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# GLOBAL ATTRACTOR, INERTIAL MANIFOLDS AND STABILIZATION OF NONLINEAR DAMPED BEAM EQUATIONS

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**Abstract**

In this paper we use energy estimates to prove that a nonlinear damped beam equation, which is a model of an extensible elastic cantilever beam, has a compact global attractor and a flat inertial manifold, even though the spectral gap condition is not satisfied. An explicit bound for the dimension of the inertial manifold is obtained in terms of the physical parameters. As an application, it is shown that for any extensibility coefficient there exists a finite dimensional linear feedback control that achieves an exponential stabilization.

**Keywords.** Nonlinear beam equation, asymptotic dynamics, global attractor, inertial manifold, stabilization, hyperbolic evolution equation.

**AMS Subject Classification.** 35B40, 35G25, 35L70, 34C35, 73D35, 73H05.

## §1. Introduction

The objective of this paper is to show the existence of a global attractor and an inertial manifold for a weakly damped nonlinear hyperbolic evolution equation. This equation does not satisfy the spectral gap condition, which is a common difficulty in constructing inertial manifolds for hyperbolic equations.

In recent years it has been shown (cf. [6], [9-11], [22] and references therein) that for many parabolic dissipative nonlinear partial differential equations the long-time behavior of the solutions in a Hilbert space is completely determined by a finite dimensional inertial manifold, that is, a manifold which is positively invariant and exponentially attractive.

For hyperbolic evolution equations including nonlinear wave equations, the existence of a global attractor with finite Hausdorff and fractal dimensions has been studied under damping dissipation (cf. [11] and [22] and the references therein). A pioneering work on inertial manifolds in the hyperbolic case was [15], in which it was shown that if a spectral gap condition is satisfied, namely, the spectrum of the linearization has a suitably large gap comparable with the uniform Lipschitz constant of the nonlinearity, then the nonlinear wave equation considered has an inertial manifold. By a special renorming of the energy space, [15] proved that a suitably large damping coefficient ensures that the spectral gap condition holds. However, for many hyperbolic evolution equations with distributed or boundary damping, such a spectral gap condition fails to hold. Recently, [3] and [24] reported new existence results on inertial manifolds for elastic beams. In [3], a nonlinear inextensible strongly damped beam is shown to have a flat inertial manifold; in [24], a model similar to ours with hinged ends and an external load is shown to have an inertial manifold.

In this paper we consider a nonlinear damped elastic beam equation:

$$\frac{\partial^2 u}{\partial t^2} + \alpha \frac{\partial^4 u}{\partial x^4} + \delta \frac{\partial u}{\partial t} - (\beta + \int_0^1 u_x^2(\xi, t) d\xi) \frac{\partial^2 u}{\partial x^2} = 0, \tag{1.1}$$

where $u(x,t)$ is the transverse deflection of the beam, $\alpha$ and $\delta$ are positive constants, while $\beta \in \mathbf{R}$ is a not necessarily positive constant. This equation is a mathematical model for the small (transversal vibrations) of an extensible beam subject to an axial internal force. The nonlinear term in (1.1) represents the change of tension in the beam due to its extensibility.

This model equation was originally proposed by [23], and has been studied along with some analogous equations by [1], [4-5], [7-8], [11], [13-14], [18-21]. The existence of solutions was studied via Galerkin approximation and the energy decay of solutions was proved under a common condition that $\beta > -\nu_1$, where $\nu_1$ is the smallest eigenvalue of $\alpha A\varphi = \nu A^{1/2}\varphi$ with A being the fourth order spatial differential operator together with the homogeneous boundary conditions. For general $\beta$, [11] proved the existence of a global attractor for (1.1) under clamped boundary conditions by using a Lyapunov functional. Besides, there are no global results on the dynamics for general $\beta$ as far as we are aware. These previous works dealt only with clamped or hinged endpoints, which are qualitatively different from the cantilever boundary conditions below.

The cantilever assumption means that the left endpoint at x = 0 is clamped at rest and the right endpoint at x = 1 is free of transversal force and of bending torque. The boundary conditions are given by

$$u(0,t) = u_x(0,t) = u_{xx}(1,t) = u_{xxx}(1,t) = 0 \text{ for } t \geq 0. \tag{1.2}$$

As indicated in §2, this nonlinear damped beam system does not satisfy the spectral gap condition. However, we shall show that this system, for any real value of the parameter

$\beta$, has the absorbing property, a compact gloabal attractor, and a flat inertial manifold whose finite dimension can be bounded explicitly in terms of the physical parameters. The approach is via *a priori* estimates. The result on inertial manifolds is then applied to a control problem. For any real value of $\beta$, we show that there exists a finite dimensional linear feedback control that exponentially stabilizes the nonlinear beam.

We remark that the main results in this paper also hold for other homogeneous boundary conditions. The only difference lies in the characteristic equation for the distribution of the eigenvalues of the main linear operator $A$.

It is expected that the approach and results can be extended to nonlinear plate systems and can be adapted to deal with the global dynamics of elastic systems with boundary damping.

In §2, we formulate (1.1)-(1.2) as an abstract evolution in a Hilbert space, and calculate the spectrum of the major operator. In the remaining sections §3-§6, we prove the absorbing property, and the existence of a global attractor, an inertial manifold, and the feedback stabilization.

## §2. Abstract Evolution Equation

In this section we formulate the nonlinear damped beam equation (1.1) with the homogeneous boundary conditions (1.2) as an abstract nonlinear evolution equation and prove the local existence and uniqueness of solutions in the energy space.

The initial-boundary value problem is

$$u_{tt} + \alpha u_{xxx} + \delta u_t - (\beta + |u_x|^2)u_{xx} = 0, \quad t \geq 0, x \in (0,1)$$

$$u(0,t) = u_x(0,t) = u_{xx}(1,t) = u_{xxx}(1,t) = 0, \quad t \geq 0 \tag{2.1}$$

$$u(x,0) = u_0(x), u_t(x,0) = u_1(x), x \in [0,1],$$

where $|\cdot|$ denotes the norm of $L^2(0,1)$.

Define a linear operator $A$: $\mathcal{D}(A) \to H = L^2(0,1)$ by

$$\mathcal{D}(A) = \hat{H}^4(0,1) = \{\varphi \in H^4(0,1) : \varphi(0) = \varphi\prime(0) = \varphi\prime\prime(1) = \varphi\prime\prime\prime(1) = 0\}$$

(2.2) $$A\varphi = \frac{d^4\varphi}{dx^4}(\text{ the distributional derivative}).$$

Clearly $A$ is densely defined in H, and standard arguments show that $A$ is self-adjoint and positive definite and has a compact resolvent $A^{-1}$. The spectrum $\sigma(A)$ of $A$ consists of eigenvalues $\{\lambda_j\}_{j=1}^{\infty}$ given by $\lambda_j = \mu_j^4$, where $\{\mu_j > 0\}_{j=1}^{\infty}$ are the ordered positive roots of the following transcendental equation

(2.3) $$\cos\mu + \operatorname{sech}\mu = 0, \mu > 0.$$

Further, each eigenvalue $\lambda_j$ has multiplicity one.

A simple argument shows that the countable positive roots of the equation (2.3) are distributed pair by pair successively in the intervals $((2i+\frac{1}{2})\pi), (2i+\frac{3}{2})\pi), i = 0, 1, 2, \cdots$.

