ON THE STABILITY OF THE SHEARING FLOW BETWEEN PIPES

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IMA Preprint Series # 598

October 1989

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CONTENTS

§ 1. INTRODUCTION

This note is devoted to the investigation of both linear and nonlinear stability of a newtonian liquid between mutually sliding pipes.

Although this problem falls into the category of spiral flows already considered by Joseph [1],and Joseph & Munson [2], the analysis given there does not cover what happens in the case of pure sliding. We did not find any other reference in which the above problem had been investigated. This circumstance was a little surprising particularly after having devoted a recent paper [3] of ours to the more complex problem of the stability of the flow of *two immiscible* newtonian liquids between sliding pipes. We think that the results of the present paper may find application in piston-cylinder systems and in coating technology,when a feeder region, serving as a flow strightener, is used to produce a smooth surface on a moving wire.

In the present paper we give theoretical estimates and solve numerically the eigenvalue problem coming out from the energy theory using a collocation method based on Chebyshev polynomials. Our analysis shows convergence of all neutral curves to those of Couette flow when the gap between the cylinders reduces to zero. When the radii ratio increases away from one an improvement of the region of stability is obtained as shown by figure 1. Finally we considered the problem from the point of view of the linear theory and we never found instability.

§ 2. NONLINEAR STABILITY ANALYSIS

2 a. *Universal stability*

We consider the problem of a Newtonian incompressible liquid flowing between two cylinders, the inner one sliding along the common axis with given velocity $\mathbf{V} = V\mathbf{e}_x$. Here all functions and vectors are evaluated in cylindrical coordinates (r,θ,x) . The two pipes are infinitely long with common axis at $r=0$ and radii R_1, R_2 $(R_1 < R_2)$.

The equations of motion are

$$\rho \frac{dU}{dt} = - \nabla P + \mu \nabla U , \qquad \nabla . U = 0$$

$$U = Ve_x \quad \text{at} \quad r = R_1$$

$$U = 0 \quad \text{at} \quad r = R_2$$

The basic flow with vanishing pressure gradient $\frac{\partial P}{\partial x} = 0$ is

$$U = (0,0,W(r))$$

with

$$W(r) = V\left(1 - \frac{\log(r/R_1)}{\log(R_2/R_1)}\right)$$

We perturb this flow with $\hat{U} = (u,v,w+W)$, $\hat{p} = P + \pi$ and introduce dimensionless parameters

$$b = \frac{R_2}{R_1} \,, \quad \mathcal{R} = \frac{\rho\, V\, R_1}{\mu}$$

Also we scale the lengths with R_1, velocities with V , pressures with ρV^2, times with R_1/V .

If we use the same symbols for dimensional and dimensionless variables, the nonlinear equations which govern the disturbances (u,π) become

$$\begin{aligned} &\partial_t u + (U + u) \,.\, \nabla u + u \,.\, \nabla U = -\nabla\pi + \frac{1}{\mathcal{R}}\, \Delta u \\ &\nabla . u = 0 \\ &u_{|\partial\Omega} = 0 \end{aligned} \tag{1}$$

The standard energy analysis gives

$$\dot{E} \leq \left(\int_\Omega \xi(r)\, u^2 - \frac{1}{\mathcal{R}} \int_\Omega (\nabla u)^2 \right) \tag{2}$$

where Ω is the region of motion and $-\xi(r)$ is a lower bound for the characteristic values of

$$D = \tfrac{1}{2}\left(\nabla U + (\nabla U)^T \right)$$

and $E = \frac{1}{2}\int_\Omega u^2$ is the energy. Since the characteristic values of D are 0 and $\pm \frac{1}{2\, r \log b}$, (2) gives

$$\dot{E} \le \frac{1}{2\ \log b}\int_\Omega \frac{u^2}{r} - \frac{1}{\mathfrak{R}}\int_\Omega (\nabla u)^2 \le \left(\frac{b \log b}{2\pi^2} - \frac{1}{\mathfrak{R}}\right)\int_\Omega |\nabla u|^2 \tag{3}$$

where we have used the inequality

$$\int_\Omega \frac{u^2}{r} = \int_\Omega r\,\frac{u^2}{r^2} \le b\int_\Omega \frac{u^2}{r^2} \le \gamma \int_\Omega (\nabla u)^2 \tag{4}$$

where (see [1] for example) $\gamma = \frac{b \log^2 b}{\pi^2}$.

Thus the basic flow is nonlinearly stable if $\mathfrak{R} < \frac{2\pi^2}{b \log b}$ or equivalently, if $V < \frac{2\nu\pi^2}{R_2 \log(R_2/R_1)}$. It is well known however that this estimate is too rough and we might find larger values of admissible V if we try to determine the critical value of $\mathfrak{R}$ via a variational argument.

Actually $\dot{E} \le 0$ if

$$\mathfrak{R} < \mathfrak{R}_c \tag{5}$$

where $\mathfrak{R}_c{}^{-1}$ is the lowest eigenvalue λ of the problem

$$\begin{aligned} u \,.\, D &= -\nabla\hat{\pi} + \lambda\, \Delta u \\ \nabla \,.\, u &= 0 \end{aligned} \tag{6}$$

$$\mathbf{u} = 0 \qquad \text{on } \partial\Omega$$

Here u is meant to be a solution (if there exists one) of the variational problem

$$\max_{u \in S} \left\{ \frac{-\int_\Omega u \,.\, D \,.\, u}{\int_\Omega (\nabla u)^2} \right\} \tag{7}$$

where

$$S = \left\{ u \in W_0^{1,2}(\Omega) \;/\; \nabla \,.\, u = 0 \right\}$$

2 b. *Variational approach*

Using normal modes in cylindrical coordinates,that is

$$\begin{bmatrix} \hat{u} \\ \hat{v} \\ \hat{w} \\ \hat{\pi} \end{bmatrix} = \begin{bmatrix} i\,u(r) \\ v(r) \\ w(r) \\ \pi(r) \end{bmatrix} \exp(i\alpha x + in\theta) \tag{8}$$

and assuming $\alpha \neq 0$, the linear eigenvalue problem (6) leads , after simple algebraic manipulations, to the following equations in u and v only

$$
\begin{aligned}
&r^4 u'''' + 2\, r^3 u''' - (2\alpha^2 r^2 + n^2 + 3)\, r^2 u'' - (2\alpha^2 r^2 - n^2 - 3)\, r u' + \\
&+ (\alpha^4 r^4 + (n^2+2)\, \alpha^2 r^2 + 3n^2 - 3)\, u + n\, r^3 v''' - 2nr^2 v'' - \\
&-(\alpha^2 r^2 + n^2 - 3)\, nrv' + 3(\alpha^2 r^2 + n^2 - 1)\, nv = \frac{i\alpha\lambda^{-1} r^2}{2 \log b}\, (2ru' + nv) \\
&nr^3 u''' + 2nr^2 u'' - (\alpha^2 r^2 + n^2 + 1)\, nru' - (3\alpha^2 r^2 + n^2 - 1)\, nu + (\alpha^2 r^2 + n^2)\, r^2 v'' \\
&+ (\alpha^2 r^2 - n^2)\, rv' - [\alpha^4 r^4 + (2n^2 + 1)\, \alpha^2 r^2 + n^2 (n^2 - 1)]\, v = \frac{i\alpha\lambda^{-1} r^2}{2 \log b}\, nu
\end{aligned}
\tag{9}
$$

In the axisymetric case (n=0) system (9) splits into two uncoupled equations:

$$
\begin{aligned}
&(L_1 - \alpha^2)^2\, u = \frac{i\alpha\lambda^{-1}}{\log b}\, \frac{u'}{r} \\
&L_1 v = \alpha^2 v
\end{aligned}
\tag{10}
$$

where

$$L_n = (.)'' + \frac{1}{r}\,(.)' - \frac{n^2}{r^2}\,(.)$$

To (10) we append the boundary conditions

$$u(1) = u'(1) = u(b) = u'(b) = v(1) = v(b) = 0. \tag{11}$$

The equation for v is irrelevant to the computation of the eigenvalues.

