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ASYMPTOTIC CONDITIONS FOR THE SOLVABILITY OF A FOURTH ORDER BOUNDARY VALUE PROBLEM WITH PERIODIC BOUNDARY CONDITIONS

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Abstract. This paper concerns the existence of solutions of the fourth order periodic boundary value problem

$$-\frac{d^4u}{dx^4}+f(u(x))u'(x)+g(x,u(x))=e(x), x\in[0,2\pi],$$
$$u(0)-u(2\pi)=u'(0)-u'(2\pi)=u''(0)-u''(2\pi)=u'''(0)-u'''(2\pi)=0,$$

under some non-uniform resonance and non-resonance conditions on the asymptotic behavior of $u^{-1}g(x,u)$ for $|u|\to\infty$.

1. Introduction. Fourth order boundary value problems arise in the study of the equilibrium of an elastic beam under an external load, (e.g., see [1], [2], [5], [6], [16]) where the existence, uniqueness and iterative methods to construct the solutions have been studied extensively. The author studied in [7] the following fourth order boundary value problems with periodic boundary conditions:

(1.1)
$$\frac{d^4u}{dx^4}+f(u)u'+g(x,u)=e(x), x\in[0,2\pi],$$
$$u(0)-u(2\pi)=u'(0)-u'(2\pi)=u''(0)-u''(2\pi)=u'''(0)-u'''(2\pi)=0$$

and

(1.2)
$$-\frac{d^4u}{dx^4}+\alpha u'+g(x,u)=e(x), x\in[0,2\pi],$$
$$u(0)-u(2\pi)=u'(0)-u'(2\pi)=u''(0)-u''(2\pi)=u'''(0)-u'''(2\pi)=0,$$

where $f:\mathbf{R}\to\mathbf{R}$ is continuous, $g:[0,2\pi]\times\mathbf{R}\to\mathbf{R}$ satisfies Caratheodory's conditions, $e\in L^1[0,2\pi]$ and $\alpha\in\mathbf{R}$. The purpose of this paper is to study the analogue of (1.2) when α is replaced by $f(u)$; viz. the boundary value problem

$$-\frac{d^4u}{dx^4}+f(u)u'+g(x,u)=e(x),\quad x\in[0,2\pi],$$
$$u(0)-u(2\pi)=u'(0)-u'(2\pi)=u''(0)-u''(2\pi)=u'''(0)-u'''(2\pi)=0. \tag{1.3}$$

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under more general conditions on the asymptotic behaviour of $u^{-1}g(x,u)$ relative to the two first eigen-values 0 and 1 of the linear problem

$$(1.4)\qquad \begin{aligned} &-\frac{d^4u}{dx^4}+\lambda u=0,\\ &u(0)-u(2\pi)=u'(0)-u'(2\pi)=u''(0)-u''(2\pi)=u'''(0)-u'''(2\pi)=0.\end{aligned}$$

Instead of assuming, like in [7], that $\limsup u^{-1}g(x,u)\le\beta<1$, $(\beta\in\mathbf{R})$ uniformly for a.e. $x\in[0,2\pi], |u|\to\infty$ we assume in this paper that there exists a function $\Gamma:[0,2\pi]\to\mathbf{R}$ with $\Gamma=\Gamma_0+\Gamma_1+\Gamma_\infty$ where $\Gamma_0(x)\le 1$ for a.e. $x\in[0,2\pi]$ with strict inequality on a subset of $[0,2\pi]$ of positive measure, $\Gamma_1\in L^1[0,2\pi]$, $\Gamma_\infty\in L^\infty[0,2\pi]$ with $|\Gamma_1|_{L^1}$ and $|\Gamma_\infty|_{L^\infty}$ sufficiently small such that

$$(1.5)\qquad \limsup_{|u|\to\infty} u^{-1}g(x,u)\le\Gamma(x)$$

uniformly for a.e. $x\in[0,2\pi]$. Accordingly, the expression $\limsup_{|u|\to\infty}u^{-1}g(x,u)$ can cross any number of eigenvalues n^4 of the linear problem (1.4) as far as those crossing take place in subsets $[0,2\pi]$ of sufficiently small measure.

The methods and results of this paper are motivated by the paper of Gupta-Mawhin ([8] (see also [12], [13]) for the second order boundary value problem with periodic boundary conditions:

$$(1.6)\qquad \begin{aligned} &\frac{d^2u}{dx^2}+f(u)u'+g(x,u)=e(x),\quad x\in[0,2\pi],\\ &u(0)-u(2\pi)=u'(0)-u'(2\pi)=0.\end{aligned}$$

We present in section 2 some lemmas giving a priori inequalities that are needed to apply degree-theoretic arguments to obtain existence of solutions for the problem (1.3). In section 3, non-resonance conditions for the existence of solutions of (1.3) are studied and in section 4 we study (1.3) when it is at resonance. We study in section 5 the boundary value problem (1.2) when g satisfies asymptotic conditions (1.5) and obtain a theorem which partially extends the theorem of section 4. This requires a rather different lemma, similar to the second order case ([8]), which makes use of an inequality of E. Schmidt [15] for periodic absolutely - continuous functions. The result of section 5 in an improvement over the result of section 4 when $\Gamma_0=\Gamma_\infty=0$ and $f\equiv\alpha$; but still is not as sharp as Theorem 2.4 of [7] when applied to the case of a constant Γ. But then theorem 3 of section 5 allows $u^{-1}g(x,u)$ to cross nfinitely many eigen-values of (1.4).

