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LINEAR PROGRAMMING: THE CONCEPTS WITHOUT THE SIMPLEX

by

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Introduction

This paper addresses what the author perceives as a common misunderstanding by students of the basic concepts underlying the linear programming model, solution generation, and solution interpretation. In particular, the pedagogic approach recommended in this paper was created to be used in place of the teaching (and learning?) of the algebraic simplex method. Fewer and fewer introductory management science courses utilize the simplex method to demonstrate the generation of improved solutions and the manipulation of the optimum solution in an attempt to perform "what if" sensitivity analysis.

To replace this void, this paper presents a three "chapter" approach which utilizes an easy to understand two-dimensional linear programming problem as the delivery vehicle for the concepts of basis, basic variables, active constraints, positive and negative shadow prices, right-hand-side ranges, microeconomic marginal revenue and marginal cost analysis, activity analysis, internal resource allocation analysis, decreasing returns to scale, and linear programming as an iterative boundary "search" process. With these concepts clearly demonstrated and firmly in hand, students are well prepared to run linear programming software and fully and correctly utilize the complete parametric output which is commonly provided.

As the reader will note, the body of this paper consists of Chapters 3 through 5. These chapters report on the actual linear programming solution methodology and constrained resource sensitivity analysis. The omitted Chapters 1 and 2 present the students with a typical introduction to management science and a "generic" problem which can be modeled and solved with linear programming. This problem, originally found in Bierman, Bonini, and Hausman's Quantitative Analysis for Business Decisions, 3rd edition (1969), is paraphrased and modified below:

Generic Problem for Linear Programming

A firm is trying to determine the quantities to produce of its two products, product A and product B. Its answer to this question is constrained by a limited supply of the resources necessary to produce the products, and its desire to find a solution that maximizes its profit for the production period. What we know is that the production of each product (A and B) competes for the use of two resources. Let us call these resources production time on machine 1 and production time on machine 2. In particular, each unit of product A requires 2 hours of machine 1 time and 2 hours of machine 2 time, and each unit of product B requires 3 hours of machine 1 time and 1 hour of machine 2 time. This is clearly a scarce resource allocation problem, as we have only 12 hours of machine 1 capacity available for use in the production of products A and B, and only 8 hours of machine 2 time is available for similar use.

We also know that it costs $5 per hour for materials and labor to operate machine 1, and $7 per hour to operate machine 2. The wholesale price that we receive for a unit of product A is $30 and the wholesale price that we receive for a unit of product B is $29.

The Constrained Model

If we let X_1 and X_2 represent the unknown quantities of products A and B, the generic problem can be formatted as follows:

$$\text{MAX:} \quad Z = \$6X_1 + 7X_2$$

$$\text{S.T.:} \quad 2X_1 + 3X_2 \leq 12$$

$$2X_1 + 1X_2 \leq 8$$

Solution Method and Output Interpretation

Chapter 3 presents a simplex-compatible iterative technique for finding the optimum solution.

Chapter 4 introduces the concepts of shadow prices and right-hand-side ranges, along with a preview of marginal revenue and marginal cost analysis.

Chapter 5 extends the parametric analysis of the resource constraints through the derivation of the firm's demand function for each resource. Negative shadow prices are discussed, along with revenue and cost analysis of outputs (activity analysis).

Chapter 3

MARGINAL BENEFIT ANALYSIS

Refer to Figure 1. This geometric representation of the machine 1 and machine 2 constraints is called a geometric simplex. You can look upon the task of the decision maker (you!) as that of finding the best, feasible, production plan on this graph. Note that this graph is scaled in units of Product A (X_1) and units of Product B (X_2). This means that all points on this graph represent different product mixes of A and B (i.e., each point represents a different production plan). Which one is the best?

This type of problem is quite common in management science. The general name given to this kind of decision situation is that of a "search problem." This name makes sense if you think of our task as that of "searching" the positive quadrant (Figure 1) for the best, feasible, production plan.

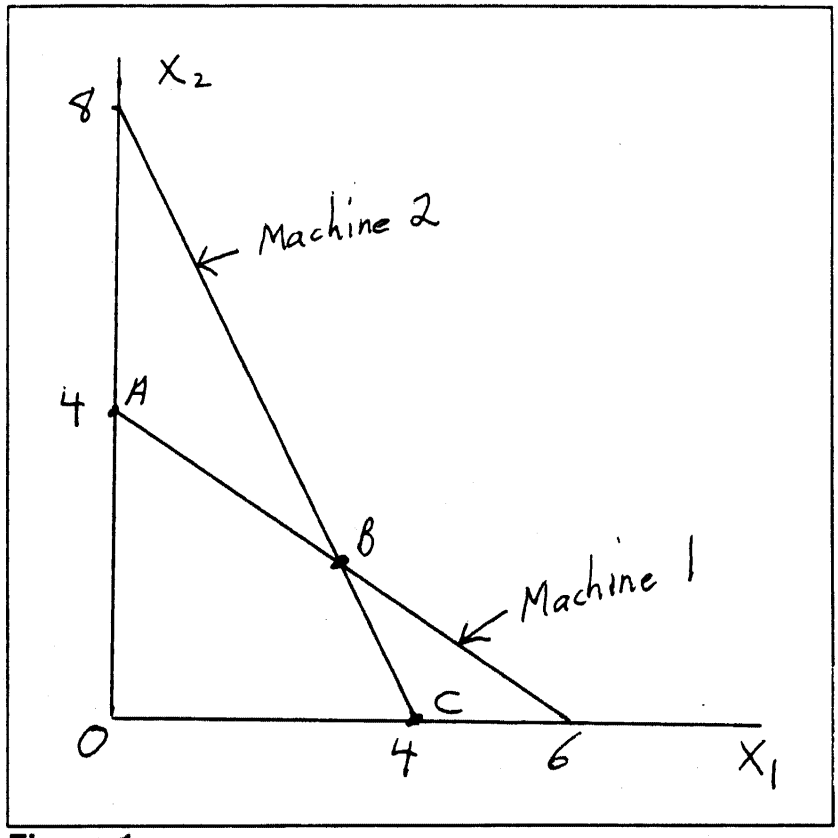

Figure 1

In proceeding with our analysis of this problem, we will utilize a geometric approach. This geometric method is called "marginal benefit analysis" and parallels almost exactly the algebraic "simplex method" that is implemented in most linear programming software packages. Both methods

start with the solution (i.e., production plan) where both X's equal zero, and then "search" for improved, higher profit, solutions to the problem. Let us see how this method works.

The First Iteration

Recalling what we learned in Chapter 2 about how the best solutions will never fall within the area formed by the feasible boundaries, marginal benefit analysis searches for improved solutions by analyzing new solutions that lie on these boundaries. Starting our boundary search with the solution at the origin, we note that there are two feasible boundaries that are candidates for our improved solution search (the O towards C boundary and the O towards A boundary). Each one of these boundaries must be evaluated separately.

Arbitrarily beginning our analysis with the O towards C boundary, the first question that we must answer in this marginal analysis is which of our two variables, X_1 or X_2, is the marginal variable. The marginal variable is defined as that variable that is increasing in value for the boundary move under consideration. It is clear that in moving from point O to point C variable X_1 is increasing and thus is the marginal variable. What is happening to the value of X_2?

The second question that we must answer is what is the incremental profit (what we also call "marginal benefit") brought about by a one unit increase (i.e., a marginal increase) in the marginal variable? This incremental profit is calculated by noting that at the origin (point O) profit is equal to \$0.00. If we move from point O towards point C far enough to make X_1 take on a value of 1, the new value of profit will be \$6.00. This represents an increase in profit, or marginal benefit, of \$6.00. Note that this \$6.00 marginal benefit represents the incremental profit of searching for improved solutions along the O towards C boundary. It is this figure that we will use to compare the relative goodness of searching on this boundary as compared to searching along the other feasible boundary that also leaves from the customary starting point of the origin.

Performing the above analysis for the boundary which starts at point O and ends at point A should produce X_2 as the marginal variable. The marginal benefit of increasing the value of X_2 by one unit is \$7.00. (Profit changes from a value of \$0.00 to a new value of \$7.00).

It is now decision time! Which of the two feasible boundaries looks the most promising? Strictly based upon incremental (marginal) analysis, the O towards A boundary is the winner. The marginal benefit of moving from the solution at the origin towards the solution at point A is \$7.00. Note that since this boundary is linear, the incremental profit will remain at \$7.00 per unit of X_2 for each unit of X_2 added along this boundary. It is this fact that allows us to conclude that since we wish to move in the direction of point O towards point A, we also wish to move as far in that direction as we can (i.e., as far as is feasible). The final result of this analysis is to cause us to move from the solution at point O to the solution at point A.

The work that we have just performed is referred to as the first "iteration" in marginal benefit analysis. An iteration is a set of formalized procedures that allow us to arrive at an improved solution to our problem. Later, you will see that the same iterative method, based on marginal analysis, is the heart and soul of the simplex method of linear programming. What is another way of knowing that the solution at point A is better than the solution at point O?

The next iteration follows the same steps as the first iteration, with the one difference that now we will start our analysis from the best solution found so far (point A).

The Second Iteration

Keeping in mind that our methodology is one of searching along feasible boundaries for improved solutions, we note that there are two feasible boundaries which leave from point A. We can move from point A towards point B or from point A back towards point O. The latter movement is one that we do not have to formally analyze. Can you see why that is true? (If incremental profit was positive in our

move from point O to point A, it must be negative for a move from point A to point O. Please demonstrate this to your own satisfaction.)

To analyze the move from point A towards point B, the first task is to identify the marginal variable. Since we have defined the variable that is increasing in value to be the marginal variable, X_1 is our candidate. (Note that X_2 is decreasing in value as we move from point A towards point B.)

The second task is to calculate what the impact on the profit function will be if, starting at point A, we allow X_1 to increase by one unit. Referring to Figure 2, it is clear that if we start from point A and move far enough along the A-B line segment, the value of X_2 will decrease in value. The question that we must answer is if we move far enough along the A-B segment to allow X_1 to increase by <u>one</u> unit, exactly how much does X_2 decrease? Once we have calculated the answer to this question, we can compute what the impact on profit will be of making this change in the values of X_1 and X_2. <u>That change in profit will represent the marginal benefit (or marginal loss) of moving from point A towards point B.</u> Referring to the geometry of this question (Figure 2), we must determine the change in coordinates of moving from point A to point P.

