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UTILITARIAN CHOICE RULES

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A choice set S is a subset of R^2 representing the utility levels, measured in von Neumann-Morgenstern utility scales, attainable by two agents through some joint action. A bargaining problem is a pair (S,d) , where S is a choice set and d is a distinguished alternative of S strictly dominated by some other element of S . This alternative, called the "status quo" or the "disagreement point" is interpreted as the outcome that would achieve in the absence of a compromise between the agents. Given a class Σ of choice sets, a choice rule f defined on Σ associates to every S in Σ a unique point $f(S)$ of S . Similarly, given a class Σ' of bargaining problems, a bargaining solution f defined on Σ' associates to every (S,d) in Σ' a unique point $f(S,d)$ of S . (Choice rules and bargaining solutions are therefore not defined on the same domains.)

One would like choice rules and bargaining solutions to satisfy certain axioms. Nash's solution to bargaining problems [1950] involves four well-known axioms, spelled out below. There are, however, no choice rule simultaneously satisfying the counterparts for Σ of these four axioms.

It is shown here how one of them, the Invariance axiom, can be weakened so as to permit compatibility; the solution so obtained is the

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equal weight utilitarian rule. The characterization of both the Nash solution and of the utilitarian choice rules can also be achieved by replacing the Independence of Irrelevant Alternatives axiom, which involves "contractions" of choice sets, by another (or "dual") axiom which involves "expansions" of choice sets. This axiom is interpreted in a way suggested by Shapley's [4] extension of the value to games without transferable utility.

1. Notations and Definitions

$\Delta = \{p \in R^2 \mid \|p\| = 1\} \cup \{0\}$. Σ is the class of compact and strictly convex subsets of R^2 . Σ' is the class of pairs (S,d) obtained by choosing S in Σ and for each S , a point d in S strictly dominated by some other point of S . Given S in Σ and x in S , $W(S,x) = \{p \in \Delta \mid \forall y \in S, py \preceq px\}$. Note that for all S in Σ and x in S , $W(S,x)$ contains 0 . If x belongs to the interior of S , then $W(S,x) = \{0\}$. Σ'_{dif} is the subset of Σ of choice sets such that for all x on the boundary of S , $W(S,x) \setminus \{0\} \equiv \tilde{W}(S,x)$ is a singleton. Σ'_{dif} is correspondingly defined. Given $p \in R^2$ with $p > 0$ and $\|p\| = 1$, the utilitarian choice rule with weights p associates to every S in Σ the unique maximizer over S of the expression $p_1x_1 + p_2x_2$.

2. A Characterization of Utilitarian Choice Rules

First, we state Nash's axioms.

Pareto-optimality on Σ' (PO'): $\forall S' = (S,d) \in \Sigma', \forall y \in S, y \not\preceq f(S')$.

Symmetry on Σ' (Sy'): Given $x = (x_1, x_2) \in R^2$, let $M(x) = (x_2, x_1)$. Also, given $S \in \Sigma$, let $M(S) = \{y \in R^2 \mid y = M(x) \text{ for } x \in S\}$. Note that $S \in \Sigma \Rightarrow M(S) \in \Sigma$. The symmetry axiom says: $\forall S' = (S,d) \in \Sigma', [M(S) = S, M(d) = d] \Rightarrow M(f(S')) = f(S')$.

Invariance with respect to positive affine transformations on Σ' (Inv'):

Given $e = (a_1, b_1, a_2, b_2) \in R^4$ and $x = (x_1, x_2) \in R^2$, let $V_e(x) = (a_1 x_1 + b_1, a_2 x_2 + b_2)$. Also, given $S \in \Sigma$, let $V_e(S) = \{y \in R^2 \mid y = V_e(x) \text{ for } x \in S\}$. Note that $S \in \Sigma \Rightarrow V_e(S) \in \Sigma$. The invariance axiom says: $\forall S' = (S, d) \in \Sigma', \forall e = (a_1, b_1, a_2, b_2) \in R^4$, with $a_1, a_2 > 0$, $f(V_e(S), V_e(d)) = V_e(f(S'))$.

Independence of irrelevant alternatives on Σ' (IIA'): $\forall S' =$

(S, d) , $T' = (T, d) \in \Sigma'$, $[T \supset S, f(S') \in T] \Rightarrow f(S') = f(T')$.

