ON THE CONTINUATION FOR VARIATIONAL INEQUALITIES DEPENDING ON AN EIGENVALUE PARAMETER

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IMA Preprint Series # 334

August 1987

INSTITUTE FOR MATHEMATICS AND ITS APPLICATIONS

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Minneapolis, Minnesota 55455

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ON THE CONTINUATION FOR VARIATIONAL INEQUALITIES DEPENDING ON AN EIGENVALUE PARAMETER

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Abstract. In this paper we generalize recent theoretical results on the local continuation of parameter-dependent nonlinear variational inequalities. The variational inequalities are rather general and describe, for example, the buckling of beams, plates or shells subject to obstacles. Under a technical hypothesis that is verified for the simply supported beam, we obtain the existence of a continuation of both the solution and the eigenvalue with respect to a local parameter. A numerical continuation method is presented that easily overcomes turning points. Numerical results are presented for the nonlinear beam.

1. Introduction

Let V be a closed convex subset of a real Hilbert space H. We assume that $0 \in V$ and $V \neq \{0\}$. In this paper we are interested in the local continuation of solutions $(u,\lambda) \in V \times \mathbf{R}$ to the variational inequality

$$u \in V:\quad f'(u)(v-u) \geq \lambda g'(u)(v-u) \quad \text{for all} \quad v \in V . \tag{1}$$

Here the f', g' are the first Gateaux derivatives of continuous real functionals f and g defined on H. We assume that all Gateaux or Frechet derivatives which we shall need exist and are continuous, moreover that:

*The second author was supported by the Air Force Office of Scientific Research under grant AFOSR-84-0315.

AMS subject classification: Primary 35J85, 65K10, 65N25
Secondary 49A29, 58E35, 65K10

(2) f is weakly semicontinuous on V,

(3) g is weakly continuous on V,

(4) $f''(u)(v,v)$ is equivalent to the given norm $\|v\| = (v,v)^{1/2}$ on H,

(5) $g''(u)(v,v)$ is weakly continuous with respect to v.

Let $(u,\lambda) \in V \times \mathbf{R}$ be a solution of (1). The question is whether for a given ε, $|\varepsilon| \leq \varepsilon_0$, $\varepsilon_0 > 0$ small enough, there exists a solution $u(\varepsilon)$ to the eigenvalue $\lambda(\varepsilon) = \lambda + \varepsilon$. We show that there is a solution $u(\varepsilon)$ of (1) with

$$\|u(\varepsilon) - u\| \leq c\ \varepsilon ,$$

provided a certain eigenvalue criterion is satisfied. Moreover, one has for the local behavior $u(\varepsilon) = u + \varepsilon u_1 + o(\varepsilon)$, where u_1 is a solution of an associated linear variational inequality over a closed convex cone with the vertex at zero, cf. section 3.

This paper generalizes recent results for elliptic variational inequalities of second order by Conrad, Issard-Roch, Brauner and Nicolaenko [2] to more general problems. Our method is based on variational methods of Beckert [1] for equations which were extended to variational inequalities in [6,7].

In the case of equations, that is V = H, the essential assumption would be the inequality $\lambda < \Lambda$, where Λ denotes the first eigenvalue of the associated Frechet eigenvalue problem. In the case of variational inequalities one has to replace Λ by a different number due to the nonlinearity of the set V, cf. section 1.

Instead of the variational inequality (1) we have considered in [8,9] the associated free boundary problems. This method does not seem to be applicable to general higher dimensional problems. Possibly a continuation of (u,λ) as a solution of (1) combined with suitable numerical methods as in [3,4,10,11] would be a more adequate procedure.

2. Continuation of the Solution

Let $(u,\lambda) \in V\times \mathbf{R}$ be a solution of (1) and let us consider the case $\lambda > 0$. First we investigate the question of whether (u,λ) defines a strict local minimum of the functional

$$I_\lambda(v) = f(v) - \lambda g(v) .$$

Here a vector $(u,\lambda) \in V\times \mathbf{R}$ is said to be a strict local minimum of $I_\lambda(v)$ if there exist positive numbers ρ, c such that

$$I_\lambda(v) - I_\lambda(u) \geq c\|v-u\|^2$$

for all $v \in V$ with $\|v-u\| \leq \rho$.

