A STATIC NONSTATIONARY ANALYSIS OF THE INTERACTION BETWEEN MONETARY AND FISCAL POLICY

by

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I. INTRODUCTION

The literature of "static" macroeconomics is rife with confusion regarding the passage of time, which results in confusion regarding the nature of equilibrium, the distinction between stocks and flows, and the role of expectations. There are many models in which time passes for some purposes, but stands still for others. Most multiplier-accelerator models partake of that defect. The models in Money, Interest, and Prices (7) partake of that defect and exhibit a great deal of confusion regarding the distinction between stocks and flows. Also sharing that defect are recent attempts to analyze the effects of different ways of financing a government deficit, whether by money creation or by bond sales to the public (1, 2).

That and many other questions that have received considerable attention are easily answered once a rigorous treatment of the passage of time is imposed. For example, there is a considerable literature devoted to loanable funds versus liquidity preference theories of the interest rate. The distinction supposedly rests on there being different ways of answering the question: given, say, an excess supply of money, how is that divided between an excess demand for bonds and an excess demand for commodities? In the context of the models in which it was posed, the question makes no sense. In those models, there do not exist markets for stocks of commodities, so that, in them, an excess stock supply of money can only be matched by an excess stock demand for bonds. There are models with markets for stocks of commodities, models that are quite different from the standard Keynesian model, but such models contain stocks demand function for real assets, which then imply how an excess supply of money is matched by excess demands for all assets in the model.[1] Two other questions debated at length in the literature involve the indeterminacy of the price level and the non-unitary elasticity of desired nominal money holdings with respect to the price level. Both issues were taken up in the rarified atmosphere of a model with flows of many different commodities, but only one stock, the stock of money. (Commodities, presumably, were assumed to depreciate instantaneously--that is, at an infinite rate.) It should be obvious that one can create paradoxes by assuming an economy with only one stock so that whatever happens, stock excess demands or supplies of that one stock can never be matched. And, how does the market experiment of changing the stock of money ever get accomplished in such an economy? That brings us to the last question we shall remark upon here; does it matter whether a change in the stock of money is brought about by way of an exchange of assets or in some other way? But, what other way is there? Once we distinguish between alterations of a stock at a moment-in-time and a stock growing at a rate unit time (in order, say, to finance a flow of expenditures), this last question is largely resolved. While we may still wish to consider a question like — What happens if a fire, say, destroys x% of the stock of money (currency)? — it should be recognized that aside from currency reforms, it is hard to conceive of the monetary authority bringing about moment-in-time changes in the stock of money except by way of asset exchanges.

Several of the above questions seem to pose problems that call into question the logic and completeness of the standard textbook macro-model.

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[1] See (9) for a discussion of the differences between models with and without a market in the stock of real capital.

This paper might be regarded as an attempt to save that model. To save it, all that is required is an explicit treatment of the passage of time and its relationship to the nature of equilibrium, the distinction between stocks and flows, and the role of expectations.

The view of time I adopt is implied by the assumed constancy of the capital stock between alternative equilibrium positions in static models. Given that net investment is determined in the models and may be nonzero, the assumed constancy of the capital stock, if taken literally, implies that no time passes between alternative equilibria. It is that literal interpretation that we shall adopt. It follows that our static analysis takes the form: given the economy at a moment-in-time, $\bar{t}$, and given the occurrence of some disturbance or change in policy at $\bar{t}$, how does the economy respond at $\bar{t}$ given the assumptions of the model about adjustments that occur instantaneously?

Equilibrium at the point-in-time, $\bar{t}$, shall mean that certain relationships among variables hold both at $\bar{t}$ and in some neighborhood about $\bar{t}$. The latter requirement allows an analysis of the effects of an extended set of exogenous variables and the determination of an extended set of endogenous variables. For example, we shall show how to analyze the effects of changes in the stock of money that result from once-for-all open market operations, and the effects of changes in the rate of growth of the stock that result from changes in the way government deficits are financed. And, we shall describe the relationships that determine the level of output and those that determine its rate of growth, all at a point-in-time. Obviously, equilibrium in our sense does <u>not</u> imply a tendency for the values of variables to remain unchanged through time, which is one reason we call the analysis nonstationary.

In the particular model we shall present in order to illustrate those concepts, a central role is played by the way the perceived (or expected) rate of change of the price level and its time derivatives are determined. If the perceived rate, itself, is assumed exogenous, independent of the current actual rate of change of the price level --- a particular kind of recursive assumption, then among the relationships that insure equilibrium as defined above, there is an independent or recursive subset that is indistinguishable from the standard textbook macroeconomic model. That subset implies, by the way, that the level of output is independent of the rate of growth of the money stock, and, therefore, independent of the way a given government deficit is financed. Of course, independence between the expected and actual rates of change of the price level builds into the equilibrium position at $\bar{t}$ a discrepancy between the two, which might be thought of as a disturbance that will impinge at some future time. In a sense, full equilibrium at $\bar{t}$ requires that all expected quantities --- in particular, all time derivatives of the expected rate of change of the price level --- be set equal to the corresponding actual quantities, but if that is done, then, among the relationships that imply equilibrium at $\bar{t}$, there is no finite subset that constitutes a closed or complete model. In order to obtain a closed model, a recursive or ad hoc assumption about some time derivative of the perceived rate of change of the price level must be imposed.[2/] That is the more important sense in which the analysis

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[2/] Independence between the perceived and actual rates of change of the price level is implied if, for example, the perceived rate is a function only of the time path of the price level up to and including $\bar{t}$. Dependence between the perceived and actual rates is implied if people at $\bar{t}$ "see" the right-hand time derivative of the price level at $\bar{t}$. But, how is that possible? As we shall show, the model implies a value for the right-hand time derivative of the price level, whether or not people at $\bar{t}$ are assumed to "see" that value. To say that people "see" that value at $\bar{t}$ may only mean that they use the model to predict the value of the right-hand time derivative of the price level at $\bar{t}$. See (6).