Nash proved existence and uniqueness of a bargaining solution on Σ' satisfying PO', Sy', Inv', and IIA'. It consists in selecting the point $x = (x_1, x_2)$ maximizing over S the product $(x_1 - d_1)(x_2 - d_2)$. This solution is denoted $N(\cdot)$.

The counterparts of these axioms for choice rules are easily stated (for notational convenience, the counterpart on Σ of axiom A' on Σ' is denoted A):

Pareto-optimality on Σ (PO): $\forall S \in \Sigma, \forall y \in S, y \neq f(S)$.

Symmetry on Σ (Sy): $\forall S \in \Sigma, M(S) = S \Rightarrow M(f(S)) = f(S)$.

Invariance with respect to positive affine transformations on Σ

(Inv.): $\forall S \in \Sigma, \forall e = (a_1, b_1, a_2, b_2) \in R^4$, with $a_1, a_2 > 0$, $f(V_e(S)) = V_e(f(S))$.

Independence of irrelevant alternatives on Σ (IIA): $\forall S, T \in \Sigma$, $[T \subset S, f(S) \in T] \Rightarrow f(S) = f(T)$.

The incompatibility of these axioms is proved in Sen [3] and Myerson [1]. Note, however, that the equal-weight utilitarian choice rule satisfies all of them except Inv. It is natural to attempt a reformulation of this axiom that would permit compatibility. This is the object of the following definition and of Proposition 1.

Translation invariance (T Inv.): $\forall S \in \Sigma, \forall b_1, b_2 \in R$, if $e = (1, b_1, 1, b_2)$, $f(V_e(S)) = V_e(f(S))$. (Note that $\text{Inv.} \Rightarrow \text{T Inv.}$)

Proposition 1: A choice rule defined on Σ satisfies PO, T Inv and IIA iff it is a utilitarian choice rule. It satisfies Sy in addition iff it is the equal weight utilitarian choice rule.

Proof: The sufficiency of the first statement is straightforward. To prove its necessity, let f satisfy PO, T Inv and IIA. First, we observe that by PO, for all S in Σ , $\tilde{W}(S, f(S)) \neq \emptyset$. Next, we show that

$$(1) \quad \forall S, T \in \Sigma_{\text{dif}}, \tilde{W}(S, f(S)) = \tilde{W}(T, f(T)) .$$

To prove (1), note that for all S in Σ_{dif} , $\tilde{W}(S, f(S))$ contains a unique element; let us denote it p^S , and let us assume by way of contradiction that there exist S, T in Σ_{dif} with $p^S \neq p^T$; say $p_1^S > p_1^T$. Let $x = f(S)$, $y = f(T)$, and p' in Δ with $p_1^S > p'_1 > p_1^T$; for $\lambda > 0$, let $z_\lambda = x + \lambda(-p'_2, p'_1)$ and $T_\lambda = T + \{z_\lambda - y\}$.

$$\tilde{W}(S, x) \neq \{p'\} \Rightarrow \exists \lambda_1 > 0 \quad \text{s.t.} \quad \forall \lambda \in [0, \lambda_1], z_\lambda \in S .$$

Also,

$$\tilde{W}(T, y) \neq \{p'\} \Rightarrow \exists \lambda_2 > 0 \quad \text{s.t.} \quad \forall \lambda \in [0, \lambda_2], x \in T_\lambda .$$

Let $\lambda_0 = \min\{\lambda_1, \lambda_2\}$, $U = S \cap T_{\lambda_0}$. Clearly, $U \in \Sigma$. By T Inv, $z_{\lambda_0} = f(T_{\lambda_0})$. Since $z_{\lambda_0} \in U$ and $U \subset T_{\lambda_0}$, IIA implies $f(U) = z_{\lambda_0}$. Since $x \in U$ and $U \subset S$, IIA also implies $f(U) = x$. Finally, $\lambda_0 \neq 0$ implies $x \neq z_{\lambda_0}$. This contradiction proves (1), which implies the existence of p in $\Delta \setminus \{0\}$ such that, for all S in Σ_{dif} , $\tilde{W}(S, f(S)) = \{p\}$.

Now let S be arbitrary in Σ , and x be its unique Pareto-optimal point such that $p \in \tilde{W}(S, x)$. One can find T in Σ_{dif} with

$T \subset S$, x Pareto-optimal for T , and $\tilde{W}(T,x) = \{p\}$. By (1), $f(T) = x$.
By IIA, $f(S) = x$. This concludes the proof of the first statement of
Proposition 1.

