EXACT REACHABILITY OF FINITE ENERGY STATES FOR AN ACOUSTIC WAVE/PLATE INTERACTION UNDER THE INFLUENCE OF BOUNDARY AND LOCALIZED CONTROLS

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Abstract

In this work, we derive a result of exact controllability for a structural acoustic partial differential equation (PDE) model, comprised of a three-dimensional interior acoustic wave equation coupled to a two-dimensional Kirchoff plate equation, with the coupling being accomplished across a boundary interface. For this PDE system, we show that by means of boundary controls, the interior wave and Kirchoff plate initial data can be steered to an arbitrary *finite energy* state. In this work, key use is made of recent, microlocally-derived, $L^2 \times H^{-1}$ "recovery" estimates for wave equations with Dirichlet boundary data. Moreover, the coupling of the disparate acoustic wave/Kirchoff plate dynamics is reconciled by means of sharp regularity estimates which are valid for hyperbolic equations of second order.

1 Introduction and Statement of Main Results

1.1 Statement of the Controllability Problem

Let $\Omega \subset \mathbb{R}^3$ be a open, bounded set, with sufficiently smooth (two-dimensional) boundary Γ. Moreover, we asume that $\Gamma = \Gamma_1 \cup \Gamma_0$, where Γ_0 is a flat surface. For this geometry, we have the following controlled partial differential equation (PDE) model:

$$\begin{array}{l} z_{tt} = \Delta z \ \text{ on } (0,T)\times\Omega \\ \left\{ \begin{array}{l} \dfrac{\partial z}{\partial \nu} = u_1 \ \text{ on } (0,T)\times\Gamma_1 \\ \dfrac{\partial z}{\partial \nu} = v_t \ \text{ on } (0,T)\times\Gamma_0 \end{array} \right. \\ \\ v_{tt} - \gamma\Delta v_{tt} + \Delta^2 v + z_t|_{\Gamma_0} = a(x)u_0 \ \text{ on } (0,T)\times\Gamma_0 \\ v|_{\partial\Gamma_0} = \Delta v|_{\partial\Gamma_0} = 0 \ \text{ on } (0,T)\times\partial\Gamma_0 \\ \\ \left[z(0), z_t(0), v(0), v_t(0)\right] = \left[\vec{z}_0, \vec{v}_0\right]. \end{array} \tag{1}$$

Here, the functions u_i are the "controls" of the system, invoked here so as to influence the behaviour of the dynamics in a way we shall describe presently. Also, $a(x)$ is a C^2-coefficient which is locally distributed on Γ_0, with moveover $\partial\Gamma_0 \subset \text{supp}(a)$. In addition, we specify that $0 \leq a(x) \leq 1$, with $a(x)$ being identically 1 on a connected portion of $\overline{\Gamma_0}$, with this portion containing $\partial\Gamma_0$. (One might think of a as a "smoothing out" of a characteristic function, with respect to this connected portion of $\overline{\Gamma_0}$.) Moreover, the "moment of rotational inertia parameter" γ which appears in the plate component of (1) is strictly positive, in which case the PDE, when $u_1 = 0$ and $u_0 = 0$, manifests *hyperbolic* dynamics.

Our primary goal in this paper is to study reachability properties of the controlled model (1) in the relevant *finite energy space* of initial data. Our emphasis here that we steer arbitrary data of finite energy, is not only due to physical considerations, but also to the implications of such controllability properties on other control theoretic questions related to structural acoustic flow. For example, exact controllability for finite energy data is directly applicable to optimal control theory, inasmuch as the validation of such a reachability property means that the so-called "finite cost condition", or sufficient condition for optimizability, is satisfied. With this goal in mind, our space of well-posedness $\mathbf{H}$ is here given to be

$$\mathbf{H} = H_1 \times H_0, \tag{2}$$

where

$$H_1 \equiv H^1(\Omega) \times L^2(\Omega) \text{ and } H_0 \equiv \left[H^2(\Gamma_0) \cap H_0^1(\Gamma_0)\right] \times H_0^1(\Gamma_0). \tag{3}$$

Therewith, our objective in this paper is to find controls u_1, u_0 which will steer, exactly, the given (finite energy) initial data to an arbitrary target state, at time $T > 0$. In this work, we are invoking the classical definition of exact controllability:

Definition 1 *We say that the PDE (1) is exactly controllable at time $T > 0$ if, for given initial data $[\vec{z}_0, \vec{v}_0] \in \mathbf{H}$ and arbitrary terminal state $[\vec{z}_T, \vec{v}_T] \in \mathbf{H}$, there are functions $u_1 \in L^2(\Sigma_1)$ and $u_0 \in L^2(0, T; H^{-1}(\Gamma_0))$ such that the corresponding solution of (1) satisfies $[\vec{z}(T), \vec{v}(T)] = [\vec{z}_T, \vec{v}_T]$.*

Remark 2 *Our choice here of the control space $L^2(\Sigma_1)$ (resp., $L^2(0, T; H^{-1}(\Gamma_0))$) is optimal, inasmuch as it is optimal with respect to the control of the uncoupled wave (resp., Kirchoff plate) equation.*

1.2 The History and Significance of the Problem

The PDE system (1) is an example of a *structural acoustic interaction*; namely, it comprises an acoustic wave equation (in variable z) on the interior of Ω, coupled to a structural plate equation (in variable v) on the flat boundary portion Γ_0. Although the PDE modelling of structural acoustic interactions has been an ongoing enterprise for decades–see e.g., [33],[13]–the last fifteen years in particular have seen a flourishing in the literature of those studies concerned with structural acoustic PDE's. The principal reason for this increased interest is the recent development of engineering technologies which have allowed the implementation of practicable schemes for the active control of external noise, as it enters a given structural acoustic system. Such recent contributions to the literature deal with various topics; e.g., optimal control, stability, controllability, regularity, numerical computation ([11],[5],[6],[3],[29],[19],[30],[15],[8],[9],[1],[22],[21],[14],[17]).

In these papers, a wave equation is typically invoked in the modelling, so as to describe an interior acoustic field; on the other hand, either wave, plate, beam or shell equations, subject to various degrees of damping, have been utilized so as to model the structural component of the coupled PDE system. But amongst all these distinct structural acoustic PDE examples, there is one unifying complication which makes the analysis of structural acoustic interactions a worthy and challenging field of endeavor: namely, the mode of coupling between the acoustic wave and structural PDE dynamics is by means of *unbounded trace terms* on the boundary interface. It is the need to reconcile the presence of these acoustic wave trace terms–for which the classical Sobolev

Trace Theorem cannot be applied–which has essentially dictated the analysis which underpins many of the known control theoretic results for this model (see e.g., [5] and [21]).

In one particular control application associated with structural acoustic interactions, *pointwise* control–namely, a linear combination of derivatives of delta functions–is implemented on the "active" portion of the boundary interface (which is Γ_0 in the present paper), so as to attenuate the disrupting effects of external noise which enters into a given structural acoustic system. By means of an analysis of the role played by the unbounded trace coupling–an analysis which necessitates a departure into sharp regularity studies of second order hyperbolic equations under the action of boundary data of precisely specified smoothness–it is shown in [5] that a quadratic regulator problem, which is wellposed in familiar spaces of finite energy, can be formulated with respect to the aforesaid pointwise control actions. Subsequently, the optimal control for this problem can be characterized by means of operator solutions to an associated *Riccati equation*; regularity properties of these operators depended critically on the analysis undertaken for wave traces.

This previous work in [5] is concerned then with optimization; alternatively, in the present work, we will be concerned with the *exact controllability problem* associated with the PDE (1), as it is explicitly defined in Definition 1. This definition underscores our prime concern to work within the finite energy state space $\mathbf{H}$, rather than in other spaces with perhaps non-intrinsic and operator/generator-dependent topologies (see e.g., [30]). Our interest in $\mathbf{H}$ is certainly motivated by the physical relevance of finite energy initial data; but more than that, reachability results attained with respect to $\mathbf{H}$ can be subsequently applied to other areas of control theory, such as optimization and optimal control design. As an example of a "finite energy" reachability result in structural acoustics, a property of boundary controllability is stated in [10] for a canonical wave-wave interaction–i.e., a wave equation on Γ_0 is invoked in [10], in place of a Kirchoff plate–with this property being valid for *arbitrary* initial data in the relevant finite energy space. In particular, in [10], precise geometrical configurations are given which will ensure exact controllability for the both acoustic and structural wave states, in the case that control is implemented *only* on the active portion of the boundary (which, again, is Γ_0 in the present work). (This specification of the geometry is no violation of the now classic work [12], which gives sufficient conditions for the boundary control of waves on a bounded open set.) However, as seen in [10], the price to pay for controlling the given wave-wave interaction on Γ_0 only, is that the controls must be very "rough" with respect to time.

But while we wish in the here and now to obtain reachability results in the basic space $\mathbf{H}$, the work required to obtain such results is mathematically demanding; for, in such an effort, one is foregoing the liberty of "tailoring" a topology in order to fit the regularity of certain unbounded quantities–in particular, boundary traces–which tend to appear during the course of analysis, and which result from the interaction of the respective components (wave and plate) of the system. In fact, the main contribution in this paper is the approach developed therein so as to deal with the unboundedness of these boundary traces. This approach partly involves an exploiting of recent results in both the "hidden regularity" of Kirchoff plates, and "sharp trace theory" for solutions of wave equations under the Neuman boundary conditions (i.e., the "non-Lopatinski" case). The estimates provided by this regularity results really constitute the supporting pillars of the overall analysis.

Again, our intent here is to study reachability properties of the controlled 3-D PDE (1), with controls being enacted on the boundary Γ_0 and Γ_1; here, the Kirchoff plate component should constitute a more "faithful" realization of the structural component of a structural acoustic interaction, than the canonical wave equation presented in [10]. As in the earlier [10], we are after a simple and direct statement of exact controllability, which addresses *all* finite energy initial data. Accordingly, the imposition of control on Γ_1 is unavoidable. We could proceed to specify geometrical conditions for which control on Γ_1 is limited to just a portion thereof, by invoking, say, the precise statements in [24] and [27]; but this would detract from the principle aim of the paper: namely, to validate the exact controllability property given in Definition 1, with the *locally distributed* control coefficient

$a(x)$ in place.

By way of highlighting the technical difficulties associated with the stated exact controllability problem for (1), we recall the earlier work [9], wherein the controllability of (1) is considered, with $a(x) \equiv 1$ therein. In this paper, a subtle decoupling of the problem is undertaken by means of the known sharp regularity for the boundary traces $z|_{\Gamma}$ and $z|_{\Gamma_0}$ (see Theorem 4 and Lemma 6 below. As we said earlier, this appeal to "esoteric" sharp boundary trace regularity theory is a *sine qua non* for the attainment of many, if not most, structural acoustic results of any interest). Because of this decoupling, an algorithmic approach is adopted in [9] which sequentially takes into account the known controllability results for wave equations under Neumann boundary control, and Kirchoff plate equations under *fully distributed* internal control (the latter being of course, in and of itself, less than technically demanding).