In empirical work, we have not as yet got beyond the simplest kind of recursive ad hoc assumptions about the determination of expected quantities, even though there is evidence to suggest that such formulations are inadequate. See (8).

is nonstationary, or if you like, partial equilibrium. As suggested above, the standard textbook model should be viewed as containing an ad hoc assumption about the perceived rate of change of the price level, itself. Under that assumption, and contrary to the claims of Christ (1,2) and Mundell (5), the standard textbook model is a complete, consistent model, although a nonstationary model as noted above. One of the things we shall do in this paper, though, is show what happens if such an ad hoc assumption is imposed only on expectations of higher order time derivatives of the price level.

Before turning to the model we use, we can illustrate our view of time and introduce the notation we use with the aid of Figures 1 and 2. Figure 1 shows a number of time paths of the stock of money. Suppose path 0 is the initial path. If at $\bar{t}$ the stock of money remains unchanged, but its rate of growth increases, there results path 1. In that case, we say that dM, the total differential of M, is zero, and that $d\dot{M} = d(\partial M/\partial t)$, the total differential of its rate of change is positive. (Since our variables are not differentiable functions of time at $\bar{t}$, the symbol $\dot{x}$ for any variable x stands either for the left-hand or the right-hand time derivative of x at $\bar{t}$.) If, alternatively, the stock increases at $\bar{t}$, but the rate of growth remains unchanged, there results path 1'. In that case, dM is positive and $d\dot{M}$ is zero. In Figure 2 we illustrate the notation for output. If a disturbance at $\bar{t}$ gives rise to path 1, dY is zero and $d\dot{Y}$ is positive; while if it gives rise to path 1', dY is positive and $d\dot{Y}$ is zero. In general, any disturbance or change in policy at $\bar{t}$ can be expected to affect both Y and $\dot{Y}$, and higher order time derivatives of Y.[3/]

[3/] This paper uses only elementary calculus. Nevertheless, in order to dispel doubts some readers may have about our use of "total" differentiation and its relationship to differentiation with respect to time, an appendix that justifies those operations is attached.

Figure 1

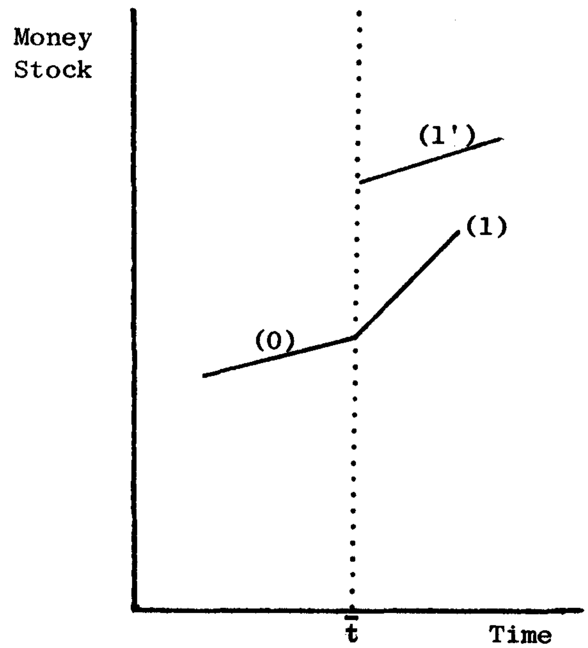
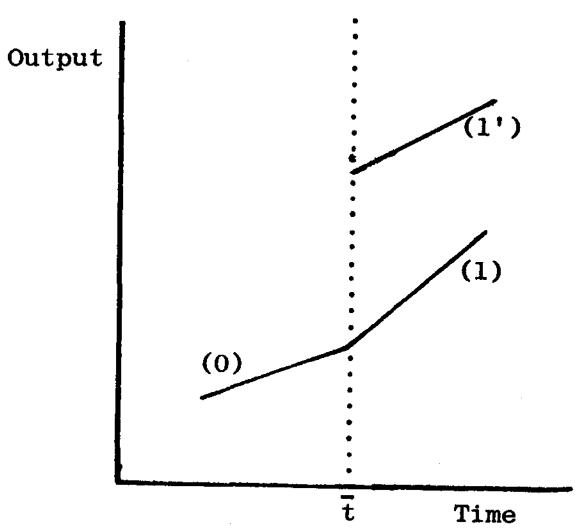


Figure 2



II. The Model

All the analysis shall be carried out within a model of a closed, one-commodity, one-paper-earning asset economy. The behavior relationships duplicate, in large measure, those in textbook models, since our main purpose is to illustrate a kind of analysis. The model contains three sectors: firms, households, and the government. We assume that production of the good is carried out by competitive firms each of which faces the same homogeneous-of-degree-one production function with labor and capital as the only two factors of production. Imperfect flexibility of the money wage, described below, prevents automatic attainment of full employment.