We note that in addition to using the classical spaces $C[0,2\pi]$, $C^k[0,2\pi]$, $L^k[0,2\pi]$ and $L^\infty[0,2\pi]$ of continuous, k-times continuously differentiable, measurable real-valued functions whose k-th power of the absolute value is Lebesgue integrable or measurable functions that are essentially bounded on $[0,2\pi]$; we shall use the Sobolev-spaces $H^k[0,2\pi]$, $(k = 2,3$ or $4)$ defined by

$$H^k[0,2\pi] = \{u : [0,2\pi] \to \mathbf{R} | u^{(j)} \text{ abs. cont. on } [0,2\pi], \\ j = 0,1,\ldots,k-1, u^{(k)} \in L^2[0,2\pi]\}.$$

with the inner product defined by

$$(u,v)_{H^k} = \sum_{j=1}^{k} \frac{1}{2\pi} \int_0^{2\pi} u^{(j)}(x) v^{(j)}(x) dx \\ + \left(\frac{1}{2\pi} \int_0^{2\pi} u(x) dx \right) \left(\frac{1}{2\pi} \int_0^{2\pi} v(x) dx \right),$$

and the corresponding norm by $|\cdot|_{H^k}$. We also define, for the sake of convenience, the norm in $L^k[0,2\pi]$ by

$$|u|_{L^k} = \left(\frac{1}{2\pi} \int_0^{2\pi} |u(x)|^k dx \right)^{\frac{1}{k}}.$$

We also use the Sobolev-space $W^{4,1}[0,2\pi]$ defined by $W^{4,1}[0,2\pi] = \{u : [0,2\pi] \to \mathbf{R}/u, u', u'', u'''$ abs. cont. on $[0,2\pi]\}$ with norm

$$|u|_{W^{4,1}} = \sum_{j=0}^{4} \int_0^{2\pi} |u^{(j)}(t)| dt.$$

2. A Priori Inequalities. For $u \in L^1[0,2\pi]$, let us write

$$\overline{u} = \frac{1}{2\pi} \int_0^{2\pi} u(x) dx, \quad \tilde{u}(x) = u(x) - \overline{u}, \tag{2.1}$$

so that $\int_0^{2\pi} \tilde{u}(x) dx = 0$. Let $\tilde{H}^2[0,2\pi] = \{u \in H^2[0,2\pi] | \overline{u} = 0\}$.

LEMMA 1. *Let $\Gamma \in L^1[0,2\pi]$ be such that, for a.e. $x \in [0,2\pi]$,*

$$\Gamma(x) \leq 1, \tag{2.2}$$

with the strict inequality holding on a subset of $[0,2\pi]$ of positive measure. Then there exists a $\delta = \delta(\Gamma) > 0$ such that for all $\tilde{u} \in \tilde{H}^2[0,2\pi]$ with $\tilde{u}(0) - \tilde{u}(2\pi) = \tilde{u}'(0) - \tilde{u}'(2\pi) = 0$,

$$B_\Gamma(\tilde{u}) = \frac{1}{2\pi} \int_0^{2\pi} [(\tilde{u}''(x))^2 - \Gamma(x) \tilde{u}^2(x)] dx \geq \delta |\tilde{u}|^2_{H^2}. \tag{2.3}$$

Proof. Using (2.2) and Wirtinger's inequality [3], we see that, for all $\tilde{u} \in \tilde{H}^2[0, 2\pi]$ with $\tilde{u}(0) - \tilde{u}(2\pi) = \tilde{u}'(0) - \tilde{u}'(2\pi) = 0$,

$$B_\Gamma(\tilde{u}) \geq \frac{1}{2\pi} \int_0^{2\pi} [(\tilde{u}''(x))^2 - \tilde{u}^2(x)]dx \geq 0, \tag{2.4}$$

and, moreover,

$$B_\Gamma(\tilde{u}) = 0. \tag{2.5}$$

if and only if

$$\tilde{u}(x) = A \sin(x + \theta), \tag{2.6}$$

for some $A, \theta \in \mathbf{R}$. But then by (2.5), (2.6) we get

$$\begin{aligned} 0 = B_\Gamma(\tilde{u}) &= \frac{1}{2\pi} \int_0^{2\pi} [1 - \Gamma(x)]\tilde{u}^2(x)dx \\ &= \frac{A^2}{2\pi} \int_0^{2\pi} [1 - \Gamma(x)] \sin^2(x + \theta)dx, \end{aligned}$$

so that by our assumption (2.2) on Γ we have $A = 0$ and hence $\tilde{u} = 0$.

Let us next assume that the conclusion of the lemma is false. Then there exists a sequence $\{\tilde{u}_n\}$, $\tilde{u}_n \in \tilde{H}^2[0, 2\pi]$ for every $n = 1, 2, 3, \ldots$ such that

$$\begin{aligned} &B_\Gamma(\tilde{u}_n) \to 0 \text{ as } n \to \infty, \\ &|\tilde{u}_n|_{H^2} = 1, \text{ for every } n = 1, 2, \ldots \end{aligned} \tag{2.7}$$

It now follows from (2.7) and the compact imbedding $H^2[0, 2\pi] \subset\hookrightarrow C^1[0, 2\pi]$ that there exists a $\tilde{u} \in \tilde{H}^2[0, 2\pi]$ such that

$$\begin{aligned} &\tilde{u}_n \to \tilde{u} \text{ weakly in } H^2[0, 2\pi], \\ &\tilde{u}_n \to \tilde{u} \text{ in } C^1[0, 2\pi]. \end{aligned} \tag{2.8}$$

Now (2.8) implies that $\tilde{u}(0) - \tilde{u}(2\pi) = \tilde{u}'(0) - \tilde{u}'(2\pi) = 0$ and $|\tilde{u}|_{H^2} \leq \liminf_{n\to\infty} |\tilde{u}_n|_{H^2}$. Hence we get that

$$0 \leq B_\Gamma(\tilde{u}) \leq \liminf_{n\to\infty} B_\Gamma(\tilde{u}_n) = 0. \tag{2.9}$$

It, now, follows from (2.9) and the first part of this proof that $\tilde{u} = 0$. Also (2.7)-(2.9) imply that

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} [\tilde{u}_n''(x)]^2 dx &= B_\Gamma(\tilde{u}_n) + \frac{1}{2\pi} \int_0^{2\pi} \Gamma(x)\tilde{u}_n^2(x)dx \\ &\to \frac{1}{2\pi} \int_0^{2\pi} \Gamma(x)\tilde{u}^2(x)dx) = \frac{1}{2\pi} \int_0^{2\pi} [\tilde{u}''(x)]^2 dx, \end{aligned}$$

so that $\tilde{u}_n \to \tilde{u}$ in $H^2[0, 2\pi]$ and $|\tilde{u}|_{H^2} = 1$. We have thus arrived at a contradiction.