On the other hand, in the present paper, we assume that the coefficient $a(x)$ in (1) has its localized support situated about the boundary. Thus, after a decoupling of the system, to be done as in [9] by the usage of microlocally-derived trace regularity results, we will ultimately be confronted with the problem of controlling the Kirchoff plate by means of locally distributed control, with respect to the *relevant space of finite energy* (as well as the aforementioned wave equation under Neumann control); this controllability of the plate component is the heart of the matter. Stated broadly, our work here will chiefly involve generating the continuous observability inequality which is dual with respect to exact controllability, as defined in Definition 1 (see the inequality (14) below). We should say, at this point, that in handling the controlled Kirchoff component, one cannot simply appeal to the known results in [26] which are applicable to boundary controlled (and uncoupled) Kirchoff plates. In point of fact, the relevant observability inequality in [26] is obtained for solutions in a finer topology–i.e., "one unit higher"–then H_0, as defined in (3). With this higher norm estimate in hand, one can subsequently use duality with the fact that the underlying Kirchoff semigroup will evolve for a sliding scale of spaces $D(A_D^{2\alpha}) \times D(A_D^{\alpha})$ (where the ["Dirichlet Laplacian"] A_D is as defined in (5) below), so as to infer the corresponding controllability property for the *uncoupled* Kirchoff plate under boundary (or locally distributed) control. On the other hand, for the generator which is associated with the structural acoustic flow (1), maximal dissipativity apparently obtains only for initial data in $\mathbf{H}$; i.e., there is generally no sliding scale for structural acoustic initial data. (See however, [8], wherein wellposedness for a related structural acoustic interaction is derived for initial data "one-half unit higher" [plus a compatibility condition], with respect to the acoustic wave component. This result, however, is microlocal in derivation, and not at all a product of semigroup theory.)

To obtain the continuous observability inequality for the exact controllability of (1) (in particular, to handle the contribution from the Kirchoff component), we will invoke a inverse-type of estimate which was derived in [4], and which is valid for wave equations under Dirichlet control and forcing terms of a particular form (The use of this estimate is allowed by the assumption that moment of parameter $\gamma > 0$). In particular, this recovery estimate reconstructs the initial energy of the Dirichlet wave (*read* transformed Kirchoff plate) energy from measurements of boundary terms. Subsequently, in order to handle these boundary terms, it will become necessary to invoke the classic paper [23], which gives interior and boundary regularity results (by means of the multiplier method) for Dirichlet-controlled wave equations with "suitable" forcing terms.

To conclude our introductory remarks, we cite the main technical ingredients of the present work. They are:

(i) Reconstruction/observability estimates in negative ("dual") norms, with applications to boundary controllability of Kirchoff plates;

(ii) Sharp trace regularity theory, corresponding to the wave equation with Neumann (non-Lopatinski) boundary conditions;

(iii) Hidden regularity of boundary traces in negative norms for Kirchoff plates with essential boundary data.

1.3 Statement of the Main Result

We assert that to achieve the exact controllability result which is posted in Definition 1, it is enough to consider the exact controllability question for the following PDE system, again within the class of controls $L^2(\Sigma_1) \times L^2(0,T;H^{-1}(\Gamma_0))$:

$$\begin{aligned}
& z_{tt} = \Delta z \quad \text{on } (0,T)\times\Omega \\
& \begin{cases} \dfrac{\partial z}{\partial \nu} + z = u_1 \quad \text{on } (0,T)\times\Gamma_1 \\ \dfrac{\partial z}{\partial \nu} = v_t \quad \text{on } (0,T)\times\Gamma_0 \end{cases} \\
& v_{tt} - \gamma\Delta v_{tt} + \Delta^2 v + z_t|_{\Gamma_0} = a(x)u_0 \quad \text{on } (0,T)\times\Gamma_0 \\
& v|_{\partial\Gamma_0} = \Delta v|_{\partial\Gamma_0} = 0 \quad \text{on } (0,T)\times\partial\Gamma_0 \\
& [z(0), z_t(0), v(0), v_t(0)] = [\vec{z}_0, \vec{v}_0]\,.
\end{aligned} \tag{4}$$

By way of justifying our claim, we start by deriving the following regularity result for solutions of (4), corresponding to given initial and boundary data:

Lemma 3 *For given data $\{[\vec{z}_0,\vec{v}_0], u_1, u_0\} \in \mathbf{H} \times L^2(\Sigma_1) \times L^2(0,T;H^{-1}(\Gamma_0))$, the solution $[\vec{z},\vec{v}]$ of (4) satisfies the following estimate:*

$$\|z, z_t\|_{C([0,T];H^{\frac{3}{5}-\epsilon}(\Omega)\times H^{-\frac{2}{5}-\epsilon}(\Omega))} + \|[v,v_t]\|_{C([0,T];H_0)} \leq C_T\left(\|[\vec{z}_0,\vec{v}_0], u_0, u_1\|_{\mathbf{H}\times L^2(\Sigma_1)\times L^2(0,T;H^{-1}(\Gamma_0))}\right).$$

Proof of Lemma 3: Let $A_D : D(A_D) \subset L^2(\Gamma_0) \to L^2(\Gamma_0)$ (resp., $P_\gamma : D(A_D) \subset L^2(\Gamma_0) \to L^2(\Gamma_0)$) be the positive definite, self-adjoint operator defined by

$$\begin{aligned} A_D g &= -\Delta g, \;\; D(A_D) = H^2(\Gamma_0)\cap H_0^1(\Gamma_0); \\ P_\gamma g &= (I+\gamma A_D)\,, \; D(P_\gamma) = D(A_D). \end{aligned} \tag{5}$$

From the now classic results in [16], one has the following characterizations:

$$D(A_D) \approx H^2(\Gamma_0)\cap H_0^1(\Gamma_0);\; D(\sqrt{P_\gamma}) \approx H_0^1(\Gamma_0).$$

These Sobolev space identifications of $D(A_D)$ and $D(\sqrt{P_\gamma})$ allow the H_0-norm in (3) to be given by

$$\left\| \begin{array}{c} v_0 \\ v_1 \end{array} \right\|_{H_0} = \|v_0\|_{D(A_D)} + \|v_1\|_{D(\sqrt{P_\gamma})}\,.$$

Now, if we let $A_0 : D(A_0) : H_0 \to H_0$ be defined by

$$A_0 = \begin{bmatrix} 0 & I \\ -P_\gamma^{-1}A_D^2 & 0 \end{bmatrix}, \text{ with } D(A_0) = D(A_D^{\frac{3}{2}})\times D(A_D),$$

then it is readily seen that A_0 generates a C_0-group $\{e^{A_0 t}\}_{t\in\mathbb{R}}$ on H_0. Accordingly, for $0\leq s\leq T$, the plate component of the solution of (4) may be written as

$$\begin{aligned}
\begin{bmatrix} v(s) \\ v_t(s) \end{bmatrix} &= e^{A_0 s}\vec{v}_0 - \int_0^s e^{A_0(s-\tau)} \begin{bmatrix} 0 \\ P_\gamma^{-1}\left(z_t|_{\Gamma_0} - au_0\right) \end{bmatrix} d\tau \\
&= e^{A_0 s}\vec{v}_0 + A_0\int_0^s e^{A_0(s-\tau)} \begin{bmatrix} 0 \\ P_\gamma^{-1}\, z|_{\Gamma_0} \end{bmatrix} d\tau + \int_0^s e^{A_0(s-\tau)} \begin{bmatrix} 0 \\ P_\gamma^{-1}(au_0) \end{bmatrix} d\tau - \begin{bmatrix} 0 \\ P_\gamma^{-1}\, z|_{\Gamma_0} \end{bmatrix}\Bigg|_0^s
\end{aligned} \tag{6}$$

Estimating this will yield now, for all $0 \leq s \leq T$,

$$\|[v(s), v_t(s)]\|_{H_0} \leq C_T \left(\left\| z|_{\Gamma_0} \right\|_{L^\infty(0,s;L^2(\Gamma_0))} + \|u_0\|_{L^2(0,T;H^{-1}(\Gamma_0))} + \|[\vec{z}_0, \vec{v}_0]\|_{H_1 \times H_0} \right). \tag{7}$$

At this point, we recall the following result in [36] (see also [25]):

Theorem 4 *(see Theorem 3.1 of [36].) Let ϕ satisfy the following wave equation on $(0, s) \times \Omega$, where Ω is a smooth bounded domain:*

$$\begin{array}{l} \phi_{tt} = \Delta\phi \quad \text{on } (0,s) \times \Omega \\ \frac{\partial \phi}{\partial \nu} + \alpha(x)\phi = g \ \text{ on } (0,s) \times \Gamma \quad (\text{where } L^\infty\text{-coefficient } \alpha(x) \geq 0 \text{ on } \Gamma) \\ [\phi(0), \phi_t(0)] = \vec{z}_0. \end{array}$$

Then we have, continuously, $\{\vec{z}_0, g\} \in H_1 \times L^2((0,s) \times \Gamma) \Rightarrow [\phi, \phi_t] \in C([0,s]; H^{\frac{3}{5}-\epsilon}(\Omega) \times H^{-\frac{2}{5}-\epsilon}(\Omega))$.