In the same way, when $n \neq 0$ we can write the system of equations as

$$- \frac{in^2 f}{2\lambda r^2 \log b} = \alpha r \, (Lf)' + (\alpha^2 r^2 + n^2) \, Lw + 2\alpha^2 r w'$$

$$\frac{i\, n^2 w}{2\lambda r \log b} = L^2 f + \alpha^2 \, Lf + \alpha r \, (Lw)' + 4\alpha \, Lw + 2\alpha^3 w \tag{12}$$

where $L = L_n - \alpha^2(.)$ and $f = ru$; this system, for $\alpha = 0$, reduces to the single equation

$$L_n(r^2 L_n^2(f)) + \frac{n^2 f}{4\lambda^2 r^2 \log^2 b} = 0 \tag{13}$$

with the boundary conditions

$$f(1) = f'(1) = f(b) = f'(b) = L_n^2(f)|_{r=1,b} = 0 \tag{14}$$

2 c. *Numerical results.*

We studied system (12) with vanishing boundary conditions (and (10)-(11) (13)-(14) as special cases) numerically using a pseudo spectral method. This is a collocation method based on Chebyshev polynomials. Following Orzag [4], we expand the eigenfunctions in terms of

$$T_K(r) = \cos(K \arccos r) \, , \quad K = 0, 1, \ldots\ldots, N$$

where N is a truncation number. To use this rapresentation we must map the interval $r \in [1,b]$ onto $y \in [-1,1]$ by $y = y(r)$. In this interval we define interpolation operators by

$$I_N\, g(y) = \sum_{K=0}^{N} \hat{g}_K\, T_K(y)$$

and

$$I_N\, g(y_j) = g(r_j)$$

where

$$y_j = \cos\frac{\pi j}{N}\ , \qquad j = 0\ , \ldots\ldots, N$$

are the collocation points and $g(y)$ is an arbitrary vector-valued eigenfunctions. Via standard arguments (see for example [3]) we end up to a linear algebric eigenvalue problem in the $2N-2$ unknowns $\hat{g}_K$ $(K=1, \ldots.., N-1)$ which has been solved using either the IMSL routines EIGZC or EIGZF.

A cross-checking of convergence of the above method has been made using equations (9) and (12) . We achieved convergence within the fourth significant digit when $N \geq 12$. Figure 1 shows convergence to the well known results of stability of plane Couette flow to two and three-dimensional disturbances when b goes to 1 . When b is less than 1.9 the discrepancy of $\mathcal{R}_c$ with respect to $\mathcal{R}_{Couette}$ (the critical Reynolds number of the Couette flow) is less than 1% ($\mathcal{R}_{Couette}=82.7$ for three-dimensional disturbances and $\mathcal{R}_{Couette}=177.6$ for two-dimensional ones,see [5] and [6]).

The most dangerous disturbance is always the azimuthal one $(\alpha=0)$ with n depending on the following ranges of b

n	b_{min}	b_{max}
1	12.45	∞
2	3.73	12.45
3	2.5	3.73
4	2.02	2.5
5	1.82	2.02

Table 1. Dependence of the critical value of n on the ranges of b.

These values are computed from figure 1.

Figure 2 shows the critical values of $\mathcal{R}$ obtained via the energy estimate $\mathcal{R} < \frac{2\pi^2}{b \log b}$ with the most dangerous mode $(\alpha=0)$ and the axisymmetric mode $(n=0)$ of figure 1. The critical curve (the heavy line) is formed by the lower branches of the discrete family of curves shown in figure 1.

INSERT FIGURES 1 AND 2

§3. LINEAR STABILITY

3 a. *A priori estimates of* $\mathcal{R}_c$

The critical values of $\mathcal{R}$ obtained via the energy analysis can be largely improved using the method of linear stability. As is well-known this method provides sufficient conditions on $\mathcal{R}$ for the (nonlinear) *instability* of the basic flow.

The system of equations to consider is now the following

$$\partial_t u + W\, \partial_x u = -\partial_r \pi + \mathcal{R}^{-1} (\tilde{\Delta} u - \frac{u}{r^2} - \frac{2}{r^2}\, \partial_\theta v)$$

$$\partial_t v + W\, \partial_x v = -\frac{1}{r}\, \partial_\theta \pi + \mathcal{R}^{-1} (\tilde{\Delta} v - \frac{v}{r^2} + \frac{2}{r^2}\, \partial_\theta u)$$

$$\partial_t w + u\, W' + W\, \partial_x w = -\partial_x \pi + \mathcal{R}^{-1}\, \tilde{\Delta} w$$

$$\frac{1}{r}\, \partial_r (ru) + \frac{1}{r}\, \partial_\theta v + \partial_x w = 0$$

$$u = v = w = 0 \qquad \text{for} \quad r = 1, b$$

where

$$\tilde{\Delta} \equiv \frac{1}{r}\,\partial_r(r\partial_r) + \frac{1}{r^2}\,\partial^2_{\theta\theta} + \partial^2_{xx}.$$

We look for solutions of type

$$\begin{bmatrix} \hat{u} \\ \hat{v} \\ \hat{w} \\ \hat{\pi} \end{bmatrix} = \begin{bmatrix} i\,u(r) \\ v(r) \\ w(r) \\ \pi(r) \end{bmatrix} \exp(in\theta + i\alpha x - \sigma t)$$

To this aim define $C = \frac{\sigma}{i\alpha}$ .Then the system for the amplitude functions of the normal mode expansion turns out to be the following

$$\begin{aligned}
&\alpha\,(W - C)\,u = \pi' - i\mathcal{R}^{-1}\,(\mathcal{L}u - \frac{2n}{r^2}v) \\
&\alpha\,(W - C)\,v = -\frac{n}{r}\,\pi - i\mathcal{R}^{-1}\,(\mathcal{L}v - \frac{2n}{r^2}\,u) \\
&\alpha\,(W - C)\,w + W'\,u = -\alpha\pi - i\mathcal{R}^{-1}\,\mathcal{L}w \\
&u' + \frac{u}{r} + \frac{n}{r}\,v + \alpha\,w = 0
\end{aligned} \tag{15}$$

with vanishing boundary conditions at $r=1$, b . Here

$$\mathcal{L} = (.)'' + \frac{1}{r}(.)' - (\alpha^2 + \frac{n^2+1}{r^2})(.)$$

For system (15) to be consistent, all parameters C, n, α, $\mathcal{R}$, b need to satisfy a characteristic equation of the form $F(c,n,\alpha,\mathcal{R},b) = 0$. For a given steady flow $\mathcal{R}$ and b are assigned. Thus $F = 0$ determines C as a function of n and α . Before calculating numerically $C = C(n,\alpha)$ we want to obtain qualitative informations about C at least in the simpler cases $n = 0$ and $\alpha = 0$, without solving the eigenvalue problem explicitely. To this aim let us consider the eigenvalue problem for $n = 0$. Notice that $(15)_2$ rewrites