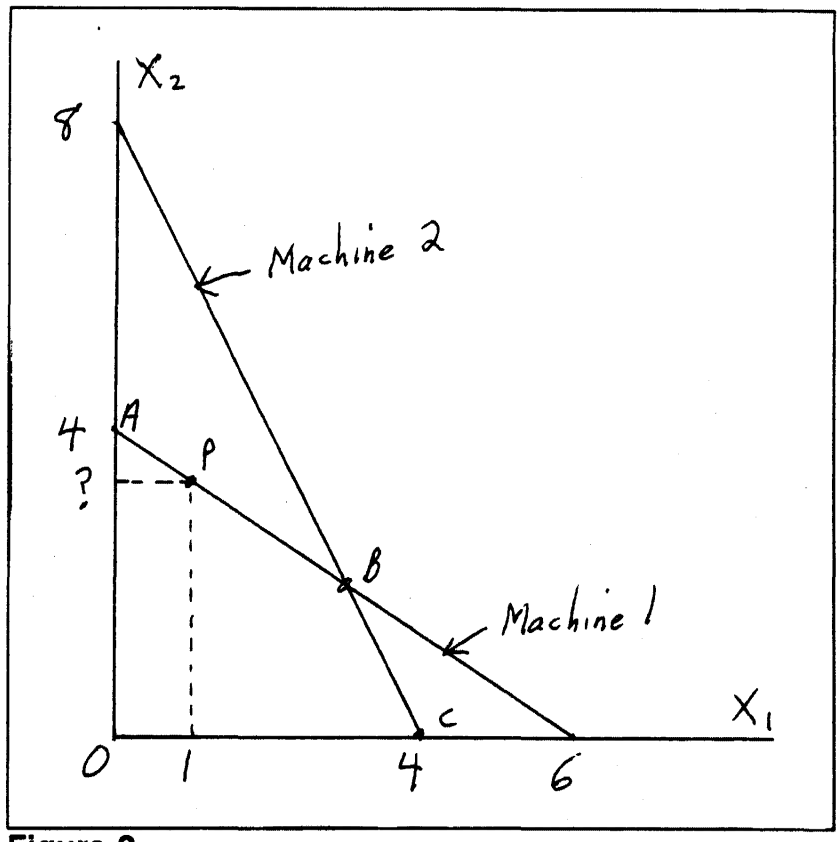

Figure 2

The algebra involved in answering the preceding question is based upon the fact that we are interpreting the impact of moving on a straight line. Since the equation for that line is given by the formula $2X_1 + 3X_2 = 12$, and since we already know what is happening to the value of X_1 (it is the marginal variable and thus its value automatically increases by one unit), we can concern ourselves with using this equation to solve for the unknown variable (X_2). Please note the following algebraic manipulations:

$$2X_1 + 3X_2 = 12$$

$$3X_2 = 12 - 2X_1$$

$$X_2 = 4 - (2/3)X_1$$

Translating this algebraic statement into English, we are told that for all points on this line, the value of X_2 equals 4 minus 2/3 times the value of X_1. Therefore, if the value of X_1 increases by one unit, the value of X_2 will decrease by 2/3 units. Figure 2 simply confirms this analysis.

As mentioned earlier, the final calculation is to convert this change in production into a change in profit. The change in profit equals the change in the quantity of X_1 times its unit profit plus the change in the quantity of X_2 times its unit profit. These calculations are shown below:

$$(+1)(\$6.00) + (-2/3)(\$7.00) = \$1.33$$

We conclude that the marginal benefit of moving from point A towards point B is a positive \$1.33. This means that total profit will increase by \$1.33 for each marginal "step" that we move in the direction of point B. (Remember that a straight line has a fixed slope; this is a mathematical way of saying that the production trade-off between product A and product B will remain at its +1 and -2/3 values all of the way from point A to point B.) Thus, we conclude that we should move "as far as we can" on this boundary. We cannot move further than point B without leaving the feasible set of

production possibilities. The final result of this iteration is to move us from the solution at point A to the solution at point B.

The Third Iteration

At this point, as is true at the end of each iteration, we are faced with the question as to whether improved solutions can be discovered by searching further along the feasible boundaries. Improved solutions are indicated when our "search" turns up a positive marginal benefit. A negative marginal benefit, on the other hand, indicates that only reductions in the value of the objective function will occur if we move in that direction. Such moves will not be taken for obvious (?) reasons. This time, we will start with the solution at point B since it represents the best solution found so far. (Can you discuss, using English, how you know that point B is the best production plan discovered so far?)

Marginal benefit analysis works by first identifying the variable that will be allowed to increase by one unit in our profit analysis (i.e., the "marginal" variable). Moving from point B towards point C, it should be clear that variable X_1 is increasing and that variable X_2 is decreasing in value. As was true in the second iteration, the question is by how much will X_2 decline if X_1 is allowed to increase by one unit. This question is answered, again, by taking the equation for the constraint on which we are moving (the second machine's constraint) and solving it for the unknown X_2:

$$2X_1 + 1X_2 = 8$$

$$1X_2 = 8 - 2X_1$$

In English, this says that X_2 equals 8 minus 2 times the value of X_1. This means that if X_1 increases by one unit, the value of X_2 will decrease by two units. The calculations necessary to find the profitability of these changes are shown below:

$$(+1)(\$6.00) + (-2)(\$7.00) = -\$8.00$$

Since the marginal benefit is a negative eight dollars, total profit would drop by eight dollars for each "step" that we take along the B-C line segment. When we arrive at this type of situation, where each feasible boundary that leaves from the best point found so far has a negative marginal benefit associated with it, we may stop iterating. We have found the best production plan. (How do you know that moving from point B towards point A has a negative marginal benefit?)

Once we have identified the optimum point, we can solve for the values of X_1 and X_2 by solving simultaneously the two equations which pass through that point. That bit of algebra should produce values of 3 for X_1 and 2 for X_2. The value of profit for this production plan is $32.00 (found by substituting the optimum values for X_1 and X_2 into the profit function).

Concluding Thoughts

Students many times believe that the answer to this production problem is intuitively obvious by simply examining Figure 1 and noting that point B is furthest distance from the origin. Since both X_1 and X_2 represent products with positive profit contributions, doesn't it make sense to operate as far away from the 0,0 production plan as possible? Think about this for a minute before reading further.

The "key," of course, is to realize that the graph in Figure 1 is scaled in units of X_1 and X_2. Units of profit appear nowhere on this graph. That is why it is necessary in each iteration of marginal benefit analysis to convert units of X^1 and units of X_2 into units of profit. For those who still doubt the necessity of this, consider the following alteration in our basic problem:

> Change the profit contribution of product B to $12.00 per unit. Keep everything else the same. Now, referring back to iteration number two, the incremental profit calculations of moving from point A towards point B, will reveal a negative marginal benefit for that move:
>
> (+1)($6.00) + (-2/3)($12.00) = -$2.00

It is not "good" to trade 2/3 units of X_2 for one additional unit of X_1 if the profit contribution of X_2 is \$12.00.

You should be able to demonstrate, using marginal benefit analysis, that the production plan represented by point C becomes the optimum plan if, for instance, the profit contribution of product A were to increase to \$15.00 per unit. (In this case, and the one illustrated in the previous paragraph, a change in the relative profitability of our products calls for complete specialization in the production of either one product or the other.)

As an additional challenge, return to the original objective function and change the profit contribution of product B to a new value of \$9.00 per unit. Your marginal benefit analysis calculations in iteration number 2 should now turn up a marginal benefit of \$0.00. What does this mean? Should you move to point B or remain at point A?

Finally, let us introduce two new words into your management science vocabulary. Whenever you are examining a linear programming solution, the constraints that pass through that solution point are called "active" constraints. Likewise, constraints that do not pass through that point are referred to as being "inactive." Applying these definitions to the above problem, we can say that in the solution at point A, constraint #1 is active and constraint #2 is inactive. At point B, both constraints are active. At point C, constraint #1 is inactive while constraint #2 is active. (Which constraints are active at point 0?)

Chapter 4

POSTOPTIMUM ANALYSIS

This chapter looks at topics which are typically addressed after the optimum solution has been found. In many applications of linear programming, the answers to these questions are as meaningful (if not more so) than finding the solution to the original problem. Note that some textbooks and journals refer to these procedures as "sensitivity analysis." This name is justified in that many of the questions addressed refer to checking how "sensitive" the present optimum solution may be to changes in the original model. In such cases, we may correctly refer to these procedures as part of a "what if" analysis.

Introduction

"What if" management has just been informed that it would be possible to increase the available capacity of machine 1 by one hour (new capacity would be 13 hours rather than 12 hours). What kinds of questions do you think management would be insistent on asking before making a decision as to whether to obtain the additional machine 1 hour?

Though probably not an exhaustive set, questions such as what will be the machine 1 operating cost for this one additional "unit of capacity" (i.e., hour), what will be the impact on our production plan (i.e., how will we use this new capacity in the production of product A and product B), and what will be the impact on our bottom line if we obtain and use this additional capacity, certainly are some of the questions that would be raised.

Shadow Prices

In the above scenario, let us keep all of the assumptions of the original problem. In particular, we will assume that the operating cost of machine 1 will remain at its previous level of $5.00 per hour. It is this operating cost that has produced the profit coefficients in the original objective function, thus these coefficients will not change.

What will be the impact on the original total profit of $32.00 if we expand the capacity of machine 1 and use the new thirteenth hour in manufacturing products A and B? This incremental profit (loss), which we will calculate below, is referred to as the "shadow price" of the machine 1 resource. Since this shadow price is found by calculating the new product mix and its new profit, it is sometimes also called the "imputed value" of a resource (i.e., the value of a resource as "imputed" from the value of the output it is responsible for producing). It is important to note that the concept of a shadow price is marginal in nature. That is, the shadow price of machine 1 is "the change in profit per unit (i.e., one hour) change in machine 1 capacity."

Referring to Figure 1, we can picture what is happening to the optimum solution. Because the new location for the machine 1 constraint has shifted above (and parallel to) the original location, the new set of feasible boundaries consists of OA'B'C.

At this juncture, challenge yourself to solve this problem by using marginal benefit analysis. You should find that the marginal benefit of moving from point 0 to point A' is still $7.00, the marginal benefit of moving from point A' to point B' is the same as previously found for the point A to point B move (+$l.33), and the marginal benefit of moving from point B' to point C is the same as was earlier found for the move from point B to point C (-$8.00). Thus, we conclude that point B' represents the new profit maximizing production plan. (Why have the marginal benefits not changed from those found in the original version of this problem?)

Solving for the coordinates of point B' (the new machine one equation solved simultaneously with the old machine two equation) yields a new X_1 value of 2.75 and a new X_2 value of 2.5. When these values are substituted into the objective function, we find that the new profit is $34.00. Since this profit is $2.00 greater than the best profit possible when we had only 12 machine 1 hours, we conclude that the shadow price of machine 1 is $2.00 per hour. Note that this shadow price is truly a "value added" concept. The $5.00 machine 1 operating cost is already netted out of the profit coefficients in the objective function. The net return of having a thirteenth hour of machine 1 capacity is to increase profit revenue by $2.00.