Appealing to this result with,

$$g(t,x) = \begin{cases} u_1(t,x), & x \in \Gamma_1 \\ v_t(t,x), & x \in \Gamma_0, \end{cases}$$

the estimate (7) becomes, for all $0 \leq s \leq T$,

$$\|[v(s), v_t(s)]\|_{H_0}^2 \leq C_T \left(\int_0^s \|v_t\|_{L^2(\Gamma_0)}^2 dt + \|u_1\|_{L^2(\Sigma_1)}^2 + \|u_0\|_{L^2(0,T;H^{-1}(\Gamma_0))}^2 + \|[\vec{z}_0, \vec{v}_0]\|_{\mathbf{H}}^2 \right).$$

Gronwall's inequality now gives, for $0 \leq t \leq T$,

$$\|[v(t), v_t(t)]\|_{H_0}^2 \leq C_T \left(\|u_1\|_{L^2(\Sigma_1)}^2 + \|u_0\|_{L^2(0,T;H^{-1}(\Gamma_0))}^2 + \|[\vec{z}_0, \vec{v}_0]\|_{\mathbf{H}}^2 \right). \tag{8}$$

In turn, combining this estimate with Theorem 4 gives

$$\|[z, z_t]\|_{C([0,T];H^{\frac{3}{5}-\epsilon}(\Omega) \times H^{-\frac{2}{5}-\epsilon}(\Omega))} \leq C_T \left(\|u_1\|_{L^2(\Sigma_1)}^2 + \|u_0\|_{L^2(0,T;H^{-1}(\Gamma_0))}^2 + \|[\vec{z}_0, \vec{v}_0]\|_{\mathbf{H}}^2 \right). \tag{9}$$

The estimates (8) and (9) complete the proof of Lemma 3. $\square$

With Lemma 4 in hand, suppose now that the controls $[u_1, u_0] \in L^2(\Sigma_1) \times L^2(0, T; H^{-1}(\Gamma_0))$ are such that the corresponding solution of (4) satisfies $[\vec{z}(T), \vec{v}(T)] = [\vec{z}_T, \vec{v}_T]$, for given target data $[\vec{z}_T, \vec{v}_T] \in \mathbf{H}$. Then from Lemma 3 and the Sobolev Embedding Theorem, we have (conservatively) $z|_\Gamma \in L^2(\Sigma)$. Subsequently, if we set $[u_1^*, u_0^*] = [u_1 - z|_{\Gamma_1}, u_0]$, then $[\vec{z}, \vec{v}]$ satisfies the PDE (1), with controls $[u_1^*, u_0^*]$ in place, and as we said, satisfy the reachability property $[\vec{z}(T), \vec{v}(T)] = [\vec{z}_T, \vec{v}_T]$.

The reason we will consider the problem (4), instead of (1), is that on the afore-defined space $\mathbf{H}$, one will avoid the acoustic wave "steady states" which are inherent in the original problem (1), inasmuch as pure Neumann boundary conditions are being considered for the z-variable. In fact, if $\mathbf{H}$ is topologized through the inner product

$$\begin{aligned} ([\vec{z}, \vec{v}], [\vec{\varsigma}, \vec{v}])_{\mathbf{H}} &= \int_\Omega \nabla z_0 \cdot \nabla \varsigma_0 d\Omega + (z_0|_{\Gamma_1}, \varsigma_0|_{\Gamma_1})_{L^2(\Gamma_1)} + (z_1, \varsigma_1)_{L^2(\Omega)} \\ &+ (A_D v_0, A_D v_0)_{L^2(\Gamma_0)} + (P_\gamma^{\frac{1}{2}} v_0, P_\gamma^{\frac{1}{2}} v_0)_{L^2(\Gamma_0)}, \end{aligned} \tag{10}$$

then an invocation of the Lumer-Phillips Theorem will yield that the structural acoustic flow, described in (4), is associated with the generator of a C_0-group on $\mathbf{H}$. Consequently, this semigroup generation will yield the wellposedness,

$$\{[\vec{z}_0, \vec{v}_0] \in \mathbf{H}, \, u_1 = 0, \, u_0 = 0\} \Longrightarrow [\vec{z}, \vec{v}] \in C([0,T]; \mathbf{H}). \tag{11}$$

On the other hand, when boundary data is present in the model (4), Lemma 3 states that the spatial regularity of the wave component is below the level of finite energy. This circumstance is not unexpected, since it is essentially "inherited" from uncoupled wave equations subject to $L^2(\Sigma)$-boundary data (see [25]).

But as we said, our principle intent here is to investigate reachability properties of the structural acoustic model (1) for *arbitrary* initial data $[\vec{z}_0, \vec{v}_0]$ in $\mathbf{H}$, by means of controls $[u_1, u_0]$ in $L^2(\Sigma_1) \times L^2(0,T;H^{-1}(\Gamma_0))$. In turn, by the reasoning which we gave above, it suffices to study the associated exact controllability problem for the (Robin) system (4). In this connection, our main result in this paper is as follows:

Theorem 5 *For terminal time $T > 0$ large enough, the PDE (4) is exactly controllable in the sense of Definition 1.*

The proof of Theorem 5 hinges on ascertaining the surjectivity of the *control to terminal state* map, which is associated with the exact controllability of (4). We could proceed to flesh out this abstract map by generating the necessary (and cumbersome) operator theoretic quantities; but we desist from doing so, and instead direct the reader, if he or she is unfamiliar with the classic functional analytical argument relating exact controllability to its "dual problem", to [34] and [24]. (See also [7] which deals with exact controllability relative to a different coupled PDE system). The relevant observability inequality, which is dual to the exact controllability of (4), is as follows: Consider the homogenous PDE,

$$\begin{array}{l} \phi_{tt} = \Delta\phi \quad \text{on } (0,T)\times\Omega \\ \begin{cases} \frac{\partial\phi}{\partial\nu} + \phi = 0 \quad \text{on } (0,T)\times\Gamma_1 \\ \frac{\partial\phi}{\partial\nu} = \omega_t \quad \text{on } (0,T)\times\Gamma_0 \end{cases} \\ \\ \omega_{tt} - \gamma\Delta\omega_{tt} + \Delta^2\omega + \phi_t|_{\Gamma_0} = 0 \quad \text{on } (0,T)\times\Gamma_0 \\ \phi|_{\Gamma_0} = \Delta\phi|_{\Gamma_0} = 0 \quad \text{on } (0,T)\times\Gamma_0 \\ \\ [\phi(T), \phi_t(T), \omega(T), \omega_t(T)] = [\phi_0, \phi_1, \omega_0, \omega_1] \in \mathbf{H}. \end{array} \tag{12}$$

With respect to this structural acoustic system, we make the denotation

$$\mathcal{E}(t) \equiv \mathcal{E}_\phi(t) + \mathcal{E}_\omega(t),$$

where the respective acoustic wave and Kirchoff plate energies are given by

$$2\mathcal{E}_\phi(t) = \int_\Omega \left[|\nabla\phi(t)|^2 + \phi_t^2(t)\right] d\Omega + \int_{\Gamma_1} \left(\phi(t)|_{\Gamma_1}\right)^2 d\Gamma_1; \;\; 2\mathcal{E}_\omega(t) = \|A_D\omega(t)\|^2_{L^2(\Gamma_0)} + \|P_\gamma\omega_t(t)\|^2_{L^2(\Gamma_0)}.$$

Note that one can verify directly that, as expected, the energy of the system (12) is *conserved*; i.e.,

$$\mathcal{E}(t) \equiv \mathcal{E}(s), \;\; \text{for all } 0 \le s,t \le T. \tag{13}$$

With the backwards problem (12) in mind, the PDE (4) will be exactly controllable, within the class of controls $L^2(\Sigma_1)\times L^2(0,T;H^{-1}(\Gamma_0))$, if the solution variables $[\phi, \phi_t, \omega, \omega_t]$ of (12) satisfy the following inequality, for $T > 0$ large enough:

$$\mathcal{E}(T) \le C_T \left(\int_0^T \int_{\Gamma_1} \phi_t^2 d\Gamma_1 dt + \int_0^T \left\| P_\gamma^{\frac{1}{2}}(a\omega_t) \right\|^2_{L^2(\Gamma_0)} \right) dt \tag{14}$$

The rest of the paper is accordingly geared towards the derivation of the inequality (14). In the course of our work, we will propagate many so-called "lower order terms". By lower terms terms, denoted throughout as l.o.t.$(\vec{\phi},\vec{\omega})$, we mean

$$\text{l.o.t.}(\vec{\phi},\vec{\omega}) = \mathcal{O}\left(\left\| \left[\vec{\phi},\vec{\omega}\right] \right\|_{C([0,T];H^{1-\epsilon}(\Omega)\times H^{-\epsilon}(\Omega)\times H^{2-\epsilon}(\Gamma_0)\times H^{1-\epsilon}(\Gamma_0))} \right), \tag{15}$$

where again, $\left[\vec{\phi},\vec{\omega}\right]$ is the solution component of (12).

2 Concerning the Wave Component of (12) (First Part of the Proof of Theorem 5)

2.1 An Energy Relation for the Wave Component of (12)

As we said above, the proof of Theorem 5 is tantamount to deriving the inequality (14), to which end we will devote our energies. To start, we will derive a relation for the interior acoustic variables $[\phi,\phi_t]$, which we will find useful throughout. To wit, multiplying the interior wave component of (12) by ϕ_t, and integrating by time and space, we have for all $0 \le t \le T$,

$$\mathcal{E}_\phi(T) = \mathcal{E}_\phi(t) + \int_t^T \langle \omega_t, \phi_t \rangle_{H^{\frac{1}{2}}(\Gamma_0)\times H^{-\frac{1}{2}}(\Gamma_0)}\, dt. \tag{16}$$

The fact that the duality pairing on the right hand side may be taken in the $H^{\frac{1}{2}}(\Gamma_0)\times H^{-\frac{1}{2}}(\Gamma_0)$-topology is a consequence of the result in [31], a work which addresses the regularity of second order hyperbolic equations, under the action of Neumann data:

Lemma 6 *(See Theorem 3, p. 443 of [31].) Let $[\varphi,\varphi_t]$ solve the following boundary value problem on $(0,T)\times\Omega$:*

$$\begin{aligned}
&\varphi_{tt} - \Delta\varphi = 0 \quad \text{on } (0,T)\times\Omega,\\
&\frac{\partial\varphi}{\partial\nu} + \alpha(x)\phi = g \in L^2(0,T;H^{\frac{1}{2}}(\Gamma)) \quad \text{on } (0,T)\times\Gamma \quad (\text{where } L^\infty\text{-coefficient } \alpha(x)\ge 0 \text{ on } \Gamma)\\
&[\varphi(0),\varphi_t(0)] = [\phi_0,\varphi_1] \in H^1(\Omega)\times L^2(\Omega).
\end{aligned}$$

Then we have the estimate

$$\int_0^T \|\varphi_t\|^2_{H^{-\frac{1}{2}}(\Gamma)}\, dt \le C_T \left\{ \|[\phi_0,\varphi_1]\|^2_{H^1(\Omega)\times L^2(\Omega)} + \int_0^T \|g\|^2_{H^{\frac{1}{2}}(\Gamma)}\, dt \right\}.$$ [1]

2.2 The Main Estimate for the Wave Component of (12)

Throughout, we will invoke the denotations $Q_\delta \equiv (\delta,T-\delta)\times\Omega$, $\Sigma_\delta = (\delta,T-\delta)\times\Gamma$ and $\Sigma_{i,\delta} = (\delta,T-\delta)\times\Gamma_i$. Let $h(x)$ be a $\left[C^2(\overline{\Omega})\right]^n$ vector field which is to b specified below. An invocation of the classical wave multiplier $h\cdot\nabla\phi$ (e.g., see [32],[35],[20],[40]) to the wave component in (12) gives

$$\begin{aligned}
&\int_{Q_\delta} H(x)\nabla\phi\cdot\nabla\phi dQ_\delta = \\
&\int_{\Sigma_{0,\delta}} \omega_t\,(h\cdot\nabla\phi)\,d\Sigma_{0,\delta} - \int_{\Sigma_{1,\delta}} \phi\,(h\cdot\nabla\phi)\,d\Sigma_{1,\delta} - \frac{1}{2}\int_{\Sigma_\delta} \phi_t^2 h\cdot\nu d\Sigma_\delta \\
&-\frac{1}{2}\int_{\Sigma_\delta} |\nabla\phi|^2 h\cdot\nu d\Sigma_\delta + \frac{1}{2}\int_{Q_\delta}\left(|\nabla\phi|^2 - \phi_t^2\right) div(h) dQ_\delta - \left[(\phi_t, h\cdot\nabla\phi)_{L^2(\Omega)}\right]_\delta^{T-\delta}.
\end{aligned} \tag{17}$$

[1] This estimate is now known not to be sharp with respect to the boundary datum g (see [36],[25],[38],[2]). But since this estimate will suffice for the present paper, we beg off from citing the pluperfect boundary trace regularity.