$$\mathcal{L}_\alpha v + i\alpha\mathcal{R}(W-C)v = 0 \tag{16}$$

where

$$\mathcal{L}_\alpha = L_1 - \alpha^2(\,.)$$

The unstable eigenvalues are determined from the equations for u only. In fact the equation for v gives rise only to stable eigenvalues; this can be easily seen by moltiplying (16) by the complex conjugate $r\bar{v}$ of rv , integrating over [1,b] and bearing in mind the boundary conditions. Indeed we have

$$\int_1^b \left(rv''\,\bar{v} + v'\,\bar{v} - \frac{v^2}{r}\right) dr = -\int_1^b \left(r\,v'^2 + \frac{v^2}{r}\right) dr$$

$$\int_1^b [\alpha^2 + i\alpha\mathcal{R} \, (W-C)] \, rv^2 \, dr = \int_1^b [\alpha^2 + \alpha\mathcal{R}C_i - i\alpha\mathcal{R} \, (W-C_r)] \, rv^2 \, dr;$$

therefore,by taking the real part,

$$\alpha\mathcal{R}C_i \int_1^b rv^2 \, dr = - \int_1^b [r \, v'^2 + (\alpha^2 r^2 + 1) \frac{v^2}{r}] \, dr$$

which implies $C_i < 0$.

We now eliminate w and π between $(15)_1$,$(15)_3$ and $(15)_4$ to get

$$r^4 u'''' + 2r^3 u''' - (3 + 2\alpha^2 r^2 + h) \, r^2 u'' + (3 - 2\alpha^2 r^2 - h) \, ru' +$$
$$+ [h \, (\alpha^2 r^2 + 1) + \alpha^4 r^4 + 2\alpha^2 r^2 - 3 - 2i\alpha\mathcal{R} \, r^3 W') \, u = 0 \tag{17}$$

where

$$h(r) = i\alpha\mathcal{R} \, (W(r)-C) \, r^2$$

The boundary conditions to append to (17) are

$$u(1) = u'(1) = u(b) = u'(b) = 0 \, . \tag{18}$$

Let us now multiply (17) by $r^{-m} \, \bar{u}$ and integrate over (1,b). An integration by parts yields

$$\int_1^b \Big\{ r^{4-m} u''^2 + 2\,(3-m)\, r^{3-m}\, u''\, \bar{u}' + [3 - (3-m)\,(2-m) +$$

$$2\alpha^2 r^2 + h(r)]\; r^{2-m}\, u'^2 +$$

$$+ [(3 - (3-m)\,(2-m)](2-m) + 2\,(3-m)\, \alpha^2 r^2 + (1-m)\, h(r) + \tag{19}$$

$$h'(r)\, r + 3]\; r^{1-m}\, u'\, \bar{u} +$$

$$+ [h(r)(\alpha^2 r^2 + 1) + \alpha^4 r^4 + 2\alpha^2 r^2 - 3 - 2i\alpha \mathfrak{R}\, r^3 W']\, \frac{u^2}{r^m} \Big\}\, dr = 0$$

Let I be the left hand side of (19) ; let also $\mathfrak{J}^{+} = I + \bar{I}$ and $\mathfrak{J}^{-} = I - \bar{I}$. We get

$$\mathfrak{J}^{+} = \int_1^b \Big\{ 2\, r^{4-m}\, u''^2 + [-4m^2 + 22\, m - 24 + 4\alpha^2 r^2 + 2\alpha \mathfrak{R}\, r^2 C_i]\; r^{2-m}\, u'^2 +$$

$$+ [m^4 - 8m^3 + 20\, m^2 - 16\, m - 3 + 2(-m^2 + 6m - 7)\, \alpha^2 r^2 + 2\alpha^4 r^4 +$$

$$+ (2\alpha^2 r^2 - 7 + 6m - m^2)\, \alpha \mathfrak{R}\, r^2 C_i]\, \frac{u^2}{r^m} +$$

$$+ [(3 - m)\, i\alpha \mathfrak{R}\, r^2 (W - C_r) + i\alpha \mathfrak{R}\, r^3 W']\; r^{1-m} (u'\, \bar{u} - u\, \bar{u}') \Big\}\, dr$$

$$\mathfrak{J}^{-} = \int_1^b \Big\{ 2\,(3-m)\, r^{3-m}\, (u''\, \bar{u}' - u'\, \bar{u}'') + 2i\alpha \mathfrak{R}\, r^2 (W - C_r)\; r^{2-m}\, u'^2 +$$

$$+ [m^3 - 7m^2 + 13\,m - 3 + 2\,(3-m)\,\alpha^2 r^2]\, r^{1-m}(u'\,\bar{u} - u\,\bar{u}') +$$

$$+ (3 - m)\,\alpha\mathcal{R}\, C_i\, r^{3-m}(u'\,\bar{u} - u\,\bar{u}') + [i\alpha\mathcal{R}\, r^2\,(W - C_r)\,.$$

$$\cdot\,(2\alpha^2 r^2 - m^2 + 6m - 7)\,(W - C_r) + 2(m-5)\,i\alpha\mathcal{R}\, r^3 W']\,\frac{u^2}{r^m}\Big\}\,dr$$

For $m=3$, since $W'\,r = -1/\log b$, the real and the imaginary part of (19) give

$$\frac{\mathfrak{g}^+}{2} = \int_1^b \Big\{ r\,u''^2 + (2\alpha^2 r^2 + 3)\,\frac{u'^2}{r} + (\alpha^4 r^4 + 2\alpha^2 r^2 - 3)\,\frac{u^2}{r^3} -$$

$$- \frac{i\alpha\mathcal{R}}{\log b}\,\frac{u'\,\bar{u} - u\,\bar{u}'}{2} + \alpha\mathcal{R}\, C_i\,[r\,u'^2 + (\alpha^2 r^2 + 1)\,\frac{u^2}{r}]\,dr = 0$$

$$\frac{\mathfrak{g}^-}{2} = \int_1^b \Big\{ (W - C_r)\, r\, u'^2 + [(\alpha^2 r^2 + 1)\,(W - C_r) - 2r\,W']\,\frac{u^2}{r} \Big\}\,dr = 0$$

Therefore it turns out that C_i is negative if and only if

$$\int_1^b \Big\{ r u''^2 + (2\alpha^2 r^2 + 3)\,\frac{u'^2}{r} + (\alpha^4 r^4 + 2\alpha^2 r^2 - 3)\,\frac{u^2}{r^3} \Big\}\,dr > \tag{20}$$

$$\frac{i\alpha\mathcal{R}}{\log b} \int_1^b \frac{u'\,\bar{u} - u\,\bar{u}'}{2}\,dr$$

First notice that

$$\left| \frac{i\alpha\mathcal{R}}{\log b} \int_1^b \frac{u'\,\bar{u} - u\,\bar{u}'}{2}\,dr \right| \le \frac{\alpha\mathcal{R}}{\log b} \int_1^b |u|\,|u'| \le \frac{\alpha\mathcal{R}}{\log b}\,\|u\|\,\|u'\| \le \tag{21}$$

$$\frac{\alpha \mathcal{R}}{\log b} \frac{(b-1)}{\pi} \, ||u'||^2$$

where we used the well known fact (see [1] for example) that the minimum value of the ratio

$$\int_1^b u'^2 \, dr \Big/ \int_1^b u^2 \, dr$$

among functions for which $u(1) = u(b) = 0$ is $\pi^2/(b-1)^2$.