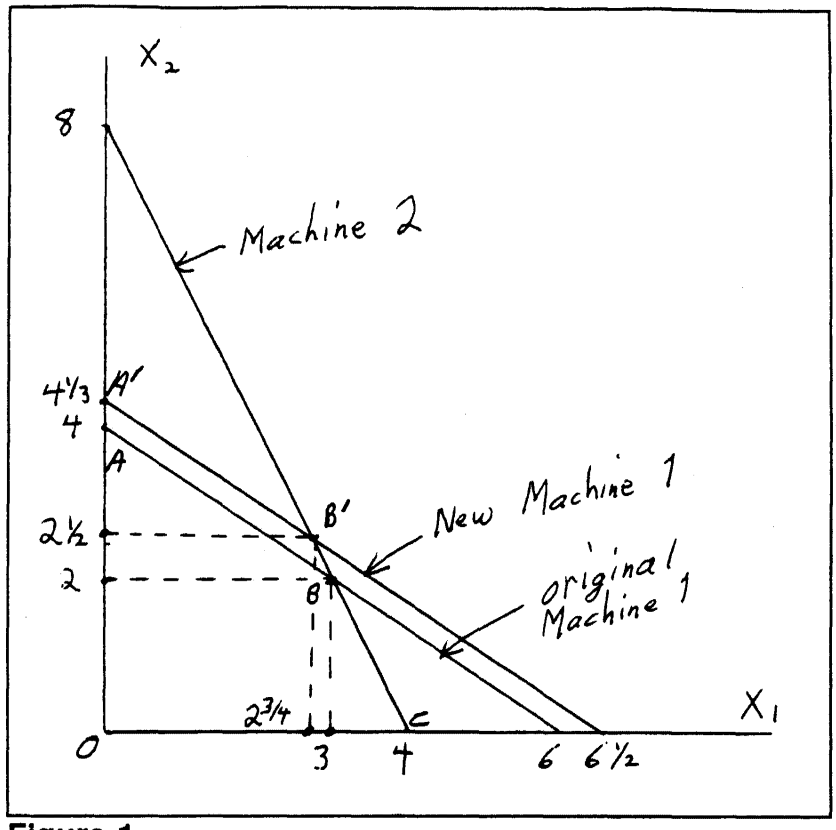

Figure 1

Carefully examine the new production plan for products A and B. The changes in production levels that are required in order for the firm to continue to maximize profits would surprise many people. After all, faced with an increase in the available capacity of one of your resources (and no change in the other), you are told to reduce the scheduled production of one of your two outputs! Can you explain this apparent contradiction to management?

Look closely at the revised output levels for products A and B. Product A is being reduced by .25 units. As a result, we will "lose" $1.50 in product A profit (25% of $6.00) However, we must combine this effect with what is happening to the output level of product B. Our analysis shows that product B will increase by one-half unit

over its original value. That additional .5 unit of product B will produce $3.50 in new profit! What is the net effect? Combining the gain of $3.50 with the loss of $1.50 produces a net gain of $2.00! This analysis has calculated the production impacts and their incremental profit generated by having one additional hour of capacity on machine 1; this is an alternative way of calculating the shadow price of a resource.

Here is a challenge question! Demonstrate that if the capacity of machine 1 were to decline by one hour, the production of product A should be increased by .25 units and the production of product B should be reduced by .5 units. Your calculations, of course, should also indicate that profit will go down to $30.00 as a result of these changes. Machine 1 clearly has a marginal value to the firm of $2.00 per hour!

Finally, let us calculate the shadow price of the second machine. Referring to Figure 2, the impact of having one more hour of machine 2 capacity is to shift the machine 2 constraint outward from its original location. The new set of feasible boundaries is 0AB'C'. Can you demonstrate with marginal benefit analysis that point B' is the new optimum solution? (You should find that the marginal benefits of 0 towards A, A towards B', and B' towards C' are unchanged from the original problem. Do you know why?) Solving the new machine 2 constraint (machine 2 has 9 hours of capacity now) simultaneously with the original machine 1 constraint,

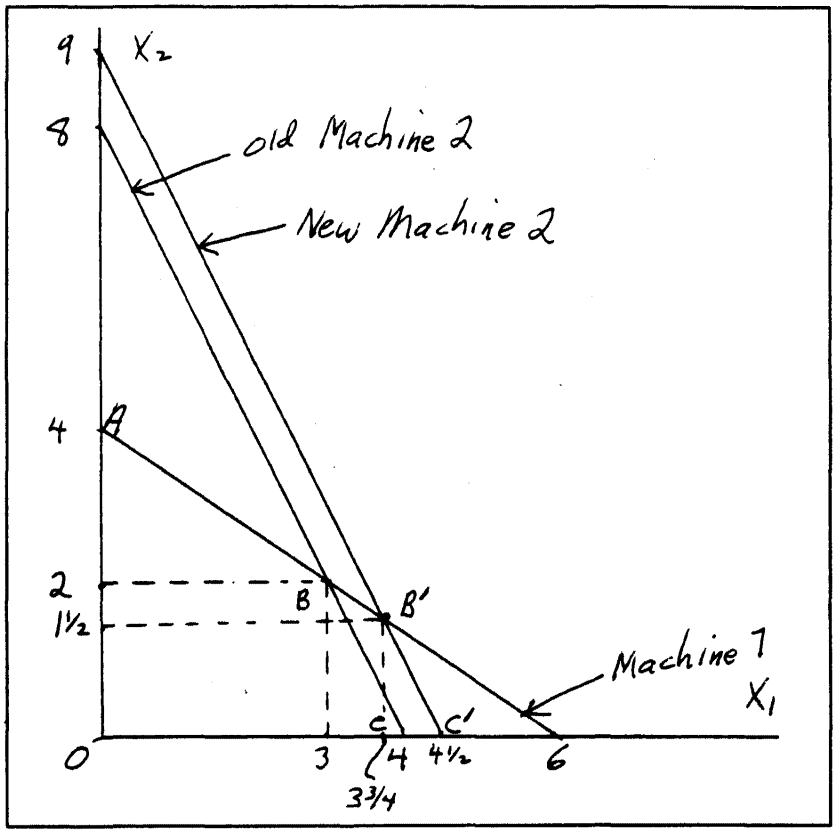

Figure 2

produces a solution where X_1 equals 3.75 and X_2 equals 1.5. This production plan has a profit of $33.00. Since the original production plan had a profit of $32.00, we conclude that the shadow price of machine 2 is $1.00 per hour!

An "incremental" output analysis indicates that the impact of having the ninth machine 2 hour is to reduce the output level of product B by .5 units (which reduces product B profit by $3.50) and increases the output level of product A by .75 units (which increases product A's profit by $4.50). Not too surprisingly, the net profit of these output tradeoffs is an increase of $1.00. Imputed values are revenue values of inputs (i.e., "resources") derived from the revenue value of the outputs these resources are responsible for producing! Do you see this?

Now its your turn. Demonstrate that the effect of having one less hour of machine 2 capacity is to cause the scheduled output of product A to decrease by .75 units and the scheduled output of product B to increase by .5 unit. The marginal impact on profit is to reduce profit by $1.00. Hopefully, these results are not surprising.

Marginal and Total Revenue and Cost Analysis

One of the several uses of the information generated in the previous section is to allocate total revenue amongst the various resources responsible for producing it. For instance, we know that the profit produced by the optimum production plan is $32.00. If we calculate the total imputed value of the resources allocated to this production plan, we find that this value also equals $32.00:

(machine 1 capacity used in the production plan)x(machine 1 shadow price per hour) + (machine 2 capacity used in the production plan)x(machine 2 shadow price per hour) =
(12 hours)x($2.00) + (8 hours)x($1.00) = $32.00

This result should not be surprising. After all, a resource's shadow price is a "value added above cost" concept, and the value of the objective function for our problem is a net revenue (i.e., profit) concept. These concepts are identical, and thus we would expect their valuations to be identical. As demonstrated earlier, we have again demonstrated that a resource shadow price is a value that is imputed from the value of the output it is responsible for producing.

The final topic in this section deals with an important idea that you learned in microeconomics. That is, a firm (or any entity) is at an optimum level of operations if the marginal cost of each activity it is engaged in is equal to the marginal revenue of that activity (MR = MC).

Our firm is engaged in two "activities": the production of product A and the production of product B. The marginal revenue of producing product A is $6.00 (i.e., the increase in profit per unit increase in the production of product A is $6.00), and the marginal revenue of producing product B is $7.00. The cost (or internalized valuation) of the resources required in the production of one unit of product A can be calculated as follows:

(units of machine 1 resource required)x(shadow price of machine 1) + (units of machine 2 resource required)x(shadow price of machine 2):

(2 M_1 hours)($2.00/hr) + (2 M_2 hours)($1.00/hr) = $6.00

Indeed, marginal revenue does equal marginal cost for product A. Similar calculations for product B reveal:

(3 M_1 hours)($2.00/hr) + (1 M_2 hours)($1.00/hr) = $7.00

Again, marginal revenue equals marginal cost for the current level of product B "activity." Since each activity that this firm is engaged in has MR = MC, we may conclude that this firm is operating at an optimum level. In the next chapter, we will

examine how these calculations will differ when the firm is operating at a nonoptimum solution.

Right-Hand-Side Ranges for the Constraints

In an earlier section of this chapter, entitled "Shadow Prices", we discovered that machine 1 was returning $2.00 in revenue above its $5.00 cost. Management can find several uses for this information. For instance, assume that management can obtain additional hours of capacity on machine 1 for the present marginal operating cost of $5.00. Should it seriously consider this capacity expansion?

Superficially, the answer would appear to be "yes," since the firm will net a $2.00 profit for each hour of expanded capacity at machine 1. This answer, however, avoids the question of "how many" hours of expanded machine 1 capacity will continue to yield a shadow price of $2.00. (Do you remember the "law of diminishing returns" from your microeconomics course? How does it relate to the situation being posed to our firm?)

Figure 3 is a more extensive analysis of the impacts of expanding machine 1 capacity. As noted earlier in this chapter, increasing machine 1 capacity to 13 hours caused the optimum production plan to shift to point B'. The move from point B to B' caused the production of product A to drop by .25 units and the production of product B to increase by .5 units; it was this production "trade-off" that produced an increase in profit of $2.00. What do you think the impacts on production and profit will be of increasing machine 1 capacity to 14 hours?

Point B" represents the optimum solution to the problem when 14 hours of machine 1 time are available. (Make certain that you can discuss how we know that B" is the new optimum!) Point B" represents an additional cutback of .25 units of product A and an additional .5 units of product B. Thus, the shadow price of the 14th hour of machine 1 capacity remains at $2.00. It is the fixed slope of the machine 2 line between points B and B' and between points B' and B" that causes the unchanging tradeoff between products A and B. As long as that tradeoff remains

unchanged, the shadow price of expanded machine 1 capacity will remain at $2.00 per hour! Do you see that the machine 1 line can shift upwards with these same tradeoffs until it passes through point B"'? (The machine 2 line leaves the positive quadrant at that point, and thus the shifting point B can no longer shift along the slope of the machine 2 line).