The asset structure of the model is also very simple. In addition to real capital, all of which is held by firms, there are three paper assets; money, equities, and call loans. I assume that firms hold no money; all is held by individuals - an assumption which helps preserve the fiction of only two factors of production, since if firms hold money, it makes even less sense than it otherwise does to assume that output is independent of the stock of real money holdings. Firms issue as liabilities only equities. Call loans, the third paper asset, may be issued and held by individuals or the government. They are assumed, however, not to be distinguishable by issuer. The system is further simplified by the assumption that from the point of view of individuals, call loans and equities are perfect substitutes when their real yields -- defined to take account of the perceived time rate of change of the price level -- are equal. That assumption, in effect, reduces the number of paper earning assets to one.[4/]

1. Firms, Factor Inputs, and Production

We assume that each firm faces a perfectly elastic supply curve of labor at the real wage, W/P, and a perfectly inelastic supply curve of capital. Firms can, however, engage in investment, that is, in nonzero rates of accumulation.

Let π^i be the ith firm's real profits at a point in time;

$$\pi^i = f(K^i, L^i) - (W/P)L^i - (r + \delta - \theta)K^i$$

where f is the common differentiable production function, K^i and L^i are the firm's capital and labor inputs, r is the nominal interest rate, δ is a constant depreciation rate, and θ is the perceived (percentage) rate of change of the price level, assumed common to everyone in the economy.

The firm is able to alter the flow of profits at a point in time only by altering its labor input. I assume that it chooses L^i so as to maximize profits, from which is obtained the familiar necessary condition,

$\pi^i_L = 0$, or,

(1) $f_L(K^i, L^i) = W/P$

Given the fixed value of K^i, (1) is a relationship between the real wage and desired employment of firm i; firm i's demand function for labor. I shall assume that that relationship is always satisfied.

If the firm could buy and sell stocks of real capital at the price P or could rent it at the rental $(r+\delta-\theta)P$, a second necessary condition

4/ Everything we do also holds for the case of a constant risk differential between the real yields of equities and call loans.

for profit maximization would be $\pi^i_K = 0$.[5/] Since such dealings are ruled out, I shall assume the firm's desired rate of accumulation of capital to be an increasing function of π^i_K, in particular,

(2) $$\dot{K}^i = g^i(\pi^i_K) = g^i(f_K - \delta - r + \theta);\ g^{i\prime} > 0$$

where $\dot{K}^i$ is desired net investment, assumed always equal to actual net investment of firm i. According to this formulation, investment is an increasing function of the difference between the net marginal product of capital, $f_K - \delta$, and the real rate of interest, $r-\theta$, where f_K is a function of employment, and hence, output.

In a sense, the demand function for labor and the investment function completely describe the firm's activities; the firm does not engage in portfolio operations, since, having assumed that it holds only one asset, real capital, and only one liability, equity, fixity of its stock of capital implies fixity of its outstanding stock of equities. Changes in the value of outstanding equity are not ruled out, however, nor are nonzero rates of additions or subtractions from the stock of equity, which, since I assume that all earnings are distributed, finance the firms' net investment, $\dot{K}^i$.

The real value of the firm's outstanding equities, denoted $(V/P)^i$, is assumed equal to real earnings capitalized at the real rate of interest:

$$(V/P)^i = [f(K^i, L^i) - (W/P)L^i - \delta K^i]/(r-\theta)$$

Then, given that f is homogeneous of degree one and that $f_L = W/P$,

(3) $$(V/P)^i = K^i(f_K - \delta)/(r-\theta),$$

[5/] See (9) for a discussion of the static model that results from assuming a perfect market in existing capital.

so that the market value of the firm is greater than, equal to, or less than the replacement value of its capital stock as the net marginal product of capital is greater than, equal to, or less than the real rate of interest. Since, nothing prevents $(f_K-\delta)/(r-\theta)$ from changing at a point-in-time, $(V/P)^i$ can also change.

Under the assumption that firms share a common production function, the superscript i can be dropped from equations (1)-(3). The resulting equations are to be interpreted as aggregate relationships among the total capital stock, K, total employment, L, and the total market value of all firms, V/P.

2. Consumer Saving and Portfolio Decisions

The assumption that call loans and equities are perfect substitutes when their real yields are equal implies that there is only one portfolio decision, that involving the allocation of marketable wealth between money and paper-earning assets. That being the case, equilibrium in the market for all paper assets is guaranteed by equality between the the supply of and demand for real money balances, which we write as

$$M/P = m(Y,r) \quad ; \; m_Y > 0, \; m_r < 0$$

where m is the demand function for money balances. The absence of wealth from the function m implies that holding income and the rate of interest constant, any change in wealth is assumed to change desired holdings of earning assets by the full change in wealth; that is, letting b be the demand function for paper-earning assets, and $Z = V/P + B/P + M/P$ be wealth - where B is the net interest bearing debt of the government, the above implies $V/P + B/P = b(Y, r, Z)$, where $b_Y = -m_Y$, $b_r = -m_r$, and $b_Z = 1$.

For the consumption-saving decision, we assume

$$C = c(Y - \delta K - T) \quad ; \; 1 > c' > 0,$$

where C is real consumption expenditures and T is real tax revenue minus transfers. This specification may seem overly simple in a number of respects. First, $Y-\delta K-T$ is not equal to actual disposable income if the latter is defined as the sum of consumer expenditures and the rate of change of wealth or saving. Actual disposable income is[6]

$$Y^D = Y - \delta K - T-(\dot{P}/P)(M+B)/P + K\partial[(f_K- \delta)/(r-\theta)]/\partial t + \dot{K}[1- (f_K-\delta)/(r-\theta)]$$

Second, there are good reasons for including the rate of interest and wealth as additional arguments of c. We choose to ignore both refinements, because they would complicate the presentation, while leaving our main conclusions unaffected. Note that the allocation of total saving, $Y^D - C$, between rates of accumulation of real money holdings and real paper-earning-asset holdings is determined by the time derivative of the function m.

