k, \ell) := \frac{p(k)}{\ell+2}$. We will prove inductively that

$$\|F_\ell\|_{\mathcal{F}_{q(k,\ell)}(Q_T)} + \|P_{\ell+1} - P_\ell\|_{\mathcal{K}_{q(k,\ell)}} \leq C(k, M). \tag{4.2.40}$$

Hölder's inequality immediately implies that $\|F_0\|_{\mathcal{F}_{\frac{p(k)}{2}}(Q_T)} \leq M^2$. Since $P_1 - P_0 = -L(F_0)$, the estimate $\|P_1 - P_0\|_{\mathcal{K}_{\frac{p(k)}{2}}(Q_T)} \leq C(k) M^2$ follows from Lemma 4.2.1. Let us now assume that we have the estimate (4.2.40) for some $0 \leq \ell < k$. We observe the identity

$$F_{\ell+1} = P_{\ell+1} \otimes (P_{\ell+1} - P_\ell) + (P_{\ell+1} - P_\ell) \otimes P_\ell, \tag{4.2.41}$$

so that $\|F_{\ell+1}\|_{\mathcal{F}_{q(k,\ell+1)}(Q_T)} \leq C(k,M)$ due to Hölder's inequality and (4.2.32). Now recall that $P_{\ell+2} - P_{\ell+1} = -L(F_{\ell+1})$. Lemma 4.2.1 implies

$$\|P_{\ell+2} - P_{\ell+1}\|_{\mathcal{K}_{q(k,\ell+1)}(Q_T)} \leq C(k,M). \tag{4.2.42}$$

This completes the induction. It is clear that (4.2.38) follows from (4.2.40) with $\ell = k$, and (4.2.39) is obtained from (4.2.38) by Hölder's inequality. Lastly, (4.2.37) concerning $P_{k+1} - P_k$ follows from the classical energy estimate for the Stokes equations and that $F_k \in L^2(Q_T)$. $\square$

Finally, we prove a linear estimate concerning solutions of the time-dependent Stokes equations in the space $L^\infty_t(\dot{B}^{s_p}_{p,\infty})_x(Q_T)$. By Young's convolution inequality, all tempered distributions f on $\mathbb{R}^3$ satisfy

$$\|Sf\|_{L^\infty_t(\dot{B}^{s_p}_{p,\infty})_x(Q_\infty)} \leq c\|f\|_{\dot{B}^{s_p}_{p,\infty}(\mathbb{R}^3)} \tag{4.2.43}$$

for a constant $c > 0$ independent of $p \in [1,\infty]$. Let us show that for all $q \in [1,\infty[$ and $p \in [q,\infty]$, there exists a constant $c := c(p,q) > 0$ such that for all $0 < T \leq \infty$ and $F \in \mathcal{F}_q(Q_T)$,

$$\sup_{t\in]0,T[} \|L(F)(\cdot,t)\|_{\dot{B}^{s_p}_{p,\infty}(\mathbb{R}^3)} \leq c(p,q)\|F\|_{\mathcal{F}_q(Q_T)}. \tag{4.2.44}$$

After extending F by zero, it suffices to consider $T = \infty$. By Sobolev embedding for Besov spaces, we need only consider the case $p = q$. For $t, \tau > 0$,

$$\begin{aligned}\tau^{-\frac{s_p}{2}}\|S(\tau)L(F)(\cdot,t)\|_{L^p(\mathbb{R}^3)} &\leq \tau^{-\frac{s_p}{2}}\int_0^{t+\tau}\|S(t+\tau-s)\mathbb{P}\mathrm{div}\, F(\cdot,s)\|_{L^p(\mathbb{R}^3)}ds \\ &\leq \frac{c(p)\tau^{\frac{-s_p}{2}}}{(t+\tau)^{\frac{-s_p}{2}}}\|F\|_{\mathcal{F}_p(Q_\infty)} \leq c(p)\|F\|_{\mathcal{F}_p(Q_\infty)}.\end{aligned} \tag{4.2.45}$$

By the heat flow characterisation of Besov norms with negative upper index, we see that (4.2.45) implies (4.2.44). We refer the reader to [9, Lemma 8] for the analogous estimate on $L\colon \mathcal{Y}_T \to L^\infty_t \mathrm{BMO}^{-1}_x(Q_T)$.

4.3 Weak Besov solutions

This section contains the general theory of the weak Besov solutions introduced in Definition 4.1.5.

4.3.1 Basic properties

First, let us describe the singular integral representation of the pressure used throughout the paper.

Remark 4.3.1 (Associated pressure)**.** Let v be as in Definition 4.1.5 with pressure $q \in L^{\frac{3}{2}}_{\mathrm{loc}}(Q_T)$, $T > 0$. There exists a constant function of time $c \in L^1_{\mathrm{loc}}(]0,T[)$ such that for a.e. $(x,t) \in Q_T$,

$$q(x,t) = (-\Delta)^{-1} \operatorname{div} \operatorname{div} v \otimes v - F + c(t). \tag{4.3.1}$$

Since the Navier–Stokes equations and local energy inequality (4.1.22) do not depend on the choice of constant, we will assume in the sequel that $c \equiv 0$. The resulting pressure is known as the *associated pressure*.

Proof. For now, assume that $u_0 \in \dot{B}^{s_p}_{p,\infty}(\mathbb{R}^3)$ is divergence-free and $F \in \mathcal{F}_q(Q_T)$ for some $p \in]3,\infty[$ and $q \in]3,p]$. By the Calderón-Zygmund estimates, certainly $\widetilde{q} \in L^1_{\mathrm{loc}}(Q_T)$. If $(v,\widetilde{q})$ solves the Navier–Stokes equations with forcing term $\operatorname{div} F$ in the sense of distributions, then $\nabla q = \nabla \widetilde{q}$, which implies (4.3.1). Therefore, to complete the proof, we need only verify that $(v,\widetilde{q})$ is a solution.

1. Picard iterate. Let π_k denote the pressure associated to the kth Picard iterate,

$$\pi_k(u_0,F) := (-\Delta)^{-1} \operatorname{div} \operatorname{div}[P_{k-1} \otimes P_{k-1} - F]. \tag{4.3.2}$$

By the Calderón-Zygmund estimates, $\pi_k \in L^\infty_{t,\mathrm{loc}} L^p_x(Q_T) + L^\infty_{t,\mathrm{loc}} L^q_x(Q_T)$. Recall that the kth Picard iterate is constructed as a solution of the heat equation

$$\partial_t P_k - \Delta P_k = \mathbb{P} \operatorname{div}[F - P_{k-1} \otimes P_{k-1}] \text{ in } Q_T. \tag{4.3.3}$$

Since $-\nabla \pi_k = (I - \mathbb{P}) \operatorname{div}[F - P_{k-1} \otimes P_{k-1}]$, we may add this term back into (4.3.3) to obtain the time-dependent Stokes equations with RHS $\operatorname{div}[F - P_{k-1} \otimes P_{k-1}]$. Hence, π_k is a valid pressure for the kth Picard iterate.

2. Correction term. Next, let p denote the pressure associated to the correction term,

$$p := (-\Delta)^{-1} \operatorname{div} \operatorname{div}[u \otimes u + P_k \otimes u + u \otimes P_k + F_k]. \tag{4.3.4}$$

The Calderón-Zygmund estimates imply that $p \in L^{\frac{3}{2}}(Q_T) + L^2_{t,\mathrm{loc}} L^2_x(Q_T)$. Recall that $u \in L^\infty_t L^2_x \cap L^2_t \dot{H}^1_x(Q_T)$ is a distributional solution of

$$\left.\begin{aligned} \partial_t u - \Delta u &= -\nabla \widetilde{p} - \operatorname{div} \widetilde{F} \\ \operatorname{div} u &= 0 \end{aligned}\right\} \text{ in } Q_T, \tag{4.3.5}$$

where $\widetilde{F} = u \otimes u + P_k \otimes u + u \otimes P_k + F_k$ for some pressure $\widetilde{p}$ in the class of tempered distributions. As in Step 1, if u solves the heat equation with RHS $-\operatorname{div} \mathbb{P}\widetilde{F}$, then p is a valid pressure for u, i.e., (u,p) solves (4.3.5).

The multiplier associated to the Leray projector $\mathbb{P}$ is not smooth at the origin, so we truncate it. Let $\varphi \in C^\infty_0(\mathbb{R}^3)$ such that $\varphi \equiv 1$ outside $B(2)$ and $\varphi \equiv 0$ inside $B(1)$. Consider the operators $T_\varepsilon := \varphi(D/\varepsilon)$ and $\mathbb{P}_\varepsilon := \mathbb{P} T_\varepsilon$ for all $\varepsilon > 0$. Applying $\mathbb{P}_\varepsilon$ to (4.3.5), we obtain

$$(\partial_t - \Delta)(T_\varepsilon u) = -\mathbb{P}_\varepsilon \operatorname{div} \widetilde{F} \text{ in } Q_T. \tag{4.3.6}$$

In the limit $\varepsilon \downarrow 0$, we have $T_\varepsilon u \to u$ and $\mathbb{P}_\varepsilon \operatorname{div} \widetilde{F} \to \mathbb{P} \operatorname{div} \widetilde{F}$ in the sense of tempered distributions. Hence, the desired heat equation is satisfied.[14]

3. Conclusion. Let $\widetilde{q} = \pi_k + p$. Combining Steps 1 and 2 for π_k and p gives that $(v, \widetilde{q})$ is a distributional solution of the Navier–Stokes equations on Q_T with forcing term $\operatorname{div} F$.

In the general case $u_0 \in \mathrm{BMO}^{-1}(\mathbb{R}^3)$ divergence-free and $F \in \mathcal{F}(Q_T)$, the above analysis remains valid with the following caveat. Since the kth Picard iterate only belongs to the L^∞-based space $\mathcal{X}_T$, $(-\Delta)^{-1} \operatorname{div} \operatorname{div} P_k \otimes P_k$ may only belong to $L^\infty_{t,\mathrm{loc}} \mathrm{BMO}_x(Q_T)$. Hence, $\pi_k(u_0, F)$ and q may only be well defined up to the addition of a constant function of time $c \in L^1_{\mathrm{loc}}(]0,T[)$. Even so, we still refer to "the associated pressure" with the understanding that

[14]This method is valid under quite mild assumptions on u and F. Certainly some assumptions are necessary in order to exclude certain "parasitic" solutions $u = c(t)$, $p = -c'(t) \cdot x$, $F = 0$ of the time-dependent Stokes equations. For such solutions, $\widehat{u}$ is supported at the origin in frequency space, so $T_\varepsilon u \to 0$ as $\varepsilon \downarrow 0$.

A different method is to take the curl of the time-dependent Stokes equations with RHS $\operatorname{div} F$ and initial data u_0 and compare it to the curl of the solution of the heat equation with RHS $\mathbb{P} \operatorname{div} F$ and initial data u_0. By well-posedness for the heat equation, the two vorticities are equal, and hence their velocities are equal according to the Biot-Savart law (that is, under mild assumptions).

our computations will not rely on the particular choice of representative. □

Our next order of business is the following proposition:

Proposition 4.3.2 (Energy inequalities for u)**.** *Let $T > 0$ and v be a weak Besov solution on Q_T as in Definition 4.1.5. Let $p := q - \pi_k$. Then u obeys the* local energy inequality *for every $t \in]0,T]$ and all non-negative test functions $0 \leq \varphi \in C_0^\infty(Q_\infty)$:*

$$\begin{aligned}\int_{\mathbb{R}^3} \varphi(x,t)|u(x,t)|^2\,dx + 2\int_0^t \int_{\mathbb{R}^3} \varphi|\nabla u|^2\,dx\,dt' \\ \leq \int_0^t \int_{\mathbb{R}^3} |u|^2(\partial_t\varphi + \Delta\varphi) + [|u|^2(u+P_k) + 2pu]\cdot\nabla\varphi + 2[P_k \otimes u + F_k] : \nabla(\varphi u)\,dx\,dt'.\end{aligned} \tag{4.3.7}$$

Hence, u satisfies the global energy inequality

$$\begin{aligned}\|u(\cdot,t_2)\|^2_{L^2(\mathbb{R}^3)} + 2\int_{t_1}^{t_2}\int_{\mathbb{R}^3} |\nabla u(x,t')|^2\,dx\,dt' \\ \leq \|u(\cdot,t_1)\|^2_{L^2(\mathbb{R}^3)} + 2\int_{t_1}^{t_2}\int_{\mathbb{R}^3} [(P_k \otimes u) + F_k] : \nabla u\,dx\,dt'\end{aligned} \tag{4.3.8}$$

for almost every $t_1 \in]0,T[$ and all $t_2 \in]t_1,T]$.

Remark 4.3.3. By adapting the proof of Proposition 4.3.2, it is also possible to show the following. Let v be as in Definition 4.1.5 *except* that v is not assumed to satisfy the local energy inequality (4.1.22). Instead, assume that u satisfies its corresponding local energy inequality (4.3.7). Then v satisfies (4.1.22). In particular, v is a weak Besov solution on Q_T. This fact will be useful in Proposition 4.3.14.

The proof is based on an identity that appears in the classical proof of weak-strong uniqueness and is useful in obtaining an energy inequality for the difference of two solutions of the Navier–Stokes equations. Let $f, g \in C_0^\infty(\overline{Q_\infty})$ and $0 \leq t_1 < t_2 < \infty$. Then

$$\begin{aligned}\int_{\mathbb{R}^3} f(x,t_2)\cdot g(x,t_2)\,dx + 2\int_{t_1}^{t_2}\int_{\mathbb{R}^3} \nabla f : \nabla g\,dx\,dt - \int_{\mathbb{R}^3} f(x,t_1)\cdot g(x,t_1)\,dx \\ = \int_{t_1}^{t_2}\int_{\mathbb{R}^3} (\partial_t f - \Delta f)\cdot g + f\cdot(\partial_t g - \Delta g)\,dx\,dt.\end{aligned} \tag{4.3.9}$$

We also need an analogous identity for the local energy inequality:

$$
\begin{aligned}
&\int_{\mathbb{R}^3} \varphi(x,t) f \cdot g(x,t)\,dx + 2\int_0^t \int_{\mathbb{R}^3} \varphi \nabla f : \nabla g\,dx\,dt' \\
&\quad = \int_0^t \int_{\mathbb{R}^3} (\partial_t \varphi + \Delta\varphi) f \cdot g + \varphi(\partial_t f - \Delta f)\cdot g + \varphi f \cdot (\partial_t g - \Delta g)\,dx\,dt',
\end{aligned}
\tag{4.3.10}
$$

for all $\varphi \in C_0^\infty(Q_\infty)$ and $t > 0$. These identities may be obtained for a larger class of functions by approximation.

Proof. 1. Local energy inequality for u. Recall from Definition 4.1.5 that v is assumed to satisfy the local energy inequality (4.1.22) for every $t \in]0,T]$ and all $0 \le \varphi \in C_0^\infty(Q_\infty)$. Using aforementioned properties of P_k and F, together with the fact that Riesz transforms are Calderón-Zygmund singular integral operators, we see that P_k, $P_{k-1}\otimes P_{k-1}$, F and π_k all belong to $L^2_{\mathrm{loc}}(Q_T)$. Using a mollification argument in the same spirit as in [126, p. 160-161], one can show that P_k satisfies the local energy equality

$$
\begin{aligned}
&\int_{\mathbb{R}^3} \varphi(x,t)|P_k(x,t)|^2\,dx + 2\int_0^t \int_{\mathbb{R}^3} \varphi|\nabla P_k|^2\,dx\,dt' \\
&\quad = \int_0^t \int_{\mathbb{R}^3} |P_k|^2(\partial_t\varphi + \Delta\varphi) + 2\pi_k P_k \cdot \nabla\varphi + 2[P_{k-1}\otimes P_{k-1} - F] : \nabla(P_k\varphi)\,dx\,dt'.
\end{aligned}
\tag{4.3.11}
$$

Next, one combines the identities $|u|^2 = |v|^2 - 2v\cdot P_k + |P_k|^2$ and $|\nabla u|^2 = |\nabla v|^2 - 2\nabla v : \nabla P_k + |\nabla P_k|^2$ with the local energy estimates (4.1.22) and (4.3.11) to obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^3} \varphi(x,t)|u(x,t)|^2\,dx + 2\int_0^t \int_{\mathbb{R}^3} \varphi|\nabla u|^2\,dx\,dt' \\
&\quad \le \int_0^t \int_{\mathbb{R}^3} [|v|^2 + |P_k|^2](\partial_t\varphi + \Delta\varphi) + 2[vq + P_k\pi_k]\cdot\nabla\varphi\,dx\,dt' \\
&\qquad + \int_0^t \int_{\mathbb{R}^3} |v|^2 v\cdot\nabla\varphi + 2P_{k-1}\otimes P_{k-1} : \nabla(P_k\varphi) - 2F : \nabla((v+P_k)\varphi)\,dx\,dt' \\
&\qquad\quad - 2\int_{\mathbb{R}^3} \varphi(x,t) v\cdot P_k(x,t)\,dx - 4\int_0^t \int_{\mathbb{R}^3} \varphi\nabla v : \nabla P_k\,dx\,dt'.
\end{aligned}
\tag{4.3.12}
$$

According to the weak-strong identity (4.3.10), we may write

$$\begin{aligned}
&\int_{\mathbb{R}^3} \varphi(x,t) v \cdot P_k(x,t)\, dx + 2\int_0^t \int_{\mathbb{R}^3} \varphi \nabla v : \nabla P_k \, dx\, dt' \\
&\quad = \int_0^t \int_{\mathbb{R}^3} (\partial_t \varphi + \Delta\varphi) v \cdot P_k - \nabla q \cdot P_k \varphi - \nabla \pi_k \cdot \varphi v \, dx\, dt' \\
&\qquad + \int_0^t \int_{\mathbb{R}^3} v \otimes v : \nabla(P_k \varphi) + P_{k-1} \otimes P_{k-1} : \nabla(\varphi v) - F : \nabla((v + P_k)\varphi)\, dx\, dt'.
\end{aligned} \tag{4.3.13}$$

Substitute (4.3.13) into the final line of (4.3.12) and collect various terms:

$$\begin{aligned}
&\int_{\mathbb{R}^3} \varphi(x,t) |u(x,t)|^2 \, dx + 2\int_0^t \int_{\mathbb{R}^3} \varphi |\nabla u|^2 \, dx\, dt' \\
&\quad \leq \int_0^t \int_{\mathbb{R}^3} |u|^2 (\partial_t \varphi + \Delta \varphi) + 2pu \cdot \nabla\varphi + \infty \, dx\, dt',
\end{aligned} \tag{4.3.14}$$

$$\infty := |v|^2 v \cdot \nabla \varphi - 2v \otimes v : \nabla(P_k\varphi) - 2P_{k-1} \otimes P_{k-1} : \nabla(\varphi u). \tag{4.3.15}$$

We now add and subtract $2P_k \otimes P_k : \nabla(\varphi u) = 2P_k \otimes (v - u) : \nabla(\varphi u)$ in the expression for ∞ in order to introduce the forcing term F_k:

$$\infty = 2[P_k \otimes u + F_k] : \nabla(\varphi u) + \mathrm{II}, \tag{4.3.16}$$

$$\mathrm{II} := |v|^2 v \cdot \nabla\varphi - 2v \otimes v : \nabla(P_k \varphi) - 2P_k \otimes v : \nabla(\varphi u). \tag{4.3.17}$$

Expanding $|v|^2 = |u|^2 + 2u \cdot P_k + |P_k|^2$ in (4.3.17) gives

$$\mathrm{II} = |u|^2 (u + P_k) \cdot \nabla\varphi + \mathrm{III}, \tag{4.3.18}$$

$$\mathrm{III} := [2u \cdot P_k + |P_k|^2] v \cdot \nabla\varphi - 2v \otimes v : \nabla(P_k\varphi) - 2P_k \otimes v : \nabla(\varphi u). \tag{4.3.19}$$

Clearly, the first and third terms of III are integrable. Let us now demonstrate that III is integrable by showing that the second term $v \otimes v : \nabla(P_k\varphi)$ is integrable. Recall that

$$F \in \mathcal{F}_q(Q_T) \subset L^\infty_{t,\mathrm{loc}} L^q_x(Q_T). \tag{4.3.20}$$

Second, from (4.2.32) we infer that

$$P_k,\ P_k \otimes P_k \in L^\infty_{t,\mathrm{loc}} L^p_x(Q_T). \tag{4.3.21}$$

Using (4.2.29)-(4.2.30) and (4.3.20)-(4.3.21), together with maximal regularity results for the heat equation, we infer that

$$\nabla P_k \in \bigcap_{1<s<\infty} (L^s_{t,\mathrm{loc}} L^q_x(Q_T) + L^s_{t,\mathrm{loc}} L^p_x(Q_T)). \tag{4.3.22}$$

Using that $v \in (L^\infty_t L^2_x)_{\mathrm{loc}}(Q_T)$, $\nabla v \in L^2_{\mathrm{loc}}(Q_T)$, and multiplicative inequalities, we see that $v \in L^{\frac{10}{3}}_{\mathrm{loc}}(Q_T)$. From this fact and (4.3.22), we infer that $v \otimes v : \nabla(P_k \varphi)$ is integrable.

It remains to prove that III integrates to zero. Expanding $v \otimes v = P_k \otimes v + u \otimes v$ in (4.3.19) and rearranging, we obtain

$$\begin{aligned} \int_0^t \int_{\mathbb{R}^3} \mathrm{III}\, dx\, dt' &= \int_0^t \int_{\mathbb{R}^3} |P_k|^2 v \cdot \nabla\varphi - 2P_k \otimes v : \nabla(P_k\varphi)\, dx\, dt' \\ &+ \int_0^t \int_{\mathbb{R}^3} [2u \cdot P_k] v \cdot \nabla\varphi - 2u \otimes v : \nabla(P_k\varphi) - 2P_k \otimes v : \nabla(\varphi u)\, dx\, dt'. \end{aligned} \tag{4.3.23}$$

This last expression vanishes, so we have verified that u satisfies the local energy inequality (4.3.7).

2. Global energy inequality for u. The global energy inequality (4.3.8) will follow from the local energy inequality (4.3.7) with a special choice of test function. Let $0 \leq \psi \in C_0^\infty(\mathbb{R}^3)$ such that $\psi \equiv 1$ in $B(1)$ and $\mathrm{supp}(\psi) \subset B(2)$. Fix $0 < t_1 < t_2 \leq T$. For each $\varepsilon, R > 0$, we define Lipschitz functions

$$\eta_\varepsilon(t) := \frac{1}{2\varepsilon} \int_{-\infty}^t \chi_{]t_1-\varepsilon, t_1+\varepsilon[}(t')\, dt', \quad t \in \mathbb{R}, \tag{4.3.24}$$

$$\Phi_{\varepsilon,R}(x,t) := \eta_\varepsilon(t)\psi(x/R), \quad (x,t) \in \mathbb{R}^{3+1}. \tag{4.3.25}$$

Technically, $\Phi_{\varepsilon,R}$ is neither smooth nor compactly supported in Q_T, but by approximation we may use it as a test function in the local energy inequality (4.3.7) when $0 < \varepsilon < \min(t_1, T -$

$t_1)/2$. For $0 < \varepsilon < \min(t_1, T - t_1, t_2 - t_1)/2$, this gives

$$\begin{aligned}
&\int_{\mathbb{R}^3} \psi(x/R)|u(x,t_2)|^2\,dx + 2\int_0^{t_2}\int_{\mathbb{R}^3} \Phi_{\varepsilon,R}|\nabla u|^2\,dx\,dt' \\
&\quad\le \frac{1}{2\varepsilon}\int_{t_1-\varepsilon}^{t_1+\varepsilon}\int_{\mathbb{R}^3} \psi(x/R)|u|^2\,dx\,dt' + 2\int_0^{t_2}\int_{\mathbb{R}^3} \Phi_{\varepsilon,R}[P_k\otimes u + F_k] : \nabla u\,dx\,dt' \\
&\quad\quad + \frac{1}{R^2}\int_0^{t_2}\eta_\varepsilon\int_{\mathbb{R}^3} |u|^2\Delta\psi(x/R)\,dx\,dt' \\
&\quad\quad\quad + \frac{1}{R}\int_0^{t_2}\eta_\varepsilon\int_{\mathbb{R}^3}[|u|^2(u+P_k) + 2pu]\cdot(\nabla\psi)(x/R)\,dx\,dt' \\
&\quad\quad\quad\quad + \frac{2}{R}\int_0^{t_2}\eta_\varepsilon\int_{\mathbb{R}^3}[P_k\otimes u + F_k] : u\otimes(\nabla\psi)(x/R)\,dx\,dt'.
\end{aligned} \tag{4.3.26}$$

Since $u \in L^\infty_t L^2_x \cap L^2_t \dot{H}^1_x(Q_T)$, $p \in L^{\frac{3}{2}}(Q_T) + L^2_{t,\mathrm{loc}}L^2_x(Q_T)$ (see Remark 4.3.1), $P_k \in L^\infty(\mathbb{R}^3\times]t_1-\varepsilon, T[)$, and $F_k \in L^2(Q_T)$, the last three lines of (4.3.26) vanish as $R \uparrow \infty$. Hence, we obtain

$$\begin{aligned}
&\|u(\cdot,t_2)\|^2_{L^2(\mathbb{R}^3)} + 2\int_0^{t_2}\eta_\varepsilon\int_{\mathbb{R}^3}|\nabla u|^2\,dx\,dt' \\
&\quad\le \frac{1}{2\varepsilon}\int_{t_1-\varepsilon}^{t_1+\varepsilon}\|u(\cdot,t')\|^2_{L^2(\mathbb{R}^3)}\,dt' + 2\int_0^{t_2}\eta_\varepsilon\int_{\mathbb{R}^3}[P_k\otimes u + F_k] : \nabla u\,dx\,dt'.
\end{aligned} \tag{4.3.27}$$

Using that $u(\cdot,t)$ is weakly L^2-continuous on $[0,T]$, we see that (4.3.27) in fact holds for all $0 < t_1 < t_2 \le T$ and $0 < \varepsilon < \min(t_1, T-t_1, t_2-t_1)/2$. Recall from the Lebesgue differentiation theorem that for a.e. $t_1 \in]0,T[$,

$$\frac{1}{2\varepsilon}\int_{t_1-\varepsilon}^{t_1+\varepsilon}\|u(\cdot,t')\|^2_{L^2(\mathbb{R}^3)}\,dt' \to \|u(\cdot,t_1)\|^2_{L^2(\mathbb{R}^3)} \text{ as } \varepsilon \downarrow 0. \tag{4.3.28}$$

Finally, the global energy inequality (4.3.8) follows from taking $\varepsilon \downarrow 0$ in (4.3.27). This completes the proof. □

The next proposition asserts that under mild hypotheses, weak Besov solutions are not highly sensitive to the order of the Picard iterate used in the splitting.