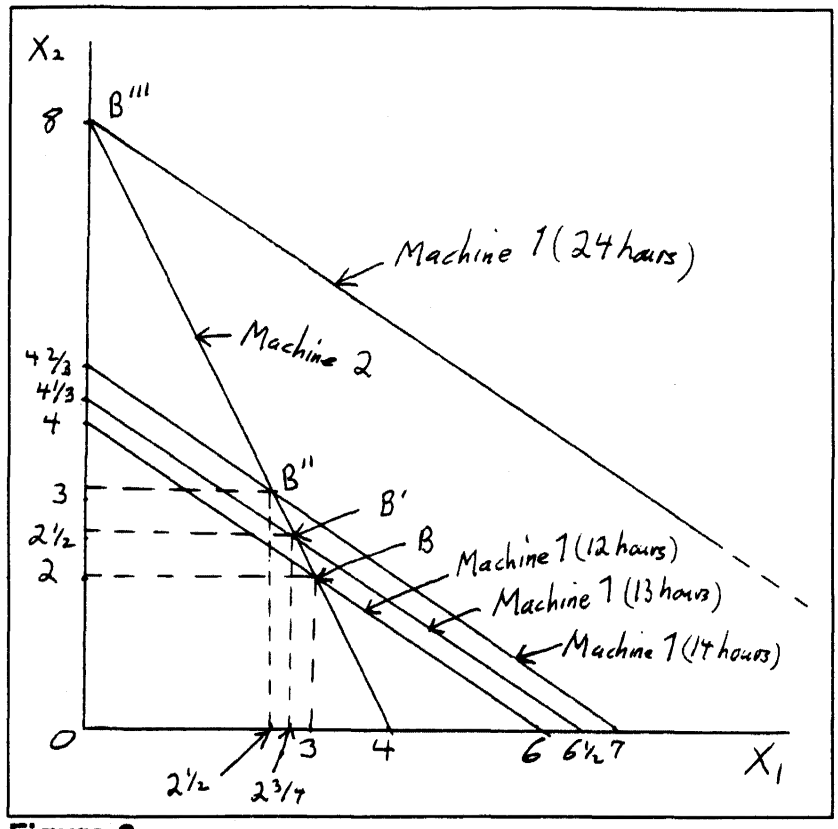

Figure 3

Back to our original question! An increase of how many machine 1 hours would cause the optimum point to shift to the B"' position? The answer is calculated by finding how many units of products A and B would be produced at point B"', and then using the machine 1 constraint to find the machine 1 capacity required to produce that quantity of products A and B. For this problem, we can read directly off of the graph that at point B"' we would be producing 8 units of product B and 0 units of product A. Substituting these values into the machine 1 constraint, we find that if 24 hours of machine 1 capacity were available, we would be operating at the B"' optimum!

(2)(**0**) + (3)(**8**) = 24 hours required for the B"' solution.

What profit can we expect from this production plan? Substituting the product A and B values into the objective function indicates that profit will have risen to $56.00. Thus, the profit at B"' is $24.00 higher than the profit at point B. This $24.00 increase in profit was brought about solely by increasing the capacity of machine 1 by

12 hours (from 12 to 24), thus generating an incremental profit per hour of $2.00. This, of course, is the shadow price of machine 1 within this range of machine 1 capacity. (As a "challenge question," can you guess what the shadow price of machine 1 would be for its 25th hour of capacity? Remember that a shadow price simply measures the impact on profit of one more unit of a resource.)

We have discovered that we can increase the capacity of machine 1 by 12 hours while maintaining its shadow price of $2.00. What if management were concerned with the opposite question. What if management had other uses for this machine in which its shadow price (value added) was greater than $2.00? Wouldn't it make economic sense to reallocate machine 1 capacity from its low return use into its higher return uses? If this action was undertaken, we would lose the $2.00 imputed value in this use and gain the higher machine 1 imputed value in the new use. It would then become a very relevant question to find out how much machine 1 capacity could be shifted out of this use before its shadow price would change from $2.00. (Remember the economic "law of diminishing returns to scale," once again. What does it tell you will happen to the imputed value of a resource if you have less and less of it to combine with the other resources? This is the reverse of the "returns to scale" question raised earlier.)

Refer to the graph presented in Figure 4. There you see the results of reducing the capacity of machine 1 by various amounts of hours. In particular, referring to point B', the results of using only 11 hours of this machine will cause the production of product A to increase by .25 units and the production of product B to decline by .5 units. (These changes are similar, though opposite in sign, from those calculated for an increase in machine 1 capacity). This production "tradeoff" will cause profit to drop by $2.00 (i.e., the shadow price of machine 1). These results were reported earlier in this chapter.

As long as this product A and B tradeoff continues to exist, the shadow price of machine 1 will continue to be $2.00. Can you see that this tradeoff continues until the machine 1 constraint intersects at point B'''? (This tradeoff exists, of course, as long as the points B, B'', B''', etc, are moving along a line of fixed slope--the machine 2 constraint line.)

What machine 1 capacity would cause the machine 1 constraint to intersect at B"'? This question is answered, as previously, by taking the product A and B production quantities at point B"' (4 units of product A and 0 units of product B, as read directly off of the graph), and substituting them directly into the machine 1 constraint:

(2)(**4**) + (3)(**0**) = 8 hours of machine 1 capacity.

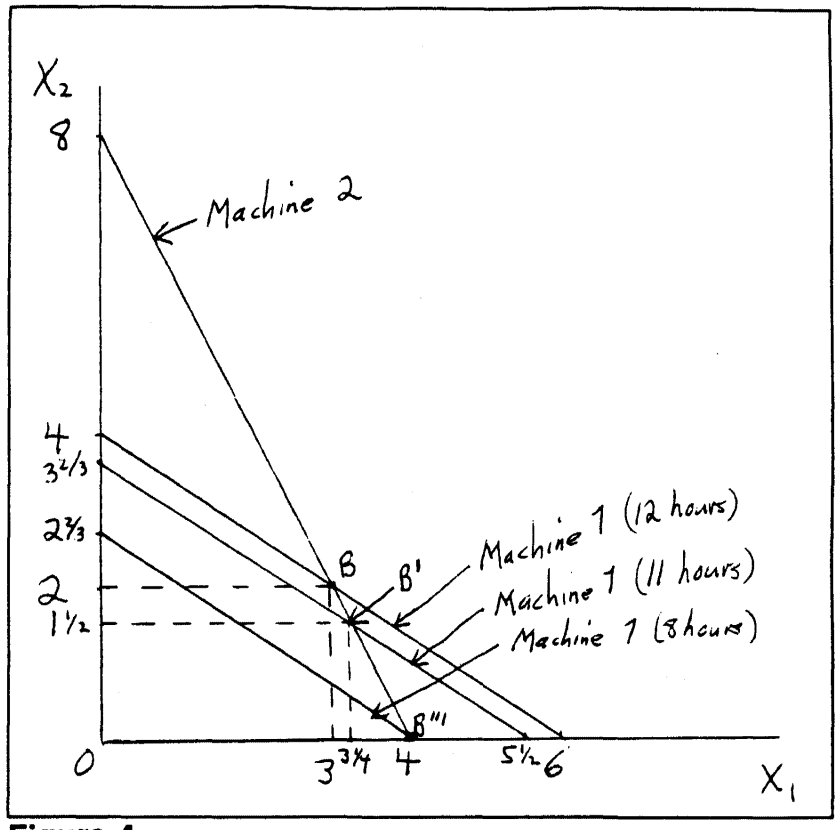

Figure 4

Note that this solution produces $24.00 in profit--a reduction of $8.00 from that at point B. This $8.00 profit reduction was caused by reducing machine 1 capacity by 4 hours. This analysis simply supports our earlier shadow price calculation. (As a "challenge question," see whether you can calculate what the new machine 1 shadow price would be if only 7 hours of capacity were available!)

We now have sufficient information to be able to inform management that as long as they allocate between 8 and 24 hours of machine 1 capacity to the production of products A and B, this resource will earn a return above cost of $2.00 per hour. (This conclusion assumes ceterus paribus pertains to all other parameters in this problem.) The importance of this information cannot be overstated in terms of helping management make intelligent resource allocation decisions. This conclusion will become even more evident when we see how shadow prices change as we alter resource usage outside of these upper and lower boundaries.

The information which we have presented above can be efficiently stated as follows:

$SP_1 = \$2.00;\ 8 \leq RHS_1 \leq 24$ hours

where RHS_1 stands for the right-hand-side of constraint number 1.

The analysis that we have just performed for the machine 1 resource we will now do for machine 2. The shadow price of machine 2 is \$1.00 per hour for the present level of machine 2 utilization (8 hours of use). How much additional machine 2 capacity could we add without changing that value-added figure? How much of that 8 hour capacity could we drop with the same result?

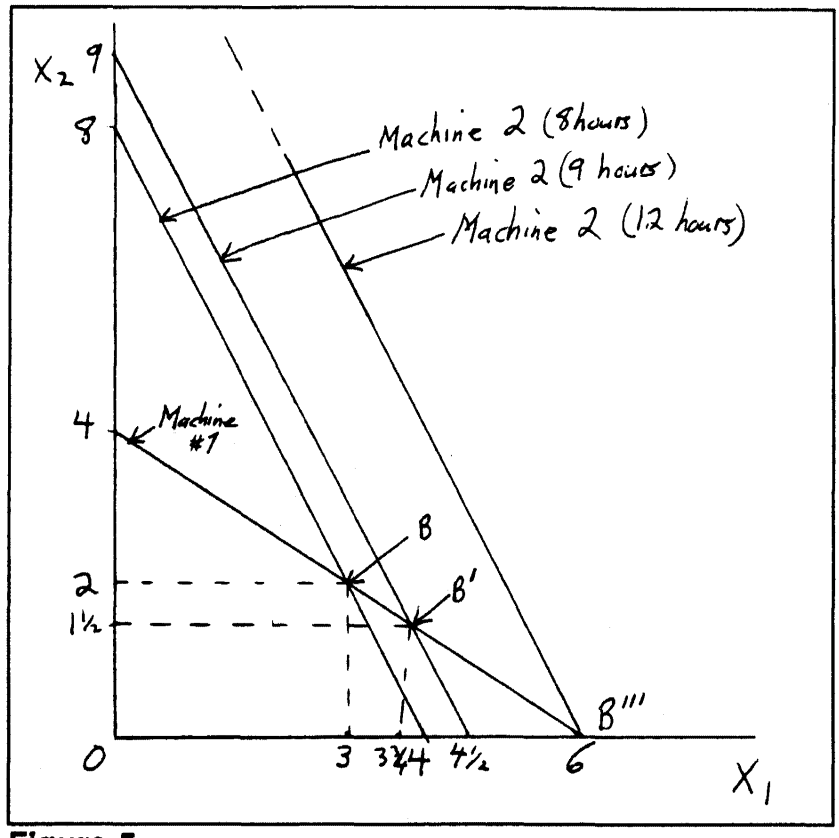

Figure 5

Referring to Figure 5, we see that as more capacity is added to machine 2, the optimum solution moves from point B to point B' to point B", etc. It is due to the fact that these shifting optima occur along a line of fixed slope that allows the shadow price of machine 2 to remain constant at \$1.00 per hour. (An earlier section in this chapter reported that for each additional hour of machine 2 capacity, the value of X_1 increased by .75 units while the value of X_2 fell by .5 units). Point B"' represents the furthest that the optimum can shift along the machine 1 line. The production plan represented by point B"' calls for 6 units of product A and 0 units of product B. The amount of

machine 2 capacity required for this production plan is 12 hours. That is, (2)(**6**) + (1)(**0**) = 12. We can add 4 hours to the 8 we already have, and each of those 4 hours will contribute $1.00 above its operating cost of $7.00 per hour.

On the other hand, if the capacity of machine 2 is to be reduced, how much can it be reduced at a loss of $1.00 per hour? The information found by examining Figure 6 indicates that the new optima will shift in just the opposite direction along the machine 1 constraint, thus causing the product A and B production tradeoffs to be just the opposite of those reported in the previous para-graph. Those tradeoffs will remain constant until the capacity of machine 2 has been reduced enough to cause point B''' to be the new optimum. The production plan at point B''' calls for the production of 0 units of product A and 4 units of product B. The machine 2 capacity required for this output mix is 4 hours. That is, (2)(**0**) + (1)(**4**) = 4.