By considering the eigen-expansion with respect to the orthonormal basis consisting of the complete eigenvectors of $A$, it can be seen that

$$A^{1/2} = -\frac{d^2\varphi}{dx^2}, \mathcal{D}(A^{1/2}) = \text{closure of } \mathcal{D}(A)(= \hat{H}^4(0,1)) \text{ in } H^2(0,1).$$

$$|A^{1/4}\varphi| = |\frac{d\varphi}{dx}|, \mathcal{D}(A^{1/4}) = \text{closure of } \mathcal{D}(A)(= \hat{H}^4(0,1)) \text{ in } H^1(0,1).$$

Then the original system (2.1) can be formulated as a second-order nonlinear evolution equation:

(2.4) $$\frac{d^2u}{dt^2} + \alpha Au + \delta\frac{du}{dt} + (\beta + |A^{1/4}u|^2)A^{1/2}u = 0, t \geq 0,$$

$$u(0) = u_0, u_t(0) = u_1,$$

where $u(t)$ denotes the abstract-valued function $u(\cdot, t)$. Let $V = \mathcal{D}(A^{1/2})$ and E = V × H, where the norm of $V$ is given by $\|v\| = |A^{1/2}v|$. Note that this norm is equivalent to the norm of $H^2(0,1)$. We assume that

$$\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \in E. \tag{2.5}$$

Equation (2.4) can be formulated as a first-order nonlinear evolution equation as follows. Define a linear operator

$$\mathcal{A} = \begin{pmatrix} 0 & I_V \\ -A & -\delta I_H \end{pmatrix} : \mathcal{D}(\mathcal{A}) \to E, \text{with} \tag{2.6}$$

$$\mathcal{D}(\mathcal{A}) = \mathcal{D}(A) \times \ \mathcal{D}(A^{1/2}) = \mathcal{D}(A) \times V,$$

in which $I_V$ and $I_H$ are the identity operators on $V$ and $H$. Next, define a nonlinear mapping $f$ by

$$f(\varphi, \psi) = \begin{pmatrix} 0 \\ (\beta + |\varphi_x|^2)\varphi_{xx} \end{pmatrix} = \begin{pmatrix} 0 \\ -(\beta + |A^{1/4}\varphi|^2)A^{1/2}\varphi, \end{pmatrix} \tag{2.7}$$

where $f : E \to E$ and $f : \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{A})$. The equation (2.4) can now be written as

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \mathcal{A}\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} + f(u(t), v(t)), \quad t \geq 0. \tag{2.8}$$

$$\begin{pmatrix} u(0) \\ v(0) \end{pmatrix} = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \in \ E. \tag{2.9}$$

We consider the mild solution of (2.8) - (2.9) as the solution of the original system (2.1), based on the following fact.

**Lemma 2.2** The operator $\mathcal{A}$ generates a $C_o$-semigroup of contractions, denoted by $T(t), t \geq 0$, and $\mathcal{A}$ has compact resolvent $\mathcal{A}^{-1} \in \mathcal{L}(E)$. Besides, the spectrum $\sigma(\mathcal{A})$ is given by

(2.10) $$\sigma(\mathcal{A}) = \{\lambda = \frac{1}{2}[\delta \pm \sqrt{\delta^2 - 4\mu^4}] : \text{where } \mu > 0 \text{ is a root of equation (2.3)}\}.$$

**Proof.** The first part of this lemma is simply a routine consequence of the properties of the operator $A$ indicated above, together with the fact that

$$\mathcal{A} = \begin{pmatrix} 0 & I_V \\ -A & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -\delta I_H \end{pmatrix},$$

where the first operator with domain $\mathcal{D}(\mathcal{A})$ generates a unitary group and has compact resolvent, while the second operator is a dissipative bounded perturbation. That $\mathcal{A}$ is invertible follows from the fact that $\sigma(\mathcal{A})$ does not contain zero.

Regarding the spectrum, it is easy to verify that $\lambda \in \sigma(\mathcal{A}) = \sigma_p(\mathcal{A})$ if and only if $-(\delta\lambda + \lambda^2) \in \sigma(A) = \sigma_p(A)$, or

(2.11) $$\lambda^2 + \delta\lambda + \mu^4 = 0, \text{ for any } \mu > 0 \text{ being a root of(2.3)}.$$

Hence such $\lambda$'s are given by (2.10). ∎

**Corollary 2.3** There is a decomposition of $\sigma(\mathcal{A})$ as follows,

(2.12) $$\sigma(\mathcal{A} = \sigma_r(\mathcal{A}) + \sigma_c(\mathcal{A}),$$

where

(2.13)
$$\sigma_r(\mathcal{A}) = \{\lambda_{2i-1} = \frac{1}{2}[-\delta + \sqrt{\delta^2 - 4\mu_i^4}] \text{ and}$$
$$\lambda_{2i} = \frac{1}{2}[-\delta - \sqrt{\delta^2 - 4\mu_i^4}] : \text{ for } \mu_i \leq \sqrt{\frac{\delta}{2}}\},$$

and

$$(2.14) \qquad \sigma_c(\mathcal{A}) = \{\lambda_{2i-1} = \frac{1}{2}[-\delta + (4\mu_i^4 - \delta^2)^{1/2}\sqrt{-1}, \text{and}$$

$$\lambda_{2i} = \frac{1}{2}[-\delta - (4\mu_i^4 - \delta^2)^{1/2}\sqrt{-1}] : \text{ for } \mu_i > \sqrt{\delta/2}\}.$$

Moreover, $\sigma_r(\mathcal{A})$ is a finite set which may even be empty, and $\sigma_c(\mathcal{A})$ is a set of infinitely many complex numbers, all with the same real part $-\delta/2$. It follows that

$$(2.15) \qquad \text{Re } \sigma(\mathcal{A}) \subset (-\delta, 0).$$

These statements follow from (2.10).

Thus the mild solution of (2.8)-(2.9) is given by

$$(2.16) \qquad \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = T(t)\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} + \int_0^t T(t-s)f(u(s), v(s))ds, \quad t \geq 0,$$

for as long as it exists. The following lemma states the local existence and uniqueness of the mild solution of (2.8).

**Lemma 2.4** For every $\binom{u_0}{u_1} \in E$, there is a $\tau = \tau(u_0, u_1) > 0$ such that the mild solution of (2.8) - (2.9) exists uniquely on the interval $[0, \tau)$. Moreover, if $\binom{u_0}{u_1} \in \mathcal{D}(\mathcal{A})$, then the mild solution $\binom{u}{v} \in C([0, \tau); E)$ of (2.15) - (2.16) is a strong solution.

**Proof.** Note that the nonlinear term $f : E \to E$ is locally Lipschitz continuous; then the conclusions follow from [17], Theorems 6.1.4 and 6.1.6, respectively. ∎

In the next section we shall prove the global existence and the absorbing property of the solutions.

## §3. Absorbing Property of the Solution Semigroup

Suppose that the mild solution of the initial value problem (2.8) - (2.9) exists uniquely in $C([0, \infty); E)$ for every $\binom{u_0}{u_1} \in E$. We can denote the solution by a nonlinear mapping:

$$S(t) : \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \to \begin{pmatrix} u(t; u_0, u_1) \\ v(t; u_0, u_1) \end{pmatrix} \in E, \quad t \geq 0.$$

The nonlinear strongly continuous semigroup $\{S(t) : t \geq 0\}$ is called the *solution semigroup* associated with the system (2.8) - (2.9).

A nonlinear semigroup $\{S(t) : t \geq 0\}$ has the *absorbing property* if there exists a bounded subset $B_0$ in $E$, such that for any bounded set $B \subset E$, there is a positive constant $t_1$, uniform for $B$, such that $S(t)B \subset B_0$ for all $t \geq t_1$. The set $B_0$ is call an *absorbing set.*

We now prove the global existence of solutions and the absorbing property for the solution semigroup of (2.8) simultaneously by *a priori* estimates.