The proof of the second statement is straightforward.

Q.E.D.

The close relationship between the Nash-solution and the equal weight
utilitarian choice rule can be heuristically understood as follows. Given
the sequence $\{d^k = (-k, -k) , k \in \mathbb{N}\}$ in \mathbb{R}^2 ,
one can define a choice rule N_k on $\Sigma_k = \{S \in \Sigma \mid \exists y \in S \text{ with } y > d^k\}$
by $N_k(S) = N(S, d^k)$ (dropping for simplicity the requirement that
 $d^k \in S$). The sequence $\{\Sigma_k, k \in \mathbb{N}\}$ forms an increasing sequence of subfamilies
of Σ . Now, given any compact set K , and any $\varepsilon > 0$, there is k
large enough so that rectangular hyperbolae with asymptotes through d^k
approximate in K a set of parallel lines with slope -1 with ε -accuracy.
As $k \rightarrow \infty$, the various N_k are not consistent with one another, in the
sense that if S belongs to Σ_k and $\Sigma_{k'}$, for $k \neq k'$, $N_k(S)$ need not
be equal to $N_{k'}(S)$. However, Proposition 1 indicates that a certain
form of asymptotic consistency holds, yielding the equal weight utilitarian
choice rule.

3. Independence of Irrelevant Expansions

To motivate this section, we paraphrase an argument due to Shapley [4].
Let a solution f over Σ'_{dif} , and let $S' = (S, d)$ in Σ'_{dif} be given.
If f satisfies PO' , $f(S')$ belongs to the boundary of S and marginal
transfers of utility are possible from $f(S')$ at the rate given by the
unique element of $\tilde{W}(S, f(S'))$. The fact that no such transfers are actually
carried out can be seen as the consequence of the right balance having been

struck at $f(S')$ between the "equity weights" given by the slope of the line connecting d to $f(S')$, and the "efficiency weights" given by the slope p of the line of support of S at $f(S')$. Let now $T' = (T, d)$ in Σ'_{dif} be given with $T \supset S$ such that T admits at $f(S')$ of the same line of support as S . At $f(S)$, the same relationship exists for T' between the equity weights and the efficiency weights. Since utility transfers can be carried out in T only at the rate p , no such transfer should actually take place, and $f(T')$ should be equal to $f(S')$.

The argument is analogous to the one usually made in the justification of IIA, as pointed out by Shapley, who appealed to that axiom in his extension of the value to games without transferable utility. There, he chose T to be a superset of S with transferable utility (i.e., with a straight line Pareto-optimal boundary), but the above reasoning suggests a more general formulation:

Independence of undominating alternatives on Σ'_{dif} (IUA'): $\forall S' = (S, d)$, $T' = (T, d) \in \Sigma'_{dif}$ with $T \supset S$, if $\exists y \in T$ with $y \geq f(S)$, then $f(T) = f(S)$.

A somewhat stronger version of this axiom appears in Thomson and Myerson [5] where its logical relationship with various monotonicity and independence axioms is studied. The axiom there is stronger in that its conclusion is stated for all pairs S', T' in Σ' (instead of Σ'_{dif}) satisfying its hypotheses. The following lemma shows the incompatibility of this stronger version with optimality.

Lemma: IUA' is incompatible over Σ' with PO'.

Proof: Let S_1 (resp S_2) be a compact and strictly convex set containing the origin in its interior and such that $(2,1)$ (resp $(1,2)$) be its unique Pareto-optimal point. Also, let S_3 be a compact and strictly convex set containing the origin in its interior and such that $(2,1)$ and $(1,2)$ be on its Pareto-optimal boundary. Then, $S'_1 = (S_1, 0)$, $S'_2 = (S_2, 0)$ and $S'_3 = (S_3, 0)$ belong to Σ' . By PO, $f(S'_1) = (2,1)$, $f(S'_2) = (1,2)$. IUA' applied to the pairs S'_1, S'_3 and S'_2, S'_3 yields $f(S'_3) = (2,1)$ and $f(S'_3) = (1,2)$, a contradiction.

Q.E.D.

The stronger version is also in violation of the interpretation given above. If $\tilde{W}(S, f(S'))$ is not a singleton, one should not conclude that the equity weights for S' at $f(S')$ bear the right relationship to any of the efficiency weights in $\tilde{W}(S, f(S'))$ but simply that they bear the right relationship to some efficiency weights in $\tilde{W}(S, f(S'))$.