Hence the lemma is true. □

LEMMA 2. *Let $\Gamma = \Gamma_0 + \Gamma_1 + \Gamma_\infty$ where $\Gamma_\infty \in L^\infty[0,2\pi]$, $\Gamma_1 \in L^1[0,2\pi]$ and $\Gamma_0 \in L^1[0,2\pi]$ is such that $\Gamma_0(x) \le 1$ for a.e. $x \in [0,2\pi]$ with strict inequality holding on a subset of $[0,2\pi]$ of positive measure. Let $\delta(\Gamma_0) > 0$ be as given by Lemma 1. Then for every $\tilde{u} \in \tilde{H}^2[0,2\pi]$ with $\tilde{u}(0) - \tilde{u}(2\pi) = \tilde{u}'(0) - \tilde{u}'(2\pi) = 0$,*

$$B_\Gamma(\tilde{u}) \ge [\delta(\Gamma_0) - \frac{\pi^2}{3}|\Gamma_1|_{L^1} - |\Gamma_\infty|_{L^\infty}]|\tilde{u}|^2_{H^2} \tag{2.10}$$

Proof. We have

$$B_\Gamma(\tilde{u}) = \frac{1}{2\pi}\int_0^{2\pi}[(\tilde{u}''(x))^2 - \Gamma_0(x)\tilde{u}^2(x)]dx - \frac{1}{2\pi}\int_0^{2\pi}\Gamma_1(x)\tilde{u}^2(x)dx - \frac{1}{2\pi}\int_0^{2\pi}\Gamma_\infty(x)\tilde{u}^2(x)dx.$$

Using, now, the fact that $H^2[0,2\pi] \subset C^1[0,2\pi]$ and the well-known inequalities (see e.g. [14])

$$|\tilde{u}|_{L^2} \le |\tilde{u}'|_{L^2} \le |\tilde{u}''|_{H^2}, \quad |\tilde{u}|_{L^\infty} \le \frac{\pi}{\sqrt{3}}|\tilde{u}'|_{L^2} \le \frac{\pi}{\sqrt{3}}|\tilde{u}|_{H^2}$$

for $\tilde{u} \in \tilde{H}^2[0,2\pi]$ with $\tilde{u}(0) - \tilde{u}(2\pi) = \tilde{u}'(0) - \tilde{u}'(2\pi) = 0$, as well as Lemma 1, we get that

$$B_\Gamma(\tilde{u}) \ge \delta(\Gamma_0)|\tilde{u}|^2_H - |\Gamma_1|_{L^1}|\tilde{u}|^2_{L^\infty} - |\Gamma_\infty|_{L^\infty}|\tilde{u}|^2_{L^2}$$
$$\ge [\delta(\Gamma_0) - \frac{\pi^2}{3}|\Gamma_1|_{L^1} - |\Gamma_\infty|^\infty_L]|\tilde{u}|^2_{H^2}.$$

Remark 1. The best value for $\delta(0)$ is easily seen to be $\frac{1}{2}$, so that $B_{\Gamma_1}(\tilde{u}) \ge (\frac{1}{2} - \frac{\pi^2}{3}|\Gamma_1|_{L^1})|\tilde{u}|^2_{H^2}$ for all $\tilde{u} \in \tilde{H}^2[0,2\pi]$ with $\tilde{u}(0) - \tilde{u}(2\pi) = \tilde{u}'(0) - \tilde{u}'(2\pi) = 0$

LEMMA 3. *Let $\gamma \in L^1[0,2\pi]$, $\Gamma = \Gamma_0 + \Gamma_1 + \Gamma_\infty$ be as in lemma 2, $\delta(\Gamma_0)$ be given by lemma 1. Then for all measurable functions $p(x)$ on $[0,2\pi]$ such that $\overline{\gamma} \le \overline{p}$, $p(x) \le \Gamma(x)$ a.e. on $[0,2\pi]$, all continuous functions $f : \mathbf{R} \to \mathbf{R}$ and all $u \in W^{4,1}[0,2\pi]$ with $u(0) - u(2\pi) = u'(0) - u'(2\pi) = u''(0) - u''(2\pi) = u'''(0) - u'''(2\pi) = 0$, we have*

$$\frac{1}{2\pi}\int_0^{2\pi}[\overline{u} - \tilde{u}(x)][-\tilde{u}^{(iv)}(x) + f(u(x))u'(x) + p(x)u(x)]dx \ge \overline{\gamma}\cdot\overline{u}^2 + [\delta(\Gamma_0) - \frac{\pi^2}{3}|\Gamma_1|_{L^1} - |\Gamma_\infty|_{L^\infty}]|\tilde{u}|^2_{H^2}. \tag{2.11}$$

Proof. For $u \in W^{4,1}[0,2\pi]$ with

$$u(0) - u(2\pi) = u'(0) - u'(2\pi) = u''(0) - u''(2\pi) = u'''(0) - u'''(2\pi) = 0$$

we have, on integrating by parts, and using lemma 2 that

$$\frac{1}{2\pi}\int_0^{2\pi}[\overline{u} - \tilde{u}(x)][-u^{(iv)}(x) + f(u(x))u'(x) + p(x)u(x)]dx$$
$$\ge \overline{p}\cdot\overline{u}^2 + \frac{1}{2\pi}\int_0^{2\pi}[(\tilde{u}''(x))^2 - p(x)\tilde{u}^2(x)]dx$$
$$\ge \overline{\gamma}\cdot\overline{u}^2 + [\delta(\Gamma_0) - \frac{\pi^2}{3}|\Gamma_1|_{L^1} - |\Gamma_\infty|_{L^\infty}]|\tilde{u}|^2_{H^2}.$$

3. Asymptotic conditions for non-resonance. Let $f : \mathbf{R} \to \mathbf{R}$ be continuous and let $g : [0, 2\pi] \times \mathbf{R} \to \mathbf{R}$ be a function satisfying Caratheodory's conditions, viz.,

(i) for each $u \in \mathbf{R}$, the function $x \in [0, 2\pi] \to g(x, u) \in \mathbf{R}$ is measurable on $[0, 2\pi]$,

(ii) for a.e. $x \in [0, 2\pi]$, the function $u \in \mathbf{R} \to g(x, u) \in \mathbf{R}$ is continuous on $\mathbf{R}$, and

(iii) for each $r > 0$, there exists a function $\alpha_r(x) \in L^1[0, 2\pi]$ such that $|g(x, u)| \leq \alpha_r(x)$ for a.e. $x \in [0, 2\pi]$ and all $u \in \mathbf{R}$ with $|u| \leq r$.