A different estimate of the term at the left hand side of (21) can be obtained this way:

$$\left| \frac{i\alpha \mathcal{R}}{\log b} \int_1^b \frac{u' \, \bar{u} - u \, \bar{u}'}{2} \, dr \right| \leq \frac{\alpha \mathcal{R}}{\log b} \, ||u|| \, ||u'|| \leq \frac{\alpha \mathcal{R} b}{\log b} \, ||\frac{u}{r}|| \, ||u'||. \tag{22}$$

Since the minimum value of the ratio

$$\int_1^b u'^2 \, dr \Big/ \int_1^b \frac{u^2}{r^2} \, dr \tag{23}$$

among functions which vanish when $r=1, b$ is equal to (see [1] for example)

$$\frac{1}{\eta^2} = \frac{1}{4} + \frac{\pi^2}{\log^2 b} \ ,$$

inequality (22) leads to

$$\left| \frac{i\alpha \mathcal{R}}{\log b} \int_1^b \frac{u' \, \bar{u} - u \, \bar{u}'}{2} \right| \leq \frac{\alpha \mathcal{R} b}{\log b} \, \eta \, ||u'||^2 \, . \tag{24}$$

Let now $b<e^{2\pi/\sqrt{3}} \simeq 37.622367$; then $\eta<1$ and we have

$$I \equiv \int_1^b r\, u''^2 + 3\, \frac{u'^2}{r} - 3\, \frac{u^2}{r^3} = \int_1^b r\, u''^2 + \frac{3}{r}\,(1-\eta^2)\, u'^2 +$$

$$+ \frac{3}{r}\left(\eta^2 u'^2 - \frac{u^2}{r^2}\right) \geq \int_1^b r\left\{u''^2 + 3\,(1-\eta^2)\,\frac{u'^2}{r^2}\right\}.$$

Since u' vanishes for $r=1, b$ we can continue this inequality as follows

$$I \geq \int_1^b \left[\frac{1}{\eta^2} + 3\,(1-\eta^2)\right] \frac{u'^2}{r^2} \geq H_\eta\, \|u'\|^2 .$$

where $H_\eta = b^{-2}\left(\frac{1}{\eta^2} + 3\,(1-\eta^2)\right)$. Moreover

$$\int_1^b 2\alpha^2 r\, u'^2 \geq 2\alpha^2 \|u'\|^2 \tag{25}$$

$$\int_1^b (\alpha^4 r^4 + 2\alpha^2 r^2)\, \frac{u^2}{r^3}\, dr \geq \left(\alpha^4 + 2\frac{\alpha^2}{b}\right)\|u\|^2 . \tag{26}$$

To satisfy (20) we consider the following possibilities:

either

$$(H_\eta b^{-2} + 2\alpha^2)\, \|u'\|^2 + \left(\alpha^4 + \frac{2\alpha^2}{b}\right) \|u\|^2 > \frac{\alpha \mathcal{R} b}{\log b}\, \eta\, \|u'\|^2 \tag{27}$$

or

$$(H_\eta b^{-2} + 2\alpha^2)\, \|u'\|^2 + \left(\alpha^4 + \frac{2\alpha^2}{b}\right) \|u\|^2 > \frac{\alpha \mathcal{R}}{\log b}\, \|u\|\, \|u'\| . \tag{28}$$

To satisfy (27) it suffices that

$$\mathcal{R} < \gamma(\alpha,\mathrm{b}) \equiv (2\alpha^2 + \mathrm{H}_\eta \mathrm{b}^{-2}) \frac{\log \mathrm{b}}{\alpha \, \mathrm{b} \, \eta} \tag{29}$$

To satisfy (28) it suffices that

$$\mathcal{R} < \delta(\alpha,\mathrm{b}) \equiv 2 \log \mathrm{b} \, (2\alpha^2 + \mathrm{H}_\eta \, \mathrm{b}^{-2})^{1/2}\left(\alpha^2 + \frac{2}{\mathrm{b}}\right)^{1/2}. \tag{30}$$

In figures 3a,b,c we have plotted γ and δ versus α for various values of b.

INSERT FIGURE 3.a, b, c

3*b*. *Axis-independent mode.*

In this section we consider system (15) of 3.a in the special case $\alpha \to 0$. To this aim we first eliminate π between $(15)_1$ and $(15)_2$ and then v by using $(15)_4$. This leads to

$$\begin{gathered}(\mathrm{r}\, \mathrm{L}_n \mathrm{f}')' - \mathrm{n}^2 \, \mathrm{L}_n\left(\frac{\mathrm{f}}{\mathrm{r}}\right) - \frac{\mathrm{f}''}{\mathrm{r}} + \frac{\mathrm{f}'}{\mathrm{r}^2} - 3\,\frac{\mathrm{n}^2}{\mathrm{r}^2}\,\mathrm{f} = -\sigma\left(\mathrm{r}\,\mathrm{f}'' + \mathrm{f}' - \mathrm{n}^2\,\frac{\mathrm{f}}{\mathrm{r}}\right) \\ \mathrm{L}_n \, \mathrm{w} - \mathrm{i}\mathcal{R}\mathrm{W}' \, \mathrm{u} = -\sigma \mathcal{R} \, \mathrm{w}\end{gathered} \tag{31}$$

with boundary conditions

$$\mathrm{u} = \mathrm{w} = \mathrm{u}' = 0 \,, \quad \text{for } \mathrm{r} = 1,\mathrm{b}. \tag{32}$$

Notice that

$$(L_n f)' = (L_0 f)' + \frac{f'}{r^2} - 2\frac{n^2}{r^3} f$$

and

$$L_n\left(\frac{f}{r}\right) = \frac{1}{r} L_n f - 2\frac{f'}{r^2} + \frac{f}{r^3}$$

Thus

$$(r\, L_0 f')' - n^2 L_0\left(\frac{f}{r}\right) = r\, L_0^2 f + \frac{f''}{r} - \frac{f'}{r^2} + 3\frac{n^2 f}{r^3} .$$

As a consequence, we finally get

$$L_n^2 f + \sigma\, \mathcal{R}\, L_n f = 0 \tag{33}$$

$$f = f' = 0 \qquad \text{at} \qquad r=1,\, b .$$

We have also to consider $(31)_2$, that is

$$L_n g + \sigma\, \mathcal{R}\, g = \frac{f}{r^2} \tag{34}$$

$$g = 0 \qquad \text{at} \qquad r=1,\, b$$

g being $\frac{iw}{\mathcal{R}}$ log b . There are, possibly, unstable eigenvalues $(\sigma_r<0)$ given by (34) which are not provided by (33).

By means of some integrations by parts and by using the boundary conditions it can be easily seen that

$$\int_1^b r^p\, \bar{f}\, Lf\, dr = \int_1^b r^p\, f\, L\bar{f}\, dr \;\Leftrightarrow\; p = 1 .$$

If we multiply $(33)_1$ by $r^\beta\, \bar{f}$ we obtain

$$I_1 + \sigma\, \mathcal{R}\, I_2 = 0 \tag{35}$$

where

$$I_1 = \int_1^b \{r^{4+\beta} (f'')^2 + 2\,(3+\beta)\, r^{3+\beta} f''\,\bar{f}' -$$

$$(\beta^2+5\,\beta+5-2\,n^2)\, r^{2+\beta}(f')^2 +$$

$$+ (3+\beta)\,(2\,n^2-\beta^2-4\beta-3)\, r^{1+\beta} f'\,\bar{f} + n^2\,(n^2-4)\, r^{\beta} f^2\}\, dr$$

and

$$I_2 = -\int_1^b \left\{r^{4+\beta}\,(f')^2 + (3+\beta)\, r^{3+\beta} f'\,\bar{f} + n^2\, r^2\, f^2\right\} dr$$

If we multiply $(34)_1$ by $r^{\gamma}\,\bar{g}$ $(g \equiv \tilde{w})$ we get

$$I_3 + \sigma\, \mathfrak{R}\, I_4 = 0 \tag{36}$$

where

$$I_3 = -\int_1^b \left\{r^{2+\gamma}\,(g')^2 + (1+\gamma)\, r^{1+\gamma}\, g'\,\bar{g} + n^2\, g^2 - f\,\bar{g}\right\} dr$$

$$I_4 = \int_1^b r^{2+\gamma}\, g^2\, dr\,.$$

Notice that if $\beta \neq -3$, (35) contains both the real and the imaginary parts of σ . Therefore we choose $\beta = -3$.