For a given $t > 0$ we set

$$V_t(u) = \{w \in H;\ u + tw \in V\}$$

and assume that $V_t(u) \neq \{0\}$ for all t with $0 < t \leq t_0$. For $w \in V_t(u)$, $I_\lambda(u+tw)$ is expanded with respect to t:

$$I_\lambda(u+tw) = I_\lambda(u) + tI_\lambda'(u)(w) + \frac{t^2}{2} I_\lambda''(u)(w,w) + O(t^3) . \tag{6}$$

If $w \in V_t(u)$, $f''(u)(w,w) = 1$ and $I'(u)(w) \geq A\,t$ for a constant A not depending on t, then (6) implies

$$I_\lambda(u+tw) - I_\lambda(u) \geq \frac{t^2}{2} \{2A + 1 - \lambda\, g''(w,w)\} + O(t^3).$$

Let

$$\Lambda_H^{-1} = \max_{w \varepsilon H \setminus \{0\}} \frac{g''(u)(w,w)}{f''(u)(w,w)} ,$$

which is well-defined because of the assumptions (4) and (5), and choose A such that

$$2A > \lambda\Lambda_H^{-1} - 1$$

holds, then

$$I_\lambda(u+tw) - I_\lambda(u) \geq c \frac{t^2}{2} + 0(t^3)$$

follows with a positive constant c which does not depend on t.

Now we consider such $w \varepsilon V_t(u)$, $f''(u)(w,w) = 1$ with $I'_\lambda(u)(w) \leq A\ t$. The expansion (6) yields

$$I_\lambda(u+tw) - I_\lambda(u) \geq \frac{t^2}{2} (1 - \lambda g''(u)(w,w)) + 0(t^3)$$

since $I'_\lambda(u)(w) \geq 0$ is satisfied.

Set

$$V_{t,A}(u) = V_t(u) \cap \{w \varepsilon H;\ f''(u)(w,w) \leq 1 \text{ and } I'_\lambda(u)(w) \leq A\ t\}$$

and pose the maximum problem

(7) $$\mu_t(u) = \max_{w \varepsilon V_{t,A}(u)} g''(u)(w,w) .$$

Because of the assumptions (4) and (5) there exists a solution of (7).

We define

$$C_u(V) = \{v \varepsilon H;\ u + v \varepsilon V\}$$

and

$$C_u^{\perp}(V) = \{v \in C_u(V);\ I_\lambda'(u)(v) = 0\}$$

and assume $C_u^{\perp}(V) \neq \{0\}$.

Let K denote the closed cone hull of $C_u^{\perp}(V)$, that is the closure of the set

$$\{sv;\ v \in C_u^{\perp}(V) \text{ and } s \geq 0\} .$$

The cone K is a convex cone with the vertex at zero. We make the following hypothesis:

(A_1) Let $t_n \to 0$, $t_n > 0$, and let $w_n \in V_{t_n}(u)$ be a weakly convergent sequence $w_n \rightharpoonup w$ such that $\limsup_{n\to\infty} t_n^{-1} I_\lambda'(u)(w_n) < \infty$ holds. Then it follows that $w \in K$.

Let

$$\Lambda_K^{-1} = \max_{w \in K \setminus \{0\}} \frac{g''(u)(w,w)}{f''(u)(w,w)} ,$$

then (7) and (A_1) imply

$$\limsup_{t \downarrow 0} \mu_t(u) \leq \Lambda_K^{-1} .$$

Since each $v \in V$, $v \neq u$, may be written as $u + tw$ with $t > 0$, $f''(u)(w,w) = 1$ and $u + tw \in V$, where $t^2 = f''(u)(v-u,v-u)$ and $w = t^{-1}(v-u)$, we have shown

<u>Lemma 1</u>. Suppose that (A_1) and the inequality $\lambda < \Lambda_K$ are satisfied. Then there exist positive numbers c, ρ such that

(8) $$I_\lambda(u+v) - I_\lambda(u) \geq c\|v\|^2$$

for all $v \in C_u(V)$ with $\|v\| \leq \rho$.