3. Government

The government makes both stock and flow decisions. It can exchange assets subject to $dM=-dB$, the constraint for once-for-all open market operations that simultaneously change the stock of money and the stock of outstanding government borrowings. And it makes flows decisions subject to

$$G = T + \dot{M}/P + \dot{B}/P \; ,$$

where G is real government expenditures and $\dot{M}$ and $\dot{B}$ are the time rates of

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[6] This expression for Y^D is obtained from the definition

$$Y^D = C + Z = C + (\dot{M/P}) + (\dot{B/P}) + (\dot{V/P})$$

after substituting into it the time derivative of the expression for V/P given in (3); the government flow constraint given in the text below; and the GNP identity by purchaser, equation (9) below.

change of the money stock and of government debt respectively, T is assumed exogenous and to include real interest payments on the outstanding government debt; if the government contracts additional call loans $(dB > 0)$, then unless we specify that T changes, it is assumed that there accompanies the open market operation either a decline in other kinds of government transfers to the private sector or a rise in tax revenues. The government is free to choose four variables; either dM or dB, and three of the following four: G, T, $\dot{M}$, $\dot{B}$.

4. The Supply of Employees or the Imperfectly Flexible Money Wage.

The simplest assumption that would allow the model to produce changes in output and employment in the face of a given production function is that the money wage is exogenous. A more elaborate assumption is that W is determined by a relationship of the form,

$$W = w(P, L); \; w_P > 0, \; w_L > 0.$$

If the function w is to be consistent with changes in output in the face of an unchanged production function, then w must not be homogeneous of degree one in P ; it must not be reducible to a relationship between the real wage and employment. We assume that the elasticity of w with respect to P is positive, but less than one.

5. The Complete Model

At this point it is convenient to summarize the above by the following equations.

(4) $Y - f(K,L)$

(5) $\dot{K} = g(f_K - \delta - r + \theta)$

(6) $C = c(Y - \delta K - T)$

(7) $f_L = W/P$

(8) $M/P = m(Y, r)$

(9) $Y = C + \dot{K} + G + \delta K$

(10) $W = w(P, L)$

If θ is assumed determined outside the system, then equations (4)-(10) form a closed system in seven variables denoted $A^o = [Y, L, \dot{K}, r, C, W, P]$. If, alternatively, θ is assumed equal to or dependent on $\dot{P}/P$, then the above seven equations are a system in eight variables, A^o and $\dot{P}$, $\dot{P}$ being an element in what we call A^1, the set of seven variables whose members are the time derivatives of the members of A^o. In that case, we require a model that simultaneously determines A^o and $\dot{P}$. The obvious candidate is the model that consists of equations (4)-(10) and the time derivatives of those equations, a fourteen equation model. That model contains as unknowns A^o, A^1, and $\dot{\theta}$, and is, therefore, a complete model only if $\dot{\theta}$ is assumed exogenous, because if $\dot{\theta}$ is dependent on $\partial(\dot{P}/P)/\partial t$, the fourteen equation model contains fifteen unknowns, A^o, A^1, and $\partial\dot{P}/\partial t$. Clearly, we may continue in this manner to higher order time derivatives of (4)-(10) finding that a complete model is never obtained if θ and all its time derivatives are set equal to or dependent on the corresponding time derivatives of the rate of change of the price level.[7/] Therefore, we shall initially assume that θ and all its time derivatives are exogenous. It follows, then, that (4)-(10) form a closed system in A^o, and, for example, that those equations plus their first-order time derivatives form a closed system in A^o and A^1 within which (4)-(10) are an independent subset. We shall then assume that θ is dependent on $\dot{P}/P$, but that all the time derivatives of θ are exogenous. Under that assumption, (4)-(10) and the first-order time derivatives of (4)-(10) form a closed system in A^o and A^1, within which there is no independent subset.

[7/] In the "money and growth" literature, it is often assumed that $\partial^n\theta/\partial t^n = \partial^n(\dot{P}/P)/\partial t^n$ for n=0,1,2,...; but, then, analysis is limited to finding steady state growth paths on which $\partial^n\theta/\partial t^n = \partial^n(\dot{P}/P)/\partial t^n = 0$ for n=1,2,... . When attempts are made to analyze the behavior of those models off steady state growth paths, *ad hoc*, recursive assumptions about θ and its time derivatives are imposed. *See* (4,10).

III. THE INTERACTION BETWEEN MONETARY AND FISCAL POLICY

We begin our analysis of the model by deriving versions of the usual IS and LM curves. The IS curve is found by substituting into the GNP identity the equalities between desired and actual values of net investment and consumption given by equations (5) and (6). Upon making the substitutions, totally differentiating the result, using the production function, (4), to eliminate dL from the expression, and rearranging, we obtain

$$(11) \qquad (1 - c' - g'f_{KL}/f_L)dY = dG - c'dT - g'dr + g'd\theta$$

The presence of f_K in the investment function implies a positive dependence of $\dot{K}$ on Y, by way of the effect on the marginal product of capital of changes in employment. This dependence is reflected in the term $g'f_{KL}/f_L$ in the coefficient of dY, and, as is well-known, makes ambiguous the sign of the slope of the IS curve.

Our version of the LM curve, which incorporates the supply side of the model, is found by substituting into the household portfolio equilibrium condition the value of P implied by the labor marginal productivity condition, (7), after the value of W from (10) has been substituted into it. Then, upon totally differentiating the result, and using the production function to eliminate dL, the result can be written

$$(12) \qquad [m_Y + (1/f_L)(M/W)(f_{LL}-w_L/P)/(w_PP/W - 1)]dY = dM/P - m_r dr$$

Note that w_PP/W is the elasticity of the money wage with respect to the price level. The larger that elasticity within the range (0,1), the greater the slope of our LM curve as measured by dr/dY. If that elasticity were one, dr/dY would be infinite, and output and employment would be unaffected by either changes in the stock of money or shifts of the IS curve.