**Lemma 3.1** For a solution of (2.8) - (2.9), the following inequality holds for $t$ in the maximal interval of existence $(0, t_{max})$

$$|u_t|^2 + \alpha|A^{1/2}u(t)|^2 + \epsilon < u_t, u(t) > + \frac{1}{2}(|A^{1/4}u(t)|^2 + \beta)^2 \tag{3.1}$$

$$\leq e^{-\epsilon t/2}\{|u_1|^2 + \alpha|A^{1/2}u_0|^2 + \epsilon < u_1, u_0 > + \frac{1}{2}(|A^{1/4}u_0|^2 + \beta)^2\} +$$

$$+\beta^2,$$

where $\binom{u_0}{u_1} \in E$, and $\epsilon > 0$ is any constant satisfying

$$0 < \epsilon \leq \min\{1, \alpha\mu_1^4, 4\delta[3 + \frac{\delta^2}{\alpha\mu_1^4}]^{-1}\}, \tag{3.2}$$

where $\mu_1 > 0$ is the least positive root of the equation (2.3).

**Proof.** Let $\binom{u_0}{u_1} \in \mathcal{D}(\mathcal{A})$, so that the mild solution $\binom{u(t)}{v(t)}$ of the initial value problem (2.8) - (2.9) is actually a strong solution on $(0, t_{max})$. Thus, the first component $u(t)$ satisfies the second-order differential equation (2.4) almost everywhere on $(0, t_{max})$.

Let $\epsilon > 0$ be an arbitrary constant. By taking the inner product of (2.4) with $2u_t + \epsilon u$ in $H$, we obtain

$$\frac{d}{dt}\{|u_t|^2 + \alpha|A^{1/2}u|^2\} + 2\delta|u_t|^2 + (\beta + |A^{1/4}u|^2) < A^{1/2}u, 2u_t > \tag{3.3}$$

$$+\epsilon\{\frac{d}{dt} < u_t, u > -|u_t|^2 + \alpha|A^{1/2}u|^2 + \delta < u_t, u > +(\beta + |A^{1/4}u|^2)|A^{1/4}u|^2\}$$

$$= \frac{d}{dt}\{|u_t|^2 + \alpha|A^{1/2}u|^2 + \frac{1}{2}(\beta + |A^{1/4}u|^2)^2 + \epsilon < u_t, u >\}$$

$$+\{(2\delta - \epsilon)|u_t|^2 + \epsilon\alpha|A^{1/2}u|^2 + \epsilon\delta < u_t, u > +\epsilon|A^{1/4}u|^2(\beta + |A^{1/4}u|^2)\}$$

$$= \frac{d}{dt}\{|u_t|^2 + \alpha|A^{1/2}u|^2 + \frac{1}{2}(\beta + |A^{1/4}u|^2)^2 + \epsilon < u_t, u >\}$$

$$+\{(2\delta - \epsilon)|u_t|^2 + \epsilon\alpha|A^{1/2}u|^2 + \epsilon\delta < u_t, u > +\epsilon(|A^{1/4}u|^2 + \frac{\beta}{2})^2\} - \frac{\epsilon\beta^2}{4}$$

$$= 0,$$

for $t \in (0, t_{max})$. Let

$$N(t) = (2\delta - \epsilon)|u_t|^2 + \epsilon\alpha|A^{1/2}u|^2 + \epsilon\delta < u_t, u > +\epsilon(|A^{1/4}u|^2 + \frac{\beta}{2})^2. \tag{3.4}$$

Then we can write

$$N(t) = \epsilon[(\frac{2\delta - \epsilon}{\epsilon})|u_t|^2 + \alpha|A^{1/2}u|^2 + (\delta - \frac{\epsilon}{2}) < u_t, u > \tag{3.5}$$

$$+\frac{\epsilon}{2} < u_t, u > +(|A^{1/4}u|^2 + \frac{\beta}{2})^2]$$

$$\geq \epsilon[(\frac{2\delta - \epsilon}{\epsilon} - \frac{\delta - \epsilon/2}{2\eta})|u_t|^2 + \frac{\alpha}{2}|A^{1/2}u|^2 +$$

$$+\left(\frac{\alpha}{2}|A^{1/2}u|^2 - \frac{\eta}{2}(\delta - \frac{\epsilon}{2})|u|^2\right) + \frac{\epsilon}{2} < u_t, u > + \left(|A^{1/4}u|^2 + \frac{\beta}{2}\right)^2]$$

where $\eta > 0$ can be arbitrarily chosen. Take

$$\eta = \alpha\mu_1^4\delta^{-1}, \tag{3.6}$$

in which $\mu_1 > 0$ is the least positive root of the equation (2.3); then we have

$$N(t) \geq \epsilon\{\left[\frac{2\delta - \epsilon}{\epsilon} - \frac{2\delta - \epsilon}{4\alpha\mu_1^4\delta^{-1}}\right]|u_t|^2 + \frac{\alpha}{2}|A^{1/2}u|^2 + \tag{3.7}$$

$$+\frac{\epsilon}{2} < u_t, u > + \left(|A^{1/4}u|^2 + \frac{\beta}{2}\right)^2\}$$

since by our choice (3.6),

$$\frac{\alpha}{2}|A^{1/2}u|^2 - \frac{\eta}{2}\left(\delta - \frac{\epsilon}{2}\right)|u|^2 \geq \frac{1}{2}\left(\alpha\mu_1^4 - \eta\delta\right)|u|^2 = 0.$$

Then take $\epsilon > 0$ small enough, so that

$$0 < \epsilon < \min\left(1, \alpha\mu_1^4, 4\delta\left(3 + \frac{\delta^2}{\alpha\mu_1^4}\right)^{-1}\right). \tag{3.8}$$

It follows that

$$\frac{2\delta - \epsilon}{\epsilon} - \frac{2\delta - \epsilon}{4\alpha\mu_1^4\delta^{-1}} = \frac{2\delta}{\epsilon} - 1 - \frac{\delta^2}{2\alpha\mu_1^4} + \frac{\epsilon}{4\alpha\mu_1^4\delta^{-1}} \geq \frac{1}{2}. \tag{3.9}$$

Therefore, by (3.7) and (3.9), we obtain

$$N(t) \geq \frac{\epsilon}{2}\left(|u_t|^2 + \alpha|A^{1/2}u|^2 + \epsilon < u_t, u > +2\left(|A^{1/4}u|^2 + \frac{\beta}{2}\right)^2\right), \tag{3.10}$$

for $t \in [0, t_{max})$.

Furthermore, note that

$$2\left(|A^{1/4}u|^2 + \frac{\beta}{2}\right)^2 \geq \frac{1}{2}\left(|A^{1/4}u|^2 + \beta\right)^2 - \frac{\beta^2}{4}. \tag{3.11}$$

Let

$$G(t) = |u_t|^2 + \alpha|A^{1/2}u|^2 + \epsilon < u_t, u > +\frac{1}{2}(|A^{1/4}u|^2 + \beta)^2. \tag{3.12}$$

Then by (3.3), (3.10), and (3.11), it follows that

$$\frac{d}{dt}G(t) + \frac{\epsilon}{2}G(t) - \frac{\epsilon\beta^2}{8} - \frac{\epsilon\beta^2}{4} \leq \frac{d}{dt}G(t) + N(t) - \frac{\epsilon\beta^2}{4} = 0, \tag{3.13}$$

which implies that

$$\frac{d}{dt}G(t) + \frac{\epsilon}{2}G(t) \leq \frac{\epsilon\beta^2}{2}, t \in [0, t_{max}). \tag{3.14}$$

By integrating this differential inequality, we obtain

$$G(t) \leq e^{-\epsilon t/2}G(0) + \beta^2(1 - e^{\epsilon t/2}) \tag{3.15}$$

$$\leq e^{-\epsilon t/2}G(0) + \beta^2. \text{ for } t \in [0, t_{max}),$$

with $\epsilon > 0$ satisfying (3.8).