Proposition 4.3.4 (Raising and lowering). *Let $0 < T \le \infty$, $u_0 \in \mathrm{BMO}^{-1}(\mathbb{R}^3)$ be a divergence-free vector field, $F \in \mathcal{F}(Q_T)$, and $0 \le k \in \mathbb{Z}$. Suppose that v is a weak Besov solution on Q_T based on the kth Picard iterate with initial data u_0 and forcing term* $\operatorname{div} F$.

(i) Then v is a weak Besov solution based on the $(k+1)$th Picard iterate.

(ii) If $k \geq 1$ and $F_{k-1}(u_0, F) \in L^2(Q_S)$ for all finite $0 < S \leq T$, then v is a weak Besov solution based on the $(k-1)$th Picard iterate.

(iii) If $u_0 \in \dot{B}^{s_p}_{p,\infty}(\mathbb{R}^3)$ and $F \in \mathcal{F}_q(Q_T)$ for some $3 < q \leq p < \infty$, then v is a weak Besov solution based on the $k(p)$th Picard iterate, where $k(p) := \lceil \frac{p}{2} \rceil - 2$.

Proof. Proof of (i). We need only consider $T < \infty$. We must show that $\widetilde{u} := u + (P_k - P_{k+1})$ belongs to the energy space, $\widetilde{u}(\cdot, t)$ is weakly continuous as an $L^2(\mathbb{R}^3)$-valued function on $[0, T]$, and $\lim_{t \downarrow 0} \|\widetilde{u}(\cdot, t)\|_{L^2(\mathbb{R}^3)} = 0$. These conditions are already satisfied for the correction term u, so it remains to show them for $P_{k+1} - P_k$. Recall now that $F_k \in L^2(Q_T)$.[15] Since $P_k - P_{k+1} = L(F_k)$, we obtain $P_k - P_{k+1} \in C([0,T]; L^2(\mathbb{R}^3)) \cap L^2_t \dot{H}^1_x(Q_T)$ and $\lim_{t \downarrow 0} \|P_k - P_{k+1}\|_{L^2(\mathbb{R}^3)} = 0$. This completes the proof.

Proof of (ii). Once we further assume that $k \geq 1$ and $F_{k-1} \in L^2(Q_T)$, the proof is nearly identical to the proof of (i).

Proof of (iii). The proof follows from (i)–(ii) combined with the estimates on $F_k(u_0, F)$ proven in Lemma 4.2.2. $\square$

4.3.2 Uniform decay estimate

The goal of this section is to prove Proposition 4.1.6, which we restate below:

Proposition 4.3.5 (Decay property)**.** *Let $T > 0$, $p \in]3, \infty[$, $q \in]3, p]$, and $k \geq k(p) := \lceil \frac{p}{2} \rceil - 2$. Assume that v is a weak Besov solution on Q_T based on the kth Picard iterate with initial data u_0 and forcing term $\operatorname{div} F$. Let $\|u_0\|_{\dot{B}^{s_p}_{p,\infty}(\mathbb{R}^3)} + \|F\|_{\mathcal{F}_q(Q_T)} \leq M$. Then*

$$\|u(\cdot, t)\|_{L^2(\mathbb{R}^3)} \leq C(k, M, p, q) t^{\frac{1}{4}}. \tag{4.3.29}$$

Heuristically, the global energy inequality starting from the initial time should give a decay rate for $\|u(\cdot, t)\|_{L^2}$ that depends on the decay rate for $\int_0^t \int_{\mathbb{R}^3} |F_k(u_0, F)|^2 \, dx \, dt'$. However, it is not obvious whether the global energy inequality even makes sense starting from the initial time without a decay rate for $\|u(\cdot, t)\|_{L^2}$.[16] This issue is overcome by decomposing u into

[15] Definition 4.1.5 requires that $F_\ell(u_0, F) \in L^2(Q_T)$ for all $\ell \geq k$.

[16] The problem is with integrating the lower order term $\int_0^t \int_{\mathbb{R}^3} P_k \otimes u : \nabla u \, dx \, dt'$.

two parts, each of which satisfies a global energy inequality with no integrability issues, and estimating u by its parts. The method involves splitting the initial data u_0 into a subcritical part $\bar{u_0}$ and a perturbation $\widetilde{u_0}$ with finite energy as in Lemma 4.1.7. The idea is that only subcritical coefficients will enter into the energy inequality for the time-evolution of the perturbation $\widetilde{u_0}$. See [17] for similar arguments in the context of global weak $L^{3,\infty}$ solutions.

The hypotheses of Proposition 4.3.5 will be taken as standing assumptions for the remainder of the section.

For $N = 1$, we decompose u_0 according to Lemma 4.1.7 and F according to Lemma 4.5.8,

$$u_0 = \bar{u_0} + \widetilde{u_0}, \qquad F = \bar{F} + \widetilde{F}, \tag{4.3.30}$$

with $\bar{u_0}, \widetilde{u_0}, \bar{F}, \widetilde{F}$ satisfying the following properties:

$$\|\bar{u_0}\|_{\dot{B}^{s_{p_2}+\delta_2}_{p_2,p_2}(\mathbb{R}^3)} \leq C(p)M, \quad \|\widetilde{u_0}\|_{L^2(\mathbb{R}^3)} \leq C(p)M, \tag{4.3.31}$$

$$\|\bar{u_0}\|_{\dot{B}^{s_p}_{p,\infty}(\mathbb{R}^3)}, \|\widetilde{u_0}\|_{\dot{B}^{s_p}_{p,\infty}(\mathbb{R}^3)} \leq C(p)M, \tag{4.3.32}$$

$$\|\bar{F}\|_{\mathcal{F}^{s'_{p_3}+\delta_3}_{p_3}(Q_T)} \leq C(q,T)M, \quad \|\widetilde{F}\|_{L^3_t L^2_x(Q_T)} \leq C(q,T)M, \tag{4.3.33}$$

$$\|\bar{F}^N\|_{\mathcal{F}_q(Q_T)}, \|\widetilde{F}^N\|_{\mathcal{F}_q(Q_T)} \leq M. \tag{4.3.34}$$

We will use the following notation. For each $k \geq 0$, we define

$$\bar{P}_k(u_0, F) := P_k(\bar{u_0}, \bar{F}^N), \quad \widetilde{P}_k(u_0, F) := P_k(\widetilde{u_0}, \widetilde{F}), \tag{4.3.35}$$

$$\bar{F}_k(u_0, F) := \bar{P}_k \otimes \bar{P}_k - \bar{P}_{k-1} \otimes \bar{P}_{k-1}, \tag{4.3.36}$$

$$E_k(u_0, F) := P_k(u_0, F) - \bar{P}_k(u_0, F), \tag{4.3.37}$$

$$G_k(u_0, F) := P_k \otimes P_k - \bar{P}_k \otimes \bar{P}_k. \tag{4.3.38}$$

We will frequently suppress dependence on the data (u_0, F) in our notation.

In this section, we will also use the following subcritical estimates for B and L, in addition to the properties discussed in Section 4.2. Namely, for $\delta > 0$,

$$\|B(U,V)\|_{\mathcal{K}^{-1+\delta}_\infty(Q_T)} \leq cT^{\frac{\delta}{2}} \|U\|_{\mathcal{K}^{-1+\delta}_\infty(Q_T)} \|V\|_{\mathcal{K}^{-1+\delta}_\infty(Q_T)}, \tag{4.3.39}$$

$$\|B(U,V)\|_{\mathcal{K}^{-1+\delta}_{\infty}(Q_T)} \leq c(p,\delta)\min(\|U\|_{\mathcal{K}^{-1+\delta}_{\infty}(Q_T)}\|V\|_{\mathcal{K}_p(Q_T)}, \|V\|_{\mathcal{K}^{-1+\delta}_{\infty}(Q_T)}\|U\|_{\mathcal{K}_p(Q_T)}), \tag{4.3.40}$$

$$\|L(G)\|_{\mathcal{K}^{-1+\delta_2}_{\infty}(Q_T)} \leq c(p_2)\|G\|_{\mathcal{F}^{s'_{p_2}+\delta_2}_{p_2}(Q_T)}, \tag{4.3.41}$$

which follow from Lemma 4.2.1. Let $\delta := \min(\delta_2, \delta_3) > 0$. Then

$$\|\bar{P}_k\|_{\mathcal{K}^{-1+\delta}_{\infty}(Q_T)} \leq C(T,\delta,k)\, Q_k(\|\bar{P}_0\|_{\mathcal{K}^{-1+\delta}_{\infty}(Q_T)}) < \infty, \tag{4.3.42}$$

where Q_k denotes a polynomial with no constant term and degree depending only on k. By the heat characterisation of homogeneous Besov spaces, (4.3.32), and (4.3.34), we have

$$\|\bar{P}_k\|_{\mathcal{K}_p(Q_T)} + \|\bar{P}_k\|_{\mathcal{X}_T} \leq C(k,M,p,q). \tag{4.3.43}$$

The same estimate holds for P_k (see (4.2.32)). Finally, using (4.3.43), along with (4.3.40) and an induction argument, we see that

$$\|\bar{P}_k\|_{\mathcal{K}^{-1+\delta}_{\infty}(Q_T)} \leq c(\delta,k,M,p,q)\|\bar{P}_0\|_{\mathcal{K}^{-1+\delta}_{\infty}(Q_T)}. \tag{4.3.44}$$

Lemma 4.3.6. *In the above notation, for all integers $k \geq 0$, $E_k(u_0,F)$ and $G_k(u_0,F)$ obey the following properties:*

$$E_k \in C([0,T]; L^2(\mathbb{R}^3)) \cap L^2_t\dot{H}^1_x(Q_T), \tag{4.3.45}$$

$$\|E_k\|_{L^\infty_t L^2_x(Q_T)} \leq C(k,M,p,q)\|\widetilde{P}_0\|_{L^\infty_t L^2_x(Q_T)}, \tag{4.3.46}$$

$$G_k \in L^2(Q_T). \tag{4.3.47}$$

Proof. In the proof, we will make use of the following identities. In particular,

$$E_k = \widetilde{P_0} - B(E_{k-1}, P_{k-1}) - B(\bar{P}_{k-1}, E_{k-1}), \tag{4.3.48}$$

$$E_k = \widetilde{P}_0 - B(E_{k-1}, E_{k-1}) - B(\bar{P}_{k-1}, E_{k-1}) - B(E_{k-1}, \bar{P}_{k-1}). \tag{4.3.49}$$

1. Showing E_k has finite kinetic energy. We proceed by induction. Clearly, $E_0 = \widetilde{P_0}$. This, together with (4.3.31) and (4.3.33), implies that

$$E_0 \in C([0,T]; L^2(\mathbb{R}^3)) \cap L^2_t\dot{H}^1_x(Q_T). \tag{4.3.50}$$

For the inductive step assume $E_{k-1}\in L^\infty_t L^2_x(Q_T)$ ($k\geq 1$). Using (4.2.33), (4.3.43), (4.3.48), (4.3.50), and the inductive assumption, we infer that $E_k\in L^\infty_t L^2_x(Q_T)$ and

$$\begin{aligned}\|E_k\|_{L^\infty_t L^2_x(Q_T)} &\leq \|E_0\|_{L^\infty_t L^2_x(Q_T)} + C(p)\|P_{k-1}\|_{\mathcal{K}_p(Q_T)}\|E_{k-1}\|_{L^\infty_t L^2_x(Q_T)}\\ &\quad + C(p)\|\bar{P}_{k-1}\|_{\mathcal{K}_p(Q_T)}\|E_{k-1}\|_{L^\infty_t L^2_x(Q_T)}\\ &\leq \|E_0\|_{L^\infty_t L^2_x(Q_T)} + C(k,M,p,q)\|E_{k-1}\|_{L^\infty_t L^2_x(Q_T)}.\end{aligned} \tag{4.3.51}$$

From (4.3.51), we can then immediately obtain (4.3.46).

2. Showing G_k is in $L^2(Q_T)$. As previously mentioned, we have

$$G_k = E_k\otimes E_k + \bar{P}_k\otimes E_k + E_k\otimes\bar{P}_k. \tag{4.3.52}$$

Using (4.3.42) and Step 1, we see that

$$\bar{P}_k\otimes E_k + E_k\otimes\bar{P}_k\in L^2(Q_T). \tag{4.3.53}$$

Next, we use the interpolation inequality (4.2.13) together with (4.3.43) and Step 1 to obtain

$$E_k\otimes E_k\in L^2(Q_T). \tag{4.3.54}$$

Combining this with (4.3.53) gives that $G^N_k\in L^2(Q_T)$. Finally, we note that

$$E_k = S(t)\widetilde{u}_0 + L(\widetilde{F})(\cdot,t) - L(G_k)(\cdot,t). \tag{4.3.55}$$

This, together with (4.3.50) and $G_k\in L^2(Q_T)$ implies that

$$E_k\in C([0,T];L^2(\mathbb{R}^3))\cap L^2_t\dot{H}^1_x(Q_T), \tag{4.3.56}$$

and furthermore, for all $t\in]0,T]$,

$$\|E_k(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} + 2\int\limits_0^t\int\limits_{\mathbb{R}^3}|\nabla E_k|^2\,dx\,dt' = 2\int\limits_0^t\int\limits_{\mathbb{R}^3}(G_k-\widetilde{F}):\nabla E_k\,dx\,dt' + \|\widetilde{u}_0\|^2_{L^2(\mathbb{R}^3)}. \tag{4.3.57}$$

$\square$

Remark 4.3.7. Standard energy estimates for Stokes equations imply that

$$\|\widetilde{P}_0\|_{L_t^\infty L_x^2(Q_T)} \leq C\|\widetilde{u_0}\|_{L^2(\mathbb{R}^3)} + CT^{\frac{1}{6}}\|\widetilde{F}\|_{L_t^3 L_x^2(Q_T)}. \tag{4.3.58}$$

We combine the above estimate with (4.3.31), (4.3.33), and Lemma 4.3.6 to obtain

$$\|E_k\|^2_{L_t^\infty L_x^2(Q_T)} \leq C(k, M, p, q, T), \tag{4.3.59}$$

with constant $C > 0$ increasing in $T > 0$.

From now on, we will assume that $v = u + P_k(u_0, F)$ is a weak Besov solution on Q_T as in Definition 4.1.5. Moreover, we will assume that $k \geq k(p)$ in order that $F_k,\ \bar{F}_k \in \mathcal{F}_2(Q_T)$ as guaranteed by Lemma 4.2.2. We will denote

$$w_k(u_0, F) := u + E_k(u_0, F) = v - \bar{P}_k(u_0, F). \tag{4.3.60}$$

It is clear from Lemma 4.3.6 that $w_k \in L_t^\infty L_x^2 \cap L_t^2 \dot{H}_x^1(Q_T)$ and $w(\cdot, t)$ is weakly continuous as an $L^2(\mathbb{R}^3)$-valued function on $[0, T]$.

Lemma 4.3.8 (Energy inequality for w_k)**.** *In the above notation, we have*

$$\begin{aligned} &\|w_k(\cdot, t)\|^2_{L^2(\mathbb{R}^3)} + 2\int_0^t \int_{\mathbb{R}^3} |\nabla w_k|^2 \, dx \, dt' \\ &\quad \leq \|\widetilde{u}_0\|^2_{L^2(\mathbb{R}^3)} + 2\int_0^t \int_{\mathbb{R}^3} (\bar{P}_k \otimes w_k + \bar{F}_k - \widetilde{F}) : \nabla w_k \, dx \, dt', \end{aligned} \tag{4.3.61}$$

for all $t \in]0, T]$.

Note that the last integral in (4.3.61) is convergent because $\bar{P}_k \in \mathcal{K}_\infty^{-1+\delta}(Q_T)$ belongs to subcritical spaces and $\bar{F}_k, \widetilde{F} \in L^2(Q_T)$. Here, $\delta := \min(\delta_2, \delta_3) > 0$.

We omit the proof of Lemma 4.3.8, as it is similar to the proof of Proposition 4.3.2. The main idea is to "transfer" the global energy inequality from u and E_k to w_k by using the weak-strong identity (4.3.9) and the cancellation properties of the nonlinearity.

Proof of Proposition 4.3.5. Below, we use the convention that the constants $C > 0$ depend only on k, M, p, q. By a scaling argument, it suffices to obtain an estimate of the form

$$\|u(\cdot, 1)\|_{L^2(\mathbb{R}^3)} \leq C \tag{4.3.62}$$

when $T \geq 1$. Since one may truncate the interval of existence, $T = 1$, without loss of generality.

Split $u_0 \in \dot{B}^{s_p}_{p,\infty}(\mathbb{R}^3)$ and $F \in \mathcal{F}_q(Q_T)$ as in the beginning of Section 4.3.2. Using the identity $u = w_k - E_k$, we obtain the inequality

$$\|u(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} \leq 2\|E_k(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} + 2\|w_k(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} \tag{4.3.63}$$

for all $t \in]0,T]$. Recall from Remark 4.3.7 that E_k^N obeys the estimate

$$\|E_k(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} \leq C. \tag{4.3.64}$$

It remains to estimate the energy of w_k. Denote $y(t) := \|w_k(\cdot,t)\|^2_{L^2(\mathbb{R}^3)}$. By manipulating the energy inequality (4.3.61) for w_k, one obtains

$$\begin{aligned} &y(t) + \int_0^t \int_{\mathbb{R}^3} |\nabla w_k|^2 \, dx \, d\tau \\ &\quad \leq C\|\widetilde{u}_0\|^2_{L^2(\mathbb{R}^3)} + C\int_0^t \int_{\mathbb{R}^3} |\widetilde{F}|^2 + |\bar{P}_k \otimes w_k|^2 + |\bar{F}_k|^2 \, dx \, d\tau \end{aligned} \tag{4.3.65}$$

for all $t \in]0,T]$. Let us now analyze each of the terms. To begin, recall that $\|\widetilde{u}_0\|^2_{L^2(\mathbb{R}^3)} \leq C$. As a consequence of Lemma 4.2.2, (4.3.32), and (4.3.34), we have that $\|\bar{F}_k\|^2_{L^2(\mathbb{R}^3 \times]0,t[)} \leq Ct^{\frac{1}{2}}$.[17] Due to the splitting properties (4.3.31) and (4.3.33), we have that $\|\widetilde{F}\|^2_{L^2(Q_T)} \leq C$. Using (4.3.44), it is not difficult to show that

$$\|\bar{P}_k\|_{\mathcal{K}^{-1+\delta}_{\infty}(Q_{T_\sharp})} \leq C(\|\bar{u_0}\|_{\dot{B}^{s_{p_2}+\delta_2}_{p_2,p_2}(\mathbb{R}^3)} + \|\bar{F}\|_{\mathcal{F}^{s'_{p_3}+\delta_3}_{p_3}(Q_T)}) \leq C. \tag{4.3.66}$$

Substituting all the estimates into (4.3.65), we obtain that

$$y(t) \leq C(1+t^{\frac{1}{2}}) + C\int_0^t \frac{y(\tau)}{\tau^{1-\delta}} \, d\tau. \tag{4.3.67}$$

Now we apply Gronwall's lemma:

$$y(t) \leq C(1+t^{\frac{1}{2}}) \times \exp\left(Ct^{\delta}\right). \tag{4.3.68}$$

We combine (4.3.68), (4.3.64), and (4.3.63) to obtain the following estimate for each $t \in$

[17] This does not rely on $\bar{F}_k$ belonging to subcritical spaces.

$]0,T]$:

$$\|u(\cdot,t)\|_{L^2}^2 \leq C(1+t^{\frac{1}{2}}) \times \left[\exp\left(Ct^\delta\right)+1\right]. \tag{4.3.69}$$

Let $t=1$ to verify (4.3.62) and complete the proof. $\square$

Corollary 4.3.9 (Global energy inequality, revised)**.** *Under the hypotheses of Proposition 4.3.5, we have that $P_k \otimes u \in L^2(Q_T)$ and, for all finite $t \in]0,T]$,*

$$\|u(\cdot,t)\|_{L^2(\mathbb{R}^3)}^2 + 2\int_0^t \int_{\mathbb{R}^3} |\nabla u|^2 \, dx \, dt' \leq 2\int_0^t \int_{\mathbb{R}^3} [(P_k \otimes u) + F_k] : \nabla u \, dx \, dt'. \tag{4.3.70}$$

Remark 4.3.10 (On the constant in the decay estimate)**.** From the proof of Proposition 4.3.5, see (4.3.69), we may take $C = Q(M)\exp(Q(M))$ in the decay estimate (4.3.29), where Q is a polynomial with coefficients depending on k,p,q, zero constant term, and degree depending only on k. Therefore, plugging (4.3.29) back into (4.3.70), we obtain

$$\|u(\cdot,t)\|_{L^2(\mathbb{R}^3)}^2 + 2\int_0^t \int_{\mathbb{R}^3} |\nabla u|^2 \, dx \, dt' \leq Q(M)\exp(Q(M))t^{\frac{1}{2}}. \tag{4.3.71}$$

4.3.3 Weak Leray-Hopf solutions

In this subsection, we prove the equivalence of suitable weak Leray-Hopf solutions and global weak Besov solutions under certain assumptions. Let $J(\mathbb{R}^3)$ denote the space of divergence-free vector fields in $L^2(\mathbb{R}^3)$.

Proposition 4.3.11 (Equivalence of suitable weak Leray-Hopf solutions and weak Besov solutions)**.** *Let $0 < T \leq \infty$, $u_0 \in J(\mathbb{R}^3) \cap \dot{B}^{s_p}_{p,\infty}(\mathbb{R}^3)$, and $F \in L^2(Q_T) \cap \mathcal{K}_q(Q_T)$ for some $p \in]3,\infty[$ and $q \in]3,p]$. A distributional vector field v on Q_T is a suitable weak Leray-Hopf solution on Q_T with initial data u_0 and forcing term $\operatorname{div} F$ if and only if v is a weak Besov solution on Q_T with the same initial data and forcing term.*

Later, we will use Proposition 4.3.11 to prove the existence of global weak Besov solutions in Corollary 4.3.17. First, we remind the reader of the definition of suitable weak Leray-Hopf solution. Recall that $C^\infty_{0,0}(Q_T)$ denotes the space of smooth vector fields $\varphi\colon Q_T \to \mathbb{R}^3$ with compact support and $\operatorname{div}\varphi = 0$.

Definition 4.3.12 (Suitable weak Leray-Hopf solution)**.** *Let $T > 0$, $u_0 \in J(\mathbb{R}^3)$, and $F \in L^2(Q_T)$.*

We say that a distributional vector field v on Q_T is a weak Leray-Hopf solution *to the Navier–Stokes equations on Q_T with initial data u_0 and forcing term* $\operatorname{div} F$ *if v satisfies the following properties:*

First, $v \in L^\infty_t L^2_x \cap L^2_t \dot{H}^1_x(Q_T)$, and v satisfies the Navier–Stokes equations on Q_T in the sense of divergence-free distributions:

$$\int_0^T \int_{\mathbb{R}^3} v \cdot \partial_t \varphi + v \otimes v : \nabla \varphi - \nabla v : \nabla \varphi - F : \nabla \varphi \, dx \, dt = 0 \tag{4.3.72}$$

for all $\varphi \in C^\infty_{0,0}(Q_T)$. In addition, $v(\cdot, t)$ is weakly continuous on $[0,T]$ as an $L^2(\mathbb{R}^3)$-valued function, and v attains its initial data strongly in $L^2(\mathbb{R}^3)$:

$$\lim_{t \downarrow 0} \|v(\cdot, t) - u_0\|_{L^2(\mathbb{R}^3)} = 0. \tag{4.3.73}$$

Finally, it is required that v satisfies the energy inequality

$$\|v(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} + 2\int_0^t \int_{\mathbb{R}^3} |\nabla v(x,t')|^2 \, dx \, dt' \leq \|u_0\|^2_{L^2(\mathbb{R}^3)} - \int_0^t \int_{\mathbb{R}^3} F : \nabla v \, dx \, dt', \tag{4.3.74}$$

for all $t \in [0,T]$.