THEOREM 1. *Let $\gamma \in L^1[0, 2\pi]$ with $\overline{\gamma} > 0$ be given. Also let $\Gamma = \Gamma_0 + \Gamma_1 + \Gamma_\infty$ with $\Gamma_1 \in L^1[0, 2\pi]$, $\Gamma_\infty \in L^\infty[0, 2\pi]$, Γ_0 measurable on $[0, 2\pi]$, $\Gamma_0(x) \leq 1$ with strict inequality holding on a subset of $[0, 2\pi]$ of positive measure, and $\frac{\pi^2}{3}|\Gamma_1|_{L^1} + |\Gamma_\infty|_{L^\infty} < \delta(\Gamma_0)$, where $\delta(\Gamma_0)$ is given by lemma 1. Assume that the inequalities*

$$\gamma(x) \leq \liminf_{|u| \to \infty} u^{-1} g(x, u) \leq \limsup_{|u| \to \infty} u^{-1} g(x, u) \leq \Gamma(x), \tag{3.1}$$

hold uniformly for a.e. $x \in [0, 2\pi]$.

Then for every, given, $e(x) \in L^1[0, 2\pi]$ the boundary value problem

$$\begin{aligned} &-u^{(iv)}(x) + f(u(x))u'(x) + g(x, u(x)) = e(x), x \in [0, 2\pi], \\ &u(0) - u(2\pi) = u'(0) - u'(2\pi) = u''(0) - u''(2\pi) = u'''(0) - u'''(2\pi) = 0 \end{aligned} \tag{3.2}$$

has at least one solution.

Proof. Let $\eta = \frac{1}{2}\min\{\overline{\gamma}, \delta(\Gamma_0) - \frac{\pi^2}{3}|\Gamma_1|_{L^1} - |\Gamma_\infty|_{L^\infty}\} > 0$. Then, by (3.1) we can find an $r > 0$ such that for a.e. $x \in [0, 2\pi]$ and every $u \in \mathbf{R}$ with $|u| \geq r$ we have

$$\gamma(x) - \eta \leq u^{-1} g(x, u) \leq \Gamma(x) + \eta. \tag{3.3}$$

Next, define $\tilde{\gamma} : [0, 2\pi] \times \mathbf{R} \to \mathbf{R}$ by

$$\tilde{\gamma}(x, u) = \begin{cases} u^{-1} g(x, u) \text{ if } & |u| \geq r, \\ r^{-1} g(x, r) \text{ if } & 0 < u < r, \\ -r^{-1} g(x, -r) \text{ if } & -r < u < 0 \\ \Gamma(x) \text{ if } & u = 0. \end{cases}$$

Note that $\tilde{\gamma}(x, u)u$ satisfies Caratheodory's conditions and, from (3.3),

$$\gamma(x) - \eta \leq \tilde{\gamma}(x, u) \leq \Gamma(x) + \eta \tag{3.4}$$

for a.e. $x \in [0, 2\pi]$ and all $u \in \mathbf{R}$. Now, define $h : [0, 2\pi] \times \mathbf{R} \to \mathbf{R}$ by

$$h(x, u) = g(x, u) - \tilde{\gamma}(x, u)u,$$

for $x \in [0, 2\pi]$, $u \in \mathbf{R}$. We then see that

$$|h(x,u)| \leq \sup_{|u|\leq r} |g(x,u) - \tilde{\gamma}(x,u)u| \leq \alpha(x), \tag{3.5}$$

for $x \in [0, 2\pi]$, $u \in \mathbf{R}$, where $\alpha(x) \in L^1[0, 2\pi]$ depends on γ, Γ and α_r.

Now, the equation in (3.2) is equivalent to the equation

$$-u^{(iv)}(x) + f(u(x))u'(x) + \tilde{\gamma}(x, u(x))u(x) + h(x, u(x)) = e(x),$$

to which we shall apply coincidence degree theory [4,9] in a manner similar to the one used in Theorem 1 of [12]. Let $X = C^1[0, 2\pi]$, $Z = L^1[0, 2\pi]$, $\operatorname{dom} L = \{u \in W^{4,1}[0, 2\pi] | u(0) - u(2\pi) = u'(0) - u'(2\pi) = u''(0) - u''(2\pi) = u'''(0) - u'''(2\pi) = 0\}$.

$$\begin{aligned}
&L : \operatorname{dom} L \subset X \to Z, u \to -u^{(iv)},\\
&F : X \to Z, u \to f(u(\cdot))u'(\cdot),\\
&G : X \to Z, u \to \tilde{\gamma}(\cdot, u(\cdot))u(\cdot),\\
&H : X \to Z, u \to h(\cdot, u(\cdot)) - e(\cdot),\\
&A : X \to Z, u \to \tilde{\gamma}(\cdot, 0)u(\cdot) = \Gamma(\cdot)u(\cdot).
\end{aligned}$$

It is easy to check that F, G, H and A are well-defined and L-compact on bounded subsets of X and that L is a linear Fredholm mapping of index zero. (see Lemma 2.1 of [7]). We consider the homotopy $\Phi : \operatorname{dom} L \times [0, 1] \to Z$ defined by

$$\Phi(u, \lambda) \equiv Lu + \lambda Fu + (1 - \lambda)Au + \lambda Gu + \lambda Hu,$$

for $u \in \operatorname{dom} L$, $\lambda \in [0, 1]$. Now, in order to apply Theorem IV.5 of [9] (see also [10], [11]) it suffices to show that the set of possible solutions, of the family of equations

$$\begin{aligned}
&- u^{(iv)}(x) + \lambda f(u(x))u'(x) + [(1 - \lambda)\Gamma(x) + \lambda\tilde{\gamma}(x, u(x))]u(x)\\
&\qquad + \lambda h(x, u(x)) - \lambda e(x) = 0,\\
&u(0) - u(2\pi) = u'(0) - u'(2\pi) - u''(0) - u''(2\pi) = u'''(0) - u'''(2\pi),
\end{aligned} \tag{3.6}$$

is a priori bounded in $C^1[0, 2\pi]$ independently of $\lambda \in [0, 1]$. If u is a solution of (3.6), then multiplying (3.6) by $\overline{u} - \tilde{u}$, integrating over $[0, 2\pi]$ and using (3.4), (3.5) together with Lemma 3 with Γ_∞ replaced by $\Gamma_\infty + \eta$ and γ by $\gamma - \eta$, we get