Now for any initial data $\binom{u_0}{u_1} \in E$, since $\mathcal{D}(\mathcal{A})$ is dense in $E$, there is a sequence $\left[\binom{u_0^n}{u_1^n}\right]_{n=1}^{\infty}$ in $\mathcal{D}(\mathcal{A})$ such that

$$\begin{pmatrix} u_0^n \\ u_1^n \end{pmatrix} \to \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \text{ in } E, \text{ as } n \to \infty.$$

Since we have shown that (3.15) or (3.1) holds for each $\binom{u_0^n}{u_1^n}$ in $\mathcal{D}(\mathcal{A})$, by the continuous dependence of the mild solution on the initial data, it follows that (3.1) remains valid for any $\binom{u_0}{u_1} \in E$. The proof is thus completed. ∎

**Theorem 3.2.** For any $\binom{u_0}{u_1} \in E$, there exists a unique global mild solution $\binom{u}{v} \in C([0,\infty); E)$ for the initial value problem (2.8) - (2.9). The solution semigroup $S(t), t \geq 0$ has an absorbing set $B_0$ in $E$.

**Proof.** By Lemma 3.1, for the choice of $\epsilon > 0$ given by (3.2) which does not depend on the initial data, inequality (3.1) holds:

$$G(t) \leq e^{-\epsilon t/2}G(0) + \beta^2, \text{ for } t \in [0, t_{max}),$$

where $G(t)$ is defined by (3.12). Note that due to the choice (3.2) for $\epsilon > 0$, we have

$$G(t) \geq \frac{1}{2}[|u_t|^2 + \alpha|A^{1/2}u|^2 + (\beta + |A^{1/4}u|^2)^2], \tag{3.16}$$

because $\epsilon < \min\{1, \alpha\mu_1^4\}$ implies that

$$\frac{1}{2}|u_t|^2 + \frac{1}{2}\alpha|A^{1/2}u|^2 + \epsilon < u_t, u >$$

$$\geq \frac{1}{2}|u_t|^2 + \frac{1}{2}\alpha|A^{1/2}u|^2 - \frac{\epsilon}{2}(|u_t|^2 + |u|^2) \geq 0.$$

Therefore, (3.1) implies that

$$\frac{min\{1,\alpha\}}{2}\|S(t)\begin{pmatrix}u_0\\u_1\end{pmatrix}\|_E^2 \le e^{-\epsilon t/2}G(0)+\beta^2, t\in[0,t_{max}). \tag{3.17}$$

According to [17; Theorem 6.1.4], this boundedness shows that the mild solution $S(t)\binom{u_0}{u_1}$ of the system (2.8) - (2.9) exists uniquely on $[0,\infty)$.

Moreover, (3.17) indicates that

$$\lim_{t\to 0}\sup\|S(t)\begin{pmatrix}u_0\\u_1\end{pmatrix}\|_E^2 \le \frac{2\beta^2}{\min\{1,\alpha\}}. \tag{3.18}$$

Let

$$\rho_0^2=\frac{2\beta^2}{\min\{1,\alpha\}}. \tag{3.19}$$

Then it is easily seen that the ball

$$B_0=\{e\in E:\|e\|<\sqrt{2}\rho_0\} \tag{3.20}$$

is an absorbing set, which completes the proof. ■

**Remark 3.1** Some previous work ([4], [5], and [18]) proved the global existence of solutions for the same nonlinear beam equation (1.1) with different homogeneous boundary conditions (hinged or clamped on both ends) and proved the exponential decay of the solutions under a common assumption that $\beta>-\nu_1$ where $\nu_1>0$ is the smallest eigenvalue of $\alpha A\varphi=\nu A^{1/2}\varphi$.

We emphasize that Theorem 3.2 in this paper has no conditions on the sign and magnitude of the parameter $\beta$ and that the cantilever beam does not seem to have an associated Lyapunov functional as in the problems discussed in [11]. As indicated by (3.19) and (3.20), $\beta^2$ plays a role in the absorbing set.

## §4. Global Attractor

Given a nonlinear semigroup $\{S(t)\}_{t\geq 0}$ on a Banach space $Z$, a set $\mathcal{T}$ is called a *global attractor* associated with $\{S(t)\}_{t\geq 0}$ if $(i)$ $\mathcal{T}$ is invariant under $\{S(t)\}_{t\geq 0}$ in the sense that $S(t)\mathcal{T} = \mathcal{T}$ for all $t \geq 0$, hence for all $t \in \mathbf{R}$; $(ii)$ $\mathcal{T}$ is compact and uniformly attracts every bounded subset of $Z$, i.e. $dist_Z(S(t)u_0, \mathcal{T}) \rightarrow, 0$ as $t \rightarrow \infty$, for any $u_0 \in Z$, and the convergence is uniform for $u_0$ in any given bounded subset.

We state a basic theorem on the existence of global attractor as the following lemma, which is one of two alternative versions in a general theorem whose proof is given in [22; Chap. I, Theorem 1.1].

**Lemma 4.1** Let $\{S(t)\}_{t\geq 0}$ be a strongly continuous (nonlinear) semigroup on a Banach space $Z$ satisfying the following condition,

$$S(t) = S_1(t) + S_2(t), \text{ for } t \geq 0, \text{ where} \tag{4.1}$$

$S_1(\cdot)$ is uniformly compact for $t$ large, and

$S_2(\cdot)$ is continuous from $Z$ into itself such that for every bounded subset $B \subset Z$,

$$\lim_{t\rightarrow\infty} \{\sup_{\varphi \in B} \|S_2(t)\varphi\|_Z\} = 0.$$

If there exists an absorbing set $B_0$ for $\{S(t)\}_{t\geq 0}$, then there exists a global attractor $\mathcal{T} = \omega(B_0)$ where $\omega(B_0)$ is the $\omega$-limit set of $B_0$.

Note that to say that $S_1(\cdot)$ is uniformly compact for $t$ large means:

(4.2) For every bounded subset $B \subset Z$, there exists a $\tau = \tau(B)$, such that $\bigcup_{t\geq\tau} S_1(t)B$ is precompact in $Z$

In the sequel, we consider a decomposition of the solution semigroup $\{S(t)\}_{t\geq 0}$ associated with the system (2.8) - (2.9) as follows:

$$S(t) = T(t) + U(t), t \geq 0, \tag{4.3}$$

where $\{T(t)\}_{t\geq 0}$ is the linear operator semigroup generated by the operator $\mathcal{A}$. We shall prove that $S_1(t) = U(t)$ and $S_2(t) = T(t)$ satisfy condition (4.1) in E.