For an arbitrary ξ we have

$$(I_1 + \xi\, I_3) + \sigma\, \mathfrak{R}\, (I_2 + \xi\, I_4) = 0\,. \tag{37}$$

Bearing in mind that

$$\mathfrak{Re} \int_1^b r^{\xi}\, f'\,\bar{f}\, dr = -\frac{\xi}{2} \int_1^b r^{\xi-1}\, f^2\, dr$$

we easily get

$$\mathcal{R}e\ \sigma = \left\{ \int_1^b r\,[(f'')^2 + (2\,n^2+1)\,\frac{(f')^2}{r^2} + n^2\,(n^2-4)\,\frac{f^2}{r^4} + \right.$$

$$+ \xi\, r^{1+\gamma}\,(g')^2 + \xi\left(n^2 - \frac{(1+\gamma)^2}{2}\right) r^{\gamma-1}\,g^2 - \tag{38}$$

$$\left. \xi\, r^{\gamma-1}\,\mathcal{R}e\, f\bar{g}]\,dr \right\} \cdot \left\{ \int_1^b r\,[(f')^2 + n^2\,\frac{f^2}{r^2} + \xi\, r^{1+\gamma}\,g^2]\,dr \right\}^{-1}$$

Let $\mathcal{N}$ and $\mathcal{D}$ be the numerator and the denominator of the right hand side of (38). Since

$$\left| \int_1^b r^{\gamma}\,\mathcal{R}e(\bar{g}\,f)\,dr \right| \le \int_1^b \frac{r^{\gamma_1}\,f^2 + r^{\gamma_2}\,g^2}{2}\,dr$$

with any $\gamma_1, \gamma_2 > 0$ such that $2\gamma = \gamma_1 + \gamma_2$, we have

$$\mathcal{N} \ge \int_1^b \left\{ (f'')^2 + (2\,n^2+1)\,\frac{(f')^2}{r^2} + [n^2\,(n^2-4) - \frac{\xi}{2}\,r^{\gamma_1+3}]\,\frac{f^2}{r^4} + \right.$$

$$\left. + \xi\, b_\gamma^{1+\gamma}\,[(g')^2 + \frac{1}{2}\left(2\,n^2 - (1+\gamma)^2 - r^{\gamma-\gamma_1}\right)\frac{g^2}{r^2} \right\} dr$$

where

$$b_\gamma = \begin{cases} 1 & \text{if } 1+\gamma \ge 0 \\ b & \text{if } 1+\gamma < 0 \end{cases}$$

If we call $\phi \equiv 1/\eta^2$ the minimum of the ratio (23) we can write

$$\mathcal{N} \ge \frac{1}{b^2} \int_1^b [A(\phi,r)\,\frac{f^2}{r^2} + \xi\, b_\gamma^{1+\gamma}\,B(\phi,r)\,\frac{g^2}{r^2}]\,dr$$

where

$$A(\phi,r) = \phi^2 + (2\,n^2+1)\,\phi + n^2\,(n^2-4) - \frac{\xi}{2}\,r^{\gamma_1+3}$$

$$B(\phi,r) = \phi + \frac{1}{2}\,[2\,n^2 - (1+\gamma)^2 - r^{\gamma-\gamma_1}]$$

If $n \geq 2$, $\gamma = \gamma_1 = -3$ and $\frac{\xi}{2} = \phi^2 + (2n^2+1)\,\phi$, we have $A(\phi,r)$, $B(\phi,r) \geq 0$ and so also $\mathcal{N}$ and $\Re e\,\sigma$ are positive.

If $n=1$, $A(\phi,r) \geq 0$ if and only if

$$\xi \leq \frac{2}{b_\gamma^{\gamma_1+3}}\,(\phi^2 + 3\,\phi - 3)$$

Since ξ has to be positive this implies $\phi > \frac{1}{2}(\sqrt{21}-3)$ that is $b < \exp\left\{2\pi/\sqrt{2\sqrt{21}-7}\right\} \simeq 71.5$. As far as $B(\phi,r)$ is concerned, this is positive provided that γ and γ_1 are conveniently chosen (say both equal to -1)

3 c. *Numerical results.*

We now calculate $C(n,\alpha) = 0$ numerically using the same code as in section 2.b. We first obtain two different systems: the former by eliminating w and π among equations (15),in the hypothesis that $\alpha \neq 0$, to get

$$\begin{aligned}
& r^4u'''' + 2r^3u''' - [h(r) + 2\alpha^2r^2 + n^2 + 3]\,r^2u'' - [h(r) + \\
& \quad 2\alpha^2r^2 - n^2 - 3]\,ru' + \\
& + [h(r)(\alpha^2r^2 + 1) + \alpha^4r^4 + (n^2 + 2)\,\alpha^2r^2 + 3n^2 - 3]\,u - \\
& \quad 2i\alpha\Re\,W'\,r^3u + nr^3v''' - \\
& -2n\,r^2v'' - [h(r)+\alpha^2r^2+ n^2- 3]\,rv' + [h(r) - i\alpha\Re\,W'r^3 + \\
& \quad 3(\alpha^2r^2 + n^2 - 1)]\,nv = 0
\end{aligned} \tag{39}$$

$$
\begin{aligned}
& n\,r^3u''' + 2n\,r^2u'' - [h(r) + \alpha^2r^2 + n^2 + 1]\,nru' - [h(r) - i\alpha\mathcal{R}\,W'\,r^3 + \\
& \quad 3\alpha^2r^2 + n^2 - 1]\,nu + \\
& + [\alpha^2r^2 + n^2]\,r^2v'' + (\alpha^2r^2 - n^2)\,r\,v' - \\
& - [h(r)\,(\,\alpha^2r^2 + n^2) + \alpha^4r^4 + (2n^2 + 1)\,\alpha^2r^2 + \\
& \quad n^2(n^2 - 1)]\,v = 0
\end{aligned}
\tag{40}
$$

The lattersystem is obtained by eliminating v and π under the hypothesis that $n \neq 0$,that is

$$
\begin{aligned}
& \alpha(W - C)\,(r^2f''(r) + rf'(r) - n^2f(r) + \alpha r^3w' + 2\alpha r^2w) + \alpha r^2W'\,(f'(r) + \alpha rw) + \\
& + \frac{i}{\mathcal{R}}\,[r^2f''''(r) + 2rf'''(r) - (\alpha^2r^2 + 2n^2 + 1)\,f''(r) - (\alpha^2r^2 - 2n^2 - 1)\,\frac{f'(r)}{r} + \\
& + n^2(\alpha^2r^2 + n^2 - 4)\,\frac{f(r)}{r^2} + \alpha r^3w''' + 5\alpha r^2w'' - (\alpha^2r^2 + n^2 - 3)\,\alpha rw' - 2(\alpha^2r^2 + n^2)\,\alpha w] = 0
\end{aligned}
$$

$$
\begin{aligned}
& \alpha\,(W - C)\,[\alpha\,rf'(r) + (\alpha^2r^2 + n^2)\,w] + n^2\,W'\,\frac{f(r)}{r} + \frac{i}{\mathcal{R}}\,[\alpha\,rf'''(r) + \alpha\,f''(r) - \\
& - (\alpha^2r^2 + n^2 + 1)\,\alpha\,\frac{f'(r)}{r} + 2\alpha\,n^2\,\frac{f(r)}{r^2} + (\alpha^2r^2 + n^2)\,w'' + (3\alpha^2r^2 + n^2)\,\frac{w'}{r} - \\
& - (\alpha^2r^2 + n^2)^2\,\frac{w}{r^2}] = 0\,.
\end{aligned}
$$

We cross checked the convergence of our code using both the above systems.In most cases the convergence of the eigenvalue of both systems was achieved when $N=30$ (number of collocation points) within the forth significant digit , as shown in table 2.