Note, in addition, that the LM curve for the case of an exogenous money wage is found by setting $w_P = w_L = 0$; making the money wage responsive to changes in prices and employment serves to increase dr/dY for our version of the LM curve.

1. The "Level" (A^o) System with an Exogenous Expected Rate of Change of the Price Level

Putting those IS and LM curves together and assuming that dr/dY for the LM curve exceeds dr/dY for the IS curve -- a necessary condition for stability under a wide range of assumptions -- we obtain most of the usual comparative static results if we assume, as we now do, that θ is exogenous. In particular, changes in government expenditures and taxes have given effects on output, employment, the price level, and the interest rate independent of how the implied change in G-T is financed. The financing decision involves an allocation of G-T among $\dot{M}/P$ and $\dot{B}/P$, but under the assumption that θ is exogenous, neither $\dot{M}$ nor $\dot{B}$ appears as a determinant of the set A^o.

There has been confusion on this point, perhaps because of a failure to recognize that equality between the rate at which the public wishes to add to its holdings of assets (saving) and the rate at which assets in total are being created or supplied is implied by equality between output and aggregate demand, or, equivalently, by equality between, on the one hand, the sum of government expenditures and net investment, and, on the other hand, the sum of saving and taxes.[8/] Thus, most of the concern often expressed about the effect on the interest rate of Treasury

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[8/] This point has been emphasized by Tobin. (11)

bond sales implied by a deficit is misplaced. At the interest rate that equates output and aggregate demand, the public is willing to acquire assets at a rate equal to that implied by the deficit and the level of net investment. There may be a compositional effect, but only on the time derivative of the interest rate (see below), and that effect, if present, is quite different from what would result if there were a discrepancy between desired saving and the rate at which assets in total were being created.

To take an example, suppose G increases, while T is held constant. We know that the new equilibrium position is characterized by higher values of Y, P and r. We also know that $G + \dot{K}$ is higher than in the initial position, and, therefore, that the non-business public must be acquiring paper assets at a higher rate than initially.[9] They are willing to, because given the higher level of income, they wish to save (add to wealth) at a higher rate than initially. The interest rate is higher than initially not because the increase in G must be financed, but because given that money income is higher and that the stock of money is unchanged (no matter how the increase in G is financed), the public must be induced to hold that unchanged stock of money. They are not willing to at the initial interest rate. Indeed, one can regard the interest rate as rising to the new equilibrium level, because of an attempt by the public to add to its holdings of money

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[9] This is not dependent on the assumption that firms do not retain earnings. All these results follow no matter how net investment is financed.

in response to the higher level of nominal income.[10/]

In conclusion, then, if we view the standard textbook model as a moment-in-time model of the variables in A^0 and as including a recursive assumption about the expected rate of change of the price level, θ, then, as claimed above, that model is a logical, complete model. Under the assumption that θ is exogenous, there is no need to specify how a change in the deficit is financed, nor, for that matter, how a deficit in the initial position is being financed.

2. The "First Derivative" (A^1) System with an Exogenous Expected Rate of Change of the Price Level.

In order to describe the determination of the set A^1, in which appear the rates of change of output and employment, we use the time derivatives of (4)-(10) to derive rate of change analogues to the IS and LM curves found above.

From the GNP identity, we have

$$\dot{Y} = \dot{C} + \partial \dot{K}/\partial t + \dot{G} + \delta \dot{K}$$

Substituting into this the equalities between actual and desired rates of change of net investment and consumption, the time derivatives of equations (5) and (6), we get

$$\dot{Y} = c'(\dot{Y} - \dot{T}) + g'(f_{KL}\dot{L} + f_{KK}\dot{K} - \dot{r} + \dot{\theta}) + \delta \dot{K} + \dot{G}$$

[10/] It seems that Milton Friedman accedes to this view. In the Friedman-Heller debate, after discussing at length the implications of how an increase in G is financed -- a discussion that is somewhat misleading according to the model set out here, Friedman concludes, in accord with the model set out here,

> "If there is going to be any net effect [of a decrease in G], it has to be on a more sophisticated level; it has to be the indirect effect of the reduction in interest rates on other variables. In particular, it has to be a willingness on the part of the populace to hold more money, more nominal money, when the interest rate goes down. [(3) page 54.]

Upon substituting into this the value of $\dot{L}$ implied by the time derivative of the production function, (4), and rearranging, we obtain,

$$(13) \quad [1 - c' - g'f_{KL}/f_L]\,\dot{Y} = \dot{G} - c'\dot{T} - g'(\dot{r} - \dot{\theta}) + [g'(f_{KK} - f_{KL}f_K/f_L) + \delta]\,\dot{K}$$

To obtain the rate of change analogue of our LM curve, we begin by differentiating the household portfolio equilibrium condition, (8), with respect to time;

$$\dot{M}/P - (M/P)(\dot{P}/P) = m_Y\dot{Y} + m_r\dot{r}$$

This expresses the condition that the rate at which actual real money balances are changing, the left-hand side, equals the rate at which desired real money balances are changing, the right-hand side. The next step involves writing $\dot{P}$ as a function of $\dot{Y}$. First, from the time derivative of (7),

$$f_{LK}\dot{K} + f_{LL}\dot{L} = \dot{W}/P - (W/P)(\dot{P}/P)$$

Then, replacing $\dot{L}$ by its value as implied by the time derivative of the production function and $\dot{W}$ by its value as implied by the time derivative of (10), we obtain upon solving for $\dot{P}/P$,

$$(14) \quad (\dot{P}/P)[1 - w_P P/W] = -(1/f_L^2)(f_{LL} - w_L/P)\dot{Y} - (1/f_L^2)(f_Lf_{LK} - f_{LL}f_K + w_Lf_K/P)\dot{K}$$