**Lemma 4.2** For any bounded subset $B \subset E$, it holds that

$$\lim_{t\rightarrow\infty} \{\sup_{\binom{u_0}{u_1} \in B} \|T(t)\begin{pmatrix} u_0 \\ u_1 \end{pmatrix}\|_E\} = 0. \tag{4.4}$$

**Proof.** Let $\binom{u_0}{u_1} \in \mathcal{D}(\mathcal{A})$ first. Then $T(t)\binom{u_0}{u_1} = \binom{u(t)}{u_t(t)}$ is a strong solution of the linear equation

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \mathcal{A}\begin{pmatrix} u(t) \\ v(t) \end{pmatrix}, t \geq 0,$$

so that $u(\cdot)$ is a strong solution of the following second-order equation,

$$(4.5) \qquad u_{tt} + \alpha A u + \delta u_t = 0$$

$$u(0) = u_0, u_t(0) = u_1.$$

By taking inner product of (4.5) with $2u_t + \epsilon u$ and choosing $\epsilon > 0$ sufficiently small, as we did in steps (3.4) through (3.10), we can get

$$(4.6) \qquad \frac{d}{dt}[|u_t|^2 + \alpha|A^{1/2}u|^2 + \epsilon < u_t, u >]$$

$$+\frac{\epsilon}{2}[|u_t|^2 + \alpha|A^{1/2}u|^2 + \epsilon < u_t, \ u >] \leq 0, t \geq 0,$$

which in turn implies, similarly to (3.16) and with suitably small $\epsilon > 0$, that

$$(4.7) \qquad \frac{1}{2}(|u_t|^2 + \alpha|A^{1/2}u|^2) \leq |u_t|^2 + \alpha|A^{1/2}u|^2 + \epsilon < u_t, u >$$

$$\leq 2e^{-\epsilon t/2}(|u_1|^2 + \alpha|A^{1/2}u_0|^2),$$

for $t \geq 0$ and all $\binom{u_0}{u_1} \in E$ by the denseness of $\mathcal{D}(\mathcal{A})$ in $E$. This relation (4.7) directly shows that (4.4) holds. ∎

Next we shall deal with the family of operators $\{U(t)\}_{t\geq 0}$. We need a preparation lemma as follows.

**Lemma 4.3** Let $B \subset E$ be any given bounded set. For any $\binom{u_0}{u_1} \in B \cap \mathcal{D}(\mathcal{A})$, the mild solution $\binom{u(t)}{v(t)}$ of (2.8) has the following property,

$$\sup_{t \geq 0} |A^{3/4} u(t)| \leq K(B) < \infty, \tag{4.8}$$

where $K(B)$ is a uniform constant for $B$.

**Proof.** Without loss of generality, let $B = \{e \in E : \|e\| \leq r_0\}$ for some $r_0$. For any $\binom{u_0}{u_1} \in B \cap \mathcal{D}(\mathcal{A})$, we take the inner product of the equation (2.4) with $2A^{1/2}u_t + \epsilon A^{1/2}u$ to obtain

$$\begin{aligned}
&\tfrac{d}{dt}(|A^{1/4}u_t|^2 + \alpha|A^{3/4}u|^2) + 2\delta|A^{1/4}u_t|^2 \\
&+\tfrac{d}{dt}\epsilon < A^{1/4}u_t, A^{1/4}u > -\epsilon|A^{1/4}u_t|^2 + \epsilon\alpha|A^{3/4}u|^2 \\
&+\epsilon\delta < A^{1/4}u_t, A^{1/4}u > +(\beta + |A^{1/4}u|^2) < A^{1/2}u, 2A^{1/2}u_t + \epsilon A^{1/2}u > \\
&= \tfrac{d}{dt}\{|A^{1/4}u_t|^2 + \alpha|A^{3/4}u|^2 + \epsilon < A^{1/4}u_t, A^{1/4}u > \\
&+(\beta + |A^{1/4}u|^2)|A^{1/2}u|^2\} + \{(2\delta - \epsilon)|A^{1/4}u_t|^2 + \\
&+\epsilon\alpha|A^{3/4}u|^2 + \epsilon\delta < A^{1/4}u_t, A^{1/4}u > +\epsilon(\beta + |A^{1/4}u|^2)|A^{1/2}u|^2\} \\
&-|A^{1/2}u|^2 < A^{1/4}u, A^{1/4}u_t >= 0,
\end{aligned}$$

where the last term satisfies

$$\begin{aligned}
&|A^{1/2}u|^2| < A^{1/4}u, A^{1/4}u_t > | = |A^{1/2}u|^2| < A^{1/2}u, u_t > | \\
&\leq |A^{1/2}u|^3 \; |u_t| \leq c_1(B),
\end{aligned} \tag{4.10}$$

due to (3.17), where $c_1(B)$ is a positive constant depending only on B.

Following steps similar to (3.4) through (3.10),

$$M(t)=(2\delta-\epsilon)|A^{1/4}u_t|^2+\epsilon\alpha|A^{3/4}u|^2+\epsilon\delta < A^{1/4}u_t,A^{1/4}u >$$
$$+\ \epsilon(\beta+|A^{1/4}u|^2)|A^{1/2}u|^2$$
$$=\{\epsilon\left(\frac{2\delta-\epsilon}{\epsilon}\right)|A^{1/4}u_t|^2+\alpha|A^{3/4}u|^2+\left(\delta-\frac{\epsilon}{2}\right) < A^{1/4}u_t,A^{1/4}u >$$
$$+\ \frac{\epsilon}{2} < A^{1/4}u_t,A^{1/4}u > +(\beta+|A^{1/4}u|^2)|A^{1/2}u|^2\}$$
$$\geq\epsilon\left(\frac{2\delta-\epsilon}{\epsilon}-\frac{\delta-\epsilon/2}{2\eta}\right)|A^{1/4}u_t|^2+\frac{\alpha}{2}|A^{3/4}u|^2$$
$$+\ \frac{\epsilon}{2} < A^{1/4}u_t,A^{1/4}u > +\ (\frac{\alpha}{2}|A^{3/4}u|^2-\frac{\eta}{2}(\delta-\epsilon)|A^{1/4}u|^2$$
$$+(\beta+|A^{1/4}u|^2)|A^{1/2}u|^2\}.$$

By choosing $\eta=\alpha\mu_1^4\delta^{-1}$ and $\epsilon$ satisfying (3.8), from (4.11) it follows that

(4.12) $$M(t)\geq\frac{\epsilon}{2}\{|A^{1/4}u_t|^2+\alpha|A^{3/4}|^2+\epsilon < A^{1/4}u,A^{1/4}u_t > +$$
$$+\left(\beta+|A^{1/4}u|^2\right)|A^{1/2}u|^2\}-C_2(B),$$

where $C_2(B)$ is a positive constant given by

(4.13) $$C_2(B)=\frac{1}{2}[|\beta|+\frac{1}{\mu_1^2}\sup_{\binom{u_0}{u_1}\in B}\|S(t)\begin{pmatrix}u_0\\u_1\end{pmatrix}\|^2]\sup_{\binom{u_0}{u_1}\in B}\|S(t)\begin{pmatrix}u_0\\u_1\end{pmatrix}\|^2.$$

Then we use (4.9) and (4.12) to get

(4.14) $$\frac{d}{dt}\Pi(t)+\frac{\epsilon}{2}\Pi(t)\leq C_1(B)+C_2(B),t\geq 0,$$

where

(4.15) $$\Pi(t)=|A^{1/4}u_t|^2+\alpha|A^{3/4}u|^2+\epsilon < A^{1/4}u,A^{1/4}u_t > +$$

$$+(\beta+|A^{1/4}u|^2)|A^{1/2}u|^2.$$

It follows that

$$\Pi(t) \leq e^{-\epsilon t/2}\Pi(0) + C_1(B) + C_2(B), \tag{4.16}$$

and consequently

$$\alpha|A^{3/4}u(t)|^2 \leq 2[C_1(B) + 3C_2(B) + C_3(B)], \text{ for } t \geq 0, \tag{4.17}$$

where

$$C_3(B) = \sup_{\binom{u_0}{u_1} \in B} |\Pi(0)|, \tag{4.18}$$

and in the last step we have used the facts that

$$|\epsilon < A^{1/4}u, A^{1/4}u_t > | \leq \frac{1}{2}\left(|A^{1/4}u|^2 + |A^{1/4}u_t|^2\right),$$

and

$$|\left(\beta + |A^{1/4}u|^2\right)|A^{1/2}u|^2| \leq 2C_2(B).$$

Finally, let $K(B) = 2[C_1(B) + 3C_2(B) + C_3(B)]$; then the inequality (4.8) holds and the proof is completed. ∎

**Lemma 4.4.** In the decomposition (4.3) of the solution semigroup $\{S(t)\}_{t\geq 0}$, the family of operators $\{U(t)\}_{t\geq 0}$ has the property of uniform compactness for $t$ large, stated in (4.1).