We say that a distributional vector field v on Q_∞ is a weak Leray-Hopf solution on Q_∞ if it is a weak Leray-Hopf solution on Q_T for all $T > 0$. These solutions are known as global *weak Leray-Hopf solutions.*

Now let $0 < T \leq \infty$. We say that a weak Leray-Hopf solution v on Q_T is suitable *if there exists a pressure $q \in L^{\frac{3}{2}}_{\mathrm{loc}}(Q_T)$ such that (v,q) is a distributional solution of the Navier–Stokes equations on Q_T with forcing term* $\operatorname{div} F$ *and moreover satisfies the local energy inequality* (4.1.22) *for all $0 \leq \varphi \in C^\infty_0(Q_\infty)$.*

The following proposition concerning the existence of suitable weak Leray-Hopf solutions is well known (see, for instance, [95]).

Proposition 4.3.13 (Existence of suitable weak Leray-Hopf solutions)**.** *Let $u_0 \in J(\mathbb{R}^3)$ and $F \in L^2(Q_T)$ for all $T > 0$. There exists a global suitable weak Leray-Hopf solution with initial data u_0 and forcing term* $\operatorname{div} F$*.*

We now prove Proposition 4.3.11.

Proof of Proposition 4.3.11. Assume the hypotheses of the proposition. It suffices to consider the case $T < \infty$. We now record a few properties. Namely,

$$P_0(u_0, F) \in C([0,T]; L^2(\mathbb{R}^3)) \cap L_t^2 \dot{H}_x^1(Q_T) \cap \mathcal{K}_p(Q_T), \tag{4.3.75}$$

$$\lim_{t \downarrow 0} \|P_0(u_0, F)(\cdot, t) - u_0\|_{L^2(\mathbb{R}^3)} = 0. \tag{4.3.76}$$

Combining these observations with (4.2.32) and (4.2.33) gives that $P_k(u_0, F) \in L_t^\infty L_x^2(Q_T) \cap \mathcal{K}_p(Q_T)$ for all $k \geq 0$. Next, the interpolation inequality (4.2.13) implies $P_k \in L^4(Q_T)$. Finally, since $F_k(u_0, F) = P_k \otimes P_k - P_{k-1} \otimes P_{k-1}$, we obtain that $F_k \in L^2(Q_T)$ for all $k \geq 0$.

1. Forward direction. Suppose that v is a suitable weak Leray-Hopf solution on Q_T with initial data u_0 and forcing term $\operatorname{div} F$. From (4.3.75) and Definition 4.3.12, it is clear that $u = v - P_0(u_0, F) \in L_t^\infty L_x^2 \cap L_t^2 \dot{H}_x^1(Q_T)$ and $u(\cdot, t)$ is weakly continuous on $[0, T]$ with values in $L^2(\mathbb{R}^3)$. In addition, (4.3.76) implies that $\lim_{t\downarrow 0} \|u(\cdot, t)\|_{L^2(\mathbb{R}^3)} = 0$. Since $F_k(u_0, F) \in L^2(Q_T)$ for all $k \geq 0$, we conclude that v is a weak Besov solution on Q_T based on the zeroth Picard iterate.

2. Reverse direction. Suppose that v is a weak Besov solution on Q_T with initial data u_0 and forcing term $\operatorname{div} F$. As observed above, $F_k(u_0, F) \in L^2(Q_T)$ for all $k \geq 0$. Hence, Proposition 4.3.4 implies that v is a weak Besov solution on Q_T based on the zeroth Picard iterate. By (4.3.75), (4.3.76), and the properties of $u = v - P_0(u_0, F)$ in Definition 4.1.5 (with $k = 0$), we have that $v \in L_t^\infty L_x^2 \cap L_t^2 \dot{H}_x^1(Q_T)$, $v(\cdot, t)$ is weakly continuous on $[0, T]$ with values in $L^2(\mathbb{R}^3)$, and $\lim_{t\downarrow 0} \|v(\cdot, t) - u_0\|_{L^2(\mathbb{R}^3)} = 0$. It remains to verify the global energy inequality (4.3.74) for weak Leray-Hopf solutions. This is obtained from the local energy inequality (4.1.22) by similar arguments as in Step 2 of Proposition 4.3.2. The proof is complete. □

4.3.4 Weak–$*$ stability

Here is our main result concerning weak-$*$ stability.

Proposition 4.3.14 (Weak–$*$ stability). *Let $0 < T \leq \infty$ and $(v^{(n)})_{n\in\mathbb{N}}$ be a sequence of weak Besov solutions on Q_T. For each $n \in \mathbb{N}$, denote by $u_0^{(n)}$ and $\operatorname{div} F^{(n)}$ the initial data and forcing term of $v^{(n)}$, respectively. Suppose that*

$$u_0^{(n)} \overset{*}{\rightharpoonup} u_0 \textit{ in } \dot{B}^{s_p}_{p,\infty}(\mathbb{R}^3), \quad F^{(n)} \overset{*}{\rightharpoonup} F \textit{ in } \mathcal{F}_q(Q_T) \tag{4.3.77}$$

for some $p \in]3,\infty[$ and $q \in]3,p]$. There exists a subsequence (still denoted by n) converging in the following senses to a weak Besov solution v on Q_T with initial data u_0 and forcing term $\operatorname{div} F$. *Namely,*

$$v^{(n)} \overset{*}{\rightharpoonup} v \textit{ in } L^\infty_t L^2_x(B(R)\times]\delta,S[),\ \nabla v^{(n)} \rightharpoonup \nabla v \textit{ in } L^2(B(R)\times]\delta,S[), \tag{4.3.78}$$

$$v^{(n)} \to v \textit{ in } L^3(B(R)\times]\delta,S[), \quad q^{(n)} \rightharpoonup q \textit{ in } L^{\frac{3}{2}}(B(R)\times]\delta,S[), \tag{4.3.79}$$

$$v^{(n)} \to v \textit{ in } C([0,S];\mathcal{S}'(\mathbb{R}^3)), \tag{4.3.80}$$

for each $R>0$, $S\in]0,T]$ finite, and $\delta\in]0,S[$. Here, $q^{(n)}$ and q denote the pressures associated to $v^{(n)}$ and v, respectively.

First, we require an analogous result for the Picard iterates.

Lemma 4.3.15 (Weak–$*$ stability of Picard iterates)**.** *Under the hypotheses of Proposition 4.3.14, there exists a subsequence (still denoted by n) such that for each $0\le k\in\mathbb{Z}$, the Picard iterates $P_k^{(n)} := P_k(u_0^{(n)},F^{(n)})$ converge in the following senses to $P_k(u_0,F)$. Namely,*

$$P_k^{(n)} \overset{*}{\rightharpoonup} P_k \textit{ in } \mathcal{K}_p(Q_T), \quad P_k^{(n)} \to P_k \textit{ in } L^q(B(R)\times]\delta,S[), \tag{4.3.81}$$

$$\nabla P_k^{(n)} \rightharpoonup \nabla P_k^{(n)} \textit{ in } L^q(B(R)\times]\delta,S[), \tag{4.3.82}$$

$$P_k^{(n)} \to P_k \textit{ in } C([0,S];\mathcal{S}'(\mathbb{R}^3)), \tag{4.3.83}$$

for each $R>0$, $S\in]0,T]$ finite, and $\delta\in]0,S[$.

In the proofs below, we allow the implicit constants $C>0$ to depend on k, M, p, q. We will also not vary our notation when passing to subsequences.

Proof of Lemma 4.3.15. It suffices to consider the case when $T<\infty$. Due to weak-$*$ convergence, there exists a constant $M>0$ such that

$$\sup_{n\in\mathbb{N}}[\|u_0^{(n)}\|_{\dot{B}^{s_p}_{p,\infty}(\mathbb{R}^3)} + \|F^{(n)}\|_{\mathcal{F}_q(Q_T)}] \le M. \tag{4.3.84}$$

Let $k\ge 0$ be a fixed integer. From (4.2.32), we obtain

$$\sup_{n\in\mathbb{N}}[\|P_k^{(n)}\|_{\mathcal{K}_p(Q_T)} + \|P_k^{(n)}\|_{\mathcal{X}_T}] \le C(k,M,p,q). \tag{4.3.85}$$

Therefore, there exists a subsequence such that $P_k^{(n)} \overset{*}{\rightharpoonup} \widetilde{P}_k$ in $\mathcal{K}_p(Q_T)$. Eventually, we will show that $\widetilde{P}_k = P_k(u_0, F)$.

1. Strong convergence in L^q. Consider the heat equation satisfied by the Picard iterates:

$$\partial_t P_k^{(n)} - \Delta P_k^{(n)} = \mathbb{P}\operatorname{div}[F^{(n)} - P_{k-1}^{(n)} \otimes P_{k-1}^{(n)}] \text{ in } Q_T \tag{4.3.86}$$

in the sense of distributions. Interior estimates for (4.3.86) give us the following gradient estimate for $P_k^{(n)}$ on domains $Q := B(R)\times]\delta, T[$. For all $n \in \mathbb{N}$,

$$\|P_k^{(n)}\|_{L_t^q W_x^{1,q}(Q)} + \|\partial_t P_k^{(n)}\|_{L_t^q W_x^{-1,q}(Q)} \leq C(Q), \tag{4.3.87}$$

for all $R > 0$ and $\delta \in]0, T[$. Hence, we may assume that $\nabla P_k^{(n)} \overset{*}{\rightharpoonup} \nabla \widetilde{P}_k$ in $L^q(B(R)\times]\delta, T[)$. By the Aubin-Lions lemma (see, for example, Seregin's book [126, Proposition 1.1] or the paper [8]) in the function spaces

$$W^{1,q}(\Omega) \overset{\text{cpt}}{\hookrightarrow} L^q(\Omega) \hookrightarrow W^{-1,q}(\Omega) \tag{4.3.88}$$

and a diagonal argument, there exists a subsequence such that $P_k^{(n)} \to \widetilde{P}_k$ in $L^q(B(R)\times]\delta, T[)$ for all $R > 0$ and $\delta \in]0, T[$.

2. Weak continuity in time. Let φ be a vector field belonging to the Schwartz class on $\mathbb{R}^3$. Since $P_k^{(n)} \in C([0,T]; \mathcal{S}'(\mathbb{R}^3))$ for all $n \in \mathbb{N}$, we consider the family $\mathcal{F}_{k,\varphi} \subset C([0,T])$ consisting of the functionals

$$[0,T] \to \mathbb{R} : t \mapsto \langle P_k^{(n)}(\cdot, t), \varphi\rangle, \quad n \in \mathbb{N}. \tag{4.3.89}$$

Our goal is to apply the Arzelà-Ascoli theorem to the family $\mathcal{F}_{k,\varphi}$. Recall from (4.2.44) in Section 4.2 that $\sup_{n\in\mathbb{N}} \|P_k^{(n)}\|_{L_t^\infty \dot{B}^{s_p}_{p,\infty}(Q_T)} \leq C$, so using the characterisation of dual spaces for homogeneous Besov spaces (see [11, Chapter 2], for example) we obtain

$$\sup_{n\in\mathbb{N}} |\langle P_k^{(n)}(\cdot,t), \varphi\rangle| \leq C\|\varphi\|_{\dot{B}^{-s_p}_{p',1}(\mathbb{R}^3)}, \quad t \in [0,T]. \tag{4.3.90}$$

Therefore, $\mathcal{F}_{k,\varphi}$ is a bounded subset of $C([0,T])$. To prove equicontinuity, we estimate $\partial_t P_k^{(n)}(\cdot, t)$

with values in the space $\mathcal{Z} := W^{-2,p}(\mathbb{R}^3) + W^{-1,\frac{p}{2}}(\mathbb{R}^3) + W^{-1,q}(\mathbb{R}^3)$. The space $\mathcal{Z}$ is motivated by the estimate

$$\begin{aligned}\sup_{t\in]0,T[}[t^{-\frac{s_p}{2}}\|\Delta P_k^{(n)}\|_{W^{-2,p}(\mathbb{R}^3)} + t^{-s_p}\|\mathbb{P}\operatorname{div} P_{k-1}^{(n)} \otimes P_{k-1}^{(n)}\|_{W^{-1,\frac{p}{2}}(\mathbb{R}^3)} + \\ + t^{-\frac{s_q'}{2}}\|\mathbb{P}\operatorname{div} F^{(n)}\|_{W^{-1,q}(\mathbb{R}^3)}] \leq C.\end{aligned} \tag{4.3.91}$$

Let $r > 1$ such that $r \min\left(s_p, \frac{s_q'}{2}\right) > -1$. The time derivative $\partial_t P_k^{(n)}$ is estimated from the other terms in time-dependent Stokes equations (4.3.86) and (4.3.91) to obtain $\|\partial_t P_k^{(n)}\|_{L_t^r \mathcal{Z}_x(Q_T)} \leq C(r)$. This gives us equicontinuity:

$$\sup_{n\in\mathbb{N}}|\langle P_k^{(n)}(\cdot,t_2),\varphi\rangle - \langle P_k^{(n)}(\cdot,t_1),\varphi\rangle| \leq |t_2 - t_1|^{1-\frac{1}{r}}C(r,\varphi), \tag{4.3.92}$$

for all $t_1, t_2 \in [0,T]$. Hence, there exists a subsequence such that

$$\langle P_k^{(n)}(\cdot,t),\varphi\rangle \to \langle \widetilde{P}_k(\cdot,t),\varphi\rangle \text{ in } C([0,T]). \tag{4.3.93}$$

The above argument was for a single vector field φ. Let $(\varphi_m)_{m\in\mathbb{N}} \subset \mathcal{S}(\mathbb{R}^3)$ be a dense sequence of vector fields in $\dot{B}^{-s_p}_{p',1}(\mathbb{R}^3)$. By the previous reasoning and a diagonal argument, there exists a subsequence such that

$$\langle P_k^{(n)}(\cdot,t),\varphi_m\rangle \to \langle \widetilde{P}_k(\cdot,t),\varphi_m\rangle \text{ in } C([0,T]) \tag{4.3.94}$$

for all $m \in \mathbb{N}$. From the estimate (4.3.90) and the density of $(\varphi_m)_{m\in\mathbb{N}}$ in $\dot{B}^{-s_p}_{p',1}$, one may show that (4.3.93) is valid for all Schwartz vector fields φ. Moreover, $\widetilde{P}_k(\cdot,0) = u_0$.

3. Showing $\widetilde{P}_k = P_k(u_0,F)$. First, note that while the convergence arguments up to now were for a fixed $k \geq 0$, we may assume they hold for all $k \geq 0$ simultaneously by a diagonalization argument. Let us proceed inductively. For the base case, we may write $\widetilde{P}_{-1} = P_{-1}(u_0,F) = 0$. Next, suppose that $\widetilde{P}_{k-1} = P_{k-1}(u_0,F)$ for a given $k \geq 0$. Let $n \to \infty$ in (4.3.86) to obtain the following heat equation:

$$\partial_t \widetilde{P}_k - \Delta \widetilde{P}_k = \mathbb{P}\operatorname{div}[F - P_{k-1} \otimes P_{k-1}] \text{ in } Q_T \tag{4.3.95}$$

in the sense of distributions.[18] Also, $\widetilde{P}_k(\cdot,0) = u_0$. Therefore, $\widetilde{P}_k \equiv P_k(u_0,F)$ on Q_T due to the well-posedness of the heat equation in $C([0,T];\mathcal{S}'(\mathbb{R}^3))$. This completes the induction and the proof. □

Remark 4.3.16. Using (4.3.81), (4.3.85), and interpolation, we have

$$P_k^{(n)} \to P_k \text{ in } L^l(B(R)\times]\delta,S[) \tag{4.3.96}$$

for any $l \geq 1$, $R > 0$, $S \in]0,T]$ finite, and $\delta \in]0,S[$.

We are now ready to prove Proposition 4.3.14.

Proof of Proposition 4.3.14. It suffices to consider the case when $T < \infty$. Let $k := k(p)$. As in Lemma 4.3.15, exists a constant $M > 0$ such that

$$\sup_{n\in\mathbb{N}} [\|u_0^{(n)}\|_{\dot{B}^{s_p}_{p,\infty}(\mathbb{R}^3)} + \|F^{(n)}\|_{\mathcal{F}_q(Q_T)}] \leq M, \tag{4.3.97}$$

$$\sup_{n\in\mathbb{N}} [\|P_k^{(n)}\|_{\mathcal{K}_p(Q_T)} + \|P_k^{(n)}\|_{\mathcal{K}_\infty(Q_T)}] \leq C(k,M,p,q). \tag{4.3.98}$$

According to Proposition 4.3.4, $v^{(n)} = u^{(n)} + P_k(u_0,F)$ is a weak Besov solution on Q_T based on the kth Picard iterate for each $n \in \mathbb{N}$.

1. Energy estimates. Recall the uniform decay estimate from Proposition 4.3.5. Namely, there exists $\alpha > 0$ such that

$$\sup_{n\in\mathbb{N}} \|u^{(n)}(\cdot,t)\|_{L^2(\mathbb{R}^3)} \leq Ct^\alpha, \quad t \in]0,\min(T,1)[. \tag{4.3.99}$$

By combining (4.3.98), (4.3.99), and the global energy inequality (4.3.70), we obtain the following Gronwall-type estimate:

$$\sup_{n\in\mathbb{N}} \sup_{t\in]0,T[} \int_{\mathbb{R}^3} |u^{(n)}(x,t)|^2\,dx + 2\int_0^t \int_{\mathbb{R}^3} |\nabla u^{(n)}|^2\,dx\,dt' \leq C. \tag{4.3.100}$$

[18] Here, we require the following fact concerning the Leray projector. For a sequence of vector fields $f_n \rightharpoonup f$ in $L^l_t L^r_x(Q_T)$, $1 < l,r < \infty$, we also have $\mathbb{P}f_n \rightharpoonup \mathbb{P}f$ in $L^l_t L^r_x(Q_T)$ due to the observation that $\int_{Q_T} \mathbb{P}f_n \cdot \varphi\,dx\,dt = \int_{Q_T} f_n \cdot \mathbb{P}\varphi\,dx\,dt$ for all vector fields $\varphi \in L^{l'}_t L^{r'}_x(Q_T)$.

For each $n \in \mathbb{N}$, we estimate the time derivative $\partial_t u^{(n)}$ in a negative Sobolev space according to the Navier–Stokes equations:

$$\begin{aligned}\sup_{n\in\mathbb{N}}\|\partial_t u^{(n)}\|_{L^2_t H^{-\frac{3}{2}}_x(Q_T)} &\leq \sup_{n\in\mathbb{N}} C[\|\Delta u^{(n)} - \operatorname{div} F_k^{(n)} - \operatorname{div} P_k^{(n)} \otimes u^{(n)} \\ &\quad - u^{(n)} \otimes P_k^{(n)}\|_{L^2_t H^{-1}_x(Q_T)} + \|\operatorname{div} u^{(n)} \otimes u^{(n)}\|_{L^2_t H^{-\frac{3}{2}}_x(Q_T)}] \leq C.\end{aligned} \tag{4.3.101}$$

By the Banach-Alaoglu theorem, we obtain a subsequence

$$u^{(n)} \overset{*}{\rightharpoonup} u \text{ in } L^\infty_t L^2_x(Q_T), \quad \nabla u^{(n)} \rightharpoonup \nabla u \text{ in } L^2(Q_T), \tag{4.3.102}$$

$$\operatorname*{ess\,sup}_{t\in]0,\min(1,T)[} \frac{\|u(\cdot,t)\|_{L^2(\mathbb{R}^3)}}{t^\alpha} \leq C. \tag{4.3.103}$$

A standard application of the Aubin-Lions lemma implies that $u^{(n)} \to u$ in $L^2(B(R)\times]0,T[)$ for all $R > 0$. Moreover, since $\sup_{n\in\mathbb{N}}\|u^{(n)}\|_{L^{\frac{10}{3}}(Q_T)} \leq C$, we obtain

$$u^{(n)} \to u \text{ in } L^l(B(R)\times]0,T[) \tag{4.3.104}$$

for all $R > 0$ and $l \in [1, \frac{10}{3}[$, by interpolation. In addition, by the estimates (4.3.101) and arguments similar to those in Lemma 4.3.15, we have a subsequence such that

$$\int_{\mathbb{R}^3} u^{(n)}(x,t)\varphi(x)\,dx \to \int_{\mathbb{R}^3} u(x,t)\varphi(x)\,dx \text{ in } C([0,T]). \tag{4.3.105}$$

Hence, $u(\cdot,t)$ is weakly continuous as an $L^2(\mathbb{R}^3)$-valued function, and by (4.3.103), we have

$$\lim_{t\downarrow 0}\|u(\cdot,t)\|_{L^2(\mathbb{R}^3)} = 0. \tag{4.3.106}$$

2. Pressure estimates. As described in Remark 4.3.1, we may take $q^{(n)}$ to be the associated pressure:

$$q^{(n)} = q_1^{(n)} + q_2^{(n)} + q_3^{(n)} + q^{(4)}, \quad n \in \mathbb{N}, \tag{4.3.107}$$

where

$$q_1^{(n)} := (-\Delta)^{-1} \operatorname{div}\operatorname{div} u^{(n)} \otimes u^{(n)}, \tag{4.3.108}$$

$$q_2^{(n)} := (-\Delta)^{-1} \operatorname{div}\operatorname{div} P_k^{(n)} \otimes u^{(n)} + u^{(n)} \otimes P_k^{(n)}, \tag{4.3.109}$$

$$q_3^{(n)} := (-\Delta)^{-1} \operatorname{div}\operatorname{div} P_k^{(n)} \otimes P_k^{(n)}, \tag{4.3.110}$$

$$q_4^{(n)} = (-\Delta)^{-1} \operatorname{div}\operatorname{div} F^{(n)}. \tag{4.3.111}$$

By the Calderón-Zygmund estimates, for all $\delta \in]0,T[$,

$$\sup_{n\in\mathbb{N}} \|q_1^{(n)}\|_{L^{\frac{3}{2}}(Q_T)} \leq C\|u\|^2_{L^3(Q_T)} \leq C, \tag{4.3.112}$$

$$\sup_{n\in\mathbb{N}} \|q_2^{(n)}\|_{L^3(\mathbb{R}^3\times]\delta,T[)} \leq C\|u\|_{L^3(Q_T)}\|P_k^{(n)}\|_{L^\infty(\mathbb{R}^3\times]\delta,T[)} \leq C(\delta), \tag{4.3.113}$$

$$\sup_{n\in\mathbb{N}} \|q_3^{(n)}\|_{\mathcal{F}_p(Q_T)} \leq C\|P_k^{(n)}\|_{\mathcal{K}_p(Q_T)}\|P_k^{(n)}\|_{\mathcal{K}_\infty(Q_T)} \leq C, \tag{4.3.114}$$

$$\sup_{n\in\mathbb{N}} \|q_4^{(n)}\|_{\mathcal{F}_q(Q_T)} \leq C\|F^{(n)}\|_{\mathcal{F}_q(Q_T)} \leq C. \tag{4.3.115}$$

There exists a subsequence such that for all $R > 0$ and $\delta \in]0,T[$,

$$q^{(n)} \rightharpoonup q \text{ in } L^{\frac{3}{2}}(B(R)\times]\delta,T[), \tag{4.3.116}$$

and $q(\cdot,t) \in L^{\frac{3}{2}}(\mathbb{R}^3) + L^3(\mathbb{R}^3) + L^p(\mathbb{R}^3) + L^q(\mathbb{R}^3)$ for a.e. $t \in]0,T[$. Hence, $c \equiv 0$ in Remark 4.3.1. That is, q is the pressure associated to v.

3. Local energy inequality. It remains to verify that v satisfies the local energy inequality (4.1.22). It will be more convenient[19] to examine the energy inequality satisfied by $u^{(n)}$:

$$\begin{aligned}
&\int_{\mathbb{R}^3} \varphi(x,t)|u^{(n)}(x,t)|^2\,dx + 2\int_0^t \int_{\mathbb{R}^3} \varphi|\nabla u^{(n)}|^2\,dx\,dt' \\
&\quad \leq \int_0^t \int_{\mathbb{R}^3} |u^{(n)}|^2(\partial_t\varphi + \Delta\varphi) + [|u^{(n)}|^2(u^{(n)} + P_k^{(n)}) + 2p^{(n)}u^{(n)}] \cdot \nabla\varphi\,dx\,dt' \\
&\quad\quad + 2\int_0^t \int_{\mathbb{R}^3} [P_k^{(n)} \otimes u^{(n)} + F_k^{(n)}] : \nabla(\varphi u^{(n)})\,dx\,dt',
\end{aligned} \tag{4.3.117}$$

[19]This way, we avoid the problematic term $\int_0^t \int_{\mathbb{R}^3} F^{(n)} : \nabla(\varphi v^{(n)})\,dx\,dt'$ in the local energy inequality for v, since $F^{(n)}$ is only assumed to converge weakly–$*$ in $\mathcal{F}_q(Q_T)$.

for all $t \in]0,T]$ and $0 \leq \varphi \in C_0^\infty(Q_\infty)$. Each term in (4.3.117) converges to its corresponding term in (4.3.7) with u, p, P_k, F_k replacing $u^{(n)}, p^{(n)}, P_k^{(n)}, F_k^{(n)}$ *except* for the term

$$\int_0^t \int_{\mathbb{R}^3} \varphi |\nabla u^{(n)}|^2 \, dx \, dt'. \tag{4.3.118}$$

Since $\int_0^T \int_{\mathbb{R}^3} |\nabla u^{(n)}|^2 \, dx \, dt' \leq C$, there exists a subsequence such that

$$|\nabla u^{(n)}|^2 - |\nabla u|^2 \overset{*}{\rightharpoonup} \mu \text{ in } \mathcal{M}(Q_S), \tag{4.3.119}$$

where $\mathcal{M}(Q_T)$ is the Banach space of all finite Radon measures on Q_T. Moreover, since $\nabla u^{(n)} \rightharpoonup \nabla u$ in $L^2(Q_T)$, the lower semicontinuity of the L^2-norm implies that $\mu \geq 0$. Therefore,

$$\int_0^t \int_{\mathbb{R}^3} \varphi |\nabla u^{(n)}|^2 \, dx \, dt' \to \int_0^t \int_{\mathbb{R}^3} \varphi |\nabla u|^2 \, dx \, dt' + \mu \tag{4.3.120}$$

for all $0 \leq \varphi \in C_0^\infty(Q_\infty)$ and $t \in]0,T]$, and u satisfies the local energy inequality (4.3.7). By Remark 4.3.3, v satisfies its corresponding local energy inequality (4.1.22). This completes the proof. □

As a consequence of weak-$*$ stability, we obtain an existence result for global weak Besov solutions.