Furthermore, the argument given above is explicitly stated for solutions that select Pareto-optimal outcomes, whence the term "efficiency weights." In our context, optimality is required separately. We are here concerned with the possibility of transferring utility from one agent to the other at some rate, regardless of whether these transfers are made along the Pareto frontier. Consequently, we will use the term "transfer weights" instead of "efficiency weights" and formulate the axiom:

Independence of irrelevant expansions on Σ' (IIE'): $\forall S' = (S, d) \in \Sigma'$ with $x = f(S')$, $\exists p^{S'} \in W(S, x)$ such that $\forall T' = (T, d) \in \Sigma'$ with (a) $T \supset S$ and (b) $p^{S'} \in W(T, x)$, then $f(S') = f(T')$.

We now argue that this axiom is a natural "dual" of IIA'. In addition to the argument developed in Thomson and Myerson concerning a variant of it and pointing in that direction, to Shapley's reference to IIA' in his justification of yet another variant, we present Propositions 2 and 3.

If IIA' and IIE' are so closely related, it may however be puzzling that their statements look so different. To understand this, note that IIA' can also be interpreted as stating that a correct balance has been struck between equity and transfer weights. If the existence of $p^{S'}$ is explicitly stated in the formulation of IIE' but not in that of IIA', it is because whenever $S' = (S, d)$ and $T' = (T, d)$ in Σ' are such that $T \supset S$, and $f(S') \in T$, then automatically $W(T, f(S')) \supset W(S, f(S'))$ and any $p^{S'}$ in $W(T, f(S'))$ will also be in $W(S, f(S'))$. The existence of $p^{S'}$ is therefore guaranteed. To further bring out the dual roles played by IIA' and IIE', we have the following:

Proposition 2: A bargaining solution defined on Σ' satisfies PO' , Sy' , Inv' and IIE' iff it is the Nash bargaining solution.

Proof: Sufficiency is straightforward. To prove necessity, let $S' = (S, d)$ in Σ' be given and $x = N_d(S')$. Let V_e be a positive affine transformation such that $\bar{d} \equiv V_e(d)$ and $y \equiv V_e(x)$ satisfy $M(\bar{d}) = \bar{d}$ and $M(y) = y$. Finally, let $T = V_e(S)$, $T' = (T, \bar{d})$, $U = T \cap M(T)$, $U' = (U, \bar{d})$. Then U is strictly convex, since T , and therefore $M(T)$ have that property; U is compact; \bar{d} is a point of U strictly dominated by y , which is a (Pareto-optimal) point of U . Therefore, $U' \in \Sigma'$. Also, $M(U) = U$ and $M(\bar{d}) = \bar{d}$. By Sy' , $f(U') = y$.

Now let $p^{U'}$ be as in the statement of IIE'. We claim that $p^{U'} \neq 0$. Otherwise, let $R' = (R, \bar{d})$ in Σ' be such that $R \supset U$ and R contains some point \bar{y} with $\bar{y} > y$. By IIE', $f(R') = f(U') = y$, in contradiction with PO'. This proves the claim. Next, we distinguish two cases:

(a) $U \in \Sigma_{dif}$. Then $\tilde{W}(U, y)$ is a singleton, equal to $(\frac{1}{2}, \frac{1}{2})$; therefore, $p^{U'} = (\frac{1}{2}, \frac{1}{2})$. Since $T \supset U$ and $(\frac{1}{2}, \frac{1}{2}) \in W(T, y)$, IIE' implies that $f(T') = f(U') = y$. By Inv', $f(S') = x$.

(b) $U \notin \Sigma_{dif}$. It is possible that $p^{U'} \neq (\frac{1}{2}, \frac{1}{2})$ if U has a kink at y . Suppose this to be the case, and let $V' = (V, \bar{d}) \in \Sigma_{dif}$ be such that $V \supset U$ and $\tilde{W}(V, y) = \{p^{U'}\}$. Also, let $y' = N_d(V')$. By (a), $f(V') = y'$. By IIE' applied to U' and T' , $f(V') = y$. Since $y' \neq y$, a contradiction follows. Therefore, $p^{U'} = (\frac{1}{2}, \frac{1}{2})$ and we conclude as above that $f(S') = x$.

Q.E.D.

Note that this proposition in fact implies uniqueness of the transfer weights for which the conclusion of IIE' applies and that stating this axiom in a form requiring uniqueness would have led to the same characterization. Note also that equality of the transfer and equity weight results from the four axioms.