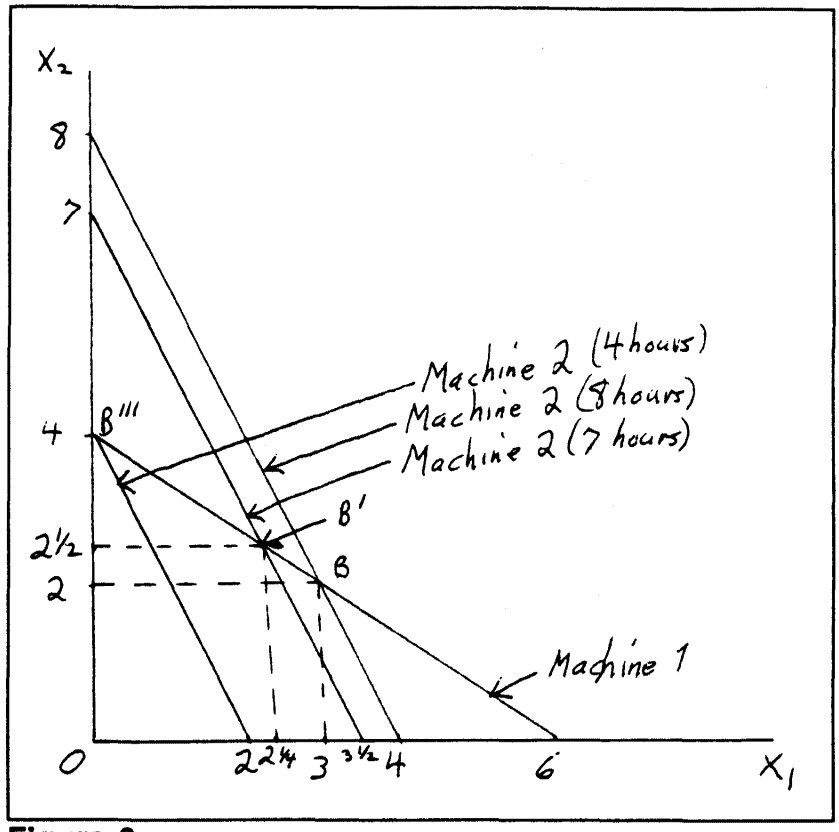

Figure 6

Our conclusion is that as long as we have between 4 and 12 hours of machine 2 capacity, each hour will return $1.00 above its operating cost of $7.00 per hour. You should challenge yourself to think about what would happen to this resource's shadow price if we had more than 12 hours or less than 4 hours available. Again:

$$SP_2 = \$1.00; \quad 8 \leq RHS_2 \leq 12 \text{ hours}$$

In summarizing the above approach to finding right-hand-side ranges for a particular constraint, note that we have traced the movement of the optimum point upward and downward on the other constraint that passes through the original optimum, until that shifting intersection "bumps" into another corner point (B''' in each of our examples). It is at that point that our ability to shift the optimum along a line of unchanging slope ends. This observation allows us to call this the "adjacent corner point rule" for finding the upward and downward stopping points for the shift.

Chapter 5

ADDITIONAL TOPICS

Now that you have learned how to derive shadow prices and right-hand-side ranges for the constraints, we can use these concepts and techniques in an extended analysis of some special topics.

Negative Shadow Prices

Is it possible to have an optimum solution to a maximization problem where the shadow prices (i.e., "values added") for one or more constraints are negative? Up until this point we have seen only positive shadow prices. Positive shadow prices, of course, mean that if we can obtain more of some constraining resource, the resource will be put to a profit-making use (at least within the RHS range). A negative shadow price means that an increase in the RHS of a constraint will cause the value of the objective function (profit?) to drop in value.

Let us create a linear programming problem that exhibits these latter characteristics:

MAX: $Z = \$60X_1 + 7X_2$

ST: $2X_1 + 3X_2 \geq 12$ hours (machine 1, again!)

$2X_1 + 1X_2 \leq 8$ hours (machine 2)

Note that this is our basic "product A and B" problem, with two important changes. Now, the profit contribution of product A has been increased to $60--loading the deck in its relative favor, compared to product B. Also, the machine 1 constraint is rewritten as a "greater than" constraint. (This could happen in a situation where management

has determined that machine 1 must be operated at least 12 hours for it to reach peak operating efficiency; any amount over 12 hours is also OK. Or, perhaps, we have a labor contract that requires that at least 12 "units" of a particular labor class be used on this job!)

Figure 1 presents the simplex schematic for this problem. Since we now have to be on or above the machine 1 constraint, and on or beneath the machine 2 constraint, the new feasible area is ABC. If you were to solve this problem with marginal benefit analysis, since the origin (point 0) is not a feasible point, it would be OK to arbitrarily choose any of the three feasible corner points for the start of the first iteration. Regardless of whether you start at point A, B, or C, you should arrive at point B as the optimum solution. (Challenge yourself by demonstrating this with your own calculations!)

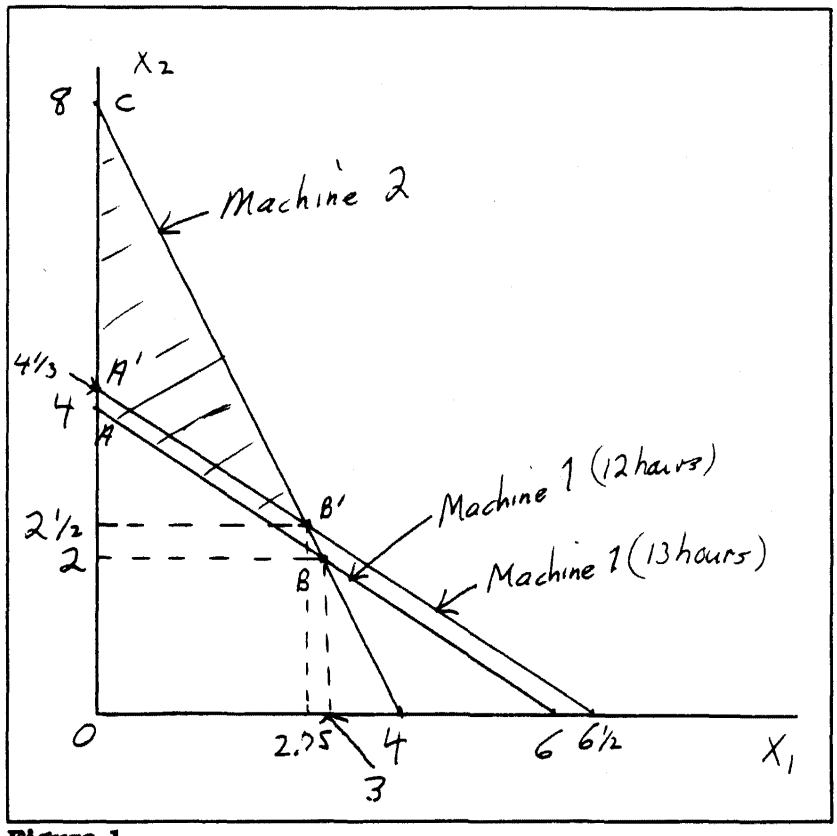

Figure 1

At point B, the values of X_1 and X_2 are 3 and 2, respectively. (How do we know this?) These values cause the objective function to assume a profit value of $194.00.

Now, observe what happens to the feasible area when we increase the RHS of machine 1 to 13 hours. Checking Figure 1, the feasible area becomes A'B'C. This is a more restrictive (i.e., smaller) feasible area than earlier. Challenge yourself to reason why you would not expect to do any better within this area, and probably

Following upon our discussions in Chapter 3, we know that the new optimum solution will be at point B'. At point B', the value of X_1 has decreased to 2.75 (a reduction of .25 units), and the value of X_2 has increased to 2.5 (an increase of .5 units). This new production plan produces a profit of $182.50, which is $11.50 less than the original profit. Our conclusion is that the shadow price of machine 1 is a negative $11.50 per hour!

Of course, the explanation of the above result is that an increase in the RHS of machine 1 reduces the amount of the highly profitable product A that we can produce. Can you see that a reduction in the RHS of this constraint will allow the optimum solution to shift downward to the right on the machine 2 line? Those solutions would represent higher levels of product A (and an increase in profit of $11.50 for each hour of reduced machine 1 required usage).

The results of this type of analysis can aid management in making resource allocation decisions. For instance, if constraint #1 were actually the result of a union contract that required management to use at least 12 hours of a particular skill class on this job, management might use their knowledge of the shadow price of this restriction to attempt to renegotiate the 12 hour requirement. Management knows that for every hour that they can reduce this requirement, they will benefit by $11.50.

Also, at this point, you should see the importance to management of knowing the right-hand-side range for this constraint. The shadow price is a negative $11.50 for RHS values falling only between 8 and 24 hours. (Be certain that you know how to calculate these numbers. Refer to Chapter 4.) As a challenge question, calculate what happens to the shadow price of constraint 1 if its RHS becomes larger than 24. Do the same for an RHS of less than 8. Comment on the "logic" of your conclusions!

Figure 2 presents the schematic of what happens to the optimum solution when the RHS of machine 2 increases to 9 hours. As can be seen, the quantity of product A increases to 3.75 units and the quantity of product B drops to 1.5 units. The impact of this new production plan on profit is to increase profit to $235.50. This is an increment of $41.50, and thus represents the shadow price of machine 2. The

corresponding RHS range for this shadow price is from 4 to 12 hours of machine 2 capacity.

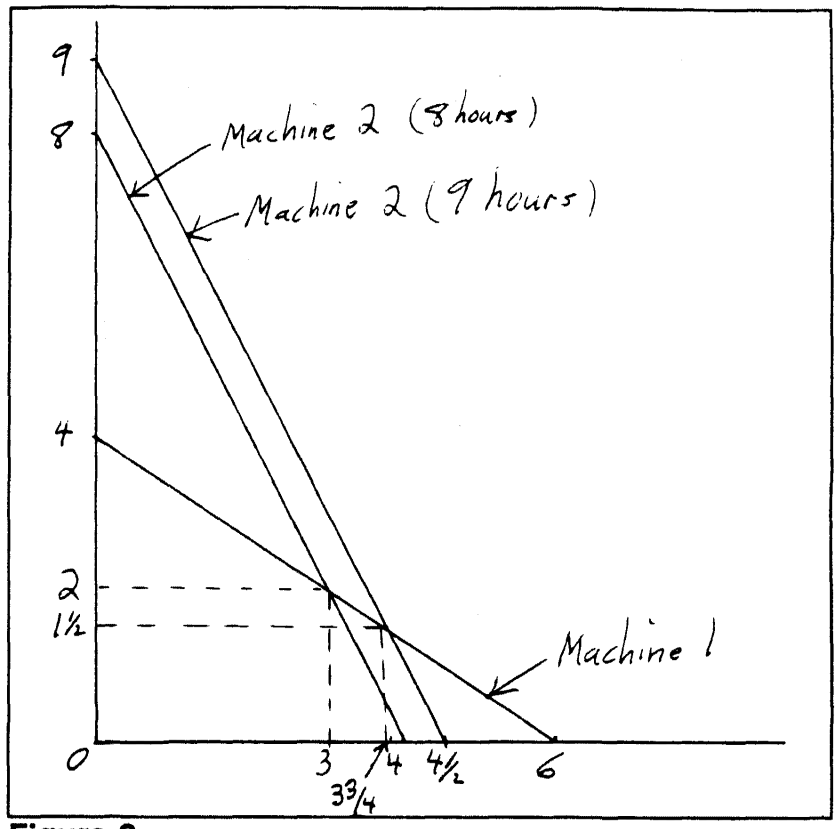

Figure 2

In Chapter 4 you were told that the <u>total</u> imputed value of your inputs will always equal the profit of the optimal solution. That conclusion holds regardless of whether inputs have positive or negative imputed values. For this problem, optimum profit equaled $194.00; therefore, it should not be surprising that:

(-$11.50)x(12 hours) + ($41.50)x(8 hours) = $194.00

Here is a great challenge question for you to contemplate! When you have a linear programming problem with an objective function that must be maximized, why is it that at the optimum solution, all "active" less-than constraints will have <u>zero or positive</u> shadow prices, all "active" greater-than constraints will have <u>zero or negative</u> shadow prices, and all strict equality constraints (we haven't seen one of these yet) will have either <u>positive, zero, or negative</u> shadow prices?