Corollary 4.3.17 (Existence)**.** *Let $0 < T \leq \infty$, $u_0 \in \dot{B}^{s_p}_{p,\infty}(\mathbb{R}^3)$ be a divergence-free vector field, and $F \in \mathcal{F}_q(Q_T)$ for some $p \in]3,\infty[$ and $q \in]3,p]$. There exists a weak Besov solution v on Q_T with initial data u_0 and forcing term* $\operatorname{div} F$.

First, we require the following lemma which we state without proof.

Lemma 4.3.18 (Density)**.** *Under the hypotheses of Corollary 4.3.17, there exist sequences $(u_0^{(n)})_{n\in\mathbb{N}} \subset \mathcal{S}(\mathbb{R}^3;\mathbb{R}^3)$ and $(F^{(n)})_{n\in\mathbb{N}} \subset C_0^\infty(Q_T;\mathbb{R}^{3\times 3})$ such that $u_0^{(n)} \overset{*}{\rightharpoonup} u_0$ in $\dot{B}^{s_p}_{p,\infty}(\mathbb{R}^3)$, $F^{(n)} \overset{*}{\rightharpoonup} F$ in $\mathcal{F}_q(Q_T)$, and for each $n \in \mathbb{N}$,* $\operatorname{div} u_0^{(n)} = 0$

Proof of Corollary 4.3.17. Let $(u_0^{(n)})_{n\in\mathbb{N}}$ and $(F^{(n)})_{n\in\mathbb{N}}$ be the approximating sequences from Lemma 4.3.18. By Proposition 4.3.13, there exists a sequence $(v^{(n)})_{n\in\mathbb{N}}$ of suitable weak Leray-Hopf solutions on Q_T with respective initial data $u_0^{(n)}$ and forcing terms $\operatorname{div} F^{(n)}$ for each $n \in \mathbb{N}$. In Proposition 4.3.11, we proved that for each $n \in \mathbb{N}$, the suitable weak Leray-Hopf solution $v^{(n)}$ is also a weak Besov solution on Q_T with initial data $u_0^{(n)}$ and forcing term $\operatorname{div} F^{(n)}$.

Finally, recall Proposition 4.3.14 regarding weak-$*$ stability of weak Besov solutions. There exists a subsequence (still denoted by n) such that $v^{(n)} \to v$ in $L^3_{\mathrm{loc}}(Q_T)$, where v is a weak Besov solution on Q_T with initial data u_0 and forcing term $\operatorname{div} F$. □

4.3.5 Weak-strong uniqueness

In this subsection, we are concerned with mild solutions of the Navier–Stokes equations and their relationship to weak Besov solutions.

Definition 4.3.19 (Mild/strong solutions)**.** *Let $T > 0$ and $F \in \mathcal{F}(Q_T)$. Assume that $u_0 \in \mathcal{S}'(\mathbb{R}^3)$ is divergence-free and $P_0(u_0, F) \in \mathcal{X}_T$. (For instance, this is satisfied when $u_0 \in \mathrm{BMO}^{-1}(\mathbb{R}^3) \cup L^\infty(\mathbb{R}^3)$ is divergence free.)*

A vector field $v \in \mathcal{X}_T$ is a mild solution *of the Navier–Stokes equations on Q_T with initial data u_0 and forcing term* $\operatorname{div} F$ *if for a.e. $t \in]0, T]$, v satisfies the integral equation*

$$v(\cdot, t) = P_0(u_0, F)(\cdot, t) - B(v, v)(\cdot, t). \tag{4.3.121}$$

A mild solution v on Q_T is a strong solution *if v is also a weak Besov solution on Q_T with initial data u_0 and forcing term* $\operatorname{div} F$*.*

We say that v is a mild (resp. strong) solution on Q_∞ with initial data u_0 and forcing term $\operatorname{div} F$ *if for all $T > 0$, v is a mild (resp. strong) solution on Q_T with initial data u_0 and forcing term* $\operatorname{div} F$*.*

Our main goal is the following theorem:

Theorem 4.3.20 (Weak-strong uniqueness)**.** *Let $0 < T \leq \infty$, $u_0 \in \dot{B}^{s_p}_{p,\infty}(\mathbb{R}^3)$ be a divergence-free vector field, and $F \in \mathcal{F}_q(Q_T)$ for some $p \in]3, \infty[$ and $q \in]3, p]$.*

There exists an absolute constant $\varepsilon_0 > 0$ with the following property.

Suppose that $v \in \mathcal{K}_\infty(Q_T)$ is a weak Besov solution on Q_T with initial data u_0 and forcing term $\operatorname{div} F$*. Moreover, assume that v satisfies*

$$\operatorname*{ess\,sup}_{0<t<S} t^{\frac{1}{2}} \|v(\cdot, t)\|_{L^\infty(\mathbb{R}^3)} < \varepsilon_0 \tag{4.3.122}$$

for some $S \in]0, T]$. If $\widetilde{v}$ is a weak Besov solution on Q_T with the same initial data and forcing term, then $v \equiv \widetilde{v}$ on Q_T.

Note that Theorem 4.3.20 proves weak-strong uniqueness until the maximal existence time of the solution in $\mathcal{K}_\infty(Q_T)$, not merely on the initial interval $]0,S[$ where the strong solution is small.

We investigate the existence of strong solutions in Proposition 4.3.22. In particular, strong solutions satisfying (4.3.122) always exist when the initial data and forcing term are sufficiently small. This observation proves Theorem 4.1.3 in the introduction.

Remark 4.3.21 (Alternative proof of small-data-uniqueness)**.** Let $\|u_0\|_{\dot{B}^{s_p}_{p,\infty}(\mathbb{R}^3)} + \|F\|_{\mathcal{F}_q(Q_T)} \leq M$. When $M \ll 1$, one may prove the uniqueness for weak Besov solutions v in the following way, which does not rely on the perturbation theory in Proposition 4.3.22.

Without loss of generality, $T = 1$. We will use Proposition 4.6.2 (ε-regularity) with $f = 0$, $q_2 = \infty$, and $p_2 = q$. Choose $0 < R \ll 1$ such that $c_0/R < \varepsilon_0$, where ε_0 is the constant in (4.3.122) and c_0 from Proposition 4.6.2.

By using the energy inequality in Remark 4.3.10, estimates on the Picard iterates in Section 4.2, and Calderón-Zygmund estimates for the pressure, one may show that

$$\sup_{x_0 \in \mathbb{R}^3} \frac{1}{R^2} \int_{1-R^2}^1 \int_{B(x_0,R)} |v|^3 + |q|^{\frac{3}{2}} \, dx \, dt < \frac{\varepsilon_{\rm CKN}}{2} \tag{4.3.123}$$

when $M \ll 1$. See the proof of Lemma 4.4.3 for similar arguments. Upon further reducing $M \ll 1$,

$$\sup_{x_0 \in \mathbb{R}^3} \frac{1}{R^\delta} \|F\|_{L^\infty_t L^q_x(B(x_0,R)\times]1-R^2,1[)} < \frac{\varepsilon_{\rm CKN}}{2}, \tag{4.3.124}$$

with $\delta = 2 - 3/q$. Combining (4.3.123)-(4.3.124) and ε-regularity, we obtain

$$\|v(\cdot,1)\|_{L^\infty(\mathbb{R}^3)} < \epsilon_0. \tag{4.3.125}$$

Using a scaling argument, one obtains (4.3.122). Finally, Theorem 4.3.20 implies the uniqueness.

Proof of Theorem 4.3.20. Let $v, \widetilde{v}$ be as in the statement of the theorem with the constant $\varepsilon_0 > 0$ in (4.3.122) to be determined. Let $k := k(p)$ and denote $u := v - P_k(u_0,F)$, $\widetilde{u} := \widetilde{v} - P_k(u_0,F)$, $w := \widetilde{u} - u$.

0. Properties of w. Observe that $w \in L^\infty_t L^2_x \cap L^2_t \dot{H}^1_x(Q_S)$ for all finite $S \in]0,T]$ solves the following Navier–Stokes-type system in the sense of distributions:

$$\left.\begin{aligned} \partial_t w - \Delta w + \operatorname{div} w \otimes w + \operatorname{div} v \otimes w + \operatorname{div} w \otimes v = -\nabla r \\ \operatorname{div} w = 0 \end{aligned}\right\} \text{ on } Q_T. \tag{4.3.126}$$

Also, $w(\cdot, t)$ is weakly continuous as an $L^2(\mathbb{R}^3)$-valued function. Due to the uniform decay estimate (4.3.29) satisfied by $u, \widetilde{u}$, we have

$$\sup_{t \in]0,T[} \frac{\|w(\cdot,t)\|^2_{L^2(\mathbb{R}^3)}}{t^{\frac{1}{2}}} < \infty. \tag{4.3.127}$$

1. Energy estimate for w. Our goal is to demonstrate that $w \equiv 0$ on Q_T. Recall that $u, \widetilde{u}$ satisfy the global energy inequality (4.3.70) starting from the initial time, see Corollary 4.3.9. (In fact, u satisfies the global energy *equality*, compare with Step 1B in Proposition 4.3.22.) As is typical in weak-strong uniqueness arguments, we combine the two energy estimates using the weak-strong identity (4.3.9) to obtain the following energy inequality for w:

$$\|w(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} + 2\int_0^t \int_{\mathbb{R}^3} |\nabla w|^2 \, dx \, dt' \leq 2\int_0^t \int_{\mathbb{R}^3} v \otimes w : \nabla w \, dx \, dt' \tag{4.3.128}$$

for all $t \in]0,T[$. The requirement $v \in \mathcal{K}_\infty(Q_T)$ together with Proposition 4.3.5 are used to make certain calculations rigorous, in particular, to ensure that the RHS of (4.3.128) is finite. See the proof of Proposition 4.3.2 for a similar argument.

2. Showing $w \equiv 0$. We will conclude with a Gronwall-type argument that crucially makes use of (4.3.127). The connection between similar decay properties and weak-strong uniqueness was observed by Dong and Zhang in [45] and was subsequently used by the second author in [15].

Manipulating (4.3.128), one obtains

$$\begin{aligned} \|w(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} &\leq c\int_0^t \int_{\mathbb{R}^3} |v|^2 |w|^2 \, dx \, ds \\ &\leq C_0 \sup_{s \in]0,t[} [s^{\frac{1}{2}} \|v(\cdot,s)\|_{L^\infty(\mathbb{R}^3)}]^2 \times t^{\frac{1}{2}} \sup_{s \in]0,t[} [s^{-\frac{1}{2}} \|w(\cdot,s)\|^2_{L^2(\mathbb{R}^3)}] \end{aligned} \tag{4.3.129}$$

for all finite $t \in]0,T]$. We may choose $\varepsilon_0 := (2C_0)^{-\frac{1}{2}}$ in the statement of the theorem. Recall the assumption (4.3.122), i.e., there exists $S \in]0,T]$ such that

$$\sup_{s\in]0,S[}[s^{\frac{1}{2}}\|v(\cdot,s)\|_{L^\infty(\mathbb{R}^3)}]^2 < \frac{1}{2C_0}. \tag{4.3.130}$$

Then (4.3.129) gives us

$$\frac{\|w(\cdot,t)\|^2_{L^2(\mathbb{R}^3)}}{t^{\frac{1}{2}}} \leq \frac{1}{2}\sup_{s\in]0,t[}[s^{-\frac{1}{2}}\|w(\cdot,s)\|^2_{L^2(\mathbb{R}^3)}] \tag{4.3.131}$$

for all $t \in]0,S[$. Hence, $w \equiv 0$ on Q_S. Now, the original energy inequality (4.3.128) gives us

$$\|w(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} \leq C\|v\|^2_{L^\infty(\mathbb{R}^3\times]S,t[)}\int_S^t \|w\|^2_{L^\infty_t L^2_x(Q_{t'})}\,dx\,dt', \tag{4.3.132}$$

for all finite $t \in]S,T]$. Finally, the standard Gronwall lemma implies that $w \equiv 0$ on Q_T. This completes the proof of weak-strong uniqueness. □

Finally, we consider the existence of strong solutions. First, we require some notation. Let $p \in]3,\infty[$ and $0 < T \leq \infty$. Define $\mathring{\mathcal{K}}_p(Q_T)$ to be the closed subspace of $\mathcal{K}_p(Q_T)$ consisting of vector fields v such that $\lim_{S\downarrow 0}\|v\|_{\mathcal{K}_p(Q_S)} = 0$ and satisfying the following additional requirement when $T = \infty$:

$$\lim_{S\uparrow\infty}\operatorname*{ess\,sup}_{t>S} t^{-\frac{s_p}{2}}\|v(\cdot,t)\|_{L^p(\mathbb{R}^3)} = 0. \tag{4.3.133}$$

Similarly, define $\mathring{\mathcal{X}}_T$ to be the closed subspace of $\mathcal{X}_T$ consisting of vector fields v such that $\lim_{S\downarrow 0}\|v\|_{\mathcal{X}_S} = 0$ and such that the following additional requirements are satisfied when $T = \infty$. Namely, $v(\cdot,t_1+\cdot) \in \mathcal{X}_\infty$ for all $t_1 > 0$, and

$$\lim_{t_1\uparrow\infty}\|v(\cdot+t_1)\|_{\mathcal{X}_\infty} = 0. \tag{4.3.134}$$

The space $\mathring{\mathcal{Y}}_T$ is defined analogously for forcing terms. Recall from Section 4.2.2 that when $q \in]3,p]$ and $F \in \mathcal{F}_q(Q_T)$, $L(F)$ belongs to $\mathcal{K}_p(Q_T) \cap \mathcal{X}_T$.

Here is our main result concerning the existence of strong solutions:

Proposition 4.3.22 (Existence of strong solutions in perturbative regime)**.** *Let* $0 < T \leq \infty$, $u_0 \in \dot{B}^{s_p}_{p,\infty}(\mathbb{R}^3)$ *be a divergence-free vector field, and* $F \in \mathcal{F}_q(Q_T)$ *for some* $p \in]3,\infty[$ *and*

$q \in]3,p]$. Suppose that $v \in \mathcal{K}_p(Q_T) \cap \mathring{\mathcal{X}}_T$ is a mild solution of the Navier–Stokes equations on Q_T with initial data u_0 and forcing term $\operatorname{div} F$. *There exists a constant $\varepsilon_0 := \varepsilon_0(v,p) > 0$ such that for all divergence-free $\widetilde{u_0} \in \dot{B}^{s_p}_{p,\infty}(\mathbb{R}^3)$ and $\widetilde{F} \in \mathcal{F}_q(Q_T)$ satisfying*

$$\|P_0(\widetilde{u_0}, \widetilde{F}) - P_0(u_0, F)\|_{\mathcal{X}_T} < \varepsilon_0, \tag{4.3.135}$$

there exists a mild solution $\widetilde{v} \in \mathcal{X}_T$ with initial data $\widetilde{u_0}$ and forcing term $\operatorname{div} \widetilde{F}$ *and such that*

$$\|\widetilde{v} - v\|_{\mathcal{X}_T} < 2\varepsilon_0. \tag{4.3.136}$$

In addition, $\widetilde{v}$ is unique amongst all mild solutions (with initial data $\widetilde{u_0}$ and forcing term $\operatorname{div} \widetilde{F}$*) that satisfy (4.3.136). Moreover, $\widetilde{v}$ is a weak Besov solution on Q_T with initial data $\widetilde{u_0}$ and forcing term* $\operatorname{div} \widetilde{F}$ *(in particular, it is a strong solution). Finally, $\widetilde{v}$ satisfies*

$$\|\widetilde{v} - v\|_{\mathcal{X}_T} \leq 2\|P_0(\widetilde{u_0}, \widetilde{F}) - P_0(u_0, F)\|_{\mathcal{X}_T}, \tag{4.3.137}$$

$$\|\widetilde{v} - v\|_{\mathcal{K}_p(Q_T)} \leq 2\|P_0(\widetilde{u_0}, \widetilde{F}) - P_0(u_0, F)\|_{\mathcal{K}_p(Q_T)}. \tag{4.3.138}$$

The method of proof is well known and goes back to the work [53] of Fujita and Kato for initial data in H^s, $s \geq 1/2$, as well as Kato's seminal paper [76] concerning small-data-global-existence for initial data in L^3. Solutions evolving from initial data in critical Besov spaces $\dot{B}^{-1+\frac{3}{p}}_{p,\infty}$, $p > 3$, were investigated by Cannone [34] and many other authors, see, e.g., the appendix of [54] and the references in [95]. Finally, solutions evolving from BMO^{-1} initial data were pioneered in [85] by H. Koch and D. Tataru.

Proof. **1. Perturbations of the zero solution**. Let us consider the case when u_0 and F are zero. As mentioned in Section 4.2.2, there exists a constant $\kappa > 0$ such that for all U and V in $\mathcal{X}_T$,

$$\|B(U,V)\|_{\mathcal{X}_T} \leq \kappa \|U\|_{\mathcal{X}_T} \|V\|_{\mathcal{X}_T}. \tag{4.3.139}$$

Furthermore, it is not difficult to show that there exists $\kappa_p > 0$ such that

$$\|B(U,V)\|_{\mathcal{K}_p(Q_T)} \leq \kappa_p \min(\|U\|_{\mathcal{X}_T} \|V\|_{\mathcal{K}_p(Q_T)}, \|U\|_{\mathcal{K}_p(Q_T)} \|V\|_{\mathcal{X}_T}), \tag{4.3.140}$$

for all $U, V \in \mathcal{X}_T \cap \mathcal{K}_p(Q_T)$. The constants are independent of $0 < T \leq \infty$. We also use (4.3.140) for $p=2$. Let us write $M\geq \|P_0(\widetilde{u_0},\widetilde{F})\|_{\mathcal{X}_T}$ and $M_p\geq \|P_0(\widetilde{u_0},\widetilde{F})\|_{\mathcal{K}_p(Q_T)}$.

1A. Existence in $\mathcal{X}_T$ and $\mathcal{K}_p(Q_T)$. Suppose that $M<(4\kappa)^{-1}$. One may verify using (4.3.139) that the Picard iterates $\widetilde{P}_k:=P_k(\widetilde{u_0},\widetilde{F})$ satisfy

$$\|\widetilde{P}_k\|_{\mathcal{X}_T}\leq 2M,\quad \|\widetilde{P}_{k+1}-\widetilde{P}_k\|_{\mathcal{X}_T}\leq 4\kappa M\|\widetilde{P}_k-\widetilde{P}_{k-1}\|_{\mathcal{X}_T} \tag{4.3.141}$$

for all integers $k\geq 0$. Hence, the sequence of Picard iterates $(\widetilde{P}_k)_{k\geq 0}$ converges to a solution $\widetilde{v}\in\mathcal{X}_T$ of the integral equation

$$\widetilde{v}(\cdot,t)=P_0(\widetilde{u}_0,\widetilde{F})-B(\widetilde{v},\widetilde{v}). \tag{4.3.142}$$

Observe that $\widetilde{v}$ is the unique solution satisfying $\|\widetilde{v}\|_{\mathcal{X}_T}<(2\kappa)^{-1}$. Now suppose that $M<(4\kappa_p)^{-1}$ is additionally satisfied. One verifies using (4.3.140) that for all integers $k\geq 0$, we have

$$\|\widetilde{P}_k\|_{\mathcal{K}_p(Q_T)}\leq 2M_p, \tag{4.3.143}$$

$$\|\widetilde{P}_{k+1}-\widetilde{P}_k\|_{\mathcal{K}_p(Q_T)}\leq 4\kappa_p M\|\widetilde{P}_k-\widetilde{P}_{k-1}\|_{\mathcal{K}_p(Q_T)}. \tag{4.3.144}$$

The sequence $(\widetilde{P}_k)_{k\geq 0}$ converges also in the space $\mathcal{K}_p(Q_T)$, so $\widetilde{v}$ additionally belongs to $\mathcal{K}_p(Q_T)$ and satisfies $\|\widetilde{v}\|_{\mathcal{K}_p(Q_T)}\leq 2M_p$.

1B. $\widetilde{v}$ is a weak Besov solution. Recall from Lemma 4.2.2 that $\widetilde{P}_{k(p)+1}-\widetilde{P}_{k(p)}\in\mathcal{K}_2(Q_T)$. Let us further assume that $M<(4\kappa_2)^{-1}$. One may demonstrate using (4.3.140) that for all $k>k(p)$,

$$\|\widetilde{P}_{k+1}-\widetilde{P}_k\|_{\mathcal{K}_2(Q_T)}\leq 4M\kappa_2\|\widetilde{P}_k-\widetilde{P}_{k-1}\|_{\mathcal{K}_2(Q_T)}. \tag{4.3.145}$$

Therefore, $\widetilde{u}:=\widetilde{v}-\widetilde{P}_{k(p)}$ belongs to $\mathcal{K}_2(Q_T)$, since

$$\|\widetilde{u}\|_{\mathcal{K}_2(Q_T)}\leq\|\widetilde{v}-\widetilde{P}_{k(p)}\|_{\mathcal{K}_2(Q_T)}\leq\sum_{k=k(p)}^{\infty}\|\widetilde{P}_{k+1}-\widetilde{P}_k\|_{\mathcal{K}_2(Q_T)}<\infty. \tag{4.3.146}$$

Let us now demonstrate that $\widetilde{u}\in C([0,S];L^2(\mathbb{R}^3))\cap L^2_t\dot{H}^1_x(Q_S)$ for all finite $S\in]0,T]$. In order to show this, we use the identity

$$\widetilde{u}(\cdot,t)=-B(\widetilde{u},\widetilde{u})(\cdot,t)-L(\widetilde{P}_k\otimes\widetilde{u}+\widetilde{u}\otimes\widetilde{P}_k+\widetilde{F}_k)(\cdot,t). \tag{4.3.147}$$

We then conclude using the following facts. Namely,

$$\|U\otimes V\|_{L^2(Q_S)}\leq S^{\frac{1}{4}}\|U\|_{\mathcal{K}_2(Q_S)}\|V\|_{\mathcal{X}_S} \tag{4.3.148}$$

($U\in\mathcal{K}_2(Q_S)$ and $V\in\mathcal{X}_S$) and the fact that $\widetilde{F}_k\in L^2(Q_S)$ for all $k\geq k(p)$, as observed in Lemma 4.2.2. Note also that since $\widetilde{u}\in\mathcal{K}_2(Q_T)$, we have that

$$\lim_{t\downarrow 0}\|\widetilde{u}(\cdot,t)\|_{L^2}=0. \tag{4.3.149}$$

It remains to prove the local energy inequality (4.1.22) for $\widetilde{v}$ with its associated pressure $\widetilde{q}:=(-\Delta)^{-1}\operatorname{div}\operatorname{div}\widetilde{v}\otimes\widetilde{v}$. Recall that $\widetilde{v}\in L^\infty(\mathbb{R}^3\times]\delta,S[)$ for all finite $S\in]0,T]$ and $\delta\in]0,S[$. By Calderón-Zygmund estimates, $\widetilde{q}\in L^\infty_t\mathrm{BMO}_x(\mathbb{R}^3\times]\delta,S[)$. Using these facts, the local energy inequality for $(\widetilde{v},\widetilde{q})$ follows by using a mollification argument in the same spirit as in [126, p. 160-161)]. Hence, the proposition is proven with $\varepsilon_0(p):=(8\max(\kappa,\kappa_p,\kappa_2))^{-1}$ in the special case that u_0 and F are zero.