Here, the matrix $H(x)$ is the quantity

$$H(x)=\begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \frac{\partial h_1}{\partial x_2} & \frac{\partial h_1}{\partial x_3} \\ \frac{\partial h_2}{\partial x_1} & \frac{\partial h_2}{\partial x_2} & \frac{\partial h_2}{\partial x_3} \\ \frac{\partial h_3}{\partial x_1} & \frac{\partial h_3}{\partial x_2} & \frac{\partial h_3}{\partial x_3} \end{bmatrix}.$$

We next apply the quantity $\phi(x)div(\tilde{h})$ to the wave component in (12), where $\tilde{h}(x)$ is any vector field in $\left[C^2(\overline{\Omega})\right]^3$ (to be eventually specified). Integrating in time and space, and subsequently using Green's Theorem and the identity $\nabla\phi\cdot\nabla\left(\phi div(\tilde{h})\right)=\phi\nabla\left(div(\tilde{h})\right)\cdot\nabla\phi+|\nabla\phi|^2\,div(\tilde{h})$, we obtain

$$\begin{aligned}\int_{Q_\delta}\left(\phi_t^2-|\nabla\phi|^2\right)div(\tilde{h})dQ_\delta &= \left[\left\langle\phi_t,\phi div(\tilde{h})\right\rangle_{H^{-\epsilon}(\Omega)\times H^{\epsilon}(\Omega)}\right]_\delta^{T-\delta} \\ &+\int_{Q_\delta}\phi\nabla\left(div(\tilde{h})\right)\nabla\phi dQ_\delta-\int_{\Sigma_{0,\delta}}\omega_t\phi div(\tilde{h})d\Sigma_{0,\delta}+\int_{\Sigma_{1,\delta}}\phi^2 div(\tilde{h})d\Sigma_{1,\delta}. \qquad (18)\end{aligned}$$

Upon estimating this relation, by means of Sobolev Trace Theory and $2ab\leq a^2+b^2$, we obtain

$$\begin{aligned}&\left|\int_{Q_\delta}\left(\phi_t^2-|\nabla\phi|^2\right)div(\tilde{h})dQ_\delta\right| \\ \leq\ & C_\epsilon\int_{\Sigma_{0,\delta}}\omega_t^2 d\Sigma_{0,\delta}+\epsilon\int_{Q_\delta}|\nabla\phi|^2\,dQ_\delta+\text{l.o.t.}\left(\vec{\phi},\vec{\omega}\right). \qquad (19)\end{aligned}$$

Setting now $h(x)=\tilde{h}(x)\equiv x-x_0$, where, say, $x_0\in\mathbb{R}^3$ is any point of Γ_0–and so necessarily $(x-x_0)\cdot\nu(x)=0$ on Γ_0–we have upon combining (17) and (19), the estimate

$$\begin{aligned}\int_{Q_\delta}|\nabla\phi|^2\,dQ &\leq C_\epsilon\left(\left|\int_{\Sigma_{0,\delta}}\omega_t\,(h\cdot\nabla\phi)\,d\Sigma_0\right|+\int_{\Sigma_{1,\delta}}\left|\frac{\partial\phi}{\partial\tau}\right|^2 d\Sigma_1+\int_{\Sigma_{1,\delta}}\phi_t^2 d\Sigma_1\right) \\ &+C_\epsilon\int_{\Sigma_{0,\delta}}|\omega_t|^2\,d\Sigma_0+C_{h,\epsilon}\mathcal{E}_\phi(T)+C_{h,\epsilon}\int_0^T\left|\langle\omega_t,\phi_t\rangle_{H^{\frac{1}{2}}(\Gamma_0)\times H^{-\frac{1}{2}}(\Gamma_0)}\right|dt+\text{l.o.t.}\left(\vec{\phi},\vec{\omega}\right) \qquad (20)\end{aligned}$$

(in deriving this estimate, we have also implicitly used the relation (16)).

To handle the first term on the right hand side: Since $h(x)=x-x_0$, with x_0 being on Γ_0, then $h(x)$ is a scalar multiple of some (unit) vector $\tau(x)$, say, where τ is a vector tangent to Γ_0. In short, we have

$$\int_{\Sigma_{0,\delta}}\omega_t\,(h\cdot\nabla\phi)\,d\Sigma_0=C\int_{\Sigma_{0,\delta}}\omega_t\frac{\partial\phi}{\partial\tau}d\Sigma_0. \qquad (21)$$

With the term $\frac{\partial\phi}{\partial\tau}$ in mind, we recall the following "sharp" regularity result for boundary traces of solutions to wave equations:

Lemma 7 *(See [36], p. 113, Theorem 3.1, Theorem 3.3(a); see also [38]). Set parameter η as*

$$\eta=\begin{cases}\frac{1}{4}, & \text{if } \Omega \text{ is a parallelepiped;} \\ \frac{1}{3}, & \text{if } \Omega \text{ is a smooth, bounded domain.}\end{cases}$$

With Γ_0 being a flat portion of the boundary of Ω, then if w solves the wave equation

$$\begin{cases} w_{tt} = \Delta w \quad \text{on } (0,T)\times\Omega \\ \dfrac{\partial w}{\partial \nu} + \alpha(x) = g \in L^2(0,T;H^\eta(\Gamma_0)) \quad \text{on } (0,T)\times\Gamma_0 \quad \textit{(where } L^\infty\textit{-coefficient } \alpha(x)\ge 0 \textit{ on } \Gamma) \\ [w(0),w_t(0)] = [w_0,w_1] \in H^1(\Omega)\times L^2(\Omega), \end{cases}$$

one has the estimate,

$$\|w\|_{L^2(0,T;H^{1-\eta}(\Gamma_0))} \le C_T\left(\|g\|_{L^2(0,T;H^\eta(\Gamma_0))} + \|[w_0,w_1]\|_{H^1(\Omega)\times L^2(\Omega)}\right).^2$$

Applying this estimate to (21) results then in

$$\begin{aligned} \int_{\Sigma_{0,\delta}} \omega_t\,(h\cdot\nabla\phi)\,d\Sigma_0 &= C\int_\delta^{T-\delta}\left\langle \omega_t, \frac{\partial\phi}{\partial\tau}\right\rangle_{H^\eta(\Gamma_0)\times H^{-\eta}(\Gamma_0)} dt \\ &\le C_T\|\omega_t\|_{L^2(0,T;H^\eta(\Gamma_0))}\left(\sqrt{\mathcal{E}_\phi(T)} + \|\omega_t\|_{L^2(0,T;H^\eta(\Gamma_0))}\right) \\ &\le \mathcal{E}_\phi(T) + \mathsf{l.o.t}\left(\vec{\phi},\vec{\omega}\right). \end{aligned} \tag{22}$$

For the second term on the right hand side of (20), we recall the following, microlocally derived, estimate:

Lemma 8 *(See Lemma 7.21 0f [27]). Let ϕ be a solution of the wave equation on $(0,T)\times\Omega$, or more generally, any second-order hyperbolic equation with smooth space dependent coefficients. Then, for all $\delta>0$, we have the estimate*

$$\int_\delta^{T-\delta}\int_{\Gamma_*}\left(\frac{\partial\phi}{\partial\tau}\right)^2 dt d\Gamma_* \le C_{T,\delta}\left(\int_0^T\int_{\Gamma_*}\phi_t^2 dt d\Gamma_* + \int_0^T\int_\Gamma\left(\frac{\partial\phi}{\partial\nu}\right)^2 dt d\Gamma\right) + \mathsf{l.o.t}\left(\vec{\phi},\vec{\omega}\right), \tag{23}$$

where Γ_* is any smooth connected segment of boundary Γ.

Applying the estimates in (22) and (23)–with $\Gamma_* = \Gamma_1$ therein–to the right hand side of (20) gives then

$$\int_{Q_\delta}|\nabla\phi|^2\,dQ \le C\left(\int_{\Sigma_1}\phi_t^2 d\Sigma_1 + \mathcal{E}_\phi(T) + \int_0^T\left|\langle\omega_t,\phi_t\rangle_{H^{\frac12}(\Gamma_0)\times H^{-\frac12}(\Gamma_0)}\right|dt\right) + \mathsf{l.o.t.}\left(\vec{\phi},\vec{\omega}\right), \tag{24}$$

where the constant C above does not depend on the terminal time $T>0$.

Subsequently, we can combine this estimate with that in (18)–where, therein, $\tilde{h}$ is any vector field satisfying $div(\tilde{h}) = 1$–so as to have

$$\int_\delta^{T-\delta}\mathcal{E}_\phi(t)dt \le C\left(\int_{\Sigma_1}\phi_t^2 d\Sigma_1 + \mathcal{E}_\phi(T) + \int_0^T\left|\langle\omega_t,\phi_t\rangle_{H^{\frac12}(\Gamma_0)\times H^{-\frac12}(\Gamma_0)}\right|dt\right) + \mathsf{l.o.t.}\left(\vec{\phi},\vec{\omega}\right),$$

where, again the constant C above does not depend on time T. Using again the acoustic wave relation (16), we have then, for T large enough,

$$(T-C-2\delta)\,\mathcal{E}_\phi(T) \le C_T\left(\int_0^T\left|\langle\omega_t,\phi_t\rangle_{H^{\frac12}(\Gamma_0)\times H^{-\frac12}(\Gamma_0)}\right|dt + \int_{\Sigma_1}\phi_t^2 d\Sigma_1\right) + \mathsf{l.o.t.}\left(\vec{\phi},\vec{\omega}\right).$$

[2] As explicitly stated in Corollary 3.4(b), Theorem 3.3(a), we have $\{g,[w_0,w_1]=0\}\in H^\eta(\Sigma) \Longrightarrow w|_\Gamma \in H^{1-\eta}(\Sigma)$. However, in the details of proof, it is evident that we have, continuously, $g\in L^2(0,T;H^\eta(\Gamma_0)) \Longrightarrow w\in L^2(0,T;H^{1-\eta}(\Gamma_0))$.