$$\begin{aligned}
0 &= \frac{1}{2\pi}\int_0^{2\pi} [\overline{u} - \tilde{u}(x)]\{-u^{(iv)}(x) + \lambda f(u(x))u'(x)\\
&\qquad + [(1 - \lambda)\Gamma(x) + \tilde{\gamma}(x, u(x))]u(x) + \lambda h(x, u(x)) - \lambda e(x)\}dx\\
&\geq (\overline{\gamma} - \eta)\overline{u}^2 + [\delta(\Gamma_0) - \frac{\pi^2}{3}|\Gamma_1|_{L^1} - |\Gamma_\infty|_{L^\infty} - \eta]|\overline{u}|^2_{H^2}\\
&\qquad - (|\alpha|_{L^1} + |e|_{L^1})|\overline{u} - \tilde{u}|_{L^\infty}\\
&\geq \frac{1}{2}\overline{\gamma}\cdot\overline{u}^2 + \frac{1}{2}[\delta(\Gamma_0) - \frac{\pi^2}{3}|\Gamma_1|_{L^\infty} - |\Gamma_\infty|]|\tilde{u}|^2_{H^2} - \beta|u|_{H^2}\\
&\geq \eta|u|^2_{H^2} - \beta|u|_{H^2}
\end{aligned}$$

and hence $|u|_{H^2} \leq \beta/\eta$ which implies that $|u|_{C^1[0,1]} \leq C$ where C is a constant independent of $\lambda \in [0,1]$, in view of the compact imbedding $H^2[0,2\pi] \subset C^1[0,2\pi]$.

This completes the proof of the Theorem. $\square$

4. Asymptotic Conditions at Resonance. Let $f : \mathbf{R} \to \mathbf{R}$ be continuous and $g : [0,2\pi] \times \mathbf{R} \to \mathbf{R}$ be a function satisfying Caratheodory's conditions.

THEOREM 2. *Let $\Gamma \in L^1[0,2\pi]$ be such that*

$$\limsup_{|u|\to\infty} \frac{g(x,u)}{u} \leq \Gamma(x), \tag{4.1}$$

uniformly a.e. in $x \in [0,2\pi]$ and $\Gamma = \Gamma_0 + \Gamma_1 + \Gamma_\infty$ where $\Gamma_\infty \in L^\infty[0,2\pi]$, $\Gamma_1 \in L^1[0,2\pi]$ and $\Gamma_0 \in L^1[0,2\pi]$ are such that $\Gamma_0(x) \leq 1$ for a.e. $x \in [0,2\pi]$, with strict inequality holding on a subset of $[0,2\pi]$ of positive measure and $|\Gamma_\infty|_{L^\infty} + \frac{\pi^2}{3}|\Gamma_1|_{L^1} < \delta(\Gamma_0)$, where $\delta(\Gamma_0)$ is given by lemma 1.

Suppose, further, that there exist real numbers a, A, r and R with $a \leq A$ and $r < 0 < R$ such that

$$g(x,u) \geq A \tag{4.2}$$

for a.e. $x \in [0,2\pi]$ and all $u \geq R$, and

$$g(x,u) \leq a \tag{4.3}$$

for a.e. $x \in [0,2\pi]$ and all $u \leq r$.

Then the periodic boundary value problem

$$\begin{gathered} -\frac{d^4u}{dx^4} + f(u(x))u'(x) + g(x,u(x)) = e(x), x \in [0,2\pi], \\ u(0) - u(2\pi) = u'(0) - u'(2\pi) = u''(0) - u''(2\pi) = u'''(0) - u'''(2\pi) = 0 \end{gathered} \tag{4.4}$$

has at least one solution for each given $e \in L^1[0,2\pi]$ with

$$a \leq \overline{e} \leq A. \tag{4.5}$$

Proof. Define $g_1 : [0,2\pi] \times \mathbf{R} \to \mathbf{R}$ by $g_1(x,u) = g(x,u) - \frac{1}{2}(a+A)$ and $e_1 \in L^1[0,2\pi]$ by $e_1(x) = e(x) - \frac{1}{2}(a+A)$, so that for a.e. $x \in [0,2\pi]$ we have by using (4.2), (4.3), (4.5)

$$g_1(x,u) \geq \frac{1}{2}(A-a) \geq 0 \text{ if } u \geq R, \tag{4.6}$$

$$g_1(x,u) \leq \frac{1}{2}(a-A) \leq 0 \text{ if } u \leq r, \tag{4.7}$$

and

$$\frac{1}{2}(e-A) \le \overline{e}_1 \le \frac{1}{2}(A-a). \tag{4.8}$$

Now, the equation in (4.4) in clearly equivalent to

$$-\frac{d^4u}{dx^4} + f(u(x))u'(x) + g_1(x, u(x)) = e_1(x). \tag{4.9}$$

Moreover, we have

$$\limsup_{|u|\to\infty} u^{-1}g_1(x,u) \le \Gamma(x),$$

uniformly a.e. in $x \in [0, 2\pi]$ and for $|u| \ge \max(R, -r)$. a.e. $x \in [0, 2\pi]$, $u^{-1}g_1(x, u) \ge 0$. So that $\Gamma(x) \ge 0$ for a.e. $x \in [0, 2\pi]$.

Let, now $\eta = \frac{1}{2}[\delta(\Gamma_0) - \frac{\pi^2}{3}|\Gamma_1|_{L^1} - |\Gamma_\infty|_{L^\infty}] > 0$. Then, there exists an $r_1 > 0$ such that for a.e. $x \in [0, 2\pi]$ and for all $u \in \mathbf{R}$, $|u| \ge r_1$, we have

$$0 \le u^{-1}g_1(x,u) \le \Gamma(x) + \eta. \tag{4.10}$$

Proceeding as in the proof of Theorem 1 (of Section 3) we can write the equation (4.9) in the equivalent form

$$-\frac{d^4u}{dx^4} + f(u(x))u'(x) + \tilde{\gamma}(x,u(x))u(x) + h(x,u(x)) = e_1(x), \tag{4.11}$$

where $0 \le \tilde{\gamma}(x,u) \le \Gamma(x) + \eta$, $|h(x,u)| \le \alpha(x)$, for a.e. $x \in [0, 2\pi]$, all $u \in \mathbf{R}$ and some $\alpha \in L^1[0, 2\pi]$. Once again, degree arguments will ensure the existence of a solution for (4.4) if the set of all possible solutions of the family of equations

$$\begin{aligned} -\frac{d^4u}{dx^4} &+ \lambda f(u(x))u'(x) + [(1-\lambda)(\Gamma(x)+\eta) + \lambda\tilde{\gamma}(x,u(x)]u(x) \\ &+ \lambda h(x,u(x)) = \lambda e_1(x), \lambda \in [0,1], \end{aligned} \tag{4.12}$$