2. Perturbations of general solutions. Now we consider the proposition in full generality.

2A. Solving the integral equation. Our goal is to solve the following integral equation:

$$z(\cdot,t)=P_0(\widetilde{u_0},\widetilde{F})(\cdot,t)-P_0(u_0,F)(\cdot,t)-B(z,z)(\cdot,t)-L_v(z)(\cdot,t), \tag{4.3.150}$$

where $L_v(z):=B(z,v)+B(v,z)$. Then $\widetilde{v}:=z+v$ will be a mild solution of the Navier–Stokes equations. The integral equation (4.3.150) is equivalent to

$$z=(I+L_v)^{-1}[P_0(\widetilde{u_0},\widetilde{F})-P_0(u_0,F)]-(I+L_v)^{-1}B(z,z), \tag{4.3.151}$$

since $I+L_v$ is invertible on $\mathcal{X}_T$ and $\mathcal{K}_p(Q_T)$, see Lemma 4.3.23. The existence and uniqueness theory for mild solutions of (4.3.151) in $\mathcal{X}_T$ and $\mathcal{K}_p(Q_T)$ is similar to that of Step 1A except that one uses Picard iterates $\bar{P}_k(u_0,F,\widetilde{u}_0,\widetilde{F})$ defined recursively by

$$\bar{P}_0(u_0,F,\widetilde{u}_0,\widetilde{F}):=(I+L_v)^{-1}[P_0(\widetilde{u_0},\widetilde{F})-P_0(u_0,F)], \tag{4.3.152}$$

$$\bar{P}_k(u_0,F,\widetilde{u}_0,\widetilde{F}):=\bar{P}_0-(I+L_v)^{-1}B(\bar{P}_{k-1},\bar{P}_{k-1}),\quad k\in\mathbb{N}. \tag{4.3.153}$$

In addition, we define

$$\varepsilon_0(v,p) := (8\max(\|(I+L_v)^{-1}\|^2_{\mathcal{X}_T}\kappa, \|(I+L_v)^{-1}\|^2_{\mathcal{K}_p(Q_T)}\kappa_p, \kappa, \kappa_p, \kappa_2))^{-1}/3, \quad (4.3.154)$$

which is less than $\varepsilon_0(p)/3$ (where $\varepsilon_0(p)$ is as in Step 1) when $v = 0$. The proof of existence and uniqueness is not difficult and follows Step 1A, so we will omit it. Let $\widetilde{v}$ denote the resulting mild solution of the Navier–Stokes equations.

2B. $\widetilde{u}$ has finite kinetic energy. Since $v \in \mathring{\mathcal{X}}_T$ and (4.3.136), there exists $\widetilde{T} \in]0,T[$ such that $\|v\|_{\mathcal{X}_{\widetilde{T}}} < 2\varepsilon_0(v,p) - \|\widetilde{v} - v\|_{\mathcal{X}_{\widetilde{T}}}$. By the triangle inequality $\|\widetilde{v}\|_{\mathcal{X}_{\widetilde{T}}} \le \|v\|_{\mathcal{X}_{\widetilde{T}}} + \|\widetilde{v} - v\|_{\mathcal{X}_{\widetilde{T}}}$ and $\varepsilon_0(v,p) < \varepsilon_0(p)/3$, we obtain

$$\|\widetilde{v}\|_{\mathcal{X}_{\widetilde{T}}} < 2\varepsilon_0(v,p) < 2\varepsilon_0(p)/3. \quad (4.3.155)$$

Since $P_0(\widetilde{u_0}, \widetilde{F}) = \widetilde{v}(\cdot,t) + B(\widetilde{v},\widetilde{v})(\cdot,t)$, we infer that

$$\|P_0(\widetilde{u_0}, \widetilde{F})\|_{X_{\widetilde{T}}} \le \|\widetilde{v}\|_{X_{\widetilde{T}}} + \kappa\|\widetilde{v}\|^2_{X_{\widetilde{T}}}. \quad (4.3.156)$$

Using (4.3.155) and the fact that $4\kappa\varepsilon_0(p) < 1$, we obtain that

$$\|P_0(\widetilde{u_0}, \widetilde{F})\|_{X_{\widetilde{T}}} < \varepsilon_0(p). \quad (4.3.157)$$

So we can construct a strong solution (with initial data $\widetilde{u_0}$ and forcing term $\operatorname{div}\widetilde{F}$) on $Q_{\widetilde{T}}$ according to Step 1. Finally, using (4.3.155), $\widetilde{v}$ agrees on $Q_{\widetilde{T}}$ with the mild solution constructed in Step 1, and in particular, $\widetilde{u} \in C([0,\widetilde{T}]; L^2(\mathbb{R}^3)) \cap \dot{L}^2_t H^1_x(Q_{\widetilde{T}})$.

To show that $\widetilde{u}$ has finite energy on Q_S for all finite $S \in]0,T]$, we appeal to Lemma 4.3.24. Specifically, after translating in time, Lemma 4.3.24 says there exists $S > 0$ and a solution $\bar{u} \in L^\infty(\mathbb{R}^3 \times]\widetilde{T}, \widetilde{T}+S[)$ of the integral equation

$$\begin{aligned} \bar{u}(\cdot,t) = S(t-\widetilde{T})\widetilde{u}(\cdot,\widetilde{T}) - \int_{\widetilde{T}}^t S(t-s-\widetilde{T})\mathbb{P}\operatorname{div} F_k(\cdot,s)\,ds \\ - \int_{\widetilde{T}}^t S(t-s-\widetilde{T})\mathbb{P}\operatorname{div}[(\bar{u}+P_k)\otimes\bar{u} + P_k\otimes\bar{u}](\cdot,s)\,ds. \end{aligned} \quad (4.3.158)$$

on $\mathbb{R}^3 \times]\widetilde{T}, \widetilde{T}+S[$. Moreover, $\bar{u}$ belongs to the energy space. Since $\bar{v} := P_k + \bar{u}$ is an L^∞ mild solution of the Navier–Stokes equations on $\mathbb{R}^3 \times]\widetilde{T}, \widetilde{T}+S[$ with initial data $\widetilde{v}(\cdot,\widetilde{T})$ and forcing term $\operatorname{div}\widetilde{F}$, the uniqueness of such solutions implies that $\widetilde{v} \equiv \bar{v}$ on $\mathbb{R}^3\times]\widetilde{T}, \widetilde{T}+S[$. Hence, $\widetilde{u} \equiv \bar{u}$ on the same domain, so we obtain that $\widetilde{u} \in C([0,\widetilde{T}+S]; L^2(\mathbb{R}^3)) \cap L^2_t\dot{H}^1_x(Q_{\widetilde{T}+S})$. We may continue in this fashion as long as the existence time is not shrinking to zero in the iteration. In light of the lower bound (4.3.159) on the existence time in Lemma 4.3.24, we conclude that $\widetilde{u} \in C([0,S]; L^2(\mathbb{R}^3)) \cap L^2_t\dot{H}^1_x(Q_S)$ for all finite $S \in]0,T]$.

2C. $\widetilde{v}$ is a weak Besov solution. The local energy inequality for $\widetilde{v}$ follows from exactly the same argument as in Step 1B. □

Lemma 4.3.23 (Spectrum of L_v)**.** *Let $0 < T \leq \infty$ and $p \in]3,\infty[$. Suppose that $v \in \mathring{\mathcal{X}}_T$ is divergence free. Then $L_v \colon \mathcal{X}_T \to \mathcal{X}_T$ and $L_v \colon \mathcal{K}_p(Q_T) \to \mathcal{K}_p(Q_T)$ defined by $L_v(z) := B(z,v) + B(v,z)$ have spectrum $\{0\}$.*

Proof. 1. L_v is not invertible. Notice that $\nabla L_v f \in L^3_{\mathrm{loc}}(Q_T)$ for all $f \in \mathcal{X}_T \cup \mathcal{K}_p(Q_T)$ due to local regularity properties of the Stokes equations. Of course, there exists elements $g_1 \in \mathcal{K}_p(Q_T)$ and $g_2 \in \mathcal{X}_T$ with $\nabla g_i \notin L^3_{\mathrm{loc}}(Q_T)$ for $i = 1,2$. Clearly, $L_v f_1 \neq g_1$ for all $f_1 \in \mathcal{K}_p(Q_T)$ and $L_v f_2 \neq g_2$ for all $f_2 \in \mathcal{X}_T$. Hence, zero belongs to the spectrum of L_v on $\mathcal{X}_T$ and $\mathcal{K}_p(Q_T)$.

2. $\lambda I - L_v$ is invertible ($\lambda \in \mathbb{C} \setminus \{0\}$). We omit the proof of invertibility, since it is nearly identical to the proof of [9, Lemma 6], in particular, p. 684-685. The main idea is to solve the linear problem $f - L_v f = g$ on a finite number of small subintervals by a perturbation argument. □

Lemma 4.3.24 (Local continuation with finite energy)**.** *Let $0 < T \leq \infty$. Assume that $a \in L^\infty(\mathbb{R}^3) \cap J(\mathbb{R}^3)$, $V \in L^\infty(Q_T)$ is a divergence-free vector field, and $G \in L^\infty(Q_T) \cap L^2(Q_T)$ with values in $\mathbb{R}^{3\times3}$. There exists a finite time $S \in]0,T]$, an absolute constant $c_0 > 0$ satisfying*

$$S \geq \frac{c_0}{(1 + \|P_0(a,G)\|_{L^\infty(Q_T)} + \|V\|_{L^\infty(Q_T)})^2}, \tag{4.3.159}$$

and a solution $u \in L^\infty(Q_S) \cap C([0,S]; L^2(\mathbb{R}^3)) \cap L^2_t\dot{H}^1_x(Q_S)$ of the following integral equation:

$$u(\cdot,t) = P_0(a,G)(\cdot,t) - B(u,u)(\cdot,t) - L_V(u)(\cdot,t), \tag{4.3.160}$$

for a.e. $t \in]0,S[$.

We omit the proof of Lemma 4.3.24, since it follows known perturbation arguments similar to those in Proposition 4.3.22.

4.4 Applications

4.4.1 Blow-up criteria

As mentioned before, the second half of this paper focuses on applications of the weak Besov solutions developed in Section 4.3. Let $\mathbb{B}$ denote the set of all divergence-free vector fields $f \in \dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3)$ satisfying

$$\lim_{\lambda \downarrow 0} \lambda f(\lambda(\cdot)) = 0 \text{ in } \mathcal{D}'(\mathbb{R}^3). \tag{4.4.1}$$

Note that $\mathbb{B}$ does not contain any non-trivial scale-invariant vector fields. We wish to prove the following theorem:

Theorem 4.4.1 (Blow-up criteria). *Let $T^* > 0$, $u_0 \in L^\infty(\mathbb{R}^3)$ be a divergence-free vector field, and $F \in L^\infty_t L^q_x(\mathbb{R}^3 \times]0,T^*[)$ for some $q \in]3,\infty[$. Suppose that $v \in L^\infty(\mathbb{R}^3 \times]0,T[)$ is a mild solution of the Navier–Stokes equations on $\mathbb{R}^3 \times]0,T[$ with initial data u_0 and forcing term* $\operatorname{div} F$ *for all $T \in]0,T^*[$. Let $p \in]3,\infty[$ and $M > 0$. There exists a constant $\varepsilon := \varepsilon(p,q,M) > 0$ with the following properties:*

(i) Suppose that $\|v(\cdot,t_1)\|_{\dot{B}^{-1+\frac{3}{p}}_{p,\infty}(\mathbb{R}^3)} \leq M$ for some $t_1 \in]0,T^*[$.[20] *If also*

$$\|v(\cdot,T^*)\|_{\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3)} + \|F\|_{\mathcal{F}_q(\mathbb{R}^3 \times]0,T^*[)} \leq \varepsilon, \tag{4.4.2}$$

then $v \in L^\infty(\mathbb{R}^3 \times]0,T^*[)$.

(ii) Suppose that there exists a sequence of times $t_n \uparrow T^*$ such that

$$\sup_{n \in \mathbb{N}} \|v(\cdot,t_n)\|_{\dot{B}^{-1+\frac{3}{p}}_{p,\infty}(\mathbb{R}^3)} \leq M. \tag{4.4.3}$$

[20] In this statement, $v(\cdot,t)$ is well-defined for each $t \in [0,T^*]$ since v belongs to $C([0,T^*];\mathcal{D}'(\mathbb{R}^3))$. One way to argue this is as follows. First, it is known that as a mild solution, v belongs to $C([0,T^*[;\mathcal{D}'(\mathbb{R}^3))$. Second, according to Proposition 4.4.5, v agrees on $\mathbb{R}^3 \times]t_1,T^*[$ with a weak Besov solution, and such a solution belongs to $C([t_1,T^*];\mathcal{D}'(\mathbb{R}^3))$.

If there exists $x^* \in \mathbb{R}^3$ such that $v(\cdot, T^*)$ satisfies

$$\operatorname{dist}(v(\cdot + x^*, T^*), \mathbb{B}) \leq \varepsilon, \tag{4.4.4}$$

where the distance is measured in the $\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3)$ norm, then v is regular at (x^*, T^*). If (4.4.4) *is satisfied for all $x^* \in \mathbb{R}^3$, then $v \in L^\infty(\mathbb{R}^3 \times]0, T^*[)$.*

Here are a few remarks concerning Theorem 4.4.1:

1. Let us mention that Escauriaza, Seregin and Šverak's result[21] was shown to hold true with the addition of certain forcing terms by Lemarié-Rieusset in [98] (specifically, Theorem 15.15, p. 527 of [98]).

2. Previously, in [40], Choe, Wolf, and Yang showed that a weak Leray-Hopf solution satisfying
$$\operatorname*{ess\,sup}_{0<t<T^*} \|v(\cdot, t)\|_{L^{3,\infty}} \leq M \tag{4.4.5}$$
is regular at (x^*, T^*), under certain additional assumptions on $v(\cdot, T^*)$, which are similar in spirit to (4.4.4).

3. The blow-up profiles that do not satisfy our assumption (4.4.4) are reminiscent of the initial data conjectured by Guillod and Šverák in [66] to give rise to non-uniqueness. It is plausible to us that there exists a global weak Besov solution v which is singular at $T^* > 0$, $\sup_{0<t<T^*} \|v(\cdot, t)\|_{\dot{B}^{s_p}_{p,\infty}(\mathbb{R}^3)} < \infty$, and such that uniqueness is lost at the singular time; that is, there exists a different global weak Besov solution $\widetilde{v}$ such that $v \equiv \widetilde{v}$ on Q_{T^*}.

From the proof of Theorem 4.4.1.(i), we obtain an analogous criterion for weak Besov solutions which we will use to prove Theorem 4.4.1.(ii).

Remark 4.4.2 (Blow-up criterion for weak Besov solutions)**.** Let $T^* > 0$, $p \in]3, \infty[$ and $q \in]3, p]$. Suppose that v is a weak Besov solution on Q_{T^*} with initial data $u_0 \in \dot{B}^{s_p}_{p,\infty}(\mathbb{R}^3)$ and forcing term $\operatorname{div} F$ ($F \in \mathcal{F}_q(Q_{T^*})$). Finally, suppose that $\|u_0\|_{\dot{B}^{s_p}_{p,\infty}(\mathbb{R}^3)} \leq M$. There exists a constant $\varepsilon := \varepsilon(p, q, M) > 0$ with the following property. Namely, if (4.4.2) is satisfied, then there exists an $\widetilde{\varepsilon} \in]0, T^*[$ such that $v \in L_\infty(\mathbb{R}^3 \times]T^* - \widetilde{\varepsilon}, T^*[)$.

[21] Specifically, they prove that if a solution belongs to $L^\infty_t L^3_x$ then it is regular. See Theorems 1.3-1.4 in [49].

Before we prove Theorem 4.4.1, we state three preliminary tools. The proofs of Lemma 4.4.3 and Proposition 4.4.4 will be postponed to the end of the section. We omit the proof of Proposition 4.4.5, since it follows perturbation arguments similar to those in Proposition 4.3.22.

Lemma 4.4.3 (Boundedness for $|x| \gg 1$)**.** *Let $T > 0$ and $q \in]3, \infty[$. Let v be a weak Besov solution (based on the kth Picard iterate, $0 \leq k \in \mathbb{Z}$) on Q_T with initial data $u_0 \in \mathrm{BMO}^{-1}(\mathbb{R}^3)$ and forcing term* $\operatorname{div} F$ *($F \in \mathcal{F}_q(Q_T)$). There exists $R := R(v, k, T, q) > 0$ such that*

$$v \in L^\infty((\mathbb{R}^3 \setminus B(R)) \times]T/2, T[). \tag{4.4.6}$$

Moreover, if $F = 0$, we have that for all $0 \leq \alpha, \beta \in \mathbb{Z}$,

$$\partial_t^\alpha \nabla_x^\beta v \in L^\infty((\mathbb{R}^3 \setminus B(R)) \times]T/2, T[). \tag{4.4.7}$$

Proposition 4.4.4 (Backward uniqueness)**.** *Let $T > 0$ and v be a weak Besov solution on Q_T with initial data $u_0 \in \dot{B}^{s_p}_{p,\infty}(\mathbb{R}^3)$, where $p \in]3, \infty[$, and zero forcing term. Furthermore, assume that $v(\cdot, T) = 0$. Then $v \equiv 0$ on Q_T.*

Proposition 4.4.5 (Strong solutions with $u_0 \in L^\infty$)**.** *Let $0 < T \leq \infty$, $u_0 \in L^\infty(\mathbb{R}^3) \cap \dot{B}^{s_p}_{p,\infty}(\mathbb{R}^3)$ be a divergence-free vector field, and $F \in L^\infty_t L^q_x(Q_T)$ for some $p \in]3, \infty[$ and $q \in]3, p]$. Suppose that $v \in L^\infty(Q_T)$ is a mild solution of the Navier–Stokes equations on Q_T with initial data u_0 and forcing term* $\operatorname{div} F$*. Then v is a weak Besov solution on Q_T with the same initial data and forcing term.*

We now prove Theorem 4.4.1 by following the rescaling procedure and backward uniqueness arguments of Seregin in [124, 121], see also the subsequent paper [16]. In turn, those arguments are adapted from the seminal work of Escauriaza, Seregin, and Šverák in [49].

Proof of Theorem 4.4.1. 0. Singular points. Let us show that to prove Theorem 4.4.1, it is sufficient to investigate potential singularities of v. Let T^*, p, q, M, v, u_0, F be as in the statement of Theorem 4.4.1, and suppose that there exists $t_1 \in]0, T^*[$ such that $\|v(\cdot, t_1)\|_{\dot{B}^{s_p}_{p,\infty}(\mathbb{R}^3)} < \infty$. We claim that $v \in L^\infty(Q_{T^*})$ provided that v has no singular points at T^*. By Proposition 4.4.5, the mild solution v is also a global weak Besov solution on $\mathbb{R}^3 \times]t_1, T^*[$ with initial data $v(\cdot, t_1)$ and forcing term $\operatorname{div} F$. By Lemma 4.4.3, there exists an $R > 0$ such that

$$v \in L^\infty(B(R)^c \times]t_1 + (T^* - t_1)/2, T^*[), \tag{4.4.8}$$

which proves the claim.

1. Proof of (i). We first discuss a few simplifications. By Sobolev embedding for homogeneous Besov spaces, we may assume that $p \geq q$. Next, by the scaling symmetry, we may assume that $T^* = 1$. Finally, we make the following observation that allows us to assume that $t_1 = 0$ in our arguments below. For the moment, suppose that v is a mild solution on Q_1 with forcing term F, as in the statement of Theorem 4.4.1.(i). Then (4.4.2) is satisfied, and $\|v(\cdot,t_1)\|_{\dot{B}^{s_p}_{p,\infty}(\mathbb{R}^3)} \leq M$ for some $t_1 \in]0,1[$. Define $\lambda := \sqrt{1-t_1}$ and

$$\bar{v}(x,t) := \lambda v(\lambda x, t_1 + \lambda^2 t), \quad \bar{F}(x,t) := \lambda^2 F(\lambda x, t_1 + \lambda^2 t). \tag{4.4.9}$$

Then $\bar{v}$ is a mild solution on Q_1 with forcing term $\operatorname{div} \bar{F}$ also satisfying the hypotheses of Theorem 4.4.1.(i) with $t_1 = 0$ and $T^* = 1$. Indeed, one may verify that $\|\bar{F}\|_{\mathcal{F}_q(Q_1)} \leq \|F\|_{\mathcal{F}_q(Q_1)}$ and

$$\|\bar{v}(\cdot,0)\|_{\dot{B}^{s_p}_{p,\infty}(\mathbb{R}^3)} \leq M, \quad \|\bar{v}(\cdot,1)\|_{\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3)} + \|\bar{F}\|_{\mathcal{F}_q(Q_1)} \leq \varepsilon. \tag{4.4.10}$$

If v is singular at $(0,1)$, then so is $\bar{v}$.

For contradiction, suppose that Theorem 4.4.1.(i) is false. Then there exists a sequence $(v^{(n)})_{n\in\mathbb{N}}$ of vector fields on Q_1 with the following properties. First, for each $n \in \mathbb{N}$, $v^{(n)} \in L^\infty(Q_T)$ is a mild solution on Q_T with initial data $u_0^{(n)} \in L^\infty(\mathbb{R}^3)$ and forcing term $F^{(n)} \in \mathcal{F}_q(Q_1)$ for all $T \in]0,1[$. Second,

$$\sup_{n\in\mathbb{N}} \|u_0^{(n)}\|_{\dot{B}^{s_p}_{p,\infty}(\mathbb{R}^3)} \leq M, \tag{4.4.11}$$

so Proposition 4.4.5 ensures that $v^{(n)}$ is a weak Besov solution on Q_1. Third,

$$\lim_{n\uparrow\infty} [\|v^{(n)}(\cdot,1)\|_{\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3)} + \|F^{(n)}\|_{\mathcal{F}_q(Q_1)}] = 0. \tag{4.4.12}$$

Finally, $v^{(n)}$ is singular at $(x^{(n)},1)$ for some $x^{(n)} \in \mathbb{R}^3$ which by the translation symmetry we may assume to be the origin.

By Proposition 4.3.14 concerning weak-$*$ stability, there exists a subsequence of $(v^{(n)})_{n\in\mathbb{N}}$ that converges to a weak Besov solution $\widetilde{v}$ on Q_1 with initial data $\widetilde{u}_0 \in \dot{B}^{s_p}_{p,\infty}(\mathbb{R}^3)$ and zero forcing term. Specifically,

$$u_0^{(n)} \overset{*}{\rightharpoonup} \widetilde{u}_0 \text{ in } \dot{B}^{s_p}_{p,\infty}(\mathbb{R}^3), \tag{4.4.13}$$

$$v^{(n)} \overset{*}{\rightharpoonup} \widetilde{v} \text{ in } (L_t^\infty L_x^2)_{\mathrm{loc}}(\overline{Q}_{\frac{1}{2},1}), \quad \nabla v^{(n)} \rightharpoonup \nabla \widetilde{v} \text{ in } L^2_{\mathrm{loc}}(\overline{Q}_{\frac{1}{2},1}). \tag{4.4.14}$$

$$v^{(n)} \to \widetilde{v} \text{ in } L^3_{\mathrm{loc}}(\overline{Q}_{\frac{1}{2},1}), \quad q^{(n)} \rightharpoonup \widetilde{q} \text{ in } L^{\frac{3}{2}}_{\mathrm{loc}}(\overline{Q}_{\frac{1}{2},1}), \tag{4.4.15}$$

$$v^{(n)}(\cdot,1) \overset{*}{\rightharpoonup} \widetilde{v}(\cdot,1) \text{ in } \mathcal{D}'(\mathbb{R}^3), \tag{4.4.16}$$

where $q^{(n)}, \widetilde{q}$ denotes the pressure associated to $v^{(n)}, \widetilde{v}$, respectively. According to Lemma 4.6.4 in the appendix, $\widetilde{v}$ also has a singular point at $(0,1)$. Furthermore, (4.4.12) and (4.4.16) imply that $\widetilde{v}(\cdot,1) = 0$. By Proposition 4.4.4, $\widetilde{v} \equiv 0$ on Q_1, which contradicts that $\widetilde{v}$ is singular. This completes the proof.

2. *Proof of (ii)* For contradiction, suppose that Theorem 4.4.1.(ii) is false. In particular, there exist T^*, p, q, M, v, u_0, F, as in the statement of Theorem 4.4.1, satisfying (4.4.3)–(4.4.4), where $\varepsilon := \varepsilon(p,q,M) > 0$ is the constant in Remark 4.4.2, and such that v is singular at (x^*, T^*) for some $x^* \in \mathbb{R}^3$. As in Step 1, we may assume that $p \geq q$, $x^* = 0$, and $T^* = 1$.