However, the following form of the expansion axiom would not have been appropriate: $\forall S' = (S, d), T' = (T, d) \in \Sigma', [T \supset S, W(T, f(S')) = W(S, f(S'))] \Rightarrow f(S') = f(T')$. The interpretation of this axiom is as follows: if $S' = (S, d)$ in Σ happens to be such that S has a kink at $f(S')$, marginal transfers of utility are possible in one direction

at one rate and in the other direction at some different rate. The axiom states invariance of the solution only for bargaining problems (T,d) with $T \supset S$ that do not allow for other marginal transfers of utility in either direction. Although the Nash solution satisfies this axiom, as well as PO' , Sy' and Inv' , it is not uniquely characterized by this list of axioms. However, any other solution coincides with the Nash solution on the subset Σ'_{dif} of Σ' .

For choice rules, we have a proposition analogous to Proposition 1: first we transcribe IIE' for Σ .

Independence of irrelevant expansions on Σ (IIE): $\forall S \in \Sigma$,
 $\exists p^S \in W(S, f(S))$ s.t. $\forall T \in \Sigma$, $[T \supset S, p^S \in W(T, f(S))] \Rightarrow f(S) = f(T)$.

Proposition 3: Same as Proposition 1 with IIA replaced by IIE.

Proof: The sufficiency of the first statement is straightforward. To prove its necessity, let f satisfy PO , $T Inv$ and IIE, and for each S in Σ , let Δ^S be the subset of $W(S, f(S))$ of prices p^S satisfying the condition appearing in the statement of IIE. An argument similar to the one appearing in the proof of Proposition 2 indicates that PO implies that for all S in Σ , and for all p in Δ^S , $p \neq 0$.

We now claim that

$$(1) \quad \forall S, T \in \Sigma, p \in \Delta^S \Rightarrow p \in \Delta^T.$$

To prove (1), suppose, by way of contradiction, that

$\exists S, T \in \Sigma, \exists p^S \in \Delta^S, p^S \notin \Delta^T$. Let $p^T \in \Delta^T$. Then $p^S \neq p^T$, say $p_1^S > p_1^T$. Let $x = f(S)$, $y = f(T)$.

For $p \in \Delta$ with $p_1^S > p_1 > p_1^T$, and for $\lambda > 0$, let $z_\lambda = x + \lambda(-p_2, p_1)$ and $T_\lambda = T + \{z_\lambda - y\}$.

S bounded $\Rightarrow \exists \lambda_1 > 0$ s.t. $\forall \lambda > \lambda_1, \forall x' \in S, p^T x' < p^T z_\lambda$.

T bounded $\Rightarrow \exists \lambda_2 > 0$ s.t. $\forall \lambda > \lambda_1, \forall y' \in T_\lambda, p^S y' < p^S x$.

Let $\lambda_0 = \max\{\lambda_1, \lambda_2\}$. One can find U in Σ with $U \supset (S \cup T_{\lambda_0})$, $p^S \in \tilde{W}(U, x)$, $p^T \in \tilde{W}(U, z_{\lambda_0})$. IIE applied to the sets S and U yields $f(U) = x$; T Inv implies $f(T_{\lambda_0}) = z_{\lambda_0}$. IIE applied to the sets T_{λ_0} and U yields $f(U) = z_{\lambda_0}$. Since $x \neq z_{\lambda_0}$, this is a contradiction, which proves (1).

To show the existence of a unique p in $\Delta \setminus \{0\}$ such that, for all S in Σ , $\Delta^S = \{p\}$, let now S in Σ_{dif} be given. Δ^S is a singleton, since $\tilde{W}(S, f(S))$ is a singleton. Then, from (1), for all T in Σ , $\Delta^T = \{p\}$. This proves the first statement of the proposition.

The proof of the second statement is straightforward.

Q.E.D.

Final Remarks

(1) All of the propositions can be formulated for an arbitrary number of agents.

(2) Strict convexity of the elements of Σ can be relaxed at the cost of introducing multi-valued solutions, except in Proposition 2, where single-valuedness of the solution holds on this wider domain. One should then speak of solution correspondences and reformulate the axioms accordingly. For such a reformulation, see Thomson [6]. In Propositions 1 and 3, the utilitarian choice rules could then be shown to be the smallest choice correspondences to satisfy the axioms. The exposition is simplified by working with a strictly convex domain without much loss in substance.

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