Applying Lemma 6, we get, finally,

$$
\begin{aligned}
& (T-C-2\delta)\,\mathcal{E}_\phi(T) \\
\leq\ & C_T\left(\int_0^T \|\omega_t\|^2_{H^{\frac{1}{2}}(\Gamma_0)}\,dt\right)^{\frac{1}{2}}\left\{\mathcal{E}_\phi(T)+\left(\int_0^T \|\omega_t\|^2_{H^{\frac{1}{2}}(\Gamma_0)}\,dt\right)\right\}^{\frac{1}{2}}+C_T\int_{\Sigma_1}\phi_t^2\,d\Sigma_1+\text{l.o.t.}\left(\vec{\phi},\vec{\omega}\right) \\
\leq\ & \delta\mathcal{E}_\phi(T)+C_T\int_{\Sigma_1}\phi_t^2\,d\Sigma_1+\text{l.o.t.}\left(\vec{\phi},\vec{\omega}\right).
\end{aligned}
$$

By these means, we have thus derived the following:

Theorem 9 *For terminal time $T>0$ large enough, the solution of the interior acoustic wave component in the coupled system satisfies the following estimate:*

$$
\mathcal{E}_\phi(T)\leq C_T\int_{\Sigma_1}\phi_t^2\,d\Sigma_1+\textit{l.o.t.}\left(\vec{\phi},\vec{\omega}\right). \tag{25}
$$

3 Concerning the Plate Component (Second Part of the Proof of Theorem 5)

In a fashion wholly analogous to that which lead to the relation (16), we have for all $0\leq s\leq t\leq T$,

$$
\mathcal{E}_\omega(t)=\mathcal{E}_\omega(s)-\int_s^t \left\langle \phi_t,\omega_t\right\rangle_{H^{-\frac{1}{2}}(\Gamma_0)\times H^{\frac{1}{2}}(\Gamma_0)}\,dt, \tag{26}
$$

where, again, the topology of the duality pairing is validated by Lemma 6. In this Section, we shall prove the following:

Theorem 10 *For terminal time $T>0$, the plate component of (12) satisfies the following estimate for all $0\leq t\leq T$:*

$$
\mathcal{E}_\omega(t)\leq C_T\left(\int_{\Sigma_1}\phi_t^2\,d\Sigma_1+\int_0^T\left\|P_\gamma^{\frac{1}{2}}(a\omega_t)\right\|^2_{L^2(\Gamma_0)}\,dt\right)+\text{l.o.t.}(\vec{\phi},\vec{\omega}). \tag{27}
$$

Proof of Theorem 10:

To start, we set $\psi\equiv\Delta\omega$, where again ω is the plate variable in (12). This gives the boundary value problem,

$$
\begin{aligned}
\gamma\psi_{tt}-\Delta\psi &= \phi_t|_{\Gamma_0}+\omega_{tt}\ \text{ on } (0,T)\times\Gamma_0 \\
\psi|_{\partial\Gamma_0} &= 0\ \text{ on } (0,T)\times\partial\Gamma_0 \\
[\psi(T),\psi_t(T)] &= [\Delta\omega_0,\Delta\omega_1]\,.
\end{aligned} \tag{28}
$$

For this boundary value problem, we recall the following "recovery estimate", which was derived in [4]:

Lemma 11 *(See Theorem 4.2 of [4]). Suppose that φ satisfies the following wave equation on $(0,T)\times\Gamma_0$, where Γ_0 is a (smooth) bounded and open set:*

$$
\gamma\varphi_{tt}-\Delta\varphi-F(\phi)=\frac{d}{dt}f_1+f_2\quad \textit{on } (0,T)\times\Gamma_0.
$$

Here, F is a first-order linear differential operator in time and space, with $L^{\infty}((0,T)\times\Gamma_0)$-coefficients. Moreover, the forcing terms $f_1 \in L^2(0,T;L^2(\Gamma_0))\cap C([0,T];H^{-\frac{1}{2}+\epsilon}(\Gamma_0))$, and $f_2 \in L^2(0,T;H^{-1}(\Gamma_0))$. Then for $T > 2diam(\Omega)$, we have the estimate for all $t \in [0,T]$,

$$\begin{aligned}
& \|\varphi(t)\|^2_{L^2(\Gamma_0)} + \|\varphi_t(t)\|^2_{H^{-1}(\Gamma_0)} \\
\leq & \left\{ \|\varphi\|^2_{L^2((0,T)\times\partial\Gamma_0)} + \left\|\frac{\partial\varphi}{\partial n}\right\|^2_{H^{-1}((0,T)\times\partial\Gamma_0))} + \|\varphi\|^2_{H^{-1}((0,T)\times\Gamma_0)} \right. \\
& \left. + \int_0^T \left[\|f_1\|^2_{L^2(\Gamma_0)} + \|f_2\|^2_{H^{-1}(\Gamma_0)}\right] dt + \|f_1\|^2_{C([0,T];H^{-\frac{1}{2}+\epsilon}(\Gamma_0))} \right\}.
\end{aligned}$$

(Here, $n(x)$ is the unit normal vector which is exterior to $\partial\Gamma_0$.)

We proceed now to apply this estimate to the wave equation in (28), with

$$f_1 \equiv \omega_t + \phi|_{\Gamma_0}\,;\ F = f_2 = 0.$$

Doing so, we obtain then for all $0 \leq t \leq T$,

$$\begin{aligned}
& \|\psi(t)\|^2_{L^2(\Gamma_0)} + \|\psi_t(t)\|^2_{H^{-1}(\Gamma_0)} \\
\leq & \ C_T \left\{ \left\|\frac{\partial\psi}{\partial n}\right\|^2_{H^{-1}((0,T)\times\partial\Gamma_0))} + \|\psi\|^2_{H^{-1}((0,T)\times\Gamma_0)} \right. \\
& \left. + \int_0^T \left\|\omega_t + \phi|_{\Gamma_0}\right\|^2_{L^2(\Gamma_0)} dt + \left\|\omega_t + \phi|_{\Gamma_0}\right\|^2_{C([0,T];H^{-\frac{1}{2}+\epsilon}(\Gamma_0))} \right\} \\
\leq & \ C_T \left\|\frac{\partial\psi}{\partial n}\right\|^2_{H^{-1}((0,T)\times\partial\Gamma_0))} + \text{l.o.t.}\left(\vec{\phi},\vec{\omega}\right)
\end{aligned} \tag{29}$$

(in writing down the last estimate, we are using the fact that $\left\|\phi|_{\Gamma_0}\right\|_{C([0,T];H^{-\frac{1}{2}+\epsilon}(\Gamma_0))}$ is bounded by lower order terms, by virtue of the Sobolev Trace Theorem).

Apparently, we must estimate the term $\left\|\frac{\partial\psi}{\partial n}\right\|_{H^{-1}((0,T)\times\partial\Gamma_0))}$. To this end, we let $\tilde{a}(x)$ be a smooth cutoff function which is *identically one* on $\partial\Gamma_0$, and moreover satisfies

$$\text{supp}(\tilde{a}) \subset \{x \in \text{supp}(a) : a(x) = 1\} \tag{30}$$

(again, $a(x)$ is the locally distributed coefficient which appears in (1) and (4)).

Subsequently, we multiply the boundary value problem in (28) by the coefficient $\tilde{a}(x)$, and make the change of variable

$$\tilde{\psi} \equiv \tilde{a}(x)\psi. \tag{31}$$

As defined, we then have that $\left[\tilde{\psi},\tilde{\psi}_t\right]$ solves the following problem:

$$\begin{aligned}
& \gamma\tilde{\psi}_{tt} - \Delta\tilde{\psi} = -\left(\psi\Delta\tilde{a} + 2\nabla\tilde{a}\cdot\nabla\psi\right) + \tilde{a}\omega_{tt} + \tilde{a}\ \phi_t|_{\Gamma_0} \quad \text{on } (0,T)\times\Gamma_0 \\
& \tilde{\psi}\Big|_{\partial\Gamma_0} = 0 \ \text{ on } (0,T)\times\partial\Gamma_0 \\
& \left[\tilde{\psi}(T), \left(\tilde{\psi}\right)_t(T)\right] = \left[\tilde{a}(x)\Delta\omega_0, \tilde{a}(x)\Delta\omega_1\right].
\end{aligned} \tag{32}$$

Concerning this boundary value problem, we recall the following regularity results:

Lemma 12 *(see [23], Theorem 2.3 and Remark 2.8, therein.) Suppose that* φ *satisfies the following wave equation on* $(0,T)\times\Gamma_0$*, where* Γ_0 *is a (smooth) bounded and open set:*

$$\begin{array}{l}\gamma\varphi_{tt}-\Delta\varphi=\frac{d}{dt}f_1+f_2 \quad on\ (0,T)\times\Gamma_0,\\ \varphi|_{\Gamma_0}=g \quad on\ (0,T)\times\partial\Gamma_0\\ [\varphi(0),\varphi_t(0)]=[\phi_0,\varphi_1]\end{array}$$

where forcing terms $f_1\in L^2(0,T;L^2(\Gamma_0))$*,* $f_2\in L^2(0,T;H^{-1}(\Gamma_0))$*, boundary data* $g\in L^2(0,T;L^2(\partial\Gamma_0))$ *and initial data* $[\phi_0,\varphi_1]\in L^2(\Omega)\times H^{-1}(\Omega)$*. Then, continuously, we have*

$$[\varphi,\varphi_t]\in C([0,T];L^2(\Gamma_0)\times H^{-1}(\Gamma_0));\ \frac{\partial\varphi}{\partial\nu}\in H^{-1}((0,T)\times\partial\Gamma_0).$$