We now zoom in the singularity to obtain a contradiction. For each $n \in \mathbb{N}$, we define $\lambda_n := (1-t_n)^{\frac{1}{2}}$, and for a.e. $(x,t) \in Q_1$,

$$v^{(n)}(x,t) := \lambda_n v(\lambda_n x, t_n + \lambda_n^2 t), \tag{4.4.17}$$

$$F^{(n)}(x,t) := \lambda_n^2 F(\lambda_n x, t_n + \lambda_n^2 t). \tag{4.4.18}$$

Proposition 4.4.5 and (4.4.3) imply that $v^{(n)}$ is a weak Besov solution on Q_1 with initial data $u_0^{(n)} := \lambda_n v(\lambda_n \cdot, t_n)$ and forcing term $\operatorname{div} F^{(n)}$. Furthermore,

$$\sup_{n\in\mathbb{N}} \|u_0^{(n)}\|_{\dot{B}^{s_p}_{p,\infty}(\mathbb{R}^3)} \leq M, \quad \lim_{n\uparrow\infty} \|F^{(n)}\|_{\mathcal{F}_q(Q_1)} = 0. \tag{4.4.19}$$

Each velocity field $v^{(n)}$ is singular at $(0,1)$. By Proposition 4.3.14 regarding weak-$*$ stability, there exists a divergence-free vector field $\widetilde{u_0} \in \dot{B}^{s_p}_{p,\infty}(\mathbb{R}^3)$ and a subsequence of $(v^{(n)})_{n\in\mathbb{N}}$ converging to a weak Besov solution $\widetilde{v}$ on Q_1 with initial data $\widetilde{u}_0$, see (4.4.13)–(4.4.16) in Step 1. Due to Lemma 4.6.4 in the appendix, $\widetilde{v}$ is singular at $(0,1)$. On the other hand, by (4.4.16) and the assumption (4.4.4), there exists $\Psi \in \dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3)$ with $\|\Psi\|_{\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3)} \leq \varepsilon$ and

$$v^{(n)}(\cdot,1) = \lambda_n v(\lambda_n \cdot, 1) \overset{*}{\rightharpoonup} \Psi = \widetilde{v}(\cdot,1) \text{ in } \mathcal{D}'(\mathbb{R}^3). \tag{4.4.20}$$

Since also $\|\widetilde{u}_0\|_{\dot{B}^{s_p}_{p,\infty}(\mathbb{R}^3)} \leq M$, Remark 4.4.2 implies that $\widetilde{v}$ is regular at $(0,1)$. This is the desired contradiction. The proof is complete. $\square$

We now prove the auxiliary results Lemma 4.4.3 and Proposition 4.4.4. Let $Q_{S,T} := \mathbb{R}^3\times]S,T[$ when $0 < S < T \leq \infty$.

Proof of Lemma 4.4.3. Using the scale-invariance of the Navier–Stokes equations, we may assume without loss of generality that $T = 1$. We will use the ε-regularity criterion for suitable weak solutions to control the equation near spatial infinity, see Proposition 4.6.2.

For $z = (x,t) \in Q_{1/2,1}$, $r \in]0,1/2[$, $R_0 > 1/2$, and $|x| \geq R_0$, we have that

$$\begin{aligned} \frac{1}{r^2}\int_{Q(z,r)} |v|^3\,dx'\,dt &\leq \frac{c}{r^2}\int_{Q(z,r)} |u|^3\,dx'\,dt + \frac{c}{r^2}\int_{Q(z,r)} |P_k|^3\,dx'\,dt \\ &\leq \frac{c}{r^2}\int_{\frac14}^{1}\int_{|x|\geq R_0-\frac12} |u|^3\,dx'\,dt + cr^3\|P_k\|^3_{L^\infty(Q_{1/4,1})}. \end{aligned} \tag{4.4.21}$$

Here, $Q(z,r) := B(x,r)\times]t-r^2,t[$ denotes a parabolic ball. Fix $r_0 := r_0(\|P_k\|_{L^\infty(Q_{1/4,1})}, \varepsilon_{\mathrm{CKN}}) > 0$ satisfying

$$cr_0^3\|P_k\|^3_{L^\infty(Q_{1/4,1})} \leq \frac{\varepsilon_{\mathrm{CKN}}}{8}. \tag{4.4.22}$$

Since v is a weak Besov solution on Q_1, we have that $u \in L^\infty_t L^2_x \cap L^2_t \dot{H}^1_x(Q_1)$. This implies $u \in L^3(Q_1)$. Hence, there exists $R_0 := R_0(u, r_0, \epsilon_0) > 1/2$ such that

$$\frac{c}{r_0^2}\int_{\frac14}^{1}\int_{|x|\geq R_0-\frac12} |u|^3\,dx'\,dt \leq \frac{\varepsilon_{\mathrm{CKN}}}{8}. \tag{4.4.23}$$

Hence, for $z = (x,t) \in Q_{1/2,1}$ and $|x| \geq R_0$, we have that

$$\frac{1}{r_0^2}\int_{Q(z,r_0)} |v|^3\,dx'\,dt \leq \frac{\varepsilon_{\mathrm{CKN}}}{4}. \tag{4.4.24}$$

Similarly, after possibly adjusting r_0 and R_0, one may obtain that for $z = (x,t) \in Q_{1/2,1}$ and

$|x| \geq R_0$,

$$
\begin{aligned}
&\frac{1}{r_0^2}\int_{Q(z,r_0)} |q - [q]_{x,r_0}(t)|^{\frac{3}{2}}\, dx' dt' \\
&\quad \leq \frac{c}{r_0^2}\int_{Q(z,r_0)} |p|^{\frac{3}{2}}\, dx'\, dt' + \frac{c}{r_0^2}\int_{Q(z,r_0)} |\pi_k - [\pi_k]_{x,r_0}(t)|^{\frac{3}{2}}\, dx'\, dt' \\
&\quad \leq \frac{c}{r_0^2}\int_{\frac{1}{4}}^{1}\int_{|x'|\geq R_0-\frac{1}{2}} |p_1|^{\frac{3}{2}}\, dx'\, dt' + \frac{c}{r_0^{\frac{3}{4}}}\Big(\int_{\frac{1}{4}}^{1}\int_{|x'|\geq R_0-\frac{1}{2}} |p_2|^2\, dx'\, dt'\Big)^{\frac{3}{4}} \\
&\qquad + c r_0^3 \|\pi_k\|^{\frac{3}{2}}_{L_t^\infty \mathrm{BMO}_x(Q_{1/4,1})} \\
&\quad \leq \frac{\varepsilon_{\mathrm{CKN}}}{4},
\end{aligned}
\tag{4.4.25}
$$

where $[q]_{x,r}(t') := |B(x,r)|^{-1}\int_{B(x,r)} q(x',t')\, dx'$. In (4.4.25), we have used the fact that $p \in L^{\frac{3}{2}}(Q_1) + L^2_{t,\mathrm{loc}}L^2_x(Q_1)$ (see the proof of Remark 4.3.1).

Clearly, there exists $\widetilde{q} > 1$ such that

$$
F \in L^{\widetilde{q}}_{t,\mathrm{loc}}L^q_x(Q_1) \ \text{ with } \ \frac{2}{\widetilde{q}} + \frac{3}{q} = 2 - \delta \ \text{ and } \ \delta > 0. \tag{4.4.26}
$$

Since $\widetilde{q}$ and q are finite, we may adjust R_0 to obtain the following for $z = (x,t) \in Q_{1/2,1}$ and $|x| \geq R_0$. Namely,

$$
r_0^\delta \|F\|_{L_t^{\widetilde{q}}L_x^q(Q(z,r_0))} \leq r_0^\delta \|F\|_{L_t^{\widetilde{q}}L_x^q(\mathbb{R}^3\setminus B(R_0-1/2)\times]1/4,1[)} \leq \frac{\varepsilon_{\mathrm{CKN}}}{2}. \tag{4.4.27}
$$

Using Proposition 4.6.2, (4.4.24), (4.4.25), and (4.4.27) gives the desired conclusion. $\square$

Proof of Proposition 4.4.4. 0. Properties of v. It is sufficient to show that $v \equiv 0$ in $\mathbb{R}^3\times]T/2,T[$. A repeated application then gives $v \equiv 0$ on Q_T.

By rescaling the problem, we may assume that $T = 1$. From Definition 4.1.5, there exists an integer $k \geq 0$ and $u \in L_t^\infty L_x^2 \cap L_t^2 \dot{H}_x^1(Q_1)$ such that

$$
v = P_k(u_0) + u \tag{4.4.28}
$$

and satisfies certain additional properties, including the local energy inequality (4.1.22). Observe that $P_k(u_0) \in L_t^\infty L_x^p(\mathbb{R}^3\times]\delta,1[)$ and the associated pressure $\pi_k := (-\Delta)^{-1}\operatorname{div}\operatorname{div} P_{k-1}\otimes P_{k-1} \in L_t^\infty L_x^{\frac{p}{2}}(\mathbb{R}^3\times]\delta,1[)$ for all $\delta \in]0,1[$. Also, $u \in L^3(Q_1)$ and $p \in L^{\frac{3}{2}}(Q_1)$. Hence, the velocity field satisfies

$$v \in L_t^\infty L_x^p(\mathbb{R}^3 \times]\delta, 1[) + L^3(\mathbb{R}^3 \times]\delta, 1[), \quad \delta \in]0, 1[. \tag{4.4.29}$$

Let $\omega := \operatorname{curl} v$ denote the vorticity.

1. Suffices to prove $\omega \equiv 0$. To complete the proof, it is sufficient to prove that $\omega \equiv 0$ on $Q_{\frac{1}{2},1} := \mathbb{R}^3 \times]1/2, 1[$. In such case, the velocity field v is harmonic, due to the well-known identity $\Delta = \nabla \operatorname{div} - \operatorname{curl} \operatorname{curl}$ for the vector Laplacian. Then $\Delta v(\cdot, t) = 0$ while $v(\cdot, t) \in L^p(\mathbb{R}^3) + L^3(\mathbb{R}^3)$ for almost every $t \in]1/2, 1[$. Finally, the Liouville theorem for entire harmonic functions implies that $v \equiv 0$ on $Q_{\frac{1}{2},1}$ and finishes the proof.

2. Backward uniqueness: $\omega \equiv 0$ *near spatial infinity*. From Lemma 4.4.3, there exists $R := R(v, P_k(u_0)) > 0$ such that for $K = B(R)$, we have $\nabla^{\ell-1} v \in L^\infty(K^c \times]1/2, 1[)$ and $\|\nabla^{\ell-1} v\|_{L^\infty((K^c \times]1/2,1[)} \leq C(r_0, \ell)$ for all $\ell \in \mathbb{N}$. Now recall that the vorticity satisfies the equation

$$\partial_t \omega - \Delta \omega = -\operatorname{curl}(u \cdot \nabla u) = \omega \cdot \nabla u - u \cdot \nabla \omega, \tag{4.4.30}$$

from which we obtain that $\partial_t \omega \in L^\infty(K^c \times]1/2, 1[)$, and

$$|\partial_t \omega - \Delta \omega| \leq c(|\omega| + |\nabla \omega|) \text{ on } K^c \times]1/2, 1[. \tag{4.4.31}$$

Moreover, $\omega(\cdot, 1) = 0$. From Theorem 5.1 in [49] concerning backward uniqueness for the differential inequality (4.4.31), we obtain that $\omega \equiv 0$ on $K^c \times]1/2, 1[$.

3. Unique continuation: $\omega \equiv 0$ *near the spatial origin*. The proof will be complete once we demonstrate that $w \equiv 0$ in $K \times]1/2, 1[$. For the moment, let us take for granted the following claim that we prove in Step 4:

> *Claim*: There exists an open set $G \subset]0, 1[$ such that $\overline{G} = [0, 1]$ and v is smooth on $\mathbb{R}^3 \times G$.

With the claim in hand, let $t_1 \in G \cap]1/2, 1[$ and $x_0 \in K^c$. Let $t_0 \in \mathbb{R}$ be such that $[t_0, t_1] \subset G$. From the smoothness of v, we have that $\omega, \partial_t \omega, \nabla^2 \omega \in L^2(B(x_0, 2R) \times [t_0, t_1])$ for any $R > 0$, and

$$|\partial_t \omega - \Delta \omega| \leq c(|\omega| + |\nabla \omega|) \text{ on } B(x_0, 2R) \times]t_0, t_1[. \tag{4.4.32}$$

In addition, recall that $\omega \equiv 0$ in a neighborhood of (x_0, t_1). Hence, by Theorem 4.1 in [49] concerning unique continuation across spatial boundaries, $\omega \equiv 0$ in $B(x_0, R) \times \{t_1\}$. Since

$t_1 \in G\cap]1/2,1[$ and $R>0$ are arbitrary, we have that $\omega \equiv 0$ in $\mathbb{R}^3 \times (G\cap]1/2,1[)$. Moreover, by the density of G and weak-$*$ continuity of $\omega(\cdot,t)$ on $[0,1]$ in the sense of distributions on $\mathbb{R}^3$, we obtain that $\omega \equiv 0$ on $Q_{\frac{1}{2},1}$, as desired. (Another way to complete Step 3 is to use the spatial analyticity of smooth solutions of the Navier–Stokes equations.)

4. Showing v is smooth at generic times. We will now prove the claim from Step 3. Let Π denote the set of all $t_0 \in]0,1[$ such that $u(\cdot,t) \in H^1(\mathbb{R}^3)$ and the global energy inequality (4.3.8) is satisfied with initial time t_0. The second condition ensures that

$$\lim_{t\downarrow t_0} \|u(\cdot,t) - u(\cdot,t_0)\|_{L^2(\mathbb{R}^3)} = 0. \tag{4.4.33}$$

Notice that $|\Pi| = 1$, and in particular, $\overline{\Pi} = [0,1]$. We will prove that for each $t_0 \in \Pi$, there exists $t_1 := t_1(t_0) \in]t_0,1[$ such that v is smooth on $]t_0,t_1[$. Then $G := \cup_{t_0\in\Pi}]t_0, t_1(t_0)[$ will satisfy the desired properties. From the above, we see that u is a weak Leray-Hopf solution on $\mathbb{R}^3 \times]t_0,t_1[$, with initial data $u(\cdot,t_0) \in H^1(\mathbb{R}^3)$ and forcing term

$$f := -P_k \cdot \nabla u - u \cdot \nabla P_k - F_k. \tag{4.4.34}$$

One can show that f belongs to $L^2(\mathbb{R}^3 \times]t_0,1[)$ for all $k \geq k(p)$, where $k(p) = \lceil \frac{p}{2} \rceil - 2$. By unique solvability results for weak Leray-Hopf solutions,[22] we can conclude the following. Namely, we can find $t_1 := t_1(t_0,u,f) > 0$ such that

$$u, \nabla u \in L^\infty_t L^2_x(\mathbb{R}^2 \times]t_0,t_1[) \text{ and } u \in L^\infty_t L^6_x(\mathbb{R}^3 \times]t_0,t_1[). \tag{4.4.35}$$

Hence,

$$v \in L^\infty(\mathbb{R}^3 \times]t_0,t_1[) + L^\infty_t L^6_x(\mathbb{R}^3 \times]t_0,t_1[), \tag{4.4.36}$$

$$\nabla v \in L^\infty(\mathbb{R}^3 \times]t_0,t_1[) + L^2(\mathbb{R}^3 \times]t_0,t_1[). \tag{4.4.37}$$

Using known arguments due to Serrin [128], we deduce that

$$\nabla^\ell v \in L^\infty(\mathbb{R}^3 \times]t_0+\varepsilon,t_1[) \tag{4.4.38}$$

for all $0 < \varepsilon < t_1 - t_0$ and all $0 \leq \ell \in \mathbb{Z}$. Using known arguments (see Proposition 3.9, p.

[22] See Heywood's paper [67, Theorem 2'] and Sohr's book [129, Theorem 1.5.1, p. 276], for example.

160-162 of Seregin's book [126], for example), we can now show that

$$\partial_t^k \nabla^\ell v \in L^\infty(\mathbb{R}^3 \times]t_0+\varepsilon, t_1[) \tag{4.4.39}$$

for all $0<\varepsilon<t_1-t_0$ and all $0\leq k,\ell\in\mathbb{Z}$. □

4.4.2 Minimal blow-up initial data

As discussed in Section 4.1.1, global weak Besov solutions provide a convenient framework for investigating minimal blow-up problems, even when local-in-time mild solutions are no longer guaranteed to exist.

Let $\mathcal{X}$ be a critical space which continuously embeds into $\dot{B}^{-1+\frac{3}{p}}_{p,\infty}(\mathbb{R}^3)$ for some $p\in]3,\infty[$. Here, we are using the notion of critical space in Definition 4.1.10. For each $u_0\in\mathcal{X}$, we define

> $\rho_{\mathcal{X}}^{u_0} := \sup(\{0\}\cup\{\rho>0$: for all $a\in\mathcal{X}$ satisfying $\|a-u_0\|_{\mathcal{X}}\leq\rho$, any global weak Besov solution with initial condition a has no singular points$\})$.

We also define $\rho_{\mathcal{X}} := \rho_{\mathcal{X}}^0$ as in Section 4.1.1.[23]

Remark 4.4.6. If $\rho_{\mathcal{X}}<\infty$, the quantity $\rho_X^{u_0}$ may be zero (for example, when u_0 is initial data for a singular global weak Besov solution). It is guaranteed to be non-zero when additionally $u_0\in\mathrm{VMO}^{-1}(\mathbb{R}^3)$ and there exists a global mild solution $u\in\mathring{\mathcal{X}}_\infty$ with initial data u_0. In this scenario, small perturbations of u_0 also give rise to global mild solutions, see Proposition 4.3.22. For example, Theorem 4.1.3 implies that $\rho_{\mathcal{X}}>0$.

Here is our main theorem concerning minimal blow-up perturbations of global solutions, which extends Rusin's treatment in [117] for $\dot{H}^{\frac{1}{2}}$ initial data.[24]

Theorem 4.4.7 (Minimal blow-up perturbations)**.** *Let $\mathcal{X}$ be a critical space which embeds into $\dot{B}^{-1+\frac{3}{p}}_{p,\infty}(\mathbb{R}^3)$ for some $p\in]3,\infty[$. Suppose that $u_0\in\mathcal{X}$ satisfies the following property:*

> *If $(x_n,t_n)_{n\in\mathbb{N}}\subset Q_\infty$ is a sequence such that $t_n\to\infty$, $t_n\to 0$, or $|x_n|\to\infty$, then*

$$\sqrt{t_n}u_0(\sqrt{t_n}(\cdot+x_n)) \overset{*}{\rightharpoonup} 0 \textit{ in } \mathcal{D}'(\mathbb{R}^3). \tag{4.4.40}$$

[23] It is also possible to prove minimal blow-up results with non-zero forcing terms, but the setup is not as convenient owing to the fact that many natural spaces of forcing terms do not embed into each other.

[24] Rusin's paper [117] is based on profile decomposition. For minimal blow-up problems, the profile decomposition approach appears to be effective in all dimensions, whereas ours is restricted to dimension ≤ 3. The reason is that the existence theory and stability properties of suitable weak solutions are currently unknown in dimension ≥ 4.

Suppose that $\rho_{\mathcal{X}} < \infty$. Then (at least) one of the following holds:

(i) There exists a singular global weak Besov solution v with initial data $a \in \mathcal{X}$ such that $\|u_0 - a\|_{\mathcal{X}} = \rho_{\mathcal{X}}^{u_0}$.

(ii) There exists a singular global weak Besov solution v with initial data $a \in \mathcal{X}$ such that $\|a\|_{\mathcal{X}} \leq \rho_{\mathcal{X}}^{u_0}$. Hence, $\rho_{\mathcal{X}} \leq \rho_{\mathcal{X}}^{u_0}$.

Moreover,

(i') If (ii) does not hold, then there exists a compact set $K \subset Q_\infty$ and $\varepsilon_0 > 0$ such that for all $\varepsilon \in]0, \varepsilon_0[$, every singular global weak Besov solution with initial data $a \in \mathcal{X}$ and $\|a - u_0\|_{\mathcal{X}} < \rho_{\mathcal{X}}^{u_0} + \varepsilon$ has all its singularities in K. In this case, the set $\{a - u_0 : a \in \mathcal{X}$ satisfies (i)$\}$ is sequentially compact in $\mathcal{X}$ in the topology of distributions on $\mathbb{R}^3$.

(ii') If (i) does not hold, then for every compact set $K \subset Q_\infty$, there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in]0, \varepsilon_0[$, every singular global weak Besov solution with initial data $a \in \mathcal{X}$ and satisfying $\|a - u_0\|_{\mathcal{X}} < \rho_{\mathcal{X}}^{u_0} + \varepsilon$ has all its singularities outside K.

Proof. In order to prove the above theorem, we utilise the weak-$*$ stability properties of global Besov solutions, along with arguments related to those contained in [118] and [117].

Assume the hypotheses of the theorem. Suppose $(u_0^{(n)})_{n\in\mathbb{N}} \subset \mathcal{X}$ and $(v^{(n)})_{n\in\mathbb{N}}$ is an associated sequence of global weak Besov solutions such that

$$\|u_0^{(n)} - u_0\|_{\mathcal{X}} \downarrow \rho_{\mathcal{X}}^{u_0} \tag{4.4.41}$$

and for each $n \in \mathbb{N}$, $v^{(n)}$ has a singular point $(x_n, t_n) \in Q_\infty$.

Let us consider the following two mutually exclusive cases (which also exhaust all possible cases).

Case I: Suppose the sequence of singular points $(x_n, t_n)_{n\in\mathbb{N}}$ has an accumulation point $(x', t') \in Q_\infty$. By passing to a subsequence[25], we may assume that $u_0^{(n)} \overset{*}{\rightharpoonup} a$ in $\dot{B}^{s_p}_{p,\infty}(\mathbb{R}^3)$ for some $p \in]3, \infty[$, the limit a belongs to $\mathcal{X}$ with norm $\|a - u_0\|_{\mathcal{X}} \leq \rho_{\mathcal{X}}^{u_0}$, and $(v^{(n)})_{n\in\mathbb{N}}$ converge in the sense described in Proposition 4.3.14 to a global weak Besov solution v with initial data a. By Lemma 4.6.4, v has a singularity at (x', t'). According to the definition of $\rho_X^{u_0}$, we must have $\|a - u_0\|_{\mathcal{X}} = \rho_{\mathcal{X}}^{u_0}$, which verifies (i).

[25]In this proof, we will not alter our notation when passing to a subsequence.

Case II: Suppose the sequence of singular points $(x_n, t_n)_{n\in\mathbb{N}}$ has no accumulation point in Q_∞. Then there exists a subsequence such that $t_n \to 0$, $t_n \to \infty$, or $|x_n| \to \infty$. We define a sequence of singular global weak Besov solutions $(\widetilde{v}^{(n)})_{n\in\mathbb{N}}$ associated to a sequence of initial data $(\widetilde{u_0}^{(n)})_{n\in\mathbb{N}}$ by the following translation and rescaling:

$$\widetilde{v}^{(n)}(x,t) := \sqrt{t_n} v(\sqrt{t_n}(x + x_n), t_n t), \tag{4.4.42}$$

$$\widetilde{u}_0^{(n)}(x) := \sqrt{t_n} u_0^{(n)}(\sqrt{t_n}(x + x_n)). \tag{4.4.43}$$

The solutions $\widetilde{v}^{(n)}$ have singularities at the spatial origin and time $T = 1$. By passing to a further subsequence, we may assume that $\widetilde{u}_0^{(n)} \overset{*}{\rightharpoonup} a$ in $\dot{B}^{s_p}_{p,\infty}(\mathbb{R}^3)$ for some $p \in]3,\infty[$ and $a \in \mathcal{X}$ and that $(\widetilde{v}^{(n)})_{n\in\mathbb{N}}$ converges to a singular global weak Besov solution with singularity at $(x', t') = (0, 1)$. Furthermore, by the assumption on u_0 in the statement of Theorem 4.4.7, we must have

$$\widetilde{u_0}^{(n)} - \sqrt{t_n} u_0(\sqrt{t_n}(\cdot + x_n)) \overset{*}{\rightharpoonup} a \text{ in } \mathcal{D}'(\mathbb{R}^3), \tag{4.4.44}$$

so that a satisfies $\|a\|_{\mathcal{X}} \leq \rho_{\mathcal{X}}^{u_0}$. This verifies (ii).

The proof is completed by noting that if (i) does not hold, then Case I cannot occur for any minimizing sequence of initial data, and similarly, if (ii) does not hold, then Case II cannot occur. $\square$

Corollary 4.1.11 corresponds to the case $u_0 = 0$.

Remark 4.4.8 (Interpretation)**.** Suppose that Case II occurs and consider the behavior of the sequence $(x_n, t_n)_{n\in\mathbb{N}}$. One might interpret the situation

$$|x_n| \to \infty \text{ and } \inf_{n\in\mathbb{N}} t_n > \varepsilon \tag{4.4.45}$$

as meaning that u_0 has certain nice properties which cause the singularities to disappear at spatial infinity as the initial data approaches the sphere of radius $\rho_{\mathcal{X}}^{u_0}$ centered on u_0, and similarly for $t_n \to \infty$. Since $\rho_{\mathcal{X}} \leq \rho_{\mathcal{X}}^{u_0}$, one is tempted to say that, in terms of its ability to "prevent" the blow-up of nearby solutions, u_0 is at least as good as zero initial data. The case $t_n \to 0$ is perhaps not as clear. It is tempting to interpret the occurrence of singularities very close to the initial time as ill-posedness, but this conflicts with the idea that u_0 is at least as good as zero. If u_0 is "singular," as is the case for -1–homogeneous initial data giving rise to a self-similar solution, then the case $t_n \to 0$ may not be surprising.[26]

4.4.3 Forward self-similar solutions

Finally, we will prove Theorem 4.1.12. As mentioned in Section 4.1.1, we will obtain self-similar solutions evolving from rough initial data as limits of self-similar solutions evolving from $L^{3,\infty}$ initial data. The existence of such solutions was established in [26] by Galerkin approximation:

Proposition 4.4.9. ([26, Theorems 1.2–1.3]) *The conclusions of Theorem 4.1.12 are valid under the additional assumption that* $u_0 \in L^{3,\infty}(\mathbb{R}^3)$.

While the results in [26] are stated for local Leray solutions v satisfying the additional property that $\|v(\cdot,t) - S(t)u_0\|_{L^2(\mathbb{R}^3)} \leq Ct^{\frac{1}{4}}$ for all $t > 0$, it is clear from their construction that $u := v - Su_0$ belongs to the energy class. This fact, combined with the local energy inequality (4.1.22) satisfied by local Leray solutions, implies that the (discretely) self-similar solutions constructed in [26] are global weak Besov solutions.