**Proof.** Denote by

$$\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = S(t)\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \text{ and } \begin{pmatrix} \tilde{u}(t) \\ \tilde{v}(t) \end{pmatrix} = U(t)\begin{pmatrix} u_0 \\ u_1 \end{pmatrix},$$

since $U(t) = S(t) - T(t)$, the function $\binom{\tilde{u}}{\tilde{v}}$ is the mild solution of the following initial value problem:

$$\frac{d}{dt}\begin{pmatrix}\tilde{u}\\ \tilde{v}\end{pmatrix} = \mathcal{A}\begin{pmatrix}\tilde{u}(t)\\ \tilde{v}(t)\end{pmatrix} + f(u(t), v(t)), t \geq 0,$$

$$\begin{pmatrix}\tilde{u}(0)\\ \tilde{v}(0)\end{pmatrix} = 0, \tag{4.19}$$

or

$$\begin{pmatrix}\tilde{u}(t)\\ \tilde{v}(t)\end{pmatrix} = \int_0^t T(t-s)f(u(s), v(s))ds, t \geq 0. \tag{4.20}$$

For $\binom{u_0}{u_1} \in \mathcal{D}(\mathcal{A})$, the first component function $\tilde{u}$ is a strong solution of the following equation,

$$\tilde{u}_{tt} + \alpha A\tilde{u} + \delta\tilde{u}_t + (\beta + |A^{1/4}u|^2)A^{1/2}u = 0, t \geq 0, \tag{4.21}$$

$$\tilde{u}(0) = \tilde{u}_t(0) = 0.$$

Taking the inner product of the equation (4.21) with $2A^{1/2}\tilde{u}_t + \epsilon A^{1/2}\tilde{u}$, we get

$$\begin{aligned}
&\frac{d}{dt}(|A^{1/4}\tilde{u}_t|^2 + \alpha|A^{3/4}\tilde{u}|^2) + 2\delta|A^{1/4}\tilde{u}_t|^2\\
&+ \epsilon\frac{d}{dt} < \tilde{u}_t, A^{1/2}\tilde{u} > -\epsilon|A^{1/4}\tilde{u}_t|^2 + \epsilon\alpha|A^{3/4}\tilde{u}|^2\\
&+ \epsilon\delta < A^{1/4}\tilde{u}_t, A^{1/4}\tilde{u} > +(\beta + |A^{1/4}u|^2) < A^{1/2}u, 2A^{1/2}\tilde{u}_t + \epsilon A^{1/2}\tilde{u} >\\
&= \frac{d}{dt}\{|A^{1/4}\tilde{u}_t|^2 + \alpha|A^{3/4}\tilde{u}|^2 + \epsilon < A^{1/4}\tilde{u}_t, A^{1/4}\tilde{u} >\}+\\
&+ \{(2\delta - \epsilon)|A^{1/4}\tilde{u}_t|^2 + \epsilon\alpha|A^{3/4}\tilde{u}|^2 + 2\delta < A^{1/4}\tilde{u}_t, A^{1/4}\tilde{u} >\}\\
&+ (\beta + |A^{1/4}u|^2)(2 < A^{3/4}u, A^{1/4}\tilde{u}_t > +\epsilon < A^{1/2}u, A^{1/2}\tilde{u} >) = 0.
\end{aligned} \tag{4.22}$$

According to Lemma 4.3 and the absorbing property, we have shown that, for any given bounded set $B \subset E$ and any $\binom{u_0}{u_1} \in B \cap \mathcal{D}(\mathcal{A})$, there are positive constants $K_1(B)$ and $K_2(B)$ which depend only on $B$, such that

(4.23) $$|(\beta+|A^{1/4}u|^2)|A^{3/4}u|| \le K_1(B),$$

$$|(\beta+|A^{1/4}u|^2)|A^{1/2}u|| \le K_2(B).$$

Hence, if follows that

$$(\beta+|A^{1/4}u|^2)(2 < A^{3/4}u, A^{1/4}\tilde{u}_t > +\epsilon < A^{1/2}u, A^{1/2}\tilde{u} >)$$

(4.24) $$\ge -2K_1(B)|A^{1/4}\tilde{u}_t| - \epsilon K_2(B)|A^{1/2}\tilde{u}|$$

$$\ge -\frac{|K_1(B)|^2}{\eta} - \frac{\epsilon^2|K_2(B)|^2}{\eta} - \eta|A^{1/4}\tilde{u}_t|^2 - \epsilon\eta|A^{1/2}\tilde{u}|^2,$$

where $\eta > 0$ can be arbitrarily chosen. Therefore, we have

(4.25)
$$\begin{aligned}
&\frac{d}{dt}\{|A^{1/4}\tilde{u}_t|^2 + \alpha|A^{3/4}\tilde{u}|^2 + \epsilon < A^{1/4}\tilde{u}_t, A^{1/4}\tilde{u} >\}+\\
&+\epsilon\{[\frac{\delta-\epsilon}{\epsilon} - \frac{\delta-\epsilon/2}{2\gamma}]|A^{1/4}\tilde{u}_t|^2 + \frac{\alpha}{2}|A^{3/4}\tilde{u}|^2 + \frac{\epsilon}{2} < A^{1/4}\tilde{u}_t, A^{1/4}\tilde{u} >\}+\\
&+\epsilon[\frac{\alpha}{4}|A^{3/4}\tilde{u}|^2 - \frac{\gamma}{2}(\delta-\epsilon/2)|A^{1/4}\tilde{u}|^2]\\
&+\{(\delta-\eta)|A^{1/4}\tilde{u}_t|^2 + \epsilon[\frac{\alpha}{4}|A^{3/4}\tilde{u}|^2 - \eta|A^{1/2}\tilde{u}|^2]\}\\
&\le \eta^{-1}[|K_1(B)|^2 + \epsilon^2|K_2(B)|^2], t \ge 0,
\end{aligned}$$

where $\epsilon, \gamma$ and $\eta > 0$ can be arbitrarily chosen. Thus we can choose

$$0 < \eta \le \min\{\delta, \frac{\alpha}{4}\mu_1^2\},$$

(4.26) $$\gamma = \frac{1}{2}\delta^{-1}\alpha\mu_1^4, \text{and}$$

$$0 < \epsilon \le \min\{1, \alpha\mu_1^4, \delta[\frac{3}{2} + \frac{\delta^2}{\alpha\mu_1^4}]^{-1}\}.$$