Substituting this into the time derivative of the portfolio condition, we obtain upon rearranging, the rate of change analogue of our LM curve,

$$(15) \quad [m_Y - (M/P)(f_{LL} - w_L/P)/f_L^2(1 - w_P P/W)]\dot{Y} = \dot{M}/P - m_r\dot{r} +$$
$$\dot{K}(f_Lf_{LK} - f_{LL}f_K + w_Lf_K/P)/Pf_L^2(1 - w_P P/W)$$

Both (13) and (15) depend directly or indirectly on all the variables that determine the A^o variables. But, with θ exogenous the A^o variables are determined independently of $\dot{Y}$ and $\dot{r}$, and independently of $\dot{M}$, $\dot{G}$, and $\dot{T}$, so that under that assumption we may hold the A^o variables constant when inquiring into the effects on the A^1 variables of changes in $\dot{M}$, $\dot{G}$, and $\dot{T}$. In particular, assuming θ exogenous, we may hold the A^o set constant when asking about the effects of different ways of financing a given government deficit, because for given values of G and T, altering the way the implied deficit is financed involves altering $\dot{M}$ and $\dot{B}$ subject to $d\dot{M} + d\dot{B} = 0$. Moreover, since $\dot{B}$ does not appear explicitly in (13) and (15), we may examine alternative ways of financing a given deficit by imposing alternative values of $\dot{M}$ in equation (15), and jointly solving equations (13) and (15) for $\dot{Y}$ and $\dot{r}$. Given the values of the A^o variables, it is a simple matter to deduce from those equations that the higher is $\dot{M}$ the higher is $\dot{Y}$, and, then, from (14) that the higher is $\dot{M}$, the higher is $\dot{P}/P$. That implies, for example, that an increase in government expenditures financed by money creation results in higher values of $\dot{Y}$ and $\dot{P}/P$ than the same increase financed by borrowing.

3. The "Level and First Derivative" Systems with an Endogenous Expected Rate of Change of the Price Level

Here we abandon the assumption that θ is exogenous and assume that θ is directly dependent on $\dot{P}/P$, while still, however, assuming that $\dot{\theta}$ and all higher order derivatives of θ are exogenous. Then, as noted above, the set of 14 equations, (4)-(10) and the time derivatives of (4) -(10), form a closed system in 14 endogenous variables, the sets A^o and A^1. Without writing explicit solutions for those variables in terms of the exogenous variables-in general, the qualitative restrictions so far imposed on the parameters of the model are not sufficient to determine the signs of the total differentials of the endogenous variables with respect to the exogenous variables-we shall describe how we expect the model to work.

The perceived percentage rate of change of the price level, θ, connects equations (11) and (12), our versions of the IS and LM curves, to the time derivative equations, summarized by equations (13)-(15). And, since θ appears only in (11), the IS curve, it is by way of that relationship that changes in $\dot{M}$ affect the A^0 variables. Since we found above that the higher is $\dot{M}$ the higher is $\dot{P}$, and since we now assume that the higher is $\dot{P}/P$ the higher is θ, and, hence, by way of (11), the more expansionary is the position of the IS curve, it seems that an increase in government expenditures financed by money creation is more expansionary in terms of both the A^0 and A^1 variables than is the same increase financed by borrowing from the public. That is, in our model with θ directly dependent on $\dot{P}/P$, an increase in $\dot{M}$ would seem to lead to jumps in the levels of output and prices and to higher than initial rates of growth of output and prices. And, with θ an increasing function of $\dot{P}/P$, our analysis suggests that financing an increase in government expenditures by money creation results in a higher interest rate at $\bar{t}$, than would result from financing by borrowing.

The above analysis is to be contrasted with analyses in which the position of the LM curve, equation (12), depends on how a given deficit is financed. Such "period" or discrete time analyses assume a period long enough to allow the stock of money to be affected by alterations in its rate of growth. But, if government expenditures, a flow, increase to a new level, say, 1 billion per month higher than initially and if that expenditure is financed by money creation, then after a week the stock of money is .25 billion higher than it would otherwise have been, after a month 1 billion higher, and after three months 3 billion higher. Therefore, the LM curve should be shifted an amount that varies directly with the unspecified length of the period, or, more precisely, should be shifted at a rate per unit time, which brings us back to the kind of analysis outlined above. But, more to the point, if supposedly static, "period" models take account of the effects of the accumulation of some flows, in particular, the rate of money creation, they should not ignore those of other flows, for example, net investment. If the period is long enough to allow alterations in $\dot{M}$ to "significantly" affect M, it is also long enough to allow alterations in $\dot{K}$ to "significantly" affect K, because the integral of $\dot{M}$ over any period is to M as the integral of $\dot{K}$ over that period is to K. It is not legitimate to take account of the effects of the accumulation of some flows, while ignoring the accumulation of other flows.