$u(0)-u(2\pi) = u'(0)-u'(2\pi) = u''(0)-u''(2\pi) = u'''(0)-u'''(2\pi) = 0$, is, a priori, bounded in $C^1[0, 2\pi]$ independently of $\lambda \in [0, 1]$. If, now, $u(x)$ is a possible solution of (4.12) for some $\lambda \in [0, 1]$, then intgrating the equation in (4.12) over $[0, 2\pi]$ after multiplying it by $\overline{u} - \tilde{u}$, we get on using Lemma 3 with $\gamma = 0$, and Γ_∞ replaced by $\Gamma_\infty + \eta$,

$$\begin{aligned} 0 = \frac{1}{2\pi}\int_0^{2\pi} &[\overline{u} - \tilde{u}(x)]\{-\frac{d^4u}{dx^4} + \lambda f(u(x))u'(x) + [(1-\lambda)(\Gamma(x)+\eta) + \tilde{\gamma}(x,u(x))]u(x) \\ &+ \lambda h(x,u(x)) - \lambda e_1(x)\}dx \\ \ge [\delta(\Gamma_0) &- \frac{\pi^2}{3}|\Gamma_1|_{L^1} - |\Gamma_\infty|_{L^\infty} - \eta]|\tilde{u}|^2_{H^2} - (|\alpha|_{L^1} + |e_1|_{L^1})|\overline{u} - \tilde{u}|_{L^\infty} \\ \ge \eta|\overline{u}|^2_{H^2} &- \beta(|\overline{u}| + |\tilde{u}|_{H^2}), \end{aligned}$$

for some constant β, independent of $\lambda \in [0,1]$. Hence,

$$|\tilde{u}|^2_{H^2} \le (\beta/\eta)(|\overline{u}| + |\tilde{u}|_{H^2}). \tag{4.13}$$

Next, we get on integrating the equation in (4.12) over $[0, 2\pi]$,

$$\frac{1}{2\pi}(1-\lambda)\int_0^{2\pi}(\Gamma(x)+\eta)u(x)dx + \frac{1}{2\pi}\lambda\int_0^{2\pi}[g_1(x,u(x)) - e_1(x)]dx = 0. \tag{4.14}$$

If, now, $u(x) \ge R$ for all $x \in [0, 2\pi]$ then (4.6), (4.8) imply that $(1-\lambda)(\overline{\Gamma}+\eta)R \le 0$, contradicting $\overline{\Gamma}+\eta \ge \eta > 0$. Similarly $u(x) \le r$ for all $x \in [0,2\pi]$ leads to a contradiction. So there must exist a $\tau \in [0, 2\pi]$ such that

$$r < u(\tau) < R.$$

It is then easy to see from $u(x) = u(\tau) + \int_\tau^x u'(s)ds$ that

$$|\overline{u}| \le \max(R, -r) + |\tilde{u}|_{H^2} \tag{4.15}$$

(4.13) and (4.15) now imply that

$$|\tilde{u}|^2_{H^2} \le (2\beta/\eta)|\tilde{u}|_{H^2} + (\beta/\eta)\cdot \max(R, -r),$$

so that there exists a constant ρ, independent of $\lambda \in [0,1]$ such that

$$|\tilde{u}|_{H^2} \le \rho. \tag{4.16}$$

Finally (4.15) and (4.16) imply that there is a constant C, independent of $\lambda \in [0,1]$ such that

$$|u|_{H^2} \le C$$

which implies that $|u|_{C^1} \le C_1$, for some constant C_1, independent of $\lambda \in [0,1]$.

This completes the proof of Theorem 2. □

REMARK 2. *If we take $f(u) \equiv \alpha$, $\alpha \in \mathbf{R}$ and $\Gamma(x) = \beta < 1$, (i.e. $\Gamma_0 = \beta$, $\Gamma_1 = \Gamma_\infty = 0$) in theorem 2 above we get Theorem 2.4 of [7] as a corollary to theorem 2.*

5. An inequality for a linear fourth order operator with periodic boundary conditions. We obtain a partial extension of Theorem 2 of section 4 when f is a constant function and $\Gamma_0 = \Gamma_\infty = 0$. We need the following lemma which gives an inequality for a linear fourth order operator with periodic boundary conditions.

LEMMA 4. *Let $\alpha \in \mathbf{R}$, $e \in L^1[0,2\pi]$, $\Gamma \in L^1[0,2\pi]$ with $\overline{\Gamma} \geq 0$. Then every possible solution $u(x)$ of the problem*

$$(5.1)\qquad \begin{aligned} &-\frac{d^4u}{dx^4} + \alpha u'(x) + p(x)u(x) = e(x), \quad x \in [0,2\pi], \\ &u(0) - u(2\pi) = u'(0) - u'(2\pi) = u''(0) - u''(2\pi) = u'''(0) - u'''(2\pi) = 0, \end{aligned}$$

with $p \in L^1[0,2\pi]$ such that

$$(5.2)\qquad \overline{p} \leq \overline{\Gamma}, \quad 0 \leq p(x)$$

for a.e. $x \in [0,2\pi]$ satisfies the inequality

$$(5.3)\qquad (1 - \frac{\pi^2}{4}\overline{\Gamma})|\frac{d^4u}{dx^4} - \alpha u'|_{L^1}^2 \leq 2|e|_{L^1}|\frac{d^4u}{dx^4} - \alpha u'|_{L^1} + \overline{\Gamma}|u|_{L^\infty}|e|_{L^1} + 3|e|_{L^1}^2$$

Proof. Let $p \in L^1[0,2\pi]$ be as above and $u(x)$ be a solution of (5.1). Then, on multiplying the equation in (5.1) by $\frac{u(x)}{2\pi}$ and integrating over $[0,2\pi]$ we get

$$(5.4)\qquad -\frac{1}{2\pi}\int_0^{2\pi}(u''(x))^2 + \frac{1}{2\pi}\int_0^{2\pi} p(x)u^2(x)dx = \frac{1}{2\pi}\int_0^{2\pi} e(x)u(x)dx.$$