From the inequality (8) of this lemma one concludes that there exists a local continuation of the solution with respect to λ. More precisely, we have

Lemma 2. If inequality (8) holds then there exists an $\varepsilon_0 > 0$, such that for each ε, $|\varepsilon| \leq \varepsilon_0$, there is a solution $u(\varepsilon)$ of the variational inequality (1) to the eigenvalue $\lambda(\varepsilon) = \lambda + \varepsilon$. Moreover, $\|u(\varepsilon) - u\| \leq c|\varepsilon|$, where the constant c does not depend on ε.

Proof. For $\rho > 0$ set $B_\rho(u) = \{v \in H;\ \|u-v\| < \rho\}$ and $V_\rho = V \cap B_\rho(u)$. We pose the minimum problem

$$\min_{v \in V_\rho} I_{\lambda+\varepsilon}(v)$$

for a given ε .

Because of the assumptions (2) and (3) there exists a minimizer $\bar{u}$. Since

$$I_{\lambda+\varepsilon}(\bar{u}) \leq I_{\lambda+\varepsilon}(u)$$

and

$$I_{\lambda+\varepsilon}(v) = I_\lambda(v) - \varepsilon\, g(v)$$

it follows

$$I_\lambda(u+(\bar{u}-u)) - I_\lambda(u) - \varepsilon g(u+(\bar{u}-u)) + \varepsilon\, g(u) \leq 0.$$

Using the assumption of the lemma, we obtain

$$c_1\|\bar{u}-u\|^2 \leq \varepsilon\ [g(u+(\bar{u}-u)) - g(u)]$$

$$= \varepsilon\ [g'(u)(\bar{u}-u) + 0(\|\bar{u}-u\|^2)]$$

with a positive constant c_1. This implies

$$\|\bar{u}-u\| \leq c_2|\varepsilon|$$

provided $\rho > 0$ is small enough.

If we choose $\varepsilon_0 < \frac{\rho}{c_2}$, then $\|\bar{u}-u\| < \rho$ is satisfied for all ε, $|\varepsilon| \leq \varepsilon_0$.
Hence, since $\bar{u} \in V_\rho$, $\bar{u}$ is a solution of the variational inequality (1) to the eigenvalue $\lambda(\varepsilon) = \lambda + \varepsilon$. q.e.d.

Let $(u(\varepsilon),\lambda(\varepsilon))$ with $\lambda(\varepsilon) = \lambda + \varepsilon$, $|\varepsilon| \leq \varepsilon_0$, be a branch of solutions of (1) such that $u(\varepsilon) \to u$ for $\varepsilon \to 0$. We make the following hypothesis:

(A_2) For arbitrary sequences $\varepsilon_n \to 0$ and $t_n \to 0$ let $w_n \in V_{t_n}(u_n)$, where $u_n = u(\varepsilon_n)$, be a weakly convergent sequence $w_n \rightharpoonup w$ such that

$$\limsup_{n\to\infty} \quad t_n^{-1} I'_{\lambda_n}(u_n)(w_n) < \infty$$

is satisfied with $\lambda_n = \lambda + \varepsilon_n$. Then it follows that $w \in K$.

<u>Lemma 3</u>. Assume that $\lambda < \Lambda_K$ and that (A_2) holds for the continuation of Lemma 2. Then $u(\varepsilon)$ is the unique solution of (1) in V_ρ, $\rho > 0$ small enough.

<u>Proof</u>. If $u(\varepsilon)$ were not the only solution, there would exist a sequence $(\bar{u}_n,\lambda_n)$ of solutions to the variational inequality with $\bar{u}_n \neq u_n$ and $\bar{u}_n \to u$.
We have

$$I_{\lambda_n}(\bar{u}_n) - I_{\lambda_n}(u_n) = -[I_{\lambda_n}(u_n) - I_{\lambda_n}(\bar{u}_n)]$$

$$= -[I'_{\lambda_n}(\bar{u}_n)(u_n-\bar{u}_n) + \frac{1}{2} I''_{\lambda_n}(\bar{u}_n)(u_n-\bar{u}_n,u_n-\bar{u}_n)] + O(\|u_n-\bar{u}_n\|^3)$$

$$= -[I'_{\lambda}(\bar{u}_n)(u_n-\bar{u}_n) + \frac{1}{2} I''_{\lambda_n}(u_n)(u_n-\bar{u}_n,u_n-\bar{u}_n)] + O(\|u_n-\bar{u}_n\|^3) .$$

Consequently, taking

$$I'_{\lambda_n}(\bar{u}_n)(u_n-\bar{u}_n) \geq 0$$

into account, it follows that

$$I_{\lambda_n}(\bar{u}_n) - I_{\lambda_n}(u_n) \leq -\frac{t_n^2}{2} I''_{\lambda_n}(u_n)(w_n,w_n) + 0(t_n^3)$$

where $w_n = t_n^{-1}(\bar{u}_n - u_n)$ with $t_n^2 = f''(u_n)(u_n - \bar{u}_n, u_n - \bar{u}_n)$.