IV. CONCLUSION

The point-in-time interpretation of "short-run" static macroeconomic models may seem unduly restrictive in that it allows a description only of instantaneous happenings. But, as I hope I have demonstrated, that view is the only one consistent with the standard, static macroeconomic model. Furthermore, while all changes take some amount of time in order to occur, the changes assumed above to occur instantaneously can conceivably occur within any length of time no matter how short. For example, the levels of output and employment can conceivably change "almost" instantaneously, which is why those variables were assumed to be discontinuous functions of time. Finally, the point-in-time aspect of the analysis does not imply particular assumptions about what can happen instantaneously. For example, one might want to assume that firms act in a way that makes employment a continuous function of time; firms add to their work forces by hiring only at a rate per unit time, and cut back only at a rate per unit time.[11/] The point-in-time treatment leaves open the question as to what adjustments are to be treated as instantaneous.

One of the main virtues of the point-in-time approach, or for that matter, of any explicit treatment of time, is that it almost guarantees that the stock-flow distinction will be maintained. That distinction is lost in most discussions of the effects of different ways of financing government deficits, in most treatments of capital movements in models of the balance of payments or of exchange rate determination, and, in many

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[11/] In particular, one might want to treat labor and capital symmetrically from the point of view of firms, and assume that high transaction costs in the market for labor lead firms to "invest" or "disinvest" in labor only at a rate per unit time, so that, for firm i, $\dot{L}^i = h(f_L - W/P)$, and $dL^i=0$, identically. Of course, then, one might want to add hours of work as a dependent variable that can vary instantaneously.

theoretical treatments of the determinants of investment, in which the existence of a market for stocks of real capital is ignored. An explicit treatment of time makes it harder to gloss over such anomalies.

Another principle advantage of an explicit treatment of time in static models is that it allows us to make the connection --- or, rather to highlight the lack of connection --- between static analysis and various kinds of dynamic analysis. One would think that static analysis would be comprised of a moment-in-time analysis of a model, and that dynamic analysis would be comprised of analyses of the model's behavior over time. It cannot help but be disturbing to students of macroeconomics that some of the behavior relationships in the predominant class of "static" models bear little resemblance to what should be the corresponding relationships in the predominant class of "growth" models. The discrepancies center largely on the determinants of investment and on the existence or non-existence of markets for stocks of real capital. An explicit treatment of time in "static" models shows that those models have growth implications, and, therefore, emphasizes the contradictions between the two classes of models.

APPENDIX

We emerge from section II of the text with a set of, say, k structural equations, or, more accurately, with assumed signs for the first (and, in some cases, higher) order partial derivatives of k structural equations:

(1) $H_i(x(t),\ z(t)) = 0\ ;\ i = 1,2,\ldots,k$

where $x(t) = (x_1(t),\ x_2(t),\ldots\ldots,\ x_k(t))$ is the vector of k endogenous variables, $z(t)$ is the vector of N exogenous variables, and t is time. For the moment, we regard one of the z's as θ. In section III.1 we, in effect, solved the set of equations,

(2) $d[H_i(x(t),\ z(t))] = 0\ ;\ i=1,2,\ldots,k$

for $dx_1(t),\ dx_2(t),\ldots,\ dx_k(t)$ at $t = \bar{t}$. All derivatives in those equations---namely, $\partial H_i/\partial x_j(t)$ and $\partial H_i/\partial z_j(t)$ for all j --- which are functions of x(t) and z(t) were evaluated at $\lim[x(t)]$ and $\lim[z(t)]$ as $t \to \bar{t}$ from the left.

This procedure can be defended as follows. Suppose that (1) implies the reduced form

(3) $x_i(t) = G_i(z(t))\ ;\ i = 1,2,\ldots,k.$

Then in III.1 of the text we were solving for certain partial derivatives of G evaluated at $\lim[z(t)]$ as $t \to \bar{t}$ from the left. That is, under conditions satisfying the inverse function theorem, $dx_i(t)/dz_j(t)$ from (2) equals $\partial G_i/\partial z_j(t)$ from (3).

We may introduce the "time derivative" system by differentiating (3) with respect to time,

$$(4) \qquad \partial x_i(t)/\partial t = \partial G_i(z(t))/\partial t = \sum_{j=1}^{N}[\partial G_i/\partial z_j(t)][\partial z_j(t)/\partial t] \ .$$

The derivatives $\partial x_i(t)/\partial t$ and $\partial z_j(t)/\partial t$ must be defined either as lefthand or righthand time derivatives, since time derivatives at $\bar{t}$ may not exist. There are, therefore, two sets of equations like (4), one defined by evaluating all time derivatives as limits from the left, the other defined by evaluating all time derivatives as limits from the right. Both sets must hold. We are interested in determining the sign of the change in $\dot{x}$ at $\bar{t}$ defined as limit from the right of $\partial x_i(t)/\partial t$ minus limit from the left of $\partial x_i(t)/\partial t$. In general, the signs of such differences are difficult to evaluate because the partial derivatives $\partial G_i/\partial z_j(t)$ may differ for the two limiting processes. They would not differ if $dz_j=0$ for all j (the situation examined in III-2) or if the G's are linear functions. In the linear case, changes in z at $\bar{t}$ unaccompanied by changes in $\dot{z}$ do not affect $\dot{x}$. Our system is nonlinear so we may expect such effects.