To help you get started, solve the above problem two more times. For your first effort, create a new constraint that requires the production of product A to be exactly 2 units (i.e., $1X_1 + 0X_2 = 2$). Once you have created a new graph with this constraint added to the previous two machine constraints, note that to be feasible, only solutions that fall <u>on</u> this line, above the machine 1 line, and below the machine 2 line qualify. After finding the optimum solution ($X_1 = 2$, $X_2 = 4$, $Z = \$148.00$), your

shadow price analysis should find that the active machine 2 has a positive shadow price and the new equality third constraint has a positive shadow price.

For your second effort, drop the constraint that requires the production of exactly 2 units of product A. Instead, add a new third constraint that requires the production of exactly 6 units of product B (i.e., $0X_1 + 1X_2 = 6$). Your analysis should show that the new optimum requires that $X_1 = 1$, $X_2 = 6$, and $Z = \$102.00$. Shadow price computations should produce a positive shadow price for the active second constraint and a negative shadow price for the new equality third constraint. Use these two examples to help you understand what is happening to profit (and shadow prices) when we change the RHS values of the three types of constraints.

Interpreting Shadow Prices as Marginal Revenues--A Demand Concept

In Chapter 4, you learned how to derive the shadow prices of constraints at the optimum solution. Also, you learned how to calculate the range of usage of a constrained resource (the RHS range) that would keep its shadow price unchanged. Let us restate and reexamine the original product A, product B, machine 1, machine 2 problem:

MAX: $Z = \$6X_1 + \$7X_2$

S.T.: $2X_1 + 3X_2 \leq$ 12 hours of machine 1 capacity

$2X_1 + 1X_2 \leq$ 8 hours of machine 2 capacity

The graph for this problem is produced in Figure 3. The machine 1 and machine 2 constraints intersect at the optimum solution (point B).

Let us revisit our analysis of machine 1. In Chapter 4, Figure 1, we visually and mathematically derived a shadow price of $2 for this resource. The upper limit for

the RHS was found to be 24 hours, as found in Chapter 4, Figure 3. With a RHS of 24, constraint #1 would pass through the optimum solution at point D. A "challenge question" then asked for you to think about what the shadow price of constraint #1 would become <u>if we had 25 hours of machine 1 capacity.</u> It was hinted that the "law of diminishing returns to scale" had something to do with the answer to this question. Let us reason through this question together!

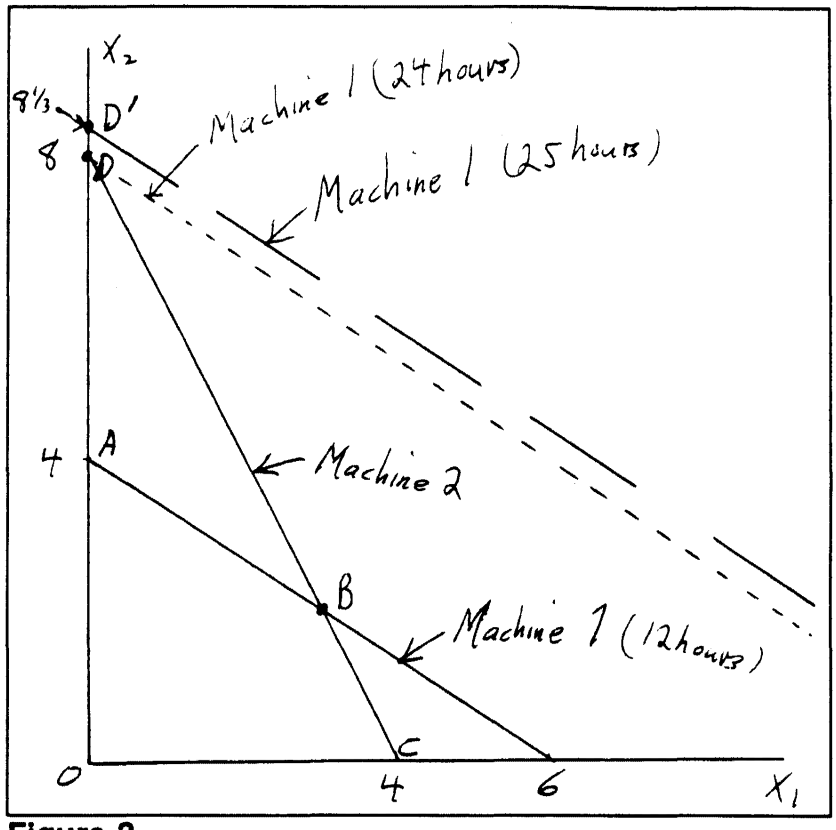

Figure 3

If the RHS of constraint #1 were 25, the location of this constraint would be the line that passes through point D'. Do you see that the feasible area <u>remains</u> the triangle 0DC? This is exactly the <u>same</u> feasible area that existed when we had only 24 hours of machine 1 capacity! (We cannot move above the machine 2 constraint without violating its capacity restriction.) What this means is that the expansion of machine 1 capacity from 24 to 25 hours will have no effect on the optimum solution. Point D will remain the optimum point and thus the optimum profit will remain at $56.00. It should now be clear that the shadow price (i.e., the incremental profit) of having more than 24 hours of machine 1 capacity is $0.00. This conclusion holds for <u>any amount</u> of machine 1 capacity above 24 hours. We have produced a clear demonstration of the "law of diminishing returns to scale."

Next, we will turn to the lower RHS boundary that maintains the shadow price of machine 1 at $2.00. Our analysis in Chapter 4 computed this lower limit to be 8 hours. (For review, refer to Chapter 4, Figure 4.)

In the graph in Figure 4, you can see that with 8 hours of machine 1 capacity, the optimum solution would be at point C. What happens to the shadow price of machine 1 if we have only 7 hours available? (Hint: Apply the reverse of the law of diminishing returns to scale!) The graph indicates that the new feasible area becomes 0A'C', with the new optimum becoming point C'.

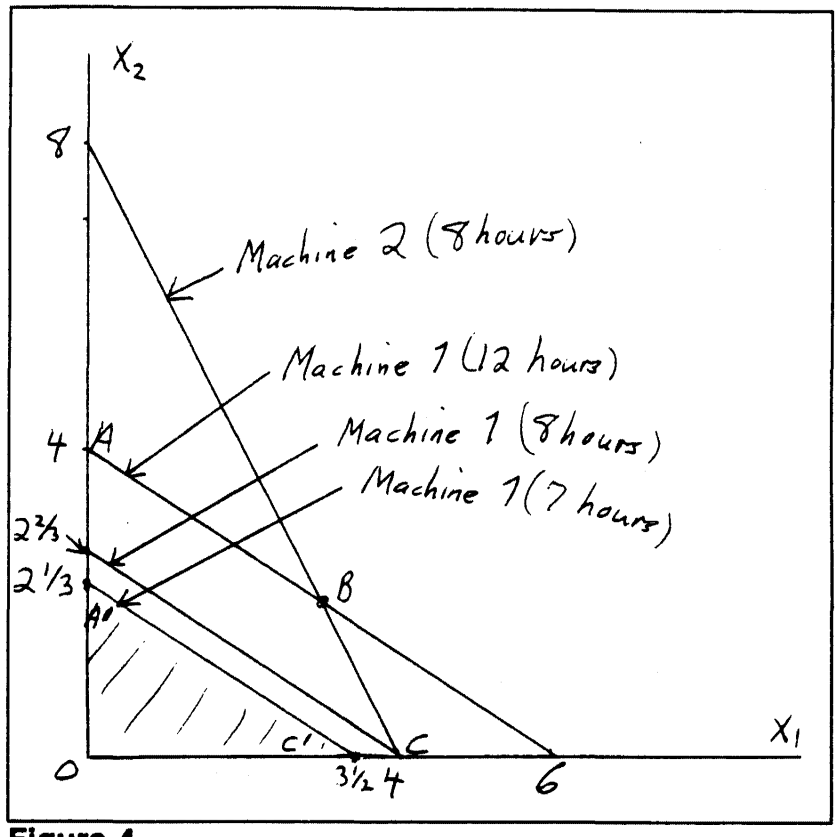

Figure 4

At point C', we are projected to produce 3.5 units of product A and 0 units of product B. (Be certain that you can verify this set of coordinates!) The profit for this production plan, of course, is $21.00. Since the profit for the production plan at point C was $24.00, we conclude that the shadow price of machine 1 increases to $3.00 per hour if we have less than 8 hours of that resource. Putting it another way, if we started out with 7 hours of machine 1 capacity and increased that to 8 hours, we would be able to produce another .5 units of product A and earn an additional $3.00 in profit.

Finally, demonstrate to yourself that if the RHS of constraint #1 fell to 6 hours, the optimum solution would move from point C' horizontally to the left along the X_1 axis. The value of product A would change to 3 units at a profit of $18.00. Five hours of machine 1 capacity would generate a solution that calls for 2.5 units of product A, 4

hours of machine 1 capacity would call for the production of 2 units of product A, etc, until a drop to 0 (zero) hours of machine 1 capacity would cause the solution to shift to the origin. The shadow price would remain fixed at \$3.00 per hour for each of these RHS changes. The conclusions of this analysis can be stated as follows:

SP_1 = \$3.00/hr.; $0 \leq RHS_1 \leq 8$ hours

SP_1 = \$2.00/hr.; $8 \leq RHS_1 \leq 24$ hours

SP_1 = \$0.00/hr.; $24 \leq RHS_1 \leq$ infinite number of hours

These relationships are graphed in Figure 5. Note that we have made the shadow price the dependent variable (a resource's return depends upon its relative scarcity--i.e., "return to scale," again). As discussed earlier, another name for shadow price is marginal revenue. Relating this concept to microeconomic theory, we have just derived the demand curve (or demand function) for the machine 1 resource!