On the other hand, the expansion (6) yields

$$I_{\lambda_n}(\bar{u}_n) - I_{\lambda_n}(u_n) = \frac{t_n^2}{2}[1 - \{\lambda_n g''(u_n)(w_n,w_n) - 2t_n^{-1} I'_{\lambda_n}(u_n)(w_n)] + 0(t_n^3) .$$

Thus, the following inequality results

$$(9) \qquad t_n^{-1} I'_{\lambda_n}(u_n)(w_n) \leq \lambda_n g''(u_n)(w_n,w_n) - 1 + 0(t_n) .$$

The right hand side remains bounded if n tends to infinity. Since $w_n \in V_{t_n}(u_n)$ because of $u_n + t_n w_n = \bar{u}_n \in V$, the assumption (A_2) implies $1 \leq \lambda g''(u)(w,w)$ for each weakly convergent subsequence $w_n \rightarrow w$. But this inequality is a contradiction to the assumption $\lambda < \Lambda_K$. q.e.d.

We summarize these considerations in

<u>Theorem 1</u>. Assume that (A_1) and $\lambda < \Lambda_K$ are satisfied, then there exists a solution $u(\varepsilon)$ of the variational inequality (1) to the eigenvalue $\lambda(\varepsilon) = \lambda + \varepsilon$, $|\varepsilon| \leq \varepsilon_0$, $\varepsilon_0 > 0$ small enough. Moreover, one has $\|u(\varepsilon) - u\| \leq c\,|\varepsilon|$ with a constant c which does not depend on ε . If in

addition (A_2) holds, then the above continuation is the unique solution of (1) in V_ρ to the eigenvalue $\lambda + \varepsilon$, provided ε_0 and ρ are small enough.

3. Local behavior of $u(\varepsilon)$

Suppose (A_1) and $\lambda < \Lambda_K$ are satisfied. From the above it is known that there exists a solution $u(\varepsilon)$ of (1) to the eigenvalue $\lambda + \varepsilon$ such that for $u(\varepsilon) = u + h(\varepsilon)$

$$\|h(\varepsilon)\| \leq c|\varepsilon| .$$

Since h is a solution of

$$h \in C_u(V): \quad I'_\lambda (u+h)(v-h) - \varepsilon g'(u+h)(v-h) \geq 0 \quad \text{for all} \quad v \in C_u(V) ,$$

one concludes

$$(10) \quad \begin{aligned} I'_\lambda(u)(v-h) + I''_\lambda(u)(h,v-h) - \varepsilon g'(u)(v-h) + \varepsilon (0(\|h\|), v-h) \\ + (0(\|h\|^2), v-h) \geq 0 \quad \text{for all} \quad v \in C_u(V) . \end{aligned}$$

Let $h_\varepsilon = |\varepsilon|^{-1}h$, $\varepsilon \neq 0$. From the above, we have $\|h_\varepsilon\| \leq c$.
Inserting a $v \in C_u^\perp(V)$ into (10), we obtain

$$(11) \quad \begin{aligned} &- |\varepsilon|^{-1}I'_\lambda(u)(h_\varepsilon) + I''_\lambda(u)(h_\varepsilon, |\varepsilon|^{-1}v - h_\varepsilon) \\ &- \operatorname{sign}(\varepsilon)g'(u)(|\varepsilon|^{-1}v-h_\varepsilon) \geq (0(|\varepsilon|),|\varepsilon|^{-1}v-h_\varepsilon) . \end{aligned}$$

For v = 0 it follows that

$$(12) \qquad I'_\lambda(u)(h_\varepsilon) \leq c|\varepsilon|$$

with a constant c which does not depend on ε .