N	$\alpha \neq 0$		$n \neq 0$	
18	.1887	$-.1273$ i	.1808	$-.1016$ i
22	.1923	$-.1234$ i	.1950	$-.1265$ i
26	.1924	$-.1238$ i	.1930	$-.1242$ i
30	.1923	$-.1238$ i	.1923	$-.1237$ i

TABLE 2 Cross-checking convergence test of σ

for $b=3$, $n=3$, $\alpha=3$, $\mathfrak{R}=1000$

Our results were disturbed by spurious eigenvalues, that is the eigenvalues of the discretized system which do not belong to the spectrum of the linear operator which controls stability (see [7] and [8]).

To distinguish them we sometimes needed to increase N up to 75 , particularly when both n and α are not vanishing.

Within the range of parameters we considered ($b \leq 10$, $\alpha \leq 10$, $n \leq 5$, $\mathfrak{R} \leq 10.000$), we never found instability. In figure 4 we plot $\mathfrak{Re}(\sigma)$ as a function of b for different $n \neq 0$ and $\alpha = 0$. The larger is n (or the inverse of the gap size), the stronger is the stability. In figure 5 we plot the immaginary part of the critical eigenvalue in the axisymmetric case, for different values of b . Also in this case the smaller is the gap size the stronger is the stability.

INSERT FIGURES 4 and 5

ACKNOWLEDGMENTS

This work was supported in part by the Institute for Mathematics and its Applications with funds provided by the National Science Foundation, by the Ministero della Pubblica Istruzione and by the Consiglio Nazionale delle Ricerche. Computer results were obtained using the computing services of the ACSS of the University of Minnesota.

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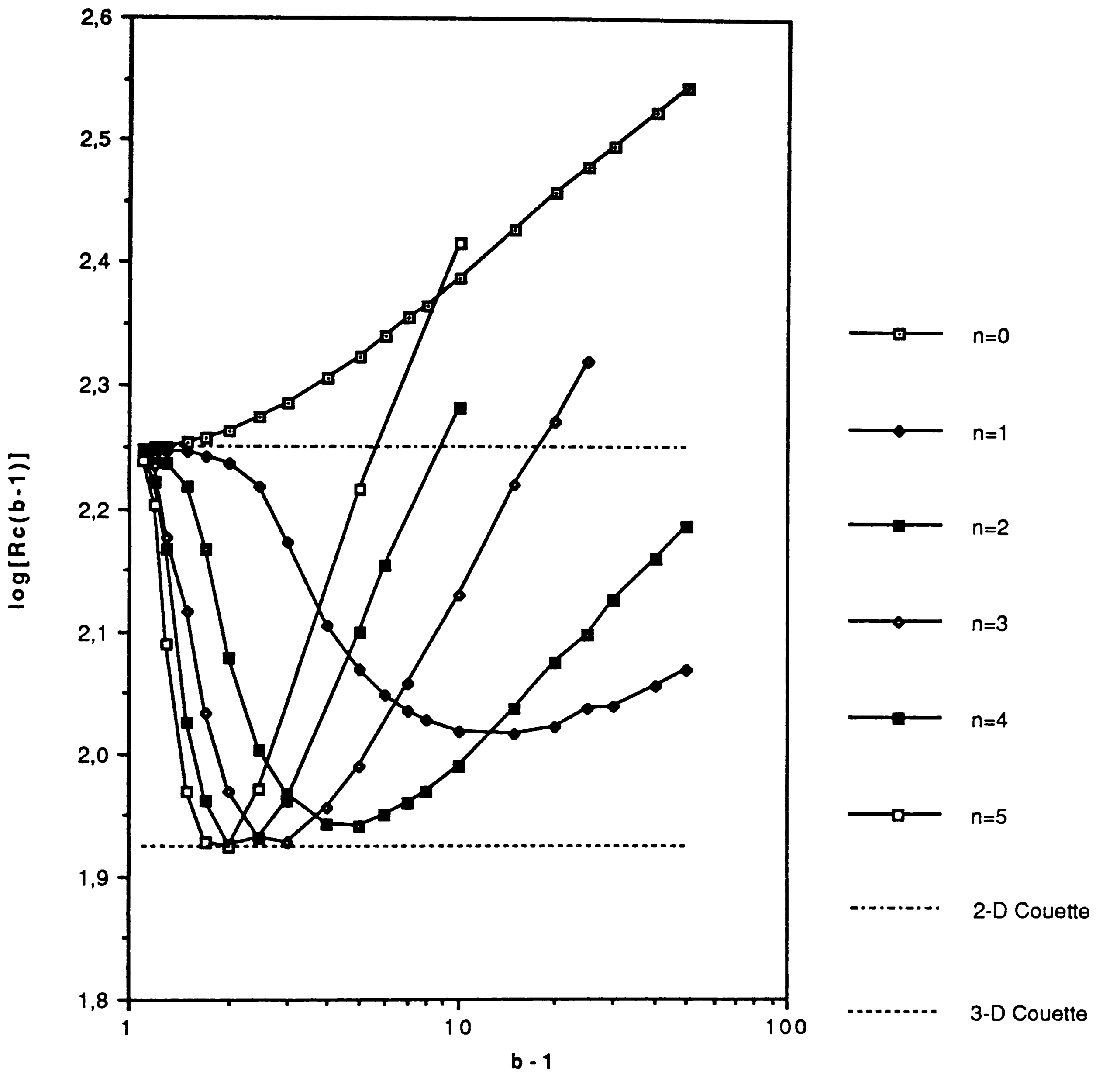

Figure 1 : Neutral curves for different values of n. As b tends to 1the critical Reynolds numbers tend to the corresponding ones of the plane Couette flow. The dar- ker curve shows the critical Reynolds number as a function of b