In any case, assuming that the $\partial G_i/\partial z_j(t)$ are differentiable functions of z(t), we can proceed more directly by totally differentiating (4). First, we write (4) as

$$\dot{x}_i = \sum_{j=1}^{N} g_{ij}(z(t))\dot{z}_j; \quad i = 1,2,\ldots,k$$

where $g_{ij}(z(t)) \equiv \partial G_i/\partial z_j(t)$. Then,

$$d\dot{x}_i = \sum_{j=1}^{N} \dot{z}_j d[g_{ij}(z(t))] + \sum_{j=1}^{N} g_{ij}(z(t))d\dot{z}_j \ ,$$

or,

$$d\dot{x}_i = \sum_{j=1}^{N} \dot{z}_j \left(\sum_{h=1}^{N} [\partial g_{ij}/\partial z_h(t)] dz_h \right) + \sum_{j=1}^{N} g_{ij}(z(t)) d\dot{z}_j$$

But, since $\partial g_{ij}/\partial z_h(t) = \partial^2 G_i/\partial z_j(t)\partial z_h(t)$,

$$(5) \qquad d\dot{x}_i = \sum_{j=1}^{N} \dot{z}_j \left(\sum_{h=1}^{N} [\partial^2 G_i/\partial z_j \partial z_h] dz_h \right) + \sum_{j=1}^{N} [\partial G_i/\partial z_j(t)] d\dot{z}_j,$$

where in (5) all derivatives are evaluated by taking limits as $t \to \bar{t}$ from the left. From (5),

$$d\dot{x}_i/dz_h \Bigg|_{\substack{d\dot{z}_j=0 \text{ all } j, \\ dz_j=0 \text{ for } j \neq h}} = \sum_{j=1}^{N} \dot{z}_j (\partial^2 G_i/\partial z_h \partial z_j)$$

assuming that the order of differentiation does not matter. It is clear that $d\dot{x}_i/dz_h$ is much harder to evaluate given only general characteristics of the G's than is

$$d\dot{x}_i/dz_h \Big|_{dz_j=0 \text{ for } j \neq h} \quad \text{or} \quad d\dot{x}_i/d\dot{z}_h \Big|_{dz_j=0 \text{ for all } j, \text{ and } d\dot{z}_j=0 \text{ for } j \neq h}.$$

From (5) the latter two total derivatives equal $\partial G_i/\partial z_h$.

Of course, since we never solve explicitly for the G's, we must proceed implicitly from (1), which is what we do in III.2 of the text. Upon differentiating (1) with respect to time, we obtain

$$(6) \quad \sum_{j=1}^{k} [\partial H_i/\partial x_j(t)] \partial x_j(t)/\partial t + \sum_{j=1}^{N} [\partial H_i/\partial z_j(t)] \partial z_j(t)/\partial t = 0; \; i = 1,2,\ldots,k.$$

Then, totally differentiating the set (6), we get

$$(7)\quad \sum_{j=1}^{k} [\partial x_j(t)/\partial t]\, d\, [\partial H_i/\partial x_j(t)] + \sum_{j=1}^{k} [\partial H_i/\partial x_j(t)]\, d\, [\partial x_j(t)/\partial t] + \sum_{j=1}^{N} [\partial z_j(t)/\partial t]\, d\, [\partial H_i/\partial z_j(t)] + \sum_{j=1}^{N} [\partial H_i/\partial z_j(t)]\, d\, [\partial z_j/\partial t] = 0;$$

where i=1,2,...,k, and where all derivatives are defined by a limiting process from the left. We may rewrite the set (7) as

$$(8)\quad \sum_{j=1}^{k} \dot{x}_j \left[\sum_{h=1}^{k} (\partial^2 H_i/\partial x_j \partial x_h) dx_h + \sum_{h=1}^{N} (\partial^2 H_i/\partial x_j \partial z_h) dz_h \right] + \sum_{j=1}^{k} [\partial H_i/\partial x_j(t)]\, d\dot{x}_j + \sum_{j=1}^{N} \dot{z}_j \left[\sum_{h=1}^{k} (\partial^2 H_i/\partial z_j \partial x_h) dx_h + \sum_{h=1}^{N} (\partial^2 H_i/\partial z_j \partial z_h) dz_h \right] + \sum_{j=1}^{N} (\partial H_i/\partial z_j) d\dot{z}_j = 0;$$

for i = 1,2,....k.

This set of k equations involves three sets of unknowns: (i) $\dot{x}_j$, defined as a limit from the left, (ii) dx_h, and (iii) $d\dot{x}_j$; where both j and h range from 1 to k. Set (i) can be eliminated by use of the linear equations (6). Set (ii) can be eliminated by use of the linear equations (2). If those substitutions are carried out, we are left with k equations in the k unknowns of set (iii), the $d\dot{x}_j$. Those equations are linear in the $d\dot{x}_j$ and in dz_h and $d\dot{z}_h$; h=1,2,...N. However, also appearing in those equations are x(t), z(t), and $\dot{z}(t)$ --- initial conditions, all defined as limits from the left. They need not enter linearly. In Section III.2, we, in effect, solved the set (8) for all dz_h=0, and, hence, by (2), all dx_h=0.

All of the above has proceeded under the assumption that θ is one of the z's, an exogenous variable. In Section III.3 of the text, we abandon that assumption and make θ a function of x_j and $\dot{x}_j$, where x_j=P. One result is a set of equations like (1), involving, however, k+1 unknowns; $x_1, x_2, \ldots x_k$

and $\dot{x}_j$. Upon totally differentiating that set, we obtain a set that corresponds to (2), however, again with k+1 unknowns; dx_1, dx_2, $\ldots dx_k$ and $d\dot{x}_j$. That set, therefore, cannot by itself be solved for the $dx_i(t)/dz_h(t)$. But, under the assumption that $\dot{\theta}$ is exogenous, set (8) provides another k equations in 2k unknowns; dx_1, $dx_2, \ldots dx_k$ and $d\dot{x}_1$, $d\dot{x}_2, \ldots d\dot{x}_k$. The two sets together comprise 2k linear equations in 2k unknowns, and, thus, can be solved. However, in our case the system is large and depends in a complicated way on initial conditions, so that we are unlikely to get very far given only our weak restrictions on the H's of (1).

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