Applying the Lemma 12 to the boundary value problem (32), with

$$f_1\equiv\tilde{a}\left(\omega_t+\phi|_{\Gamma_0}\right);\quad f_2\equiv-\left(\psi\Delta\tilde{a}+2\nabla\tilde{a}\cdot\nabla\psi\right),$$

we obtain the estimate, for all $0\le t\le T$,

$$\begin{aligned}&\left\|\left[\tilde{\psi}(t),\tilde{\psi}_t(t)\right]\right\|_{L^2(\Gamma_0)\times H^{-1}(\Gamma_0))}+\left\|\frac{\partial\tilde{\psi}}{\partial\nu}\right\|^2_{H^{-1}((0,T)\times\partial\Gamma_0)}\\ \le\ & C_T\left(\int_0^T\|\psi\Delta\tilde{a}+2\nabla\tilde{a}\cdot\nabla\psi\|^2_{H^{-1}(\Gamma_0)}dt+\int_0^T\left\|\tilde{a}\left(\omega_t+\phi|_{\Gamma_0}\right)\right\|^2_{L^2(\Gamma_0)}dt\right.\\ &\left.+\|[\tilde{a}(x)\Delta\omega_0,\tilde{a}(x)\Delta\omega_1]\|^2_{L^2(\Gamma_0)\times H^{-1}(\Gamma_0)}\right)\\ \le\ & C_T\left(\int_0^T\|\psi\Delta\tilde{a}+2\nabla\tilde{a}\cdot\nabla\psi\|^2_{H^{-1}(\Gamma_0)}dt+\|[\tilde{a}(x)\Delta\omega_0,\tilde{a}(x)\Delta\omega_1]\|^2_{L^2(\Gamma_0)\times H^{-1}(\Gamma_0)}\right)\\ &+\text{l.o.t.}\left(\vec{\phi},\vec{\omega}\right).\end{aligned}\tag{33}$$

To refine the right hand side of this inequality, we note first easily that,

$$\int_0^T\|\psi\Delta\tilde{a}\|^2_{H^{-1}(\Gamma_0)}dt\le C\int_0^T\|\psi\Delta\tilde{a}\|^2_{L^2(\Gamma_0)}dt\le C_{\tilde{a}}\int_0^T\int_{\text{supp}(\tilde{a})}(\Delta\omega)^2\,d\Gamma_0dt.\tag{34}$$

Moreover, for any $\varphi\in H_0^1(\Gamma_0)$, we have

$$\langle\nabla\tilde{a}\cdot\nabla\psi,\varphi\rangle_{H^{-1}(\Gamma_0)\times H_0^1(\Gamma_0)}=-\int_{\Gamma_0}\psi\left(\nabla\tilde{a}\cdot\nabla\varphi+\varphi\Delta\tilde{a}\right)d\Gamma_0;$$

whence we obtain the norm estimate

$$\|\nabla\tilde{a}\cdot\nabla\psi\|_{H^{-1}(\Gamma_0)}\le C_{\tilde{a}}\int_{\text{supp}(\tilde{a})}(\Delta\omega)^2\,d\Gamma_0.\tag{35}$$

Applying (34) and (35) to the right hand side of (33) gives now, for all $0\le t\le T$,

$$\begin{aligned}&\left\|\left[\tilde{\psi}(t),\tilde{\psi}_t(t)\right]\right\|_{L^2(\Gamma_0)\times H^{-1}(\Gamma_0))}+\left\|\frac{\partial\tilde{\psi}}{\partial\nu}\right\|^2_{H^{-1}((0,T)\times\partial\Gamma_0)}\\ \le\ & C_{T,\tilde{a}}\left(\int_0^T\int_{\text{supp}(\tilde{a})}(\Delta\omega)^2\,d\Gamma_0dt+\|[\tilde{a}(x)\Delta\omega_0,\tilde{a}(x)\Delta\omega_1]\|^2_{L^2(\Gamma_0)\times H^{-1}(\Gamma_0)}\right)\\ &+\text{l.o.t.}\left(\vec{\phi},\vec{\omega}\right).\end{aligned}\tag{36}$$

We proceed now to estimate the first term on the right hand side.

Proposition 13 *The plate variable of the system (12) satisfies the relation*

$$\begin{aligned}&\int_0^T \|a\Delta\omega\|^2_{L^2(\Gamma_0)}\,dt\\ \leq\;& C\left(\int_0^T \left\|P_\gamma^{\frac{1}{2}}(a\omega_t)\right\|^2_{L^2(\Gamma_0)}dt+\left|\left[\left\langle P_\gamma^{\frac{1}{2}}(a\omega_t),P_\gamma^{\frac{1}{2}}(a\omega)\right\rangle_{H^{-1}(\Gamma_0)\times H_0^1(\Gamma_0)}\right]_{t=0}^{t=T}\right|\right)+\mathsf{l.o.t.}(\vec{\phi},\vec{\omega}).\end{aligned}$$

Proof of Proposition 13: Applying the coefficient $a(x)$ to the plate component in (12), we obtain the system

$$\begin{aligned}&(a\omega)_{tt}-\gamma\Delta(a\omega)_{tt}+\Delta^2(a\omega)+a\,\phi_t|_{\Gamma_0}=\left[\Delta^2,a\right]\omega-\gamma\left(\omega_{tt}\Delta a+2\nabla a\cdot\nabla\omega_{tt}\right)\quad\text{on }\Gamma_0\\ &a\omega|_{\Gamma_0}=0;\quad \Delta(a\omega)|_{\Gamma_0}=2\nabla a\cdot\nabla\omega|_{\partial\Gamma_0}\quad\text{on }\partial\Gamma_0\\ &(a\omega(T),a\omega_t(T))=(a\omega_0,a\omega_1)\end{aligned}\tag{37}$$

(above, as usual, $[P_1,P_2]$ denotes the action of commutation with respect to two pseudodifferential operators P_1,P_2 in $OPS^{s_1}(\mathbb{R}^2)$ and $OPS^{s_1}(\mathbb{R}^2)$, respectively). Thereto, we multiply both sides of the PDE by $a\omega$ and subsequently integrate in time and space. Integrating by parts, we eventually obtain the following relation:

$$\begin{aligned}&\int_0^T \|\Delta(a\omega)\|^2_{L^2(\Gamma_0)}\,dt\\ =\;&\int_0^T\left[\left\|P_\gamma^{\frac{1}{2}}(a\omega_t)\right\|^2_{L^2(\Gamma_0)}+\left(2\nabla a\cdot\nabla\omega|_{\partial\Gamma_0},\frac{\partial(a\omega)}{\partial n}\right)_{L^2(\partial\Gamma_0)}+\left\langle\left[\Delta^2,a\right]\omega,a\omega\right\rangle_{[H^2(\Gamma_0)]'\times H^2(\Gamma_0)}\right]dt\\ &-\left[\left\langle P_\gamma^{\frac{1}{2}}(a\omega_t),P_\gamma^{\frac{1}{2}}(a\omega)\right\rangle_{H^{-1}(\Gamma_0)\times H_0^1(\Gamma_0)}-\left(a\,\phi|_{\Gamma_0},a\omega\right)_{L^2(\Gamma_0)}\right]_{t=0}^{t=T}\\ &+\int_0^T\left(\gamma(\Delta a)\omega_t+a\,\phi|_{\Gamma_0},a\omega_t\right)_{L^2(\Gamma_0)}dt+2\gamma\int_0^T\langle\nabla a\cdot\nabla\omega_t,a\omega_t\rangle_{H^{-1}(\Gamma_0)\times H_0^1(\Gamma_0)}\,dt\\ &-\left[\gamma\left((\Delta a)\omega_t,a\omega\right)_{L^2(\Gamma_0)}+2\gamma\langle\nabla a\cdot\nabla\omega_t,a\omega\rangle_{H^{-1}(\Gamma_0)\times H_0^1(\Gamma_0)}\right]_{t=0}^{t=T}.\end{aligned}$$

Estimating the right hand side of this expression, by making use of the known Sobolev regularity of the commutator–see e.g., [39]; in particular, $\left[\Delta^2,a\right]\in\mathcal{L}\left(H^1(\Gamma_0),\left[H^2(\Gamma_0)\right]'\right)$–the Sobolev Embedding Theorem and Leibniz' Formula, we have

$$\begin{aligned}\int_0^T\|a\Delta\omega\|^2_{L^2(\Gamma_0)}dt\leq\;&\epsilon\int_0^T\|a\omega\|^2_{H^2(\Gamma_0)}dt+C\int_0^T\left\|P_\gamma^{\frac{1}{2}}(a\omega_t)\right\|^2_{L^2(\Gamma_0)}dt\\ &+\left|\left[\left\langle P_\gamma^{\frac{1}{2}}(a\omega_t),P_\gamma^{\frac{1}{2}}(a\omega)\right\rangle_{H^{-1}(\Gamma_0)\times H_0^1(\Gamma_0)}\right]_{t=0}^{t=T}\right|\\ &+\int_0^T\left\|2\nabla a\cdot\nabla\omega|_{\partial\Gamma_0}+\frac{\partial(a\omega)}{\partial n}\right\|^2_{L^2(\partial\Gamma_0)}dt+\mathsf{l.o.t.}(\vec{\phi},\vec{\omega})\\ \leq\;&\epsilon\int_0^T\|a\Delta\omega\|^2_{L^2(\Gamma_0)}dt+\left|\left[\left\langle P_\gamma^{\frac{1}{2}}(a\omega_t),P_\gamma^{\frac{1}{2}}(a\omega)\right\rangle_{H^{-1}(\Gamma_0)\times H_0^1(\Gamma_0)}\right]_{t=0}^{t=T}\right|\\ &+C\int_0^T\left\|P_\gamma^{\frac{1}{2}}(a\omega_t)\right\|^2_{L^2(\Gamma_0)}dt+\mathsf{l.o.t.}(\vec{\phi},\vec{\omega}),\end{aligned}\tag{38}$$

where in the last step we have used the fact that the map $\omega\to 2\nabla a\cdot\nabla\omega|_{\partial\Gamma_0}+\frac{\partial(a\omega)}{\partial n}\in\mathcal{L}\left(H^{\frac{3}{2}+\epsilon}(\Gamma_0),H^\epsilon(\partial\Gamma_0)\right)$. This concludes the proof of Proposition 13. □

Applying the last Proposition to the estimate (36) gives now, for all $0 \le t \le T$,

$$\left\| \left[\tilde{\psi}(t), \tilde{\psi}_t(t) \right] \right\|_{L^2(\Gamma_0)\times H^{-1}(\Gamma_0))} + \left\| \frac{\partial \tilde{\psi}}{\partial \nu} \right\|^2_{H^{-1}((0,T)\times\partial\Gamma_0)} \le C_T \left\| [\tilde{a}(x)\Delta\omega_0, \tilde{a}(x)\Delta\omega_1] \right\|^2_{L^2(\Gamma_0)\times H^{-1}(\Gamma_0)}$$
$$+C_T \left(\int_0^T \left\| P_\gamma^{\frac{1}{2}}(a\omega_t) \right\|^2_{L^2(\Gamma_0)} dt + \left| \left[\left\langle P_\gamma^{\frac{1}{2}}(a\omega_t), P_\gamma^{\frac{1}{2}}(a\omega) \right\rangle_{H^{-1}(\Gamma_0)\times H_0^1(\Gamma_0)} \right]_{t=0}^{t=T} \right| \right) + \text{l.o.t.}(\vec{\phi},\vec{\omega}). \quad (39)$$