Let $h_{\varepsilon_i} \rightharpoonup \bar{h}$ be a weakly convergent subsequence with $\varepsilon_i \to 0$ and $\varepsilon_i > 0$ or $\varepsilon_i \to 0$ and $\varepsilon_i < 0$, then (11) implies

$$(13) \qquad I''_\lambda(u)(\bar{h},v-\bar{h}) - \sigma g'(u)(v-\bar{h}) \geq 0$$

for all $v \in K$, where

$$\sigma = \begin{cases} 1 & \text{if } \varepsilon_i > 0 , \ \varepsilon_i \to 0 \\ -1 & \text{if } \varepsilon_i < 0 , \ \varepsilon_i \to 0 \ . \end{cases}$$

We recall that K is the closed cone hull of $C_u^{\perp}(V)$, cf. section 2. Since (A_1) is assumed, (12) implies $\bar{h} \in K$. Again, because of hypothesis (A_1) there is a subsequence with $h_\varepsilon \to \bar{h}$. That may be shown as follows. For a subsequence h_ε we have $f''(u)(h_\varepsilon,h_\varepsilon) \to a$. Consequently, (11) implies

$$- a + f''(u)(\bar{h},v) - \lambda g''(u)(\bar{h},v-\bar{h}) - \sigma g'(u)(v-\bar{h}) \geq 0.$$

Introducing $v = \bar{h}$ into this inequality, we obtain

$$f''(u)(\bar{h},\bar{h}) \geq a \ .$$

On the other hand, it holds that

$$\lim_{\varepsilon\to 0} f''(u)(h_\varepsilon,h_\varepsilon) \geq f''(u)(\bar{h},\bar{h})$$

which implies $a = f''(u)(\bar{h},\bar{h})$.

Let H_0 be the linear space $H_0 = K - K$ and assume that

(14) $$\lambda < \Lambda_{H_0}$$

is satisfied, where $\Lambda_{H_0}^{-1} = \max_{w \in H_0 \setminus \{0\}} \dfrac{g''(u)(w,w)}{f''(u)(w,w)}$.

The assumption (14) yields $I''_\lambda(u)(v,v) \geq c\|v\|^2$

for all $v \in H_0$ with a positive constant c. It is well known, see [5], that this inequality implies the existence of a unique solution $\bar{h} \in K$ of (13) . Thus, we have proved the following theorem.

Theorem 2. Assume that hypothesis (A_1) and $\lambda < \Lambda_{H_0}$ are satisfied. Then there exist eigenfunctions $u(\varepsilon)$ of (1) to the eigenvalues $\lambda(\varepsilon) = \lambda + \varepsilon$, with

$$u(\varepsilon) = u + \varepsilon u_1 + o(\varepsilon) ,$$

where $\|\varepsilon^{-1} o(\varepsilon)\| \to 0$ as $\varepsilon \to 0$ and where $u_1 \in K$ is the unique solution of (13) with $\sigma = 1$ for $\varepsilon > 0$ and $\sigma = -1$ if $\varepsilon < 0$.

4. The Simply Supported Beam

The deflection of a simply supported beam being axially compressed by a force proportional to λ is given by (1) where (cf. [8])

$$f(u) = \frac{1}{2} \int_0^1 (1-u'^2)^{-1} u''^2 dx, \quad g(u) = - \int_0^1 (1-u'^2)^{1/2} dx ,$$

(15) $$V = \{v \in H, \ |v'| < 1 \text{ and } v \leq d \text{ on } [0,1]\} ,$$

$$H = \{v \in H^2(0,1), \ v(0) = v(1) = 0\} .$$

In this section, we show that (15) satisfies the assumptions and hypotheses of 2. and 3., while a numerical continuation method will be presented in the last section.

While it is clear that the assumptions (2)-(5) are satisfied for the above problem, it remains to show the hypothesis (A_1). Let w be the function defined in (A_1). From [6] we have for this w that

$$w(k) = w'(k) = 0, \quad w(1-k) = w'(1-k) = 0 \ ,$$

$$\text{and} \quad w \leq 0 \quad \text{on} \quad [k,1-k] \ ,$$

where $[k,1-k]$, $k \leq \frac{1}{2}$, is the interval over which the beam is in contact with the obstacle. For sufficiently small $\varepsilon > 0$ one defines

$$w_\varepsilon(x) = \begin{cases} w(\frac{k}{k-\varepsilon}x) & , \quad 0 \leq x \leq k-\varepsilon \ , \\ 0 & , \quad k-\varepsilon \leq x \leq k \ , \\ w(x) & , \quad k \leq x \leq 1-k \ , \\ 0 & , \quad 1-k \leq x \leq 1-k+\varepsilon \ , \\ w(\frac{kx-\varepsilon}{k-\varepsilon}) & , \quad 1-k+\varepsilon \leq x \leq 1 \ . \end{cases}$$