Since, now $\overline{p} \leq \overline{\Gamma}$ we have, by using Schwarz's inequality

$$(5.5)\qquad \begin{aligned} (\frac{1}{2\pi}\int_0^{2\pi}|p(x)u(x)|dx)^2 &\leq (\frac{1}{2\pi}\int_0^{2\pi} p(x)dx)\frac{1}{2\pi}\int_0^{2\pi} p(x)u^2dx) \\ &\leq \overline{\Gamma}(\frac{1}{2\pi}\int_0^{2\pi} p(x)u^2(x)dx), \end{aligned}$$

and hence, using the equation in (5.1),

$$(5.6)\qquad (\frac{1}{2\pi}\int_0^{2\pi}|e(x) + \frac{d^4u}{dx^4} - \alpha u'|dx)^2 \leq \overline{\Gamma}(\frac{1}{2\pi}\int_0^{2\pi} p(x)u^2(x)dx)$$

We next apply an inequality of E. Schmidt [15] (see also [8]) to $u''' - \alpha\tilde{u}$ to get

$$(5.7)\qquad \begin{aligned} \frac{1}{2\pi}\int_0^{2\pi}[u''' - \alpha\tilde{u}]^2dx &= \frac{1}{2\pi}\int_0^{2\pi}(u''')^2dx + \frac{\alpha^2}{2\pi}\int_0^{2\pi}\tilde{u}^2dx \\ &\leq \frac{\pi^4}{4}(\frac{1}{2\pi}\int_0^{2\pi}|\frac{d^4u}{dx^4} - \alpha u'|dx)^2. \end{aligned}$$

Now, we get from (5.4), (5.6) and (5.7) that

$$\overline{\Gamma}^{-1}(\frac{1}{2\pi}\int_0^{2\pi}|e(x)+\frac{d^4u}{dx^4}-\alpha u'|dx)^2+\frac{1}{2\pi}\int_0^{2\pi}(u''')^2dx+\frac{\alpha^2}{2\pi}\int_0^{2\pi}\tilde{u}^2dx$$
$$\leq\frac{1}{2\pi}\int_0^{2\pi}p(x)u^2(dx)+\frac{\pi^2}{4}(\frac{1}{2\pi}\int_0^{2\pi}|\frac{d^4u}{dx^4}-\alpha u'|dx)^2$$
$$=\frac{1}{2\pi}\int_0^{2\pi}(u'')^2dx+\frac{1}{2\pi}\int_0^{2\pi}e(x)u(x)dx+\frac{\pi^2}{4}|\frac{d^4u}{dx^4}-\alpha u'|_{L^1}^2.$$

Hence,

$$-\frac{\pi^2}{4}|\frac{d^4u}{dx^4}-\alpha u'|_{L^1}^2+\overline{\Gamma}^{-1}|e(x)+\frac{d^4u}{dx^4}-\alpha u'|_{L^1}^2$$
$$\leq\frac{1}{2\pi}\int_0^{2\pi}(u'')^2dx-\frac{1}{2\pi}\int_0^{2\pi}(u''')^2dx-\frac{\alpha^2}{2\pi}\int_0^{2\pi}\tilde{u}^2dx+\frac{1}{2\pi}\int_0^{2\pi}e(x)u(x)dx$$
$$\leq|e|_{L^1}\cdot|u|_{L^\infty},$$

in view of Wintinger's inequality $|u''|_{L^2}\leq|u'''|_{L^2}$. Finally, then

$$(1-\frac{\pi^2}{4}\overline{\Gamma})|\frac{d^4u}{dx^4}-\alpha u'|_{L^1}^2=|\frac{d^4u}{dx^4}-\alpha u'+e-e|_{L^1}^2-\frac{\pi^2}{4}\overline{\Gamma}|\frac{d^4u}{dx^4}-\alpha u'|_{L^1}^2$$
$$\leq|e+\frac{d^4u}{dx^4}-\alpha u^1|_{L^1}^2+2|e|_{L^1}|e+\frac{d^4u}{dx^4}-\alpha u'|_{L^1}$$
$$+|e|_{L^1}^2-\frac{\pi^2}{4}\overline{\Gamma}|\frac{d^4u}{dx^4}-\alpha u'|_{L^1}^2$$
$$\leq 2|e|_{L^1}|\frac{d^4u}{dx^4}-\alpha u'|_{L^1}+\overline{\Gamma}|e|_{L^1}\cdot|u|_{L^\infty}+3|e|_{L^1}^2.$$

Hence the lemma. □

THEOREM 3. *Let $\alpha\in\mathbf{R}$ be given and $g:[0,2\pi]\times\mathbf{R}\to\mathbf{R}$ be a function satisfying Caratheodory's conditions. Assume that there exists a $\Gamma\in L^1[0,2\pi]$ such that*

$$\limsup_{|u|\to\infty}u^{-1}g(x,u)\leq\Gamma(x)$$

uniformly a.e. on $[0,2\pi]$ and that $\overline{\Gamma}<\frac{4}{\pi^2}$. Suppose, further that there exist real numbers a,A,r,R with $a\leq A$ and $r<0<R$ such that for a.e. $x\in[0,2\pi]$, $g(x,u)\geq A$ when $u\geq R$ and $g(x,u)\leq a$ when $u\leq r$. Then the periodic boundary value problem

$$(5.8)\qquad\begin{aligned}&-\frac{d^4u}{dx^4}+\alpha u'+g(x,u(x))=e(x),\quad x\in[0,2\pi],\\&u(0)-u(2\pi)=u'(0)-u'(2\pi)=u''(0)-u''(2\pi)=u'''(0)-u'''(2\pi)=0\end{aligned}$$

has at least one solution for each given $e \in L^1[0, 2\pi]$ *with* $a \leq \overline{e} \leq A$.