To conclude the proof of Theorem 10, we must deal with the terminal data on the right hand side of (39). To this end, we multiply the wave equation in (32) by $A_D^{-1}\tilde{\psi}_t$, and integrate in time and space (where again, $\tilde{\psi} = \tilde{a}\Delta\omega$). This gives for all $0 \le s \le T$ (after an integration by parts),

$$\begin{aligned} & \frac{1}{2} \left\| [\tilde{a}\Delta\omega_0, \gamma\tilde{a}\Delta\omega_1] \right\|^2_{L^2(\Gamma_0)\times H^{-1}(\Gamma_0)} \\ = \; & \frac{1}{2} \left\| \left[\tilde{\psi}(s), \gamma\tilde{\psi}_t(s) \right] \right\|^2_{L^2(\Gamma_0)\times H^{-1}(\Gamma_0)} + \int_s^T \left\langle \tilde{a}\omega_{tt} + \tilde{a}\ \phi_t|_{\Gamma_0}, A_D^{-1}\tilde{\psi}_t \right\rangle_{H^{-1}(\Gamma_0)\times H_0^1(\Gamma_0)} d\tau \\ & - \int_s^T \left\langle \psi\Delta\tilde{a} + 2\nabla\tilde{a}\cdot\nabla\psi, A_D^{-1}\tilde{\psi}_t \right\rangle_{H^{-1}(\Gamma_0)\times H_0^1(\Gamma_0)} d\tau. \end{aligned} \quad (40)$$

Now concerning the right hand side of this expression, we have after using the wave equation in (32),

$$\begin{aligned} & \int_s^T \left\langle \tilde{a}\omega_{tt} + \tilde{a}\ \phi_t|_{\Gamma_0}, A_D^{-1}\tilde{\psi}_t \right\rangle_{H^{-1}(\Gamma_0)\times H_0^1(\Gamma_0)} d\tau \\ = \; & -\int_s^T \left\langle \tilde{a}\omega_t + \tilde{a}\ \phi|_{\Gamma_0}, A_D^{-1}\tilde{\psi}_{tt} \right\rangle_{H^{-1}(\Gamma_0)\times H_0^1(\Gamma_0)} d\tau + \left[\left(\tilde{a}\omega_t + \tilde{a}\ \phi|_{\Gamma_0}, A_D^{-1}\tilde{\psi}_t \right)_{L^2(\Gamma_0)} \right]_s^T \\ = \; & -\frac{1}{\gamma} \int_s^T \left(\tilde{a}\omega_t + \tilde{a}\ \phi|_{\Gamma_0}, \frac{\partial}{\partial t} A_D^{-1}(\tilde{a}\omega_t + \tilde{a}\ \phi|_{\Gamma_0}) - \tilde{\psi} - A_D^{-1}(\psi\Delta\tilde{a} + 2\nabla\tilde{a}\cdot\nabla\psi) \right)_{L^2(\Gamma_0)} d\tau \\ & + \left[\left(\tilde{a}\omega_t + \tilde{a}\ \phi|_{\Gamma_0}, A_D^{-1}\tilde{\psi}_t \right)_{L^2(\Gamma_0)} \right]_s^T \\ = \; & -\frac{1}{2\gamma} \left[\left\| A_D^{-\frac{1}{2}} \left(\tilde{a}\omega_t + \tilde{a}\ \phi|_{\Gamma_0} \right) \right\|^2_{L(\Gamma_0)} \right]_s^T + \int_s^T \left\langle \tilde{a}\omega_t + \tilde{a}\ \phi|_{\Gamma_0}, \tilde{a}\Delta\omega \right\rangle_{H^\epsilon(\Gamma_0)\times H^{-\epsilon}(\Gamma_0)} + \text{l.o.t.}(\vec{\phi},\vec{\omega}) \\ = \; & \text{l.o.t.}(\vec{\phi},\vec{\omega}), \end{aligned} \quad (41)$$

where in this last step, we are using the fact that $\Delta \in \mathcal{L}(H^{2-\epsilon}(\Gamma_0), H^{-\epsilon}(\Gamma_0))$ (as well as the Sobolev Trace Theorem, applied to the term $\phi|_{\Gamma_0}$).

Moreover, for the third term on the right hand side of (40),

$$
\begin{aligned}
& \left| \int_s^T \left\langle \psi \Delta \tilde{a} + 2 \nabla \tilde{a} \cdot \nabla \psi, A_D^{-1} \tilde{\psi}_t \right\rangle_{H^{-1}(\Gamma_0) \times H_0^1(\Gamma_0)} d\tau \right| \\
= & \left| \int_s^T \left\langle A_D^{-1} \left(\psi \Delta \tilde{a} + 2 \nabla \tilde{a} \cdot \nabla \psi \right), \Delta \left(\tilde{a} \omega_t \right) - \omega_t \Delta \tilde{a} - 2 \nabla \tilde{a} \cdot \nabla \omega_t \right\rangle_{H_0^1(\Gamma_0) \times H^{-1}(\Gamma_0)} d\tau \right| \\
\leq & C_{\tilde{a}} \left(\int_0^T \left(\int_{\mathrm{supp}(\tilde{a})} (\Delta \omega)^2 d\Gamma_0 \right) dt + \int_0^T \left(\int_{\mathrm{supp}(\tilde{a})} \left[|\omega_t|^2 + \gamma \left| \nabla \omega_t \right|^2 \right] d\Gamma_0 \right) dt \right) \\
& + \mathsf{l.o.t.}(\vec{\phi}, \vec{\omega}) \\
\leq & C \left(\int_0^T \left\| P_\gamma^{\frac{1}{2}} (a \omega_t) \right\|_{L^2(\Gamma_0)}^2 dt + \left| \left[\left\langle P_\gamma^{\frac{1}{2}} (a \omega_t), P_\gamma^{\frac{1}{2}} (a \omega) \right\rangle_{H^{-1}(\Gamma_0) \times H_0^1(\Gamma_0)} \right]_{t=0}^{t=T} \right| \right) \\
& + \mathsf{l.o.t.}(\vec{\phi}, \vec{\omega}), \qquad (42)
\end{aligned}
$$

after using the estimate (35) and Proposition 13. Applying (41) and (42) to the right hand side of (40) gives then the estimate, for all $0 \leq s \leq T$,

$$
\begin{aligned}
& \frac{1}{2} \left\| [\tilde{a} \Delta \omega_0, \gamma \tilde{a} \Delta \omega_1] \right\|_{L^2(\Gamma_0) \times H^{-1}(\Gamma_0)}^2 \leq \frac{1}{2} \left\| \left[\tilde{\psi}(s), \gamma \tilde{\psi}_t(s) \right] \right\|_{L^2(\Gamma_0) \times H^{-1}(\Gamma_0)}^2 \\
& + C_T \left(\int_0^T \left\| P_\gamma^{\frac{1}{2}} (a \omega_t) \right\|_{L^2(\Gamma_0)}^2 dt + \left| \left[\left\langle P_\gamma^{\frac{1}{2}} (a \omega_t), P_\gamma^{\frac{1}{2}} (a \omega) \right\rangle_{H^{-1}(\Gamma_0) \times H_0^1(\Gamma_0)} \right]_{t=0}^{t=T} \right| \right) \\
& + \mathsf{l.o.t.}(\vec{\phi}, \vec{\omega}). \qquad (43)
\end{aligned}
$$

Integrating both sides of this relation from 0 to T we subsequently have,

$$
\begin{aligned}
& \left\| [\tilde{a} \Delta \omega_0, \gamma \tilde{a} \Delta \omega_1] \right\|_{L^2(\Gamma_0) \times H^{-1}(\Gamma_0)}^2 \leq C_T \int_0^T \left\| \left[\tilde{\psi}(s), \gamma \tilde{\psi}_t(s) \right] \right\|_{L^2(\Gamma_0) \times H^{-1}(\Gamma_0)}^2 ds \\
& + C_T \left(\int_0^T \left\| P_\gamma^{\frac{1}{2}} (a \omega_t) \right\|_{L^2(\Gamma_0)}^2 dt + \left| \left[\left\langle P_\gamma^{\frac{1}{2}} (a \omega_t), P_\gamma^{\frac{1}{2}} (a \omega) \right\rangle_{H^{-1}(\Gamma_0) \times H_0^1(\Gamma_0)} \right]_{t=0}^{t=T} \right| \right) \\
& + \mathsf{l.o.t.}(\vec{\phi}, \vec{\omega}). \qquad (44)
\end{aligned}
$$

Estimating the right hand side of (44) by another invocation of Proposition 13 gives now,

$$
\begin{aligned}
& \left\| [\tilde{a} \Delta \omega_0, \gamma \tilde{a} \Delta \omega_1] \right\|_{L^2(\Gamma_0) \times H^{-1}(\Gamma_0)}^2 \\
\leq & C_T \left(\int_0^T \left\| P_\gamma^{\frac{1}{2}} (a \omega_t) \right\|_{L^2(\Gamma_0)}^2 dt + \left| \left[\left\langle P_\gamma^{\frac{1}{2}} (a \omega_t), P_\gamma^{\frac{1}{2}} (a \omega) \right\rangle_{H^{-1}(\Gamma_0) \times H_0^1(\Gamma_0)} \right]_{t=0}^{t=T} \right| \right) \\
& + \mathsf{l.o.t.}(\vec{\phi}, \vec{\omega}). \qquad (45)
\end{aligned}
$$

Now combining (29), (39) and (45), with the fact that $\tilde{a} = 1$ on $\partial\Gamma_0$, we have for all $0 \le t \le T$,

$$\begin{aligned} & \|A_D\omega(t)\|^2_{L^2(\Gamma_0)} + \left\|P_\gamma^{\frac{1}{2}}\omega_t(t)\right\|^2_{L^2(\Gamma_0)} \\ \le\ & \left(1 + \left\|P_\gamma^{\frac{1}{2}}A_D^{-1}\right\|^2_{\mathcal{L}(H^{-1}(\Gamma_0),L^2(\Gamma_0))}\right)\|\psi(t)\|^2_{L^2(\Gamma_0)} + \|\psi_t(t)\|^2_{H^{-1}(\Gamma_0)} \\ \le\ & C_T \left\|\frac{\partial\tilde{\psi}}{\partial n}\right\|_{H^{-1}((0,T)\times\partial\Gamma_0)} + \mathsf{l.o.t.}(\vec{\phi},\vec{\omega}) \\ \le\ & \tilde{C}_T\left(\left|\left[(a\omega_t, P_\gamma(a\omega))_{L^2(\Gamma_0)}\right]_0^T\right| + \int_0^T \left\|P_\gamma^{\frac{1}{2}}(a\omega_t)\right\|^2_{L^2(\Gamma_0)} dt\right) + \mathsf{l.o.t.}(\vec{\phi},\vec{\omega}). \end{aligned} \tag{46}$$