Next, we require the following approximation lemma proven in [28].[27]

Lemma 4.4.10. ([28, Lemmas 2.2 and 5.2]) *Assume the hypotheses of Theorem 4.1.12. If* u_0 *is* λ*-DSS, then there exists a sequence* $(u_0^{(n)})_{n\in\mathbb{N}} \subset L^{3,\infty}(\mathbb{R}^3)$ *of divergence-free* λ*-DSS vector fields such that* $u_0^{(n)} \to u_0$ *in* $\dot{B}^{s_p}_{p,\infty}(\mathbb{R}^3)$. *If* u_0 *is scale-invariant, then there exists a sequence* $(u_0^{(n)})_{n\in\mathbb{N}} \subset L^{3,\infty}(\mathbb{R}^3)$ *of scale-invariant divergence-free vector fields such that* $u_0^{(n)} \to u_0$ *in* $\dot{B}^{s_p}_{p,\infty}(\mathbb{R}^3)$.

With these useful facts in hand, we now prove Theorem 4.1.12.

Proof of Theorem 4.1.12. Let u_0 be the λ-DSS (resp. scale-invariant) initial data from the statement of Theorem 4.1.12. According to Lemma 4.4.10, there exists a sequence $(u_0^{(n)})_{n\in\mathbb{N}} \subset L^{3,\infty}(\mathbb{R}^3)$ of divergence-free λ-DSS (resp. scale-invariant) vector fields such that $u_0^{(n)} \to u_0$ in $\dot{B}^{s_p}_{p,\infty}(\mathbb{R}^3)$. By Proposition 4.4.9, given such a sequence, there exist λ-DSS (resp. scale-invariant) global weak Besov solutions $v^{(n)}$, $n \in \mathbb{N}$, with initial data $u_0^{(n)}$. Proposition 4.3.14 allows us to extract a subsequence of $(v^{(n)})_{n\in\mathbb{N}}$ converging in $L^3_{\mathrm{loc}}(\mathbb{R}^3 \times \mathbb{R}_+)$ to a global weak

[26] It is interesting to note that scale-invariant solutions in quite general spaces are smooth regardless of the size of their initial data, see [72].

[27] One may also approximate by (discretely) self-similar vector fields that are smooth away from the origin.

Besov solution v with initial data u_0.[28] Since the approximating solutions $v^{(n)}$ are λ-DSS (resp. scale-invariant), the limit solution v is λ-DSS (resp. scale-invariant) as well. This completes the proof. $\square$

4.5 Appendix: Splitting lemmas

In this appendix, we prove several splitting lemmas, including Lemma 4.1.7 from the introduction.

To illustrate the key points, we consider the following simple situation. Let $1 \leq p_0 \leq p \leq p_1 \leq \infty$. For each measurable function $f \colon \mathbb{R} \to \mathbb{R}$ with $\|f\|_{L^p} < \infty$ and $N > 0$, we may write $f = f_+^N + f_-^N$, where

$$f_+^N := f\chi_{\{|f|>N\|f\|_{L^p}\}}, \quad f_-^N := f\chi_{\{|f|\leq N\|f\|_{L^p}\}}. \tag{4.5.1}$$

Then, by elementary arguments,

$$\|f_+^N\|_{L^{p_0}} \leq N^{1-\frac{p}{p_0}}\|f\|_{L^p}, \quad \|f_-^N\|_{L^{p_1}}^{p_1} \leq N^{1-\frac{p}{p_1}}\|f\|_{L^p}. \tag{4.5.2}$$

This splitting has the desirable property that it is "uniform," in the sense that $\|f_+^N\|_{L^{p_0}}$ can be made small without making $\|f_-^N\|_{L^{p_1}}$ too large, and vice versa.[29] Moreover, it satisfies the obvious estimate

$$\|f_+^N\|_{L^p}, \|f_-^N\|_{L^p} \leq \|f\|_{L^p}, \tag{4.5.5}$$

as can be seen from taking $p_0 = p_1 = p$ in (4.5.2). This is known as the "persistency property."

It is well known that uniform splittings such as (4.5.2) can be readily obtained from abstract interpolation theory. Our main reference is [22, Chapters 3-4]. Let A_0, A_1 be Banach spaces

[28]Note that weak-$*$ convergence $u_0^{(n)} \overset{*}{\rightharpoonup} u_0$ in $\dot{B}^{s_p}_{p,\infty}(\mathbb{R}^3)$ would be sufficient to apply Proposition 4.3.14.

[29]This is also a "non-dimensionalized" splitting. If one uses

$$f_+^N := f\chi_{\{|f|>N\}}, \quad f_-^L := f\chi_{\{|f|\leq N\}}, \tag{4.5.3}$$

instead, then N has the same dimensions as f, and when $p_1 < \infty$,

$$\|f_+^L\|_{L^{p_0}}^{p_0} \leq N^{p_0-p}\|f\|_{L^p}^p, \quad \|f_-^L\|_{L^{p_1}}^{p_1} \leq N^{p_1-p}\|f\|_{L^p}^p. \tag{4.5.4}$$

embedded in a Hausdorff topological vector space U. The K*-functional* is defined by

$$K(t,a) = \inf_{a=a_0+a_1} \|a_0\|_{A_0} + t\|a_1\|_{A_1}, \quad (t,a) \in \mathbb{R}_+ \times (A_0 + A_1), \tag{4.5.6}$$

where a_0, a_1 are required to belong to A_0, A_1, respectively. The function $t \mapsto K(t,a)$ is continuous. For $0 < \theta < 1$ and $1 \leq q \leq \infty$, $K_{\theta,q}$ is defined as the Banach space consisting of all $a \in \Sigma$ satisfying

$$\|a\|_{K_{\theta,q}} := \left\| t^{-\theta} K(t,a) \right\|_{L^q(\mathbb{R}_+, \frac{dt}{t})} < \infty. \tag{4.5.7}$$

By definition, for each $\varepsilon > 0$, $a \in K_{\theta,\infty}$, and $N > 0$, there exist $a_0 \in A_0$ and $a_1 \in A_1$ such that

$$\|a_0\|_{A_0} \leq N^{-\theta}\big(\|a\|_{K_{\theta,\infty}} + \varepsilon\big), \quad \|a_1\|_{A_1} \leq N^{1-\theta}\big(\|a\|_{K_{\theta,\infty}} + \varepsilon\big). \tag{4.5.8}$$

This uniform splitting property is analogous to (4.5.2). Furthermore, every Banach space $\widetilde{A} \subset A_0 + A_1$ satisfying the uniform splitting property (4.5.2) with $\|a\|_{\widetilde{A}}$ in place of $\|a\|_{K_{\theta,\infty}}$ must embed continuously into $K_{\theta,\infty}$.[30] For example, the spaces $[A_0, A_1]_\theta$ obtained from the complex interpolation method embed continuously into $K_{\theta,\infty}$.

In the sequel, *we are interested in splittings of homogeneous Besov spaces that are uniform and satisfy the persistency property*. Since the persistency property does not appear to obviously follow from the abstract real interpolation theory, our approach will be to construct such splittings explicitly.

To begin, we present the homogeneous Besov spaces as spaces of distributions modulo polynomials.

Let $d, m \in \mathbb{N}$. Let $\mathcal{S}'$ denote the space of tempered distributions on $\mathbb{R}^d$ with values in $\mathbb{R}^m$. Let $\mathcal{P} \subset \mathcal{S}'$ denote the closed subspace consisting of polynomials on $\mathbb{R}^d$ with values in $\mathbb{R}^m$. Then $\mathcal{S}'/\mathcal{P}$ denotes the space of tempered distributions *modulo polynomials* on $\mathbb{R}^d$ with values in $\mathbb{R}^m$.

Recall the operators $\dot{\Delta}_j$, $j \in \mathbb{Z}$, defined in Section 4.2.1.

For $s \in \mathbb{R}$ and $0 < p, q \leq \infty$, we define the homogeneous Besov space $\dot{B}^s_{p,q}$ as the space of

[30] In contrast, $J_{\theta,1}$ must continuously embed into every Banach space $\widetilde{A} \supset A_0 \cap A_1$ satisfying

$$\|a\|_{\widetilde{A}} \leq C\|a_0\|_{A_0}^{1-\theta}\|a_1\|_{A_1}^{\theta} \tag{4.5.9}$$

for a constant $C > 0$ independent of $a \in \widetilde{A}$. See [22, Theorem 3.5.2], for example.

tempered distributions (modulo polynomials) $u \in \mathcal{S}'/\mathcal{P}$ satisfying

$$\|u\|_{\dot{B}^s_{p,q}} := \left\| \left(2^{js} \|\dot{\Delta}_j u\|_{L^p}\right)_{j\in\mathbb{Z}} \right\|_{\ell^q} < \infty. \tag{4.5.10}$$

Note that $\sum_{j\geq J} \dot{\Delta}_j u \to u$ in $\mathcal{S}'/\mathcal{P}$ as $J \to -\infty$. Moreover, when $s < d/p$ (or $s = d/p, q \leq 1$), the sum converges in $\mathcal{S}'$ and determines a unique tempered distribution u. Hence, this definition of $\dot{B}^s_{p,q}$ is equivalent to the one given in Section 4.2.1.

Let R be the *retraction* from $\mathcal{S}'/\mathcal{P}$ to the space of $\mathcal{S}'$-valued sequences over $\mathbb{Z}$:

$$Ru = \left(\dot{\Delta}_j u\right)_{j\in\mathbb{Z}}. \tag{4.5.11}$$

Let S be the *co-retraction* from the space of $\mathcal{S}'$-valued sequences over $\mathbb{Z}$ to $\mathcal{S}'/\mathcal{P}$:

$$S(u_j)_{j\in\mathbb{Z}} = \sum_{j\in\mathbb{Z}} \widetilde{\Delta}_j u_j, \tag{4.5.12}$$

where $\widetilde{\Delta}_j = \dot{\Delta}_{j-1} + \dot{\Delta}_j + \dot{\Delta}_{j+1}$. Then $SR = I$ is the identity map on $\mathcal{S}'/\mathcal{P}$. Let $\ell^s_q L^p$ denote the space of L^p-valued sequences $(u_j)_{j\in\mathbb{Z}}$ over Z satisfying

$$\|(u_j)_{j\in\mathbb{Z}}\|_{\ell^s_q L^p} := \left\| \left(2^{js} \|u_j\|_{L^p}\right)_{j\in\mathbb{Z}} \right\|_{\ell^q} < \infty. \tag{4.5.13}$$

Then $R\colon \dot{B}^s_{p,q} \to \ell^s_q L^p$ and $S\colon \ell^s_q L^p \to \dot{B}^s_{p,q}$ are continuous maps.

The retraction/co-retraction technology allows one to "transfer" splittings in sequence spaces to splittings in Besov spaces. Therefore, we begin with two splitting lemmas in sequence spaces:

Lemma 4.5.1 (Horizontal splitting)**.** *Let $s, s_0, s_1 \in \mathbb{R}$ be distinct real numbers and $p \in]0, \infty]$. For all $u \in \ell^s_\infty L^p$ and $K > 0$, there exist $f^K \in \ell^{s_0}_1 L^p$ and $g^K \in \ell^{s_1}_1 L^p$ such that*

$$u = f^K + g^K, \tag{4.5.14}$$

$$\|f^K\|_{\ell^{s_0}_1 L^p} \leq \frac{K^{s_0 - s}}{1 - 2^{|s_0 - s|}} \|u\|_{\ell^s_\infty L^p}, \tag{4.5.15}$$

$$\|g^K\|_{\ell^{s_1}_1 L^p} \leq \frac{K^{s_1 - s}}{1 - 2^{|s_1 - s|}} \|u\|_{\ell^s_\infty L^p}. \tag{4.5.16}$$

Moreover,

$$\|f^K\|_{\ell^s_\infty L^p}, \|g^K\|_{l^s_\infty L^p} \le \|u\|_{\ell^s_\infty L^p}. \tag{4.5.17}$$

Proof. Without loss of generality, we assume that $s_0 < s < s_1$. For $\kappa = \lfloor \log_2 K \rfloor$, we define

$$f_j^K = \begin{cases} u_j & j > \kappa \\ 0 & \text{otherwise} \end{cases}, \quad g^K = u - f^K. \tag{4.5.18}$$

Hence,

$$\sum_{j\in\mathbb{Z}} 2^{js_0} \|f_j^K\|_{L^p} \le \sum_{j>\kappa} 2^{j(s_0-s)} \times \|u\|_{\ell^s_\infty L^p} \le \frac{K^{s_0-s}}{1-2^{s-s_0}} \|u\|_{\ell^s_\infty L^p}, \tag{4.5.19}$$

$$\sum_{j\in\mathbb{Z}} 2^{js_1} \|g_j^K\|_{L^p} \le \sum_{j\le\kappa} 2^{j(s_1-s)} \times \|u\|_{\ell^s_\infty L^p} \le \frac{K^{s_1-s}}{1-2^{s_1-s}} \|u\|_{\ell^s_\infty L^p}, \tag{4.5.20}$$

and the persistency property (4.5.17) is valid due to (4.5.18). $\square$

Lemma 4.5.2 (Diagonal splitting)**.** *Let $\sigma, \widetilde{s}, \bar{s} \in \mathbb{R}$, $0 < \widetilde{p} < p < \bar{p} \le \infty$, and $q, \widetilde{q}, \bar{q} \in (0, \infty]$ such that (σ, p, q) belongs to the open segment connecting $(\widetilde{s}, \widetilde{p}, \widetilde{q})$ and $(\bar{s}, \bar{p}, \bar{q})$. Then for all $g \in \ell^\sigma_q L^p$ and $N > 0$, there exist $\widetilde{g}^N \in \ell^{\widetilde{s}}_{\widetilde{q}} L^{\widetilde{p}}$ and $\bar{g}^N \in \ell^{\bar{s}}_{\bar{q}} L^{\bar{p}}$ such that*

$$g = \widetilde{g}^N + \bar{g}^N, \tag{4.5.21}$$

$$\|\widetilde{g}^N\|_{\ell^{\widetilde{s}}_{\widetilde{q}} L^{\widetilde{p}}} \le N^{1-\frac{p}{\widetilde{p}}} \|g\|_{\ell^\sigma_q L^p} \tag{4.5.22}$$

$$\|\bar{g}^N\|_{\ell^{\bar{s}}_{\bar{q}} L^{\bar{p}}} \le N^{1-\frac{p}{\bar{p}}} \|g\|_{\ell^\sigma_q L^p} \tag{4.5.23}$$

Moreover, for all $j \in \mathbb{Z}$,

$$\|\widetilde{g}_j^N\|_{L^p}, \|\bar{g}_j^N\|_{L^p} \le \|g_j\|_{L^p}. \tag{4.5.24}$$

Proof. There exists $\theta \in]0,1[$ such that

$$\sigma = \theta\widetilde{s} + (1-\theta)\bar{s}, \quad \frac{1}{p} = \frac{\theta}{\widetilde{p}} + \frac{1-\theta}{\bar{p}}, \quad \frac{1}{q} = \frac{\theta}{\widetilde{q}} + \frac{1-\theta}{\bar{q}}. \tag{4.5.25}$$

For $c, \lambda_j > 0$ to be specified below, we define

$$\widetilde{g}_j^N = g_j \chi_{\{|g_j| > cN\lambda_j \|g_j\|_{L^p}\}}, \quad \bar{g}_j^N = g_j - \widetilde{g}_j^N, \tag{4.5.26}$$

which gives

$$\|\widetilde{g}_j^N\|_{L^{\widetilde{p}}} \leq (cN)^{1-\frac{p}{\widetilde{p}}} \lambda_j^{1-\frac{p}{\widetilde{p}}} \|g_j\|_{L^p}, \quad \|\bar{g}_j^N\|_{L^{\bar{p}}} \leq (cN)^{1-\frac{p}{\bar{p}}} \lambda_j^{1-\frac{p}{\bar{p}}} \|g_j\|_{L^p}. \tag{4.5.27}$$

By elementary manipulations,

$$2^{j\widetilde{s}\widetilde{q}} \|\widetilde{g}_j^N\|_{L^{\widetilde{p}}}^{\widetilde{q}} \leq (cN)^{(1-\frac{p}{\widetilde{p}})\widetilde{q}} \lambda_j^{(1-\frac{p}{\widetilde{p}})\widetilde{q}} 2^{j\widetilde{s}\widetilde{q}} \|g_j\|_{L^p}^{\widetilde{q}} \tag{4.5.28}$$

$$2^{j\bar{s}\bar{q}} \|\bar{g}_j^N\|_{L^{\bar{p}}}^{\bar{q}} \leq (cN)^{(1-\frac{p}{\bar{p}})\bar{q}} \lambda_j^{(1-\frac{p}{\bar{p}})\bar{q}} 2^{j\bar{s}\bar{q}} \|g_j\|_{L^p}^{\bar{q}} \tag{4.5.29}$$

Let us only deal with values $j \in \mathbb{Z}$ such that $\|g_j\|_{L^p} > 0$. We define $\lambda_j > 0$ by the following (equivalent) equations:

$$\lambda_j^{(1-\frac{p}{\widetilde{p}})\widetilde{q}} = 2^{j(\sigma q - \widetilde{s}\widetilde{q})} \|g_j\|^{q-\widetilde{q}}, \quad \lambda_j^{(1-\frac{p}{\bar{p}})\bar{q}} = 2^{j(\sigma q - \bar{s}\bar{q})} \|g_j\|^{q-\bar{q}}, \tag{4.5.30}$$

whose equivalence will be justified below. Substituting (4.5.30) into (4.5.28)-(4.5.29) and summing over $j \in \mathbb{Z}$ gives

$$\|\widetilde{g}^N\|_{\ell^{\widetilde{s}}_{\widetilde{q}} L^{\widetilde{p}}} \leq (cN)^{1-\frac{p}{\widetilde{p}}} \|g\|_{\ell^{\sigma}_q L^p}^{\frac{q}{\widetilde{q}}}, \quad \|\bar{g}^N\|_{\ell^{\bar{s}}_{\bar{q}} L^{\bar{p}}} \leq (cN)^{1-\frac{p}{\bar{p}}} \|g\|_{\ell^{\sigma}_q L^p}^{\frac{q}{\bar{q}}} \tag{4.5.31}$$

Then choose $c > 0$ satisfying the following (equivalent) equations

$$c^{1-\frac{p}{\widetilde{p}}} = \|g\|_{\ell^{\sigma}_q L^p}^{1-\frac{q}{\widetilde{q}}}, \quad c^{1-\frac{p}{\bar{p}}} = \|g\|_{\ell^{\sigma}_q L^p}^{1-\frac{q}{\bar{q}}}, \tag{4.5.32}$$

and the estimates (4.5.22)-(4.5.23) are proven. The persistency property (4.5.24) follows from the definition (4.5.26) of $\widetilde{g}^N$ and $\bar{g}^N$.

Finally, we argue that the two equations in (4.5.30) are equivalent. By comparing exponents, they will be equivalent as long as

$$\frac{\sigma\frac{q}{\widetilde{q}} - \widetilde{s}}{1 - \frac{p}{\widetilde{p}}} = \frac{\sigma\frac{q}{\bar{q}} - \bar{s}}{1 - \frac{p}{\bar{p}}} \quad \text{and} \quad \frac{\frac{q}{\widetilde{q}} - 1}{\frac{q}{\bar{q}} - 1} = \frac{\frac{p}{\widetilde{p}} - 1}{\frac{p}{\bar{p}} - 1}. \tag{4.5.33}$$

Let us assume that $\widetilde{q} \neq \bar{q}$ and $\widetilde{s} \neq \bar{s}$ (otherwise, the proof simplifies). Thus,

$$\theta = \frac{\sigma - \bar{s}}{\widetilde{s} - \bar{s}} = \frac{\frac{1}{p} - \frac{1}{\bar{p}}}{\frac{1}{\widetilde{p}} - \frac{1}{\bar{p}}} = \frac{\frac{1}{q} - \frac{1}{\bar{q}}}{\frac{1}{\widetilde{q}} - \frac{1}{\bar{q}}}. \tag{4.5.34}$$

The second equation in (4.5.33) readily reduces to $(\theta-1)/\theta$ on each side, so let us only deal with the first equation, which is equivalent to

$$\frac{\sigma\frac{q}{\widetilde{q}}-\widetilde{s}}{\sigma\frac{q}{\bar{q}}-\bar{s}}=\frac{\theta-1}{\theta}. \tag{4.5.35}$$

Substituting $\sigma=\theta\widetilde{s}+(1-\theta)\bar{s}$ and employing the relationship $\frac{1}{q}=\frac{\theta}{\widetilde{q}}+\frac{1-\theta}{\bar{q}}$ in the numerator and denominator verifies (4.5.35). The proof is complete. $\square$

Lemma 4.5.1 is related to the characterization $\ell^s_\infty L^p=(\ell^{s_1}_1L^p,\ell^{s_0}_1L^p)_{\theta,\infty}$ (real interpolation, with $s=\theta s_0+(1-\theta)s_1$), while Lemma 4.5.2 is related to the characterization of $\ell^\sigma_qL^p=[\ell^{\bar{s}}_{\bar{q}}L^{\bar{p}},\ell^{\widetilde{s}}_{\widetilde{q}}L^{\widetilde{p}}]_\theta$ (complex interpolation, with θ as in the proof). See [22, Chapter 5].

Remark 4.5.3 (Generalizations)**.** 1. In Lemma 4.5.2, one may replace $L^p(\mathbb{R}^d;\mathbb{R}^m)$ by $L^p(\Omega;X)$, where Ω is a measure space and X is a Banach space, at no additional cost.

2. In the proof of Lemma 4.5.8, we will require an analogous diagonal splitting lemma in spaces of sequences (over $\mathbb{Z}_{j\le J}$, for fixed $J\in\mathbb{Z}$) with values in $L^\infty_tL^p_x$. One may verify that same proof works with almost no alteration.

3. One could replace the L^p spaces with Banach spaces X_α satisfying analogous splitting properties. In this way, one could iterate the proof of Lemma 4.5.2 to handle a variety of mixed spaces combining L^p and ℓ^s_q. One could also allow the function spaces to depend on $j\in\mathbb{Z}$.

Combining the previous two lemmas, we obtain

Proposition 4.5.4 (Non-diagonal splitting)**.** *Let $s,\widetilde{s},\bar{s}\in\mathbb{R}$ and $0<\widetilde{p}<p<\bar{p}\le\infty$ such that $(s,1/p)$, $(\widetilde{s},1/\widetilde{p})$, and $(\bar{s},1/\bar{p})$ are not colinear. There exists a unique $s_1\in\mathbb{R}$ such that $(s_1,1/p)$ belongs to the closed segment connecting $(\widetilde{s},1/\widetilde{p})$ and $(\bar{s},1/\bar{p})$. Let $s_0\in\mathbb{R}$ such that s belongs to the open segment connecting s_0 and s_1.*

For all $u\in\ell^s_\infty L^p$ and $K,N>0$, there exist $f^K\in\ell^{s_0}_1L^p$, $\widetilde{g}^{K,N}\in\ell^{\widetilde{s}}_1L^{\widetilde{p}}$, and $\bar{g}^{K,N}\in\ell^{\bar{s}}_1L^{\bar{p}}$ such that

$$u=f^K+\widetilde{g}^{K,N}+\bar{g}^{K,N}, \tag{4.5.36}$$

$$\|f^K\|_{\ell^{s_0}_1L^p}\le\frac{K^{s_0-s}}{1-2^{|s_0-s|}}\|u\|_{\ell^s_\infty L^p} \tag{4.5.37}$$

$$\|\widetilde{g}^{K,N}\|_{\ell^{\widetilde{s}}_1L^{\widetilde{p}}}\le\frac{K^{s_1-s}}{1-2^{|s_1-s|}}N^{1-\frac{p}{\widetilde{p}}}\|u\|_{\ell^s_\infty L^p} \tag{4.5.38}$$

$$\|\bar{g}^{K,N}\|_{\ell_1^{\bar{s}}L^{\bar{p}}} \leq \frac{K^{s_1-s}}{1-2^{|s_1-s|}} N^{1-\frac{p}{\bar{p}}} \|u\|_{\ell_\infty^s L^p} \tag{4.5.39}$$

Moreover, for all $j \in \mathbb{Z}$,

$$\|f_j^K\|_{L^p}, \|\widetilde{g}_j^{K,N}\|_{L^p}, \|\bar{g}_j^{K,N}\|_{L^p} \leq \|u_j\|_{L^p}. \tag{4.5.40}$$

Proof. First, apply Lemma 4.5.1 to obtain $u = f^K + g^K$. Next, apply Lemma 4.5.2 with $\sigma = s_1$ and $q = \widetilde{q} = \bar{q} = 1$ to obtain $g^K = \widetilde{g}^{K,N} + \bar{g}^{K,N}$. □

Let all indices be as in Proposition 4.5.4. Note that there exist $\theta, \phi \in]0,1[$ such that

$$\ell_\infty^s L^p = ([\ell_1^{\bar{s}} L^{\bar{p}}, \ell_1^{\widetilde{s}} L^{\widetilde{p}}]_\theta, \ell_1^{s_0} L^p)_{\phi,\infty}. \tag{4.5.41}$$

Furthermore,

$$[\ell_1^{\bar{s}} L^{\bar{p}}, \ell_1^{\widetilde{s}} L^{\widetilde{p}}]_\theta \hookrightarrow (\ell_1^{\bar{s}} L^{\bar{p}}, \ell_1^{\widetilde{s}} L^{\widetilde{p}})_{\theta,\infty}. \tag{4.5.42}$$

Thus, Proposition 4.5.4 (without the persistency property) can be obtained via the abstract interpolation theory.