As a consequence, (4.25) combined with (4.26) leads to

$$\frac{d}{dt}\{|A^{1/4}\tilde{u}_t|^2 + \alpha|A^{3/4}\tilde{u}|^2 + \epsilon < A^{1/4}\tilde{u}_t, A^{1/4}\tilde{u} >\}+ \tag{4.27}$$

$$+\frac{\epsilon}{2}\{|A^{1/4}\tilde{u}_t|^2 + \alpha|A^{3/4}\tilde{u}|^2 + \epsilon < A^{1/4}\tilde{u}_t, A^{1/4}\tilde{u} >\}$$

$$\leq \eta^{-1}[|K_1(B)|^2 + \epsilon^2|K_2(B)|^2], t \geq 0,$$

which implies, in turn, that

$$\frac{1}{2}|A^{1/4}\tilde{u}_t|^2 + \frac{\alpha}{2}|A^{3/4}\tilde{u}|^2 \leq \tag{4.28}$$

$$\leq |A^{1/4}\tilde{u}_t|^2 + \alpha|A^{3/4}\tilde{u}|^2 + \epsilon < A^{1/4}\tilde{u}_t, A^{1/4}\tilde{u} >$$

$$\leq 2\epsilon^{-1}\eta^{-1}[|K_1(B)|^2 + \epsilon^2|K_2(B)|^2], t \geq 0,$$

where we used the fact that $\tilde{u}_t(0) = \tilde{u}(0) = o$. (4.28) means that for any given bounded set $B \subset E$ and $\binom{u_0}{u_1} \in B \cap \mathcal{D}(\mathcal{A})$,

$$\|\begin{pmatrix}\tilde{u}\\ \tilde{u}_t\end{pmatrix}\|^2_{\mathcal{D}(A^{3/4})\times\mathcal{D}(A^{1/4})} \leq K_3(B), t \leq 0, \tag{4.29}$$

where

$$K_3(B) = \frac{4}{\epsilon\eta\min\{1,\alpha\}}[|K_1(B)|^2 + \epsilon^2|K_2(B)|^2]. \tag{4.30}$$

Finally, for any $\binom{u_0}{u_1} \in B$, since $f(u(t), v(t))$ is Lipschitz continuous, by [17; Theorem 4.2.9], we know that $\binom{\tilde{u}(t)}{\tilde{v}(t)}$ given by (4.20) satisfies

$$\begin{pmatrix}\tilde{u}(t)\\ \tilde{v}(t)\end{pmatrix} \in \mathcal{D}(\mathcal{A}) \subset \mathcal{D}(A^{3/4}) \times \mathcal{D}(A^{1/4}).$$

On the other hand, the denseness of $\mathcal{D}(\mathcal{A})$ in $\mathcal{D}(A^{3/4}) \times \mathcal{D}(A^{1/4})$ together with (4.29) implies that

$$(4.31) \qquad \left\| \begin{pmatrix} \tilde{u}(t) \\ \tilde{v}(t) \end{pmatrix} \right\|^2_{\mathcal{D}(A^{3/4}) \times \mathcal{D}(A^{1/4})} \leq K_3(B), t \geq 0, \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \in B.$$

By the Rellich theorem or the Sobolev imbedding theorem, $\mathcal{D}(A^{3/4}) \times \mathcal{D}(A^{3/4})$ is compactly imbedded in the space E. Hence (4.31) shows that $\cup_{t>0} U(t)B$ is precompact in $E$.

Therefore, the proof of this lemma is completed. ∎

**Theorem 4.5** The semigroup $\{S(t)\}_{t \geq 0}$ associated with (2.8) - (2.9) has a global attractor $\mathcal{T}$ which is compact and maximal in $E$.

**Proof.** Based on Lemmas 4.2 and 4.4, the decomposition $S(t) = T(t) + U(t)$, of the solution semigroup $\{S(t)\}_{t \geq 0}$ satisfies condition (4.1). Also, by Theorem 3.2, there exists an absorbing set $B_0$ in $E$ for this system. Thus by Lemma 4.1 there exists a global attractor $\mathcal{T}$. ∎

We could also estimate the uniform Lyapunov exponents to show that the global attractor has finite Hausdorff and fractal dimensions. However, since we aim to prove the existence of inertial manifolds, the dimension estimate for the global attractor will not be pursued here.

## §.5 Inertial Manifolds

In this section we shall prove the existence of an inertial manifold for the dynamical system $\{S(t)\}_{t \geq 0}$ associated with (2.8) - (2.9).

A Set $\mathcal{M} \subset E$ is called an *Inertial Manifold* (cf. [6],[9],[22]) of the dynamical system $\{S(t)\}_{t \geq 0}$ if

(i) $\mathcal{M}$ is a finite dimensional Lipschitz manifold;

(ii) $\mathcal{M}$ is positively invariant for $S(t)$, i.e. $S(t)\mathcal{M} \subset \mathcal{M}, t \geq 0$;

(iii) $\mathcal{M}$ attracts exponentially all the trajectories of (2.8) - (2.9), i.e. for any bounded subset $B$ of $E$, there is a constant C(B) such that

$$\text{dist}_E(S(t)\begin{pmatrix} u_0 \\ u_1 \end{pmatrix}, \mathcal{M}) \leq C(B)e^{-\nu t}, t \geq 0,$$

for any $\binom{u_0}{u_1} \in B$, where $\nu > 0$ is a uniform constant.

If an inertial manifold exists, it certainly contains the global attractor $\mathcal{T}$. The restriction of (2.8) - (2.9) to an inertial manifold becomes an initial value problem of an ordinary differential system which is called an *inertial form*.

The complete normalized eigenvectors $\{w_j\}_{j=1}^{\infty}$ of the operator $A : \mathcal{D}(A) \to H$ form an orthonormal basis for the real Hilbert space $H$, and also for $V = \mathcal{D}(A^{1/2})$ (with a different normalization). Their corresponding eigenvalues $\{\lambda_j = \mu_j^4\}_{j=1}^{\infty}$ are of multiplicity one and increasing.

Let

$$H_m = \text{Span}\{w_1, \cdots, w_m\} \tag{5.1}$$

where $\{w_1, \cdots, w_m\}$ are the first $m$ eigenvectors. Denote by $P_m : H \to H_m$ the orthogonal projection from $H$ onto $H_m$, and $Q_m = I_H - P_m$. In the product space $E = V \times H$, define

$$\Phi_m = \begin{pmatrix} P_m & 0 \\ 0 & P_m \end{pmatrix} : E \to H_m \times H_m, \text{ and}$$
$$\Psi_m = I - E - \Phi_m.$$

Based on the absorbing property proved in Theorem 3.2, in order to study the long-time behavior of solutions of the system (2.8) - (2.9), we can modify the equation (2.4) or the associated equation (2.8) by truncating the nonlinear term outside the absorbing ball. We briefly describe this truncation.

Let $\theta(r) : [0, \infty) \to [0, 1]$ be a fixed $C^1$-function, such that

$$\theta(r) = 1, \text{ for } 0 \le r \le 2,$$

$$\theta(r) = 0, \text{ for } 4 \le r < \infty \tag{5.2}$$

$$|\theta(r)| \le 1 \text{ and } |\theta\prime(r)| \le 1, \text{ for } r \ge 0$$

Let $\rho_0$ be given by (3.19). Define

$$\theta\rho_0(r) = \theta(r/\rho_0^2), r \ge 0, \tag{5.3}$$

and

$$F(u, v) = \theta\rho_0(\|\begin{pmatrix} u \\ v \end{pmatrix}\|^2) f(u, v), \tag{5.4}$$

where $f$ is defined by (2.7).