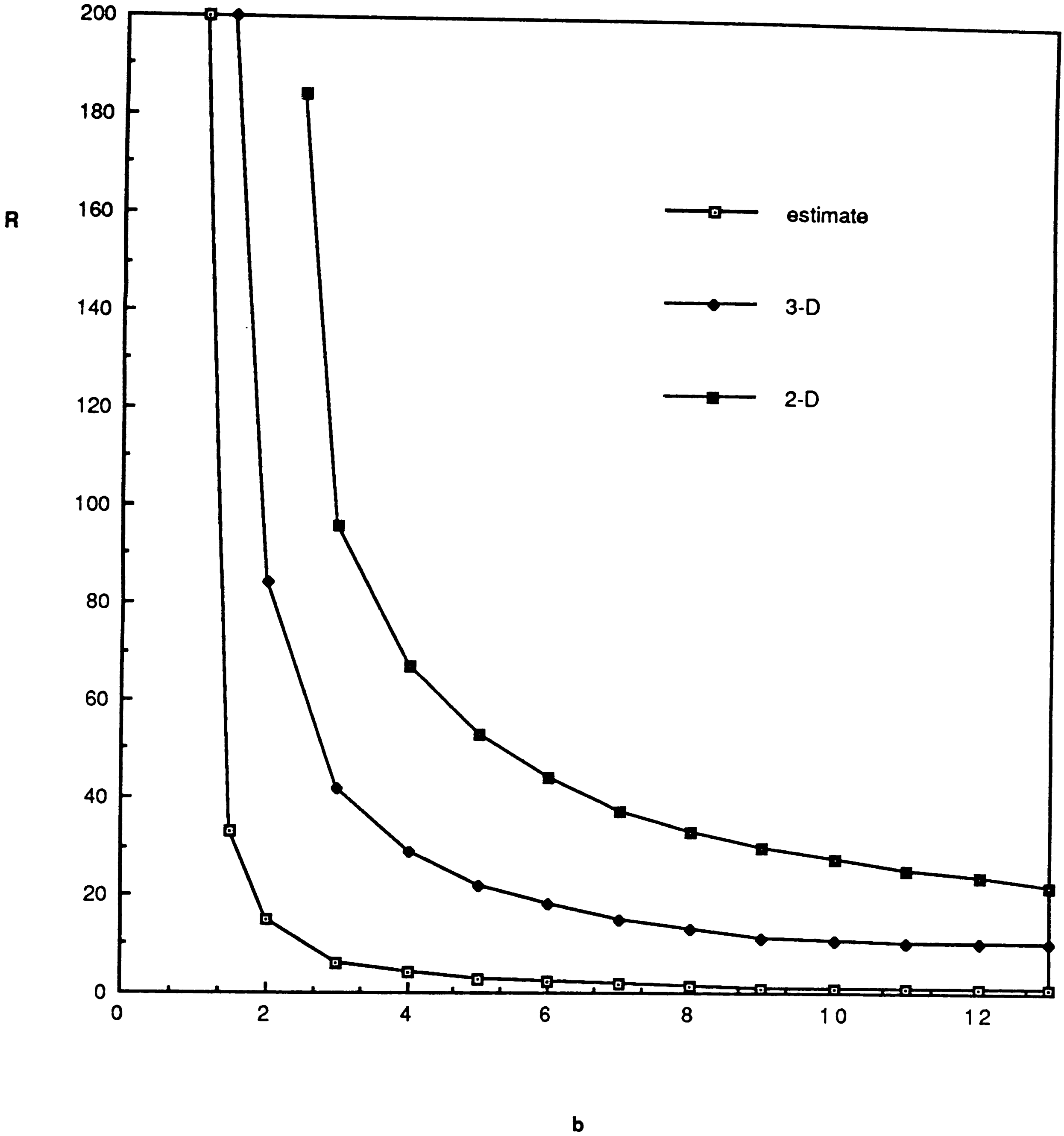

Figure 2: The flow is stable below the curve

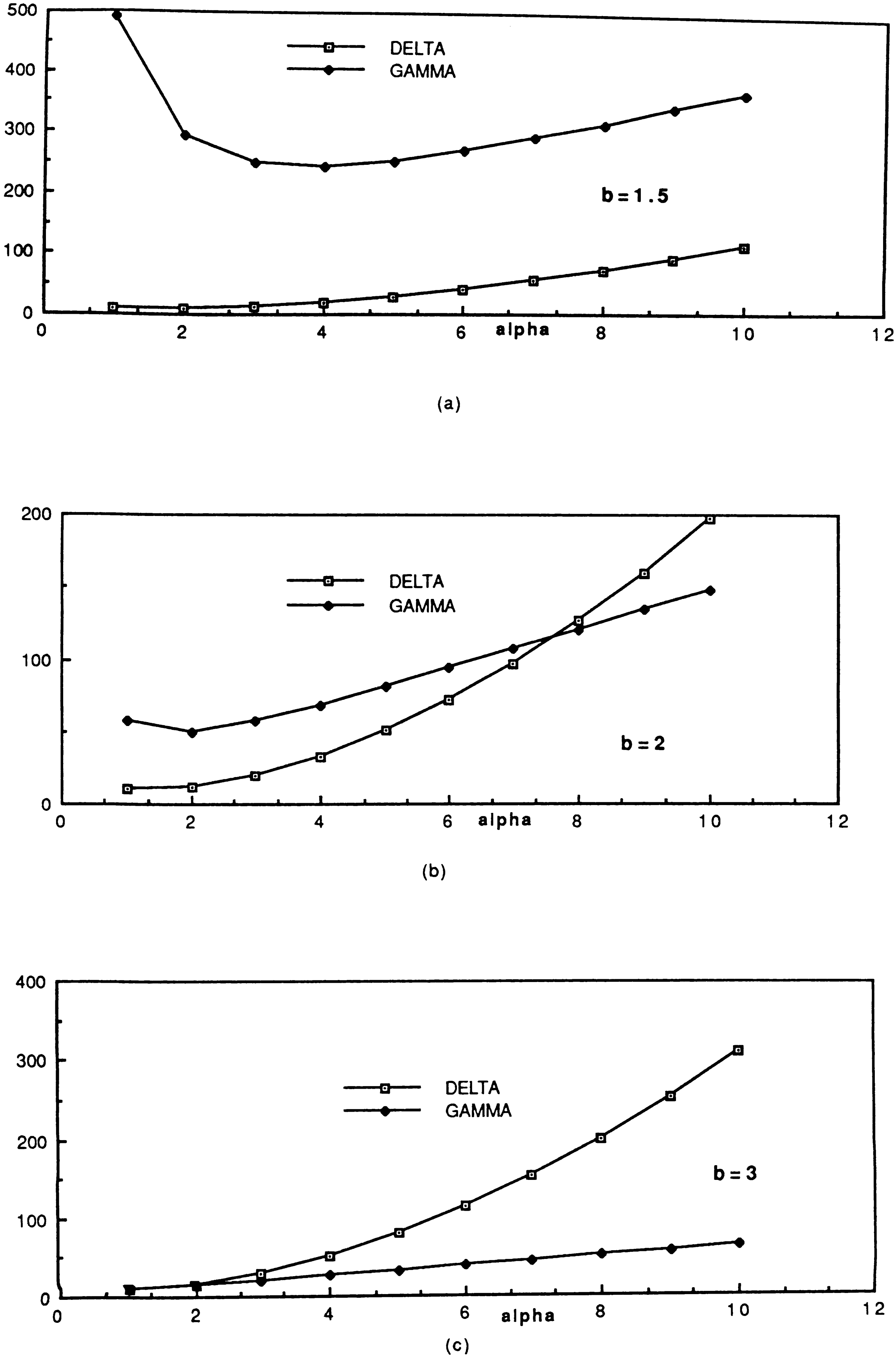

Figure 3 a,b,c. Gamma and delta vs. alpha for various values of b

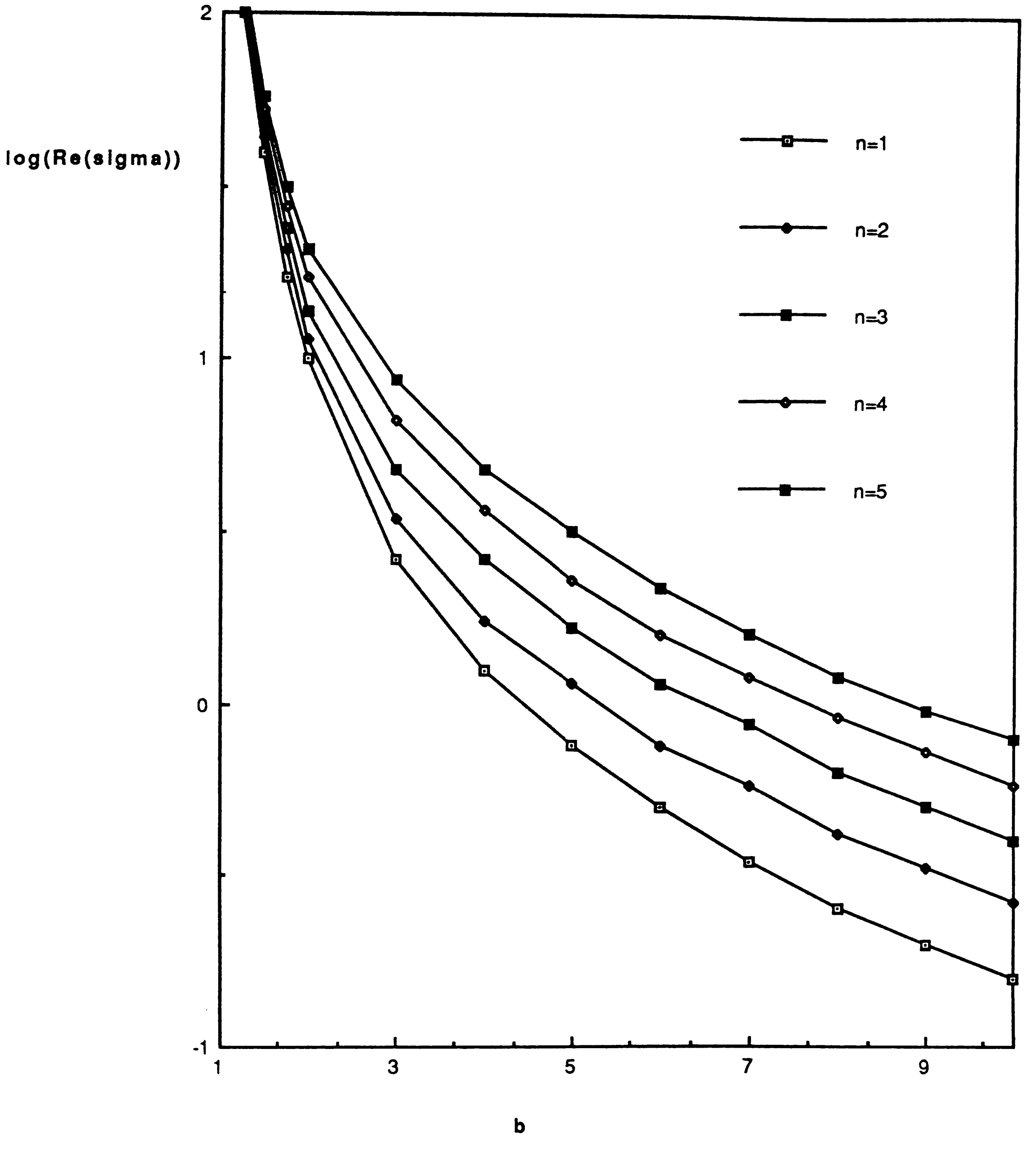