Then there is a $t_\varepsilon > 0$ sufficiently small such that $t_\varepsilon w_\varepsilon \in C_u^\perp(V)$. Thus, $w_\varepsilon \in K$. For $\varepsilon \to 0$ we have $w_\varepsilon \to w$ and since K is closed this finally yields $w \in K$.

Since the eigenvalue inequality (14) is also satisfied as long as $\lambda < \lambda_{crit}$, λ_{crit} as defined in [8], the above theorems can be applied to the problem (15), local continuation of solutions is possible and this will be exploited in the next section.

5. The Numerical Continuation Method

Only recently have continuation methods as they are frequently used for parameter-dependent boundary value problems been generalized to variational inequalities of the form (1). We refer to [3,4,11].

Typically, continuation methods proceed by predicting a new point along the solution curve and then iteratively correcting it. Since singular points as fold (turning) points and bifurcation points have to be expected, the parametrization of the solutions by λ is not always feasible. We propose to parametrize both u and λ by the arclength s. In addition, a functional r(u) of the solution will be introduced, and continuation will be done in λ and r while the arclength is only used internally.

In order to be consistent with the notations used in continuation for boundary value problems, we assume that (u,λ) are parametrized by the arclength s along the solution curve instead of by ε as, for example, in Theorem 2. We denote the first-order derivatives with respect to s at the known point (u_0,λ_0) by $(\dot{u}_0,\dot{\lambda}_0)$.

In the following, we introduce a notation suitable to describe the numerical solution of the discrete problem which is similar to that used in [10,11]. It may, for example, easily be generalized to include upper and lower bounds instead of upper bounds only as considered in the application below. We assume that the functional $I_\lambda(v)$, cf. section 2, is discretized by, for example, finite elements or finite differences, and we suppress the discretization parameter h to denote the resulting functionals again by f(u), g(u) where $u \in \mathbf{R}^n$, $n = n(h)$.

Let the closed convex subset V of $\mathbf{R}^n$ be

$$V = \{u \in \mathbf{R}^n,\ u \leq d\} \tag{16}$$

where inequalities between vectors are to be understood componentwise. Instead of (1) we then have the finite-dimensional variational inequality

$$(17) \qquad u \in V: \nabla f(u)(v-u) \geq \lambda \nabla g(u)(v-u), \text{ for all } v \in V .$$

We present now an algorithm that in the terminology of optimization is a feasible direction method. Starting from a (feasible) vector $u_0 \in V$ the method generates a sequence of iterates $\{u_k\}$, $u_k \in V$, that will converge to the solution of (17). For each k we define the set of active constraints by

$$I_k = I(u_k) = \{i \in \{1,\ldots,n\},\ u_{ki} = d_i\} .$$

Two matrices P_k and Q_k are defined by

$$P_k = (e_i)_{i \in I_k} \quad , \quad Q_k = E_n - P_k P_k^T$$

where $e_i \in \mathbb{R}^n$ is the i-th unit vector and E_n the nxn identity matrix. We further use the notation F_k, G_k for the Hessian matrices of f,g at $u = u_k$. Since we implement an active set strategy, the deactivation of constraints has to be controlled in order to assure convergence.

In the following, we attempt to continue from a given solution (u_0, λ_0) with $\|u_0\| = r$, where $\|u\| = (u^T u)^{1/2}$, to a solution (u_t, λ_t) with $\|u_t\| = r_t \neq r$. It is clear how the method could be modified to do continuation in λ or for more general normalizing conditions as used, for example, in [12]. The following version, however, permits already to continue along solution paths that have several turning points and, in fact, several points of non-smoothness (transition points) as was demonstrated in [11].