Remark 4.5.5 (Non-diagonal splitting, Besov version)**.** The non-diagonal splitting in Lemma 4.5.4 is applicable to Besov functions in the following way. Let $s, \widetilde{s}, \bar{s}, p, \widetilde{p}, \bar{p}$ be as in Lemma 4.5.4. Given $u \in \dot{B}^s_{p,\infty}$ and $K, N > 0$, we apply Lemma 4.5.4 to the retraction Ru, which belongs to $\ell_\infty^s L^p$, and obtain $Ru = f^K + \widetilde{g}^{K,N} + \bar{g}^{K,N}$, satisfying the estimates in Lemma 4.5.4 with Ru replacing u. The Besov splitting is obtained by applying the co-retraction: $u = SRu = S(f^K) + S(\widetilde{g}^{K,N}) + S(\bar{g}^{K,N})$. For the moment, we abuse notation by writing f^K instead of $S(f^K)$, etc. In summary, we have

$$u = f^K + \widetilde{g}^{K,N} + \bar{g}^{K,N}, \tag{4.5.43}$$

$$\|f^K\|_{\dot{B}^{s_0}_{p,1}} \leq C \frac{K^{s_0-s}}{1-2^{|s_0-s|}} \|u\|_{\dot{B}^s_{p,\infty}} \tag{4.5.44}$$

$$\|\widetilde{g}^{K,N}\|_{\dot{B}^{\widetilde{s}}_{\widetilde{p},1}} \leq C \frac{K^{s_1-s}}{1-2^{|s_1-s|}} N^{1-\frac{p}{\widetilde{p}}} \|u\|_{\dot{B}^s_{p,\infty}} \tag{4.5.45}$$

$$\|\bar{g}^{K,N}\|_{\dot{B}^{\bar{s}}_{\bar{p},1}} \leq C \frac{K^{s_1-s}}{1-2^{|s_1-s|}} N^{1-\frac{p}{\bar{p}}} \|u\|_{\dot{B}^s_{p,\infty}}, \tag{4.5.46}$$

and finally, the persistency property,

$$\|f^K\|_{\dot{B}^s_{p,\infty}}, \|\widetilde{g}^{K,N}\|_{\dot{B}^s_{p,\infty}}, \|\bar{g}^{K,N}\|_{\dot{B}^s_{p,\infty}} \leq C\|u\|_{\dot{B}^s_{p,\infty}}. \tag{4.5.47}$$

The constant $C > 0$ only appears when estimating the co-retraction operator and depends continuously on the parameters $s, \widetilde{s}, \bar{s}$.

The following splitting is obtained by combining Remark 4.5.5 and Sobolev embedding.

Proposition 4.5.6 (Splittings in Besov spaces)**.** *Let $1 \leq \widetilde{p} < p < \bar{p} \leq \infty$, and $s, \widetilde{s} \in \mathbb{R}$.*

Let α denote the line through $(s, 1/p)$ and $(\widetilde{s}, 1/\widetilde{p})$, β the line through $(s, 1/p)$ of slope $1/d$, and γ the horizontal line through the origin. Assume $\alpha \neq \beta$. Let $D \subset \mathbb{R}^2$ denote the interior of the compact region enclosed by α, β, and γ.

Let $\bar{s} \in \mathbb{R}$ such that $(\bar{s}, 1/\bar{p}) \in D$. Let $s_1 \in \mathbb{R}$ be the unique value such that $(s_1, 1/p)$ belongs to the open segment connecting $(\widetilde{s}, 1/\widetilde{p})$ and $(\bar{s}, 1/\bar{p})$. Let $s_0 = \bar{s} + \frac{d}{p} - \frac{d}{\bar{p}}$.

There exists a constant $C > 0$ depending continuously on the above parameters and satisfying the following properties:

For all $u \in \dot{B}^s_{p,\infty}$ and $N > 0$, there exist $\widetilde{u}^N \in \dot{B}^{\widetilde{s}}_{\widetilde{p},1}$ and $\bar{u}^N \in \dot{B}^{\bar{s}}_{\bar{p},1}$ such that

$$u = \widetilde{u}^N + \bar{u}^N, \tag{4.5.48}$$

$$\|\widetilde{u}^N\|_{\dot{B}^{\widetilde{s}}_{\widetilde{p},1}} \leq CN^{\frac{s_1 - s}{s_0 - s_1}(1-\frac{p}{\bar{p}})+(1-\frac{p}{\bar{p}})}\|u\|_{\dot{B}^s_{p,\infty}}, \tag{4.5.49}$$

$$\|\bar{u}^N\|_{\dot{B}^{\bar{s}}_{\bar{p},1}} \leq CN^{\frac{s_0 - s}{s_0 - s_1}(1-\frac{p}{\bar{p}})}\|u\|_{\dot{B}^s_{p,\infty}}. \tag{4.5.50}$$

Moreover,

$$\|\widetilde{u}^N\|_{\dot{B}^s_{p,\infty}}, \|\bar{u}^N\|_{\dot{B}^s_{p,\infty}} \leq C\|u\|_{\dot{B}^s_{p,\infty}}. \tag{4.5.51}$$

Proof. We claim that s, s_0, s_1 are distinct and that s lies on the open segment connecting s_0 and s_1. If s_0 were equal to s_1, then α would equal β. Moreover, s is strictly between s_0 and s_1 because the slope of the line between $(s, 1/p)$ and $(\bar{s}, 1/\bar{p})$ is strictly between the slope of α and the slope of β. See Figure 4.7. Hence, the hypotheses of Remark 4.5.5 are satisfied.

Let $K > 0$, to be specified below. By Remark 4.5.5, we obtain $u = f^K + \widetilde{g}^{K,N} + \bar{g}^{K,N}$ satisfying the properties described in the remark. Notice that f^K also belongs to $\dot{B}^{\bar{s}}_{\bar{p},1}$ due to

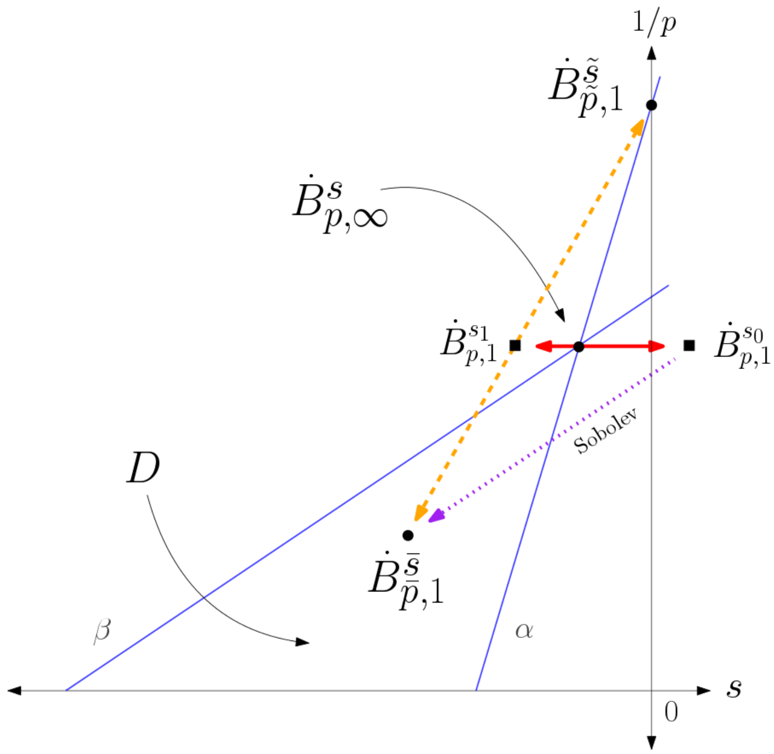

Figure 4.7: Illustration of Proposition 4.5.6. The original function $u \in \dot{B}^{s}_{p,\infty}$ is split horizontally into $f^K \in \dot{B}^{s_0}_{p,1}$ and $g^K \in \dot{B}^{s_1}_{p,1}$ along the solid red line. Next, g^K is split diagonally along the orange dashed line into $\widetilde{u}^N = \widetilde{g}^{K,N} \in \dot{B}^{\widetilde{s}}_{\widetilde{p},1}$ and $\bar{g}^{K,N} \in \dot{B}^{\bar{s}}_{\bar{p},1}$. Finally, using Sobolev embedding along the dotted purple line, f^K and $\bar{g}^{K,N}$ are combined to form $\bar{u}^N$.

Sobolev embedding and our particular choice of s_0. Moreover,

$$\|f^K\|_{\dot{B}^{\bar{s}}_{\bar{p},1}} \leq C(\bar{s}, s_0, \bar{p}, p_0)\|f^K\|_{\dot{B}^{s_0}_{p_0,1}}. \tag{4.5.52}$$

We define

$$\widetilde{u}^N = \widetilde{g}^{K,N}, \quad \bar{u}^N = f^K + \bar{g}^{K,N}. \tag{4.5.53}$$

By the triangle inequality, the estimates in Remark 4.5.5 and (4.5.52),

$$\|\widetilde{u}^N\|_{\dot{B}^{\widetilde{s}}_{\widetilde{p},1}} \leq \frac{C}{1-2^{|s_1-s|}} K^{s_1-s} N^{1-\frac{p}{\widetilde{p}}} \|u\|_{\dot{B}^{s}_{p,\infty}}, \tag{4.5.54}$$

$$\|\bar{u}^N\|_{\dot{B}^{\bar{s}}_{\bar{p},1}} \leq C\left(\frac{1}{1-2^{|s_0-s|}}+\frac{1}{1-2^{|s_1-s|}}\right)\left(K^{s_0-s}+K^{s_1-s}N^{1-\frac{p}{\bar{p}}}\right)\|u\|_{\dot{B}^s_{p,\infty}}. \quad (4.5.55)$$

Substituting $K = N^{(1-\frac{p}{\bar{p}})/(s_0-s_1)}$ gives the desired estimates. The persistency property (4.5.51) also follows from Remark 4.5.5 and the triangle inequality. □

Proof of Lemma 4.1.7. Let $d = m \geq 3$, $p > d$, $s = s_p := -1 + \frac{d}{p}$, $\widetilde{p} = 2$, $\widetilde{s} = 0$, $\bar{p} = 2p$, and $\bar{s} = (s_{2p} + \dot{s})/2$ in Proposition 4.5.6. Here, $\dot{s}$ is defined such that $(\dot{s}, 1/\bar{p}) \in \alpha$:

$$\dot{s} = \frac{\frac{1}{2p}-\frac{1}{2}}{\frac{1}{p}-\frac{1}{2}}s. \quad (4.5.56)$$

Hence, $(\bar{s}, 1/\bar{p}) \in D$, and we may apply Proposition 4.5.6 to obtain the desired splitting. The proof is completed by applying the Leray projector $\mathbb{P}$ onto divergence-free vector fields. Recall that $\mathbb{P}$ is continuous on homogeneous Besov spaces, see [11, Proposition 2.30]. □

We now state and prove an analogous splitting lemma for the forcing term.

Lemma 4.5.8 (Splitting of forcing term)**.** *Let $T > 0$ and $p \in]3,\infty[$. There exist $p_3 \in]3,\infty[$, $\delta_3 > 0$, and $C > 0$, each depending only on p, such that for each $F \in \mathcal{F}_p(Q_T)$ and $N > 0$, there exist $\bar{F}^N \in \mathcal{F}^{s'_{p_3}+\delta_3}_{p_3}(Q_T) \cap \mathcal{F}_p(Q_T)$ and $\widetilde{F}^N \in L^3_t L^2_x(Q_T) \cap \mathcal{F}_p(Q_T)$ with the following properties:*

$$F = \widetilde{F}^N + \bar{F}^N, \quad (4.5.57)$$

$$\|\widetilde{F}^N\|_{L^3_t L^2_x(Q_T)} \leq CT^{-\frac{1}{12}}N^{1-\frac{p}{2}}\|F\|_{\mathcal{F}_p(Q_T)}, \quad (4.5.58)$$

$$\|\bar{F}^N\|_{\mathcal{F}^{s'_{p_3}+\delta_3}_{p_3}(Q_T)} \leq CT^{\frac{\delta_3}{2}}N^{\frac{1}{2}}\|F\|_{\mathcal{F}_p(Q_T)}. \quad (4.5.59)$$

Furthermore,

$$\|\widetilde{F}^N\|_{\mathcal{F}_p(Q_T)}, \|\bar{F}^N\|_{\mathcal{F}_p(Q_T)} \leq \|F\|_{\mathcal{F}_p(Q_T)}. \quad (4.5.60)$$

The proof of the splitting lemma for the forcing term is easier because the function spaces in question are not homogeneous.

Proof of Lemma 4.5.8. By a scaling argument, we need only to consider the case $T = 2$.

We define a retraction R from the space of measurable tensor fields on $Q_T = \mathbb{R}^3 \times]0, T[$ to the space of sequences (over $\mathbb{Z}_{\leq 0}$) of measurable tensor fields on $\mathbb{R}^3\times]1,2[$:

$$RG = (G_j)_{j\leq 0}, \quad G_j(\cdot,t) := G(\cdot, 2^j t)\chi_{]2^j,2^{j+1}[}(2^j t), \quad t\in]1,2[,\ j\leq 0. \tag{4.5.61}$$

The retraction R is invertible, and the co-retraction S is its inverse. Namely,

$$S(G_j)(\cdot,t) = \sum_{j\in\mathbb{Z}_{\leq 0}} G_j(\cdot, 2^{-j}t)\chi_{]2^j,2^{j+1}[}(t). \tag{4.5.62}$$

For $p\in[1,\infty]$ and $s\in\mathbb{R}$, we consider the space $\ell^s_\infty(L^\infty_t L^p_x)$ consisting of sequences $(G_j)_{j\leq 0}$ of locally integrable tensor fields G_j on $\mathbb{R}^3\times]1,2[$ such that

$$\|(G_j)_{j\leq 0}\|_{\ell^s_\infty(L^\infty_t L^p_x)} := \sup_{j\leq 0} 2^{-\frac{js}{2}}\|G_j\|_{L^\infty_t L^p_x(\mathbb{R}^3\times]1,2[)} < \infty. \tag{4.5.63}$$

Note the $-1/2$ in the exponent. Then $R\colon \mathcal{F}^s_p(Q_T)\to \ell^s_\infty(L^\infty_t L^p_x)$ and $S\colon \ell^s_\infty(L^\infty_t L^p_x)\to \mathcal{F}^s_p(Q_T)$ are continuous maps with norms depending only on s.

Let $p>3$, $\sigma = s'_p := -2+3/p$, $\widetilde{p}=2$, $\bar{p}=2p$, and $q=\widetilde{q}=\bar{q}=\infty$. Let $\widetilde{s}=-7/12$ and $\bar{s}\in\mathbb{R}$ such that $(s'_p,1/p)$, $(\widetilde{s},1/2)$ and $(\bar{s},1/\bar{p})$ are colinear. Note that $\bar{s}>s'_{\bar{p}}$, since the slope of the segment from $(s'_p,1/p)$ to $(-7/12,1/2)$ is less than $1/3$.[31]

By Remark 4.5.3.2, we may apply the diagonal splitting in $\ell^s_\infty(L^\infty_t L^p_x)$ to $RF=(F_j)_{j\leq 0}$ and obtain

$$F_j = \widetilde{F}^N_j + \bar{F}^N_j, \quad j\leq 0. \tag{4.5.64}$$

Denote $\widetilde{F}^N = S(\widetilde{F}^N_j)_{j\leq 0}$ and $\bar{F}^N = S(\bar{F}^N_j)_{j\leq 0}$. Then $F=\widetilde{F}^N+\bar{F}^N$, and

$$\int_0^T \|\widetilde{F}^N(\cdot,t)\|^3_{L^2(\mathbb{R}^3)}\,dt \leq \int_0^T t^{\frac{3\widetilde{s}}{2}}\,dt\times \|\widetilde{F}^N\|^3_{\mathcal{F}^{\widetilde{s}}_2(Q_T)} \overset{\widetilde{s}=-\frac{7}{12}}{\leq} CN^{1-\frac{p}{2}}\|F\|^3_{\mathcal{F}_p(Q_T)}, \tag{4.5.65}$$

$$\|\bar{F}^N\|_{\mathcal{F}^{\bar{s}}_{\bar{p}}(Q_T)} \leq CN^{\frac{1}{2}}\|F\|_{\mathcal{F}_p(Q_T)}. \tag{4.5.66}$$

Finally, we have the persistency property:

$$\|\widetilde{F}^N\|_{\mathcal{F}_p(Q_T)}, \|\bar{F}^N\|_{\mathcal{F}_p(Q_T)} \leq \|F\|_{\mathcal{F}_p(Q_T)}. \tag{4.5.67}$$

[31]For comparison, one may verify that the slope of the segment from $(s'_p,1/p)$ to $(-1/2,1/2)$ is $1/3$.

We define $p_3 := \bar{p}$ and $\delta_3 := \bar{s} - s'_{p_3} > 0$ to complete the proof. $\square$

4.6 Appendix: ε-regularity

In this section, we recall an ε-regularity criterion for the three-dimensional Navier–Stokes equations and some of its important consequences, following [32, 105, 93, 49, 118]. In particular, we will state without proof certain results with forcing terms which we could not find in the literature and indicate what modifications are necessary to prove them.

Our main definition is adapted from the one in F. H. Lin's paper [105].

Definition 4.6.1 (Suitable weak solution)**.** *Let Q denote a parabolic ball*

$$Q(z_0, r) := B(x_0, r) \times]t_0 - r^2, t_0[\tag{4.6.1}$$

for some $z_0 = (x_0, t_0) \in \mathbb{R}^{3+1}$ and $r > 0$. Suppose that $v \in L^\infty_t L^2_x \cap L^2_t H^1_x(Q)$, $q, f \in L^{\frac{3}{2}}_{\mathrm{loc}}(Q)$, and $F \in L^2_{\mathrm{loc}}(Q)$.

We say that (v, q) is a suitable weak solution *of the Navier–Stokes equations on Q with forcing term $f + \operatorname{div} F$ if*

$$\left.\begin{aligned} \partial_t v - \Delta v + v \cdot \nabla v = -\nabla q + f + \operatorname{div} F \\ \operatorname{div} v = 0 \end{aligned}\right\} \text{ in } Q \tag{4.6.2}$$

in the sense of distributions, and the local energy inequality

$$\begin{aligned} &\int_{B(x_0,r)} |v(x,t)|^2 \varphi \, dx + 2 \int_Q |\nabla v|^2 \varphi \, dx \, dt \\ &\quad \leq \int_Q (\partial_t \varphi + \Delta \varphi)|v|^2 + (|v|^2 + 2p) v \cdot \nabla \varphi + 2 f \cdot (\varphi v) - 2F : \nabla(\varphi v) \, dx \, dt \end{aligned} \tag{4.6.3}$$

is satisfied for a.e. $t \in]t_0 - r^2, t_0[$ and all $0 \leq \varphi \in C^\infty_0(Q)$.

The following ε-regularity criterion for suitable weak solutions may be proven by copying the scheme of Ladyzhenskaya and Seregin in [93]. Higher regularity with zero forcing term was demonstrated in [111] according to the arguments in Serrin's paper [128].

Proposition 4.6.2 (ε-regularity)**.** *Let $\delta > 0$ and $p_1, p_2, q_1, q_2 \in]1, \infty]$ satisfying*

$$\frac{2}{q_1} + \frac{3}{p_1} = 3 - \delta, \quad \frac{2}{q_2} + \frac{3}{p_2} = 2 - \delta. \tag{4.6.4}$$

There exist constants $\varepsilon_{\mathrm{CKN}}, c_0 > 0$ depending on p_1, p_2, q_1, q_2 such that for all $z \in \mathbb{R}^{3+1}$, $R > 0$, and suitable weak solutions (v, q) on $Q(z, R)$ with forcing term $f + \operatorname{div} F$, $f \in L_t^{q_1} L_x^{p_1}(Q(z, R))$, $F \in L_t^{q_2} L_x^{p_2}(Q(z, R))$, the condition

$$\frac{1}{R^2} \int_{Q(z,R)} |v|^3 + |q|^{\frac{3}{2}} \, dx' \, dt' + R^\delta \|f\|_{L_t^{q_1} L_x^{p_1}(Q(z,R))} + R^\delta \|F\|_{L_t^{q_2} L_x^{p_2}(Q(z,R))} < \varepsilon_{\mathrm{CKN}} \tag{4.6.5}$$

implies that $v \in C^\alpha_{\mathrm{par}}(Q(z, R/2))$, and

$$\|v\|_{L^\infty(Q(z,R/2))} + R^\alpha [v]_{C^\alpha_{\mathrm{par}}(Q(z,R/2)} \leq \frac{c_0}{R}. \tag{4.6.6}$$

If the condition is satisfied and f, F are zero, then $\nabla^\ell v \in C^\alpha_{\mathrm{par}}(Q(z, R/2))$ for all $\ell \in \mathbb{N}$, and there exist absolute constants $c_{0,\ell} > 0$, $\ell \in \mathbb{N}$, such that

$$\|\nabla^\ell v\|_{L^\infty(Q(z,R/2))} + R^\alpha [\nabla^\ell v]_{C^\alpha_{\mathrm{par}}(Q(z,R/2)} \leq \frac{c_{0,\ell}}{R^{\ell+1}}. \tag{4.6.7}$$

The following lemma was proven without forcing terms by F. H. Lin in [105, Theorem 2.2]. In our situation, the local energy inequality (4.6.3) for the limit solution must be obtained in a slightly indirect way which is similar to the proof of Proposition 4.3.14, see below.

Lemma 4.6.3 (Weak–$*$ stability for suitable weak solutions)**.** *Let $(v^{(n)}, q^{(n)})_{n \in \mathbb{N}}$ be a sequence of suitable weak solutions on $Q := Q(z, r)$ with respective forcing terms $f^{(n)} + \operatorname{div} F^{(n)}$, $n \in \mathbb{N}$, for some $z \in \mathbb{R}^{3+1}$ and $r > 0$. Furthermore, suppose that*

$$v^{(n)} \overset{*}{\rightharpoonup} v \text{ in } L_t^\infty L_x^2(Q), \quad \nabla v^{(n)} \rightharpoonup \nabla v \text{ in } L^2(Q), \tag{4.6.8}$$

$$v^{(n)} \to v \text{ in } L^3(Q), \quad q^{(n)} \rightharpoonup q \text{ in } L^{\frac{3}{2}}(Q), \tag{4.6.9}$$

$$f^{(n)} \rightharpoonup f \text{ in } L^{\frac{3}{2}}(Q), \quad F^{(n)} \rightharpoonup F \text{ in } L^2(Q). \tag{4.6.10}$$

Let $p_1, q_1, p_2, q_2 \in]1, \infty]$ *such that*

$$\frac{2}{q_1} + \frac{3}{p_1} < 3, \quad \frac{2}{q_2} + \frac{3}{p_2} < 2. \tag{4.6.11}$$

Finally, suppose that

$$f^{(n)} \overset{*}{\rightharpoonup} f \text{ in } L^{q_1}_t L^{p_1}_x(Q), \quad F^{(n)} \overset{*}{\rightharpoonup} F \text{ in } L^{q_2}_t L^{p_2}_x(Q). \tag{4.6.12}$$

Then there exists $q \in L^{\frac{3}{2}}_{\mathrm{loc}}(Q)$ *such that* (v, q) *is a suitable weak solution on* Q *with forcing term* $f + \operatorname{div} F$*.*

We may assume that $Q \subset \mathbb{R}^3 \times \mathbb{R}_+$. To prove Lemma 4.6.3, one extends $f^{(n)}, F^{(n)}$ by zero to the whole space and defines

$$V^{(n)}(\cdot, t) := \int_0^t S(t-s)\mathbb{P} f^{(n)}(\cdot, s) + S(t-s)\mathbb{P} \operatorname{div} F^{(n)}(\cdot, s)\, ds \tag{4.6.13}$$

for each $n \in \mathbb{N}$. Each suitable weak solution is decomposed into the linear solution above and a correction term: $v^{(n)} = V^{(n)} + u^{(n)}$. Similarly, one writes $v = V + u$. Next, one "transfers" the local energy inequality (4.6.3) from the velocity field $v^{(n)}$ to obtain a local energy inequality for the correction $u^{(n)}$ on Q with lower order terms and a forcing term which converges locally strongly. This is described in the proof of Proposition 4.3.2. The local energy inequality for the corrections $u^{(n)}$ is stable under the limiting procedure, see the proof of Proposition 4.3.14. Finally, one transfers the local energy inequality from the limit correction u to obtain (4.6.3) for the velocity field v, see Remark 4.3.3.

The final proposition may be proved as in [118, Lemma 2.1].

Proposition 4.6.4 (Persistence of singularity)**.** *Assume the hypotheses of Lemma 4.6.3. Moreover, assume that* z *is a singular point of* $v^{(n)}$ *for each* $n \in \mathbb{N}$*. Then* z *is a singular point of* v*.*

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