Figure 4: The critical eigenvalue vs. b at different values of n. The azimuthal mode (alpha=0) is always stable.

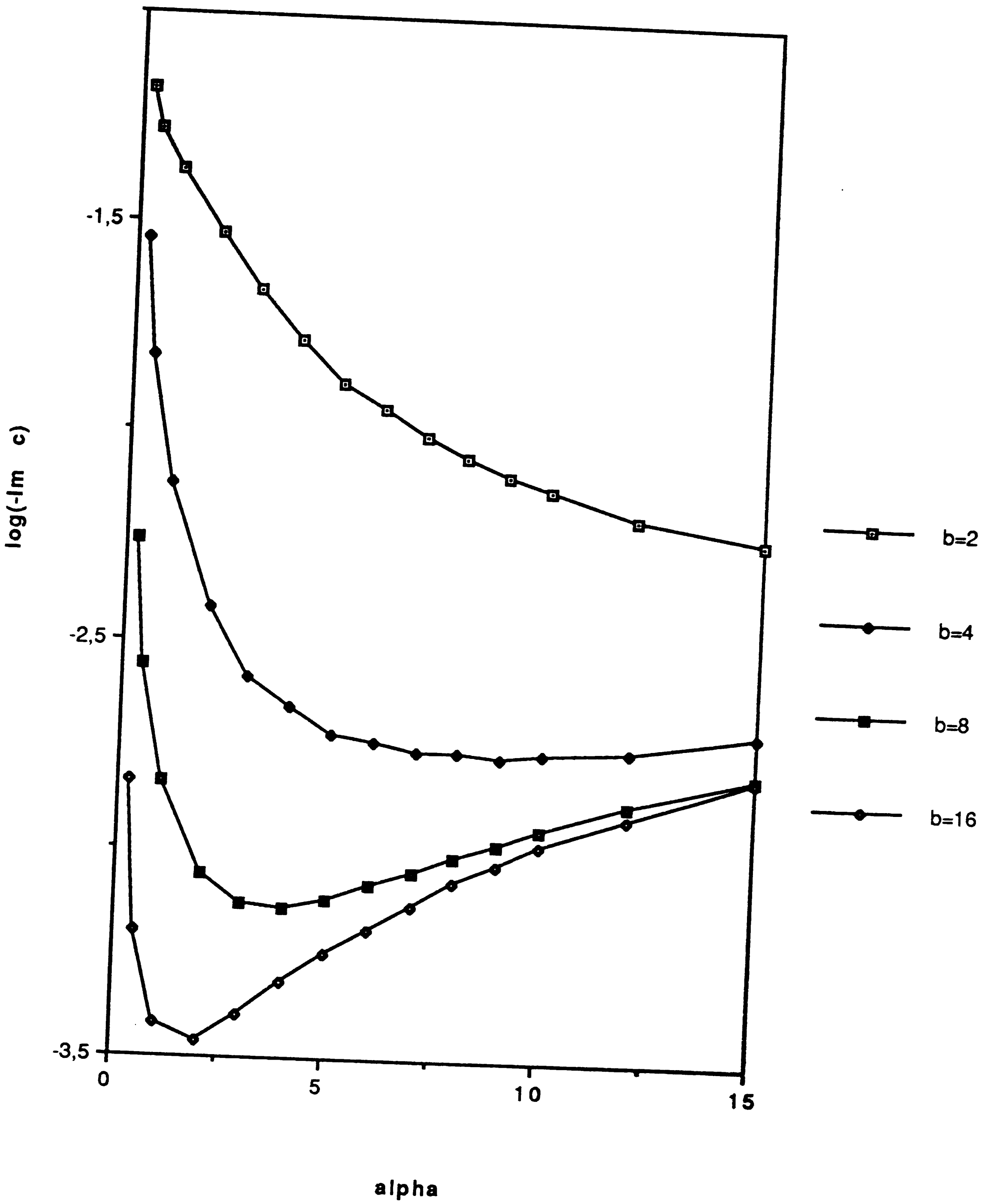

Figure 5: Im(c) vs. alpha for different values of b when R=10000 and n=0.

The axisymmetric mode is always stable