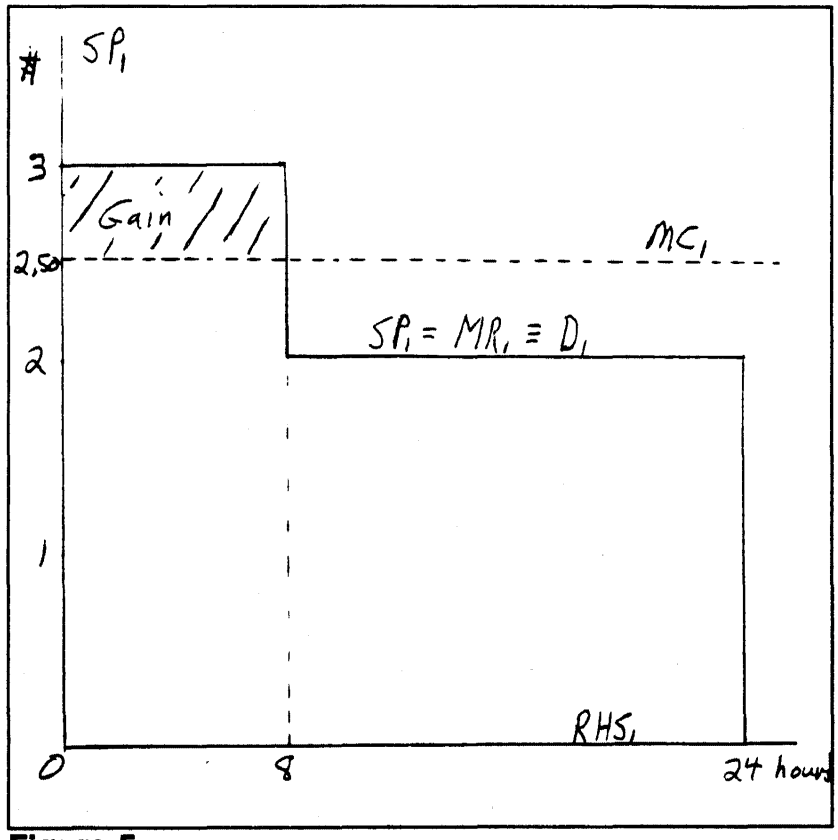

Figure 5

The information contained in this figure is of tremendous importance to management. It can be used to determine how much of a scarce resource to allocate to a particular use. For instance, in this problem, management desires to

know how many hours of machine 1 time should be used in the production of products A and B. In a purely microeconomic "theoretic" sense, a firm should utilize the quantity of a resource that causes its marginal revenue to equal its marginal cost. What is the marginal cost of machine 1?

The preceding question is certainly an interesting "challenge" question. Typically, one thinks of a firm's costs as being imposed from the outside (i.e., from the "market"). For the question at hand, however, the "cost" to the firm of allocating the machine 1 resource to the production of products A and B is the return (i.e., shadow price) that machine 1 would generate in **alternate uses** within the firm! These alternate-use shadow prices would have to be forgone if the firm allocates this scarce resource to the production of products A and B; it is in this sense that they represent costs to the firm.

The concept of allocating a scarce resource amongst its various potential uses according to its relative shadow prices in those uses is an exceedingly important use of the linear programming model. For instance, let us arbitrarily assume that machine 1 is currently being used elsewhere within the firm. In this other use, let us assume, its shadow price is $2.50. Now, what do you want to do? Do you wish to reallocate the machine 1 resource into the production of products A and B? If yes, how many hours would you wish to reallocate?

Both of the preceding questions can be answered by using the information in Figure 5. We see that the first 8 hours reallocated to the production of products A and B will return $3.00 per hour. Since this return is higher than the $2.50 shadow price of machine 1 in its alternate use, we will benefit by the difference ($.50 per hour). We give up the $2.50 return so that we can gain the $3.00 return. Now, how about reallocating more than 8 hours? Figure 5 indicates that such a decision would be economically unwise! Gaining a $2.00 imputed value here, at a "cost" of losing a $2.50 imputed value elsewhere, is clearly not to the benefit of our firm. Our conclusion is that we should reallocate exactly 8 hours of machine 1 capacity from its other use into the production of products A and B. Graphically, this conclusion follows from finding where the machine 1 marginal revenue and marginal cost curves intersect. We have demonstrated the relevance of the microeconomic "theoretic"

model! (Note that we also now know what the optimum production plan should be! In Figure 4, point C represents the optimum solution when we have 8 hours of machine 1 capacity.)

The preceding analysis has made one important assumption. As we reallocated machine 1 capacity away from its other use, we assumed that its shadow price would remain at $2.50 in that use! In practice, we would have to check the RHS range for machine 1 in its other use (that is, the RHS range that maintains its shadow price at $2.50), and make certain that a reduction of 8 hours falls at or above the lower boundary within this RHS range. What do you think would happen to the shadow price if we reallocated more hours than that specified by the lower RHS boundary?

Of course, the answer to the above question requires another visit to the law of diminishing returns to scale. The less you have of a resource, the higher the value of the remaining units (ceterus paribus). This means that if the reallocation of 8 hours of machine 1 capacity does "break" the lower RHS boundary in its other use, its shadow price will increase above $2.50 for those hours outside of the bound. If the new shadow price is higher than $3.00, you should not reallocate those marginal hours. That is, if the new marginal cost is greater than the $3.00 marginal revenue, do not reallocate those hours!

Figure 6 represents the hypothetical situation where the shadow price in the other use is $2.50 for reduced hours 1 through 4, $2.75 for reduced hours 5 and 6, and $3.50 for reallocated hours above 6. The graph indicates that we will incur a net gain in resource value-added for the first six reallocated hours, and would incur a loss of profit if we reallocated more than six hours. The total value-added is $2.50. (Be certain that you can replicate these calculations!)

Figure 6 presents a situation where the marginal revenue and marginal cost curves intersect, thus calling for a partial reallocation of machine 1 capacity. Note that there are two other possible situations. It is possible to hypothesize a marginal cost curve with all points above the marginal revenue curve. Do you see in that case no reallocation would be economically justified? Finally, it might happen that the marginal

cost curve could fall completely beneath the marginal revenue curve. In that case, the firm would reallocate out of the old use as many hours as they could utilize in the new use. (In the above problem, for instance, if the marginal cost were $1.50 per hour for any amount of hours, the firm would reallocate 24 hours into the production of products A and B.)

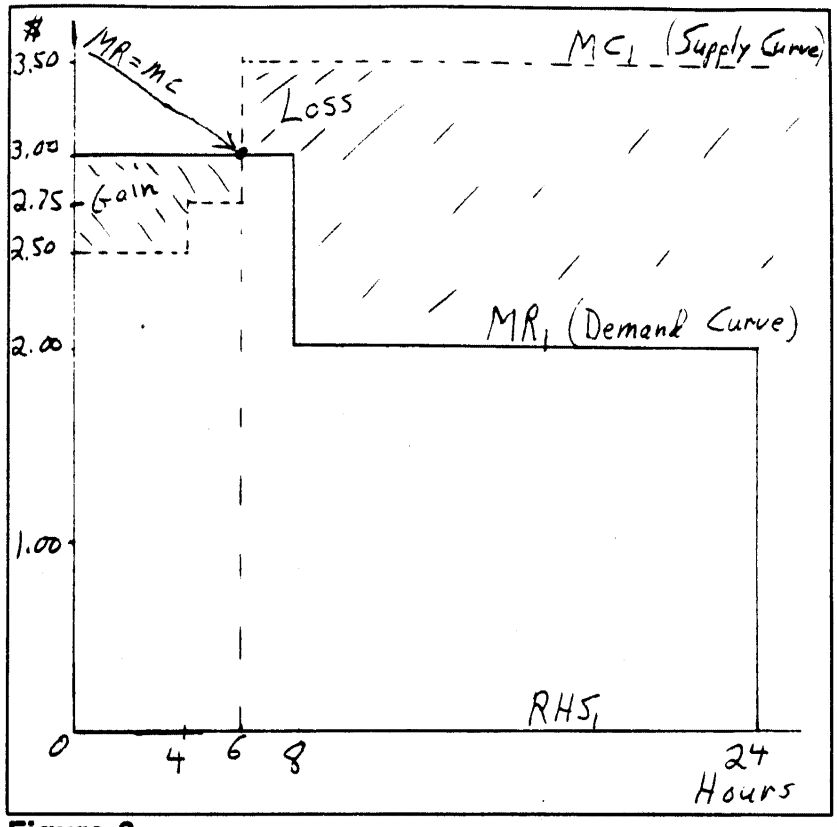

Figure 6

Finally, turning to machine 2, demonstrate with your own calculations that shadow prices and associated levels of utilization are as follows:

SP_2 = \$7.00/hour; $0 \leq RHS_2 \leq$ 4 hours of capacity

SP_2 = \$1.00/hour; $4 \leq RHS_2 \leq$ 12 hours of capacity

SP_2 = \$0.00/hour; $12 \leq RHS_2 \leq$ infinite upper limit

Revenue and Cost Analysis of Outputs (Activity Analysis)

In Chapter 4, you learned how to derive the shadow prices of constraints at the optimum solution. Does it make sense to you that constraint shadow prices can be calculated for solutions that are nonoptimum? Hopefully, after the following

demonstration, not only will you see that it is "possible" to do such calculations, but also the usefulness of that kind of information in managerial decision making.

Figure 7 reproduces the geometric simplex for the product A and B problem. Examine the solution at point A. What if management preferred that solution (a specialization in the production of product B) for reasons other than short-run revenue maximization (i.e., they might be trying to maximize product B's market share!). What are the shadow prices of machines 1 and 2 at point A?

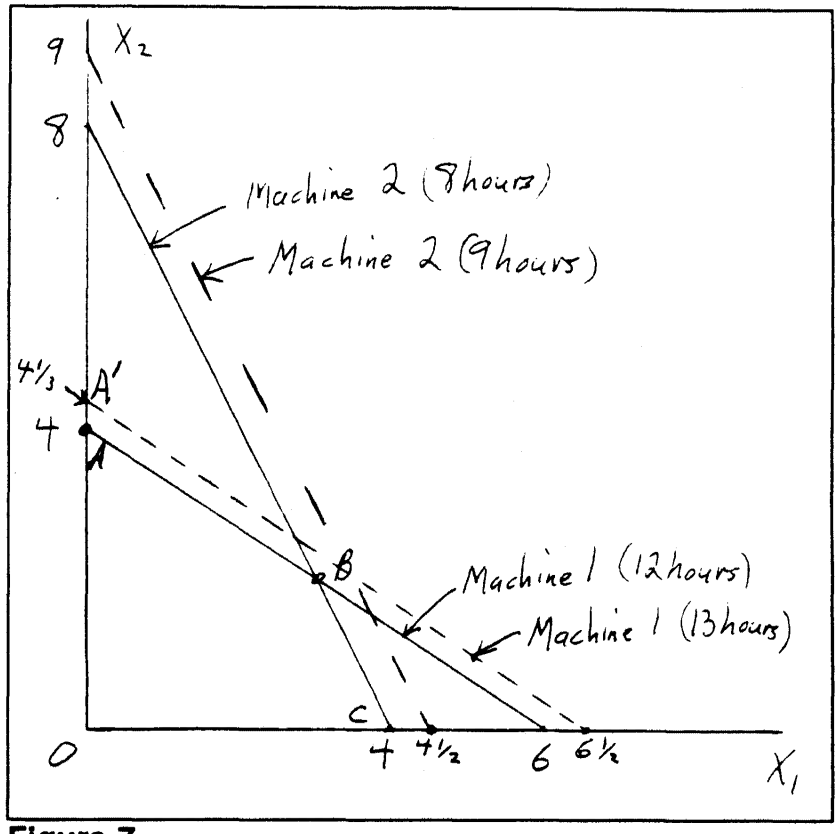

Figure 7

Let us first examine machine 2. Remembering that shadow prices represent the incremental profit of having one additional "unit" of a resource, do you see that a 9th hour of machine 2 capacity would have no impact on profit? (Plot the new machine 2 constraint on the graph!) Management would still be able to produce 4 units of product B and earn a profit of $28.00. The existence, or location, of point A would be unaffected! Another way of seeing this is to note that at point A the firm requires the use of exactly 4 hours of machine 2 capacity. Since we already have 8 hours available, we have 4 hours more than what we need. A ninth hour of machine 2 capacity would have no benefit to us <u>if our objective is to operate at a point that maximizes the output of product B</u> (point A on our graph).