To deal with the first term on the right hand side of (46): Using the inequality $ab \le \delta\frac{a^2}{2} + \frac{b^2}{2\delta}$, we have

$$\begin{aligned} & \left|\left[(a\omega_t, P_\gamma(a\omega))_{L^2(\Gamma_0)}\right]_0^T\right| \\ \le\ & C_a\left(\|A_D\omega(T)\|_{L^2(\Gamma_0)}\|\omega_t(T)\|_{L^2(\Gamma_0)} + \|A_D\omega(0)\|_{L^2(\Gamma_0)}\|\omega_t(0)\|_{L^2(\Gamma_0)}\right) \\ & + \mathsf{l.o.t.}(\vec{\phi},\vec{\omega}) \\ \le\ & \frac{\epsilon}{\tilde{C}_T}\left(\|A_D\omega(T)\|^2_{L^2(\Gamma_0)} + \|A_D\omega(0)\|^2_{L^2(\Gamma_0)}\right) + \mathsf{l.o.t.}(\vec{\phi},\vec{\omega}), \end{aligned} \tag{47}$$

where $\tilde{C}_T$ is the same constant as that in (46). Inserting this estimate into the right hand side of (46), followed by two applications of the relation (26), gives finally, for fixed t,

$$\begin{aligned} \mathcal{E}_\omega(t) \le\ & \epsilon\mathcal{E}_\omega(t) + C_T\left(\int_0^T \left|\langle\phi_t,\omega_t\rangle_{H^{-\frac{1}{2}}(\Gamma_0)\times H^{\frac{1}{2}}(\Gamma_0)}\right| dt + \int_0^T \left\|P_\gamma^{\frac{1}{2}}(a\omega_t)\right\|^2_{L^2(\Gamma_0)} dt\right) \\ & + \mathsf{l.o.t.}(\vec{\phi},\vec{\omega}). \end{aligned}$$

A final usage of Lemma 6 and Theorem 9 completes the proof of Theorem 10. □

Theorems 9 and 10 give now the (polluted) estimate, for $T > 0$ large enough,

$$\mathcal{E}(T) \le C_T\left(\int_{\Sigma_1}\phi_t^2 d\Sigma_1 + \int_0^T \left\|P_\gamma^{\frac{1}{2}}(a\omega_t)\right\|^2_{L^2(\Gamma_0)} dt\right) + \mathsf{l.o.t.}(\vec{\phi},\vec{\omega}), \tag{48}$$

which is the continuous observability inequality (14), modulo lower order terms.

4 Completion of the Proof of Theorem 5

The argument to eliminate the lower order terms in the estimate (48) is by now standard; it is an argument by contradiction which makes use of the classic Holmgren's Uniqueness Theorem (see e.g., [18]). But for the sake of completeness, we sketch out the argument here.

Lemma 14 *For T large enough, there exists a constant $C_T > 0$ such that the solution of (12) satisfies the following inequality:*

$$\left\|\left[\vec{\phi},\vec{\omega}\right]\right\|^2_{C([0,T];H^{1-\epsilon}(\Omega)\times H^{-\epsilon}(\Omega)\times H^{2-\epsilon}(\Gamma_0)\times H^{1-\epsilon}(\Gamma_0))} \le C_T\left(\|\phi_t\|^2_{L^2(\Sigma_1)}\int_0^T \left\|P_\gamma^{\frac{1}{2}}(a\omega_t)\right\|^2_{L^2(\Gamma_0)} dt\right). \tag{49}$$

Proof: Suppose the given inequality is false. Then there exists a sequence of initial data $\left\{\left[\vec{\phi}_0^{(n)}, \vec{\omega}_0^{(n)}\right]\right\} \subset \mathbf{H}$ and a corresponding solution sequence $\left\{\left[\vec{\phi}^{(n)}, \vec{\omega}^{(n)}\right]\right\}$ of the PDE (12), for which $\left\|\phi_t^{(n)}\right\|_{L^2(\Sigma_1)} < +\infty$ for all n, and which moreover satisfies

$$\begin{aligned} \left\|\left[\vec{\phi}^{(n)}, \vec{\omega}^{(n)}\right]\right\|_{C([0,T];H^{1-\epsilon}(\Omega)\times H^{-\epsilon}(\Omega)\times H^{2-\epsilon}(\Gamma_0)\times H^{1-\epsilon}(\Gamma_0))} &= 1 \ \forall n; \\ \left\|\phi_t^{(n)}\right\|^2_{L^2(\Sigma_1)} + \int_0^T \left\|P_\gamma^{\frac{1}{2}}(a\omega_t^{(n)})\right\|^2_{L^2(\Gamma_0)} dt &\to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{50}$$

Consequently, the existence of the inequality (48), for T large enough, and the convergence in (50), imply that the sequence $\left\{\left\|\left[\vec{\phi}_0^{(n)}, \vec{\omega}_0^{(n)}\right]\right\|_{\mathbf{H}}\right\}_{n=1}^{\infty}$ is bounded uniformly in n. Consequently, there is a subsequence, still denoted by $\left\{\left[\vec{\phi}_0^{(n)}, \vec{\omega}_0^{(n)}\right]\right\}_{n=1}^{\infty}$, and $\left[\vec{\phi}_0^*, \vec{\omega}_0^*\right] \in \mathbf{H}$ such that

$$\left[\vec{\phi}_0^{(n)}, \vec{\omega}_0^{(n)}\right] \to \left[\vec{\phi}_0^*, \vec{\omega}_0^*\right] \quad \text{in } \mathbf{H} \text{ weakly.} \tag{51}$$

If we denote $\left[\vec{\phi}^*, \vec{\omega}^*\right]$ as the solution pair of (12), corresponding to terminal data $\left[\vec{\phi}_0^*, \vec{\omega}_0^*\right]$, then *a fortiori*

$$\left\{\left[\vec{\phi}^{(n)}, \vec{\omega}^{(n)}\right]\right\} \to \left[\vec{\phi}^*, \vec{\omega}^*\right] \ \text{ in } L^\infty(0,T;\mathbf{H}) \text{ weak star.} \tag{52}$$

Moreover, reading off the ϕ-wave equation in (12), and using the weak convergence of $\left\{\phi^{(n)}\right\}$, we deduce the estimate

$$\left\|\phi_{tt}^{(n)}\right\|_{C([0,T];[H^1(\Omega)]')} \le C, \text{ uniformly in } n. \tag{53}$$

Similiarly, using the plate equation in (12) in conjunction with elliptic theory, Lemma 6 and the uniform boundedness of $\left\{\left\|\left[\vec{\phi}_0^{(n)}, \vec{\omega}_0^{(n)}\right]\right\|_{\mathbf{H}}\right\}_{n=1}^{\infty}$ and $\left\{\left[\vec{\phi}^{(n)}, \vec{\omega}^{(n)}\right]\right\}$, we have

$$\left\|\omega_{tt}^{(n)}\right\|_{L^2(0,T;L^2(\Gamma_0))} \le C, \text{ uniformly in } n. \tag{54}$$

From (52)–(54) and a classic compactness result of J. Simon in [37], we conclude that

$$\left[\vec{\phi}^{(n)}, \vec{\omega}^{(n)}\right] \to \left[\vec{\phi}^*, \vec{\omega}^*\right] \ \text{ strongly, in } C([0,T]; H^{1-\epsilon}(\Omega) \times H^{-\epsilon}(\Omega) \times H^{2-\epsilon}(\Gamma_0) \times H^{1-\epsilon}(\Gamma_0)).$$

In consequence, we have from (50) the equality

$$\left\|\vec{\phi}^*, \vec{\omega}^*\right\|_{C([0,T];H^{1-\epsilon}(\Omega)\times H^{-\epsilon}(\Omega)\times H^{2-\epsilon}(\Gamma_0)\times H^{1-\epsilon}(\Gamma_0))} = 1. \tag{55}$$

Furthermore, given the convergence in (50), we conclude that

$$\phi_t^*|_{\Sigma_1} = 0 \ \text{ on } \Sigma_1; \ \ \omega_t^* = 0 \ \text{ on supp}(a). \tag{56}$$

But these relations immediately yield up a contradiction by Holmgren's Uniqueness Theorem. In fact, setting $p \equiv \phi_t^*$, then p solves the overdetermined system

$$\begin{aligned} p_{tt} &= \Delta p \text{ on } Q \\ \left.\frac{\partial p}{\partial \nu}\right|_{\Gamma_1} &= 0 \text{ on } \Sigma_1 \\ p|_{\Gamma_1} &= 0 \text{ on } \Sigma_1 \\ a(x) \left.\frac{\partial p}{\partial \nu}\right|_{\Gamma_0} &= 0 \text{ on } \Sigma_0. \end{aligned}$$

By Holmgren's Uniqueness Theorem, we infer then that $\phi_t^* = p = 0$, for $T > 0$ large enough. In turn, the ellipticity of the Laplacian under the Robin boundary conditions in (12) yields that $\phi^* = 0$. Furthermore, the variable $\tilde{p} \equiv \omega_t^*$ solves the plate equation

$$\begin{aligned} (I - P_\gamma)\, \tilde{p}_{tt} + \Delta^2 \tilde{p} &= 0 \text{ on } (0,T) \times \Gamma_0 \\ \tilde{p}|_{\partial\Gamma_0} &= \Delta\tilde{p}|_{\partial\Gamma_0} = 0 \text{ on } (0,T) \times \partial\Gamma_0 \\ a(x)\tilde{p} &= 0 \text{ on } (0,T) \times \text{supp}(a). \end{aligned}$$

Again, for $T > 0$, we have that necessarily $\omega_t^* = \tilde{p} = 0$. From this and the ellipticity of A_D^2, we conclude further that $\omega^* = 0$ on Σ_0. Thus $\vec{\phi}^* = 0$ and $\vec{\omega}^* = 0$, which contradicts the relation (55). This concludes the proof of Lemma 14. □

The proof of Theorem 5 (or equivalently, the derivation of the estimate (14)), is now completed by combining the estimate (48) with Lemma 14. In turn, by the argument we gave at the begining of the paper, the Theorem 5 and Lemma 3 will yield the exact controllability of the original ("purely Neumann") problem (1).

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