Proof. We first define g_1 and e_1 as in the proof of Theorem 2(section 4) so that the equation in (5.8) can be written as

$$-\frac{d^4u}{dx^4} + \alpha u' + g_1(x, u(x)) = e_1(x), \tag{5.9}$$

with $g_1(x, u) \geq 0$ when $u \geq R$ and $g_1(x, u) \leq 0$ when $u \leq r$ for a.e. $x \in [0, 2\pi]$ and $\limsup_{|u|\to\infty} u^{-1}g_1(x, u) \leq \Gamma(x)$ uniformly for a.e. $x \in [0, 2\pi]$. Consequently for a.e. $x \in [0, 2\pi], \Gamma(x) \geq 0$. Let $\eta = \frac{1}{2}[\frac{4}{\pi^2} - \overline{\Gamma}] > 0$ so that $\overline{\Gamma} + \eta < \frac{4}{\pi^2}$ and let $r_1 > 0$ be such that

$$0 \leq u^{-1}g_1(x, u) \leq \Gamma(x) + \eta \tag{5.11}$$

for a.e. $x \in [0, 2\pi]$, $|u| \geq r_1$. Proceeding as in the proof of Theorem 1 (Section 3) we can write (5.9) in the form

$$-\frac{d^4u}{dx^4} + \alpha u' + \tilde{\gamma}(x, u(x))u(x) + h(x, u(x)) = e_1(x), \tag{5.12}$$

where $0 \leq \tilde{\gamma}(x, u) \leq \Gamma(x) + \eta$, $|h(x, u)| \leq \beta(x)$ for a.e. $x \in [0, 2\pi]$ and all $u \in \mathbf{R}$ and some $\beta \in L^1[0, 2\pi]$. The same degree arguments will imply the existence of a solution for (5.8) if the set of possible solutions of the family of equations

$$\begin{aligned} -\frac{d^4u}{dx4} + \alpha u'(x) + [(1 - \lambda)(\Gamma(x) + \eta) + \lambda\tilde{\gamma}(x, u(x))]u(x) \\ = -\lambda h(x, u(x)) + \lambda e_1(x), \end{aligned} \tag{5.13}$$

$u(0) - u(2\pi) = u'(0) - u'(2\pi) = u''(0) - u''(2\pi) = u'''(0) - u'''(2\pi) = 0$, is, a priori, bounded in $C[0, 2\pi]$ independently of $\lambda \in [0, 1]$. Let $u(x)$ be a solution of (5.13) for some $\lambda \in [0, 1]$. Since now,

$$0 \leq (1 - \lambda)(\Gamma(x) + \eta) + \lambda\tilde{\gamma}(x_1 u(x)) \leq \Gamma(x) + \eta$$

for a.e. $x \in [0, 2\pi]$, with $\tilde{\Gamma} + \eta < \frac{4}{\pi^2}$, and since

$$|e_1 - h(\cdot, u(\cdot))|_{L^1} \leq |e_1|_{L^1} + |\beta|_{L^1},$$

it follows from Lemma 4 that

$$\begin{aligned} [1 - \frac{\pi^2}{4}(\tilde{\Gamma} + \eta)]|\frac{d^4u}{dx^4} - \alpha u'|_{L^1}^2 \leq 2(|e_1|_{L^1} + |\beta|_{L^1})|\frac{d^4u}{dx^4} - \alpha u'|_{L^1} \\ + (\overline{\Gamma} + \eta)(|e_1|_{L^1} + |\beta|_{L^1})|u|_{L^\infty} \\ + 3(|e_1|_{L^1} + |\beta|_{L^1})^2 \end{aligned} \tag{5.14}$$

Also, we see as in the proof of Theorem 2 (section 4) that there exists a $\tau \in [0, 2\pi]$ such that

$$r < u(\tau) < R \tag{5.15}$$

Next, we use lemma 2.1 of [7] to deduce the existence of constants $\delta = \delta_1(\alpha) > 0$, $\delta_2 = \delta_2(\alpha) > 0$ such that

$$|\tilde{u}|_{L^\infty} \le \delta_1 |\frac{d^4\tilde{u}}{dx^4} - \alpha\tilde{u}'|_{L^1} = \delta_1 |\frac{d^4 u}{dx^4} - \alpha u'|_{L^1} \tag{5.16}$$

$$|u'|_{L^\infty} \le \delta_2 |\frac{d^4\tilde{u}}{dx^4} - \alpha\tilde{u}'| = \delta_2 |\frac{d^4 u}{dx^4} - \alpha u'|_{L^1} \tag{5.17}$$

for every $u \in C^3[0, 2\pi]$ with u''' absolutely continuous and satisfying the periodic boundary conditions in (5.13). Using, next, (5.16) in (5.14) we get

$$\begin{aligned} [1 - \frac{\pi^2}{4}(\overline{\Gamma} + \eta)] |\frac{d^4 u}{dx^4} - \alpha u'|^2_{L^1} &\le (|e_1|_{L^1} + |\beta|_{L^1})[2 + \delta_1(\overline{\Gamma} + \eta)] |\frac{d^4 u}{dx^4} - \alpha u'|_{L^1} \\ &+ (\overline{\Gamma} + \eta)(|e_1|_{L^1} + |\beta|_{L^1})|\overline{u}| + 3(|e_1|_{L^1})^2 \end{aligned} \tag{5.11}$$

Also, it follows from (5.15), (5.17) that

$$\begin{aligned} |u(x)| = |u(\tau) + \int_\tau^x u'(s)ds| &< \max(-r, R) + 2\pi |u'|_{L^\infty} \\ &\le \max(-r, R) + 2\pi\delta_2 |\frac{d^4 u}{dx^4} - 2u'|_{L^1} \end{aligned}$$

so that

$$|\overline{u}| \le \max(-r, R) + 2\pi\delta_2 |\frac{d^4 u}{dx^4} - \alpha u'|_{L^1}. \tag{5.19}$$

Finally, it follows from (5.16), (5.18), (5.19) that there exist a constant ρ, independent of $\lambda \in [0, 1]$ such that

$$|u|_{L^\infty} \le \rho.$$

This completes the proof of Theorem 3.

Remark 3. In the case when $\Gamma_0 = \Gamma_\infty = 0$ and $f \equiv \alpha$ in Theorem 2, we see that Theorem 3 improves the condition on Γ from $\overline{\Gamma} < \frac{3}{2\pi^2}$ into $\overline{\Gamma} < \frac{2}{\pi^2}$. (Note that $\delta(0) = \frac{1}{2}$ in lemma 1). In this sense Theorem 3 is an extension of Theorem 2. However, if Γ is a constant, then Theorem 3 is not as sharp as Theorem 2.

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