Recall from the concluding paragraph in Chapter 3 that a constraint that does not pass through a particular solution is called an "inactive" constraint. As a general

rule, inactive constraints will always have zero shadow prices. (A small shift in their location will have a zero impact on the solution that we are examining.)

The "active" machine 1 constraint presents us with a different situation. If we increase its RHS by a thirteenth hour, its location will shift upwards and point A will be replaced by point A'. Point A' represents 4 1/3 units of product B and a corresponding profit of $30.33. This profit represents an increase of $2.33 above the old point A profit, and thus the shadow price of machine 1 is $2.33.

Now that we have calculated both machines' shadow prices, let us use this important information in a classic economic analysis that "proves" that the solution at point A is not the economically most efficient way to use these two resources. In microeconomics, one central premise in optimization theory is that a firm is optimizing if the marginal revenue of each activity equals that activity's marginal cost. The marginal revenue for product B is simply its $7 profit contribution as found in the objective function (marginal revenue and marginal profit are synonymous). Product B's marginal cost is directly computed by noting that each unit of product B requires 3 hours of the machine 1 resource and 1 hour of the machine 2 resource. The internalized value (shadow prices!) of these resources are:

(3 hours of machine 1)($2.33) + (1 hour of machine 2)($0) = $7

Our conclusion is that the marginal revenue of product B equals the marginal cost of the resources required to produce product B at the present level of production activity.

Performing the same analysis on product A, we note that its marginal revenue is $6. The marginal cost of the resources required to produce one unit of product A is calculated as follows:

(2 hrs of machine 1)($2.33) + (2 hrs of machine 2)($0) = $4.66

Product A's marginal revenue is $1.34 greater than its marginal cost! <u>This means that if we start producing product A, each unit that we produce will add $1.34 to total profit.</u> Hopefully, this conclusion is not overly surprising to you. Do you remember that in

Chapter 3 we calculated that the "marginal benefit" of moving on the boundary from point A towards point B was $1.33? It is not by accident that we are finding the same difference between product A's marginal revenue and cost. (After all, what does it mean when we say that a product is "profitable"?)

Let's examine marginal revenues and costs for the solution at point B. For product A, its marginal revenue remains at $6.00. The marginal cost of product A is found by multiplying resource shadow prices (found in Chapter 4) times resource requirements:

(2 hrs of machine 1)($2.00) + (2 hrs of machine 2)($1.00) = $6.00

A similar analysis of product B states a marginal revenue of $7.00 and also a marginal cost of $7.00:

(3 hrs of machine 1)($2.00) + (1 hr of machine 2)($1.00) = $7.00

Our conclusion is that the solution at point B represents the optimum production plan; marginal revenue equals marginal cost for each "activity" that the firm is engaged in.

Performing marginal revenue and cost analysis for the point C solution requires that we first determine the resource shadow prices. Figure 8 lets us see that the machine 1 constraint is inactive; thus, its shadow price is $0.00. Machine 2, however, is an active constraint. If its RHS increases to 9 hours, its location shifts upward, parallel to its original location. The solution of interest becomes C', where the value of product A becomes 4.5 units. The profit of the new C' production plan is $27.00. This represents an increase of $3.00 over the profit at point C, and thus $3.00 is the shadow price of machine 2.

Performing the activity analysis produces a marginal revenue of $6.00 for product A and a marginal cost also of $6.00:

(2 hrs of machine 1)($0.00) + (2 hrs of machine 2) ($3.00) = $6.00

For product B, marginal revenue is $7.00 and marginal cost is $3.00:

(3 hrs of machine 1)($0.00) + (1 hr of machine 2) ($3.00) = $3.00

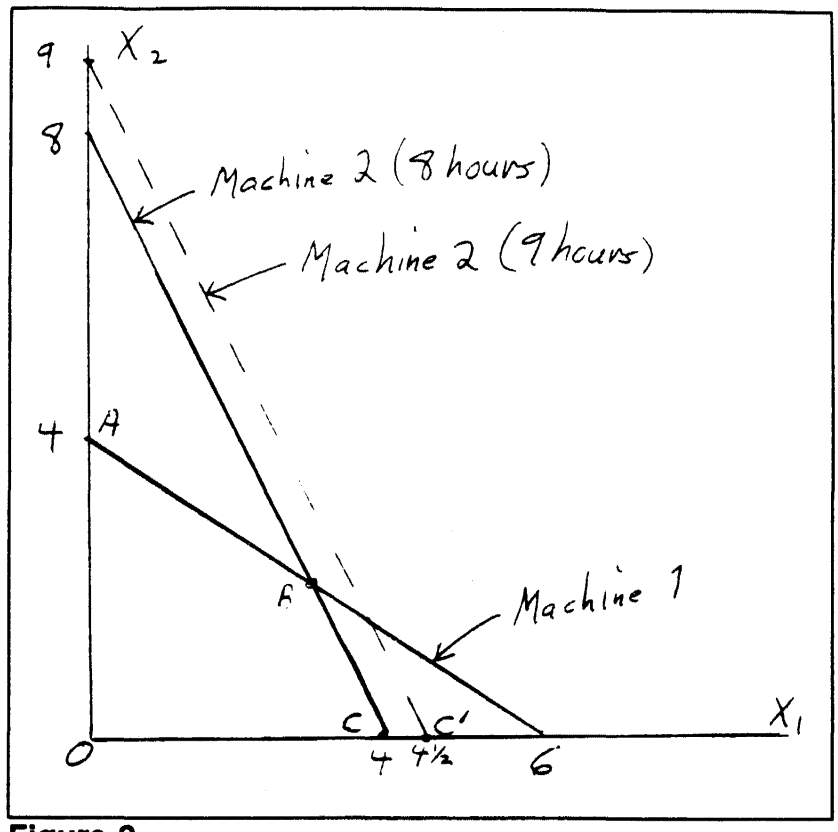

Figure 8

This information tells management that they can improve their total profit situation if they "move from" this solution in such a way that more of product B will be produced (a net gain of $4.00 per unit). The resources required in the production of product B have a lower return in their current use than the return they will achieve if **reallocated** into the production of product B! Do you see this ?

Rather than performing resource evaluations at point C, we could do an output evaluation and arrive at the same conclusion. For instance, if we perform a marginal benefit analysis for the move along the boundary from point C towards point B, we find that a one unit increase in the marginal variable (product B) causes a .5 unit reduction in product A:

$2X_1 + 1X_2 = 8$; $2X_1 = 8 - 1X_2$; therefore, $1X_1 = 4 - .5X_2$

This tradeoff produces a net gain in profit (i.e., a marginal benefit) of $4.00:

(1 unit of product B)($7) - (.5 units of product A)($6) = $4.00

Challenge yourself to reason "why" the monetary valuation results of the output analysis are identical to those of the marginal revenue and marginal cost analysis of the inputs! (Put your thoughts down on paper--it helps to commit yourself to your conclusions.)

Finally, let us return to a marginal revenue and cost analysis of the solution at the origin (point 0). Since neither constraint is active at the origin, both have zero shadow prices. Evaluation of product A, therefore, indicates that it has a marginal revenue of $6.00 and a marginal cost of resources of $0.00. Similarly, product B's marginal revenue is $7.00 and its marginal cost is also $0.00. The appropriate conclusion is that this solution is nonoptimum because we can increase total profit by increasing the production of either product.

We have just demonstrated the practical application of a basic rule of microeconomics. As long as an activity's marginal revenue does not equal its marginal cost, the firm can increase its total profit by making changes in production levels. Only when marginal revenue equals marginal cost for **each** activity is the firm maximizing its profits!

The Concept of "Slack" Resources

In the previous section, we began our analysis by examining the solution at point A (refer to Figure 7). For that production plan, it was noted that constraint #2 was inactive; we have 8 hours of machine 2 capacity available and we require only 4 hours of it to be able to produce the 4 units of product B. The "extra" 4 hours of machine 2 capacity are referred to as "slack" capacity. The fact that we may find it necessary and desirable to interpret solutions for which some constraints may be inactive, leads us to formally allow for this outcome. This is accomplished by creating a new variable that is defined as "slack." This variable is typically represented with

the letter "S" and a subscript which locates the "S" in the particular constraint for which it was created. For machine 2, we will now write:

$2X_1 + 1X_2 + S_2 = 8$ hours

Note that on the right-hand-side of the equation we list the amount of the resource available, and on the left-hand-side we list the possible ways of using those 8 hours (i.e., we can apply machine 2 capacity to produce products A and B, or we can decide to keep all or some of the 8 hours unused in slack). In any event, the left-hand-side of this constraint will now always add up to equal the value on the right-hand-side. It is because of this that we can always write constraint #2 as a strict equality!

Do you see that at points B and C the value of S_2 will be equal to zero? Those solutions require all 8 hours of machine 2 capacity for use in the production of the specified quantities of products A and B. Finally, for the solution at point 0, since no outputs are being produced, all 8 hours of machine 2 capacity will be in slack. What we have demonstrated is that for each of the four feasible corner point solutions (which are the only
solutions of interest), the machine 2 constraint can be written as a strict equality:

In general: $2X_1 + 1X_2 + S_2 = 8$ hours
At point A: $2(0) + 1(4) + 4 = 8$ hours
At point B: $2(3) + 1(2) + 0 = 8$ hours
At point C: $2(4) + 1(0) + 0 = 8$ hours
At point 0: $2(0) + 1(0) + 8 = 8$ hours

The same situation can be extended to the machine 1 constraint (and, in general, to all "less than" constraints):

In general: $2X_1 + 3X_2 + S_1 = 12$ hours
At point A: $2(0) + 3(4) + 0 = 12$ hours
At point B: $2(3) + 3(2) + 0 = 12$ hours
At point C: $2(4) + 3(0) + 4 = 12$ hours
At point O: $2(0) + 3(0) + 12 = 12$ hours

One important outcome from this exercise is that we are now working with a model that consists of all equations rather than inequalities. This will allow for efficient methods of solving the model for the optimal solution. Also, note that each solution will now not only list the values of the decision variables (i.e., the X's), but also the values of the slack variables. Knowledge of slack resources is of interest to management as is knowledge of optimum values for the X's.