The Continuation Method

Predictor Let u_0, λ_0, $\dot{u}_0$, $\dot{\lambda}_0$ be given. Compute $\overline{\delta s}$ from $\|u_0 + \overline{\delta s}\dot{u}_0\| = r_t$ and δs from

$$\delta s := \min\{\overline{\delta s}, \min_{i \notin I_k} ((d_i - u_{0i})/\dot{u}_{0i}, \dot{u}_{0i} > 0)\} .$$

Set $u_1 = u_0 + \delta s\dot{u}_0$, $\lambda_1 = \lambda_0 + \delta s\dot{\lambda}_0$, $I_1 = I(u_1)$.

Corrector Set $k = 1$.

(i) Compute $q_k = \nabla f(u_k) - \lambda_k \nabla g(u_k)$ and terminate the iteration if $P_k^T q_k \leq 0$, $\|Q_k q_k\| = 0$.

(ii) Set $|q_{kj}| := \max\{|q_{kj}|, (P_k^T q_k)_i > 0\}$.
If possible, deactivate the j-th constraint, $\tilde{I}_k = I_k - \{j\}$ and determine $\tilde{Q}_k$, otherwise let $\tilde{I}_k = I_k$, $\tilde{Q}_k = Q_k$.

(iii) Solve the linear system

$$\begin{bmatrix} F_k - \lambda_k G_k & -\nabla g(u_k) \\ u_k^T & 0 \end{bmatrix} \begin{bmatrix} \delta u_k \\ \delta\lambda_k \end{bmatrix} = \begin{bmatrix} \lambda_k \nabla g(u_k) - \nabla f(u_k) \\ \frac{1}{2}(r_t^2 - u_k^T u_k) \end{bmatrix}$$

but in the 'free' variables ($i \notin \tilde{I}_k$) only.

(iv) Determine the maximal admissible steplength $\bar{\alpha}_k$ from

$$\bar{\alpha}_k = \min_{i \notin \tilde{I}_k} ((d_i - u_{ki})/\delta u_{ki}, \delta u_{ki} > 0)$$

and set $\alpha_k = \min(\bar{\alpha}_k, 1)$, $u_{k+1} = u_k + \alpha_k \delta u_k$,

$\lambda_{k+1} = \lambda_k + \alpha_k \delta\lambda_k$. Set $k = k + 1$ and go to (i).

The corrector is a projected Newton step for the nonlinear system corresponding to (17) augmented by the normalizing condition $\|u\| = r_t$. For further comments on this method, in particular a specific deactivation strategy for (ii), we refer to [11]. The beam problem (15) was now discretized in the same way as described in detail in [8]. The numerical method given above was successfully used to compute the shape of the simply supported beam after it has buckled at the critical load $\lambda_0 = (2\pi)^2$ and is in contact with the obstacle but before it lifts off again at $\lambda = \lambda_{crit}$(cf. [8]).

In this one-dimensional example, it is preferable to solve instead the free boundary value problem over the part $[0,k]$ of the interval over which the beam is not in contact, thus avoiding having to discretize the contact region. This method was used in [8] to compute λ_{crit} and the following results were obtained using it to compute the beam in contact. $d \equiv .0125$ was used and cubic Hermite finite elements on 12 equidistant subintervals. $\lambda_{crit} = (4\pi)^2$ and $k_{crit} = 1/4$ hold for this case as was shown in [8]. For numerical results for a problem in two-dimensions and for the above continuation method we refer to [11]. Instead of plotting the deflection for various values of λ, we only list the point of first contact in Table 5.1 and the approximate values for the deflection and its derivative at the points $x_i = ik/12$, k as in Table 5.1, in Table 5.2 for a sample value of λ.

λ	50	60	70	80	90	100	110	120	130	140	150
k	.4381	.4000	.3703	.3464	.3266	.3098	.2954	.2828	.2717	.2618	.2529

Table 5.1 Point k of first contact for given load values λ for the simply supported beam.

i	$u(x_i) \times 10$	$u'(x_i) \times 10$
0	0	.2534
1	.0209	.2478
2	.0411	.2342
3	.0598	.2135
4	.0765	.1870
5	.0908	.1565
6	.1025	.1241
7	.1115	.0918
8	.1179	.0618
9	.1219	.0362
10	.1241	.0165
11	.1249	.0042
12	.125	0

Table 5.2 Approximate values of solution and derivative of simply supported beam for $\lambda = 80$

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