From now on we consider the modified equation

$$\frac{d}{dt}\binom{u}{v} = \mathcal{A}\binom{u}{v} + F(u(t), v(t)), t \geq 0, \tag{5.5}$$

and the corresponding second-order version:

$$\frac{d^2u}{dt^2} + \alpha A u + \delta\frac{du}{dt} + \theta\rho_0(\|\begin{pmatrix} u(t) \\ \frac{du}{dt}(t)\end{pmatrix}\|^2)(\beta + |A^{1/4}u|^2)A^{1/2}u = 0, \quad t \geq 0. \tag{5.6}$$

Define

$$g_u(t) = \theta\rho_0(\|\begin{pmatrix} u(t) \\ v(t)\end{pmatrix}\|^2)(\beta + |A^{1/4}u|^2); \tag{5.7}$$

Then $g_u$ is uniformly bounded for all trajectories $\binom{u(t)}{v(t)}$ of (5.5), and there exists a positive constant $d = d(\rho_0, \beta)$ such that

$$d = |\beta| + \frac{2\rho_0^2}{\mu_1^2}, \text{ and } |g_u(t)| \leq d, t \geq 0, \tag{5.8}$$

for all mild solutions of the equations (5.5).

Next, we consider an orthogonal decomposition

$$H = H_m + Q_m H \tag{5.9}$$

and the corresponding decomposition of the $H$-valued function $u(t) = p(t) + q(t)$, with $p(t) = P_m u(t)$ and $q(t) = Q_m u(t)$. Then the evolution equation (5.6) is decomposed as

$$\frac{d^2p}{dt^2} + \alpha A p + \delta\frac{dp}{dt} + g_u(t)A^{1/2}p = 0, \text{ in } H_m,$$

(5.10)

$$\frac{d^2 q}{dt^2} + \alpha A q + \delta \frac{dq}{dt} + g_u(t) A^{1/2} q = 0, \text{ in } Q_m H.$$

In the product space $E$, there is a corresponding orthogonal decomposition

$$E = (H_m \times H_m) + \Psi_m E.$$

Let $y(t) = \Phi_m \binom{u(t)}{v(t)}$ and $z(t) = \Psi_m \binom{u(t)}{v(t)}$; then (5.5) is decomposed as

$$\frac{d}{dt} y = \mathcal{A} y + \Phi_m F(u(t), v(t)), \text{ in } H_m \times H_m,$$

(5.12)

$$\frac{d}{dt} z = \mathcal{A} z + \Psi_m F(u(t), v(t)), \text{ in } \Psi_m E.$$

Obviously, if $\binom{u(t)}{v(t)} = y(t) + z(t)$ is a strong solution of (5.12), then $y(t) = \binom{p(t)}{\frac{dp}{dt}}$ and $z(t) = \binom{q(t)}{\frac{dq}{dt}}$ in which $(p(t), q(t))$ satisfies the differential equations (5.10).

We next prove the existence of an inertial manifold.

**Theorem 5.1** For a suitably large integer $m$, such that

(5.13) $$\mu_{m+1}^2 \geq \max\{\frac{4d}{\alpha}, \frac{8\rho_0^2}{\epsilon\alpha}, \frac{2|\beta|+1}{\alpha}\},$$

where

(5.14) $$\epsilon = \min\{\frac{1}{2}, 2\delta\left[3 + \frac{2\delta^2}{\alpha\mu_1^4}\right]^{-1}\},$$

$\mu_j$ is the j-th positive root of the transcendental equation (2.3), and the constant $d$ is given by (5.8), the flat manifold

(5.15) $$M_m = H_m \times H_m$$

is an inertial manifold in $E$ for the semigroup $\{S(t)\}_{t\geq 0}$ associated with (2.8) - (2.9).

**Proof.** First we see that $M_m$ given by (5.15) is a finite dimensional subspace in $E$, which of course can be regarded as a flat Lipschitz manifold.

Secondly, $M_m$ is positively invariant. In fact, for $\binom{u_0}{u_1} \in M_m \subset \mathcal{D}(\mathcal{A})$, the mild solution $\binom{u(t)}{v(t)}$ of the original system (2.8) is a strong solution, and the first component $u(t) = p(t) + q(t)$ satisfying the differential equation (2.4) with $q(0) = q_t(0)$ must be such that $q(t) \equiv 0, p(t)$ being the solution of

(5.16) $$p_{tt} + \alpha Ap + \delta p_t + (\beta + |A^{1/4}p|^2)A^{1/2}p = 0, t \geq 0,$$

$$\begin{pmatrix} p(0) \\ p_t(0) \end{pmatrix} = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \in H_m \times H_m,$$

due to uniqueness. Hence it turns out that for $\binom{u_0}{u_1} \in M_m$, the solution

$$S(t)\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \begin{pmatrix} p(t) \\ p_t(t) \end{pmatrix} \in M_m, \text{ for } t \geq 0.$$

Thirdly, we prove the exponential attraction property of $M_m$. Note that all the orbits of the original system (2.8) - (2.9) enter the absorbing ball $B_0$ at a uniform exponential rate $exp(-\epsilon t/2)$ with $\epsilon > 0$ satisfying (3.2). Therefore, we now investigate the behavior of the solutions within the absorbing ball $B_0$, where the trajectories are determined by the truncated equation (5.5) and its second-order version (5.6).

As usual we conduct *a priori* estimates for initial data $\binom{u_0}{u_1} \in \mathcal{D}(\mathcal{A})$, then extend the result to all $\binom{u_0}{u_1}$ in $E$. Taking the inner product of the second equation in (5.10) with $2q_{(t)} + 2\epsilon q$ in $H$, we get

(5.17) $$\frac{d}{dt}[|q_t|^2 + \alpha|A^{1/2}q|^2 + 2\epsilon < q_t, q > + g_u(t)|A^{1/4}q|^2]$$

$$+(2\delta - 2\epsilon)|q_t|^2 + 2\epsilon\alpha|A^{1/2}q|^2 + 2\epsilon\delta < q_t, q > + 2\epsilon g_u(t)|A^{1/4}q|^2$$

$$-\left(\frac{d}{dt}g_u(t)\right)|A^{1/4}q|^2 = 0,$$

in which, since within the absorbing ball $B_0, g_u(t) = (\beta + |A^{1/4}u|^2)$, we have

$$|\frac{d}{dt}g_u(t)| \leq 2|A^{1/2}u||u_t| \leq \|\begin{pmatrix} u \\ u_t \end{pmatrix}\|_E^2 \leq 2\rho_0^2.$$

Substitute the above fact into (5.16) and note that

$$|A^{1/2}q|^2 \geq \mu_{m+1}^2|A^{1/4}q|^2 \text{ and } |A^{1/2}q|^2 \geq \mu_{m+1}^4|q|^2, \text{ for } q \in Q_mH,$$

where $\mu_{m+1}$ is the $(m+1)$-th positive root of the equation (2.3); we then obtain

$$\frac{d}{dt}[|q_t|^2 + \alpha|A^{1/2}q|^2 + 2\epsilon < q_t, q > +(\beta + |A^{1/4}u|^2)|A^{1/4}q|^2]$$

(5.18) $$+\{(2\delta - 2\epsilon)|q_t|^2 + \frac{3}{2}\epsilon\alpha|A^{1/2}q|^2 + 2\epsilon\delta < q_t, q >$$

$$+\epsilon(\beta + |A^{1/4}u|^2)|A^{1/4}q|^2\} + \left[\frac{\epsilon\alpha}{2} - \frac{\epsilon d}{\mu_{m+1}^2} - \frac{2\rho_0^2}{\mu_{m+1}^2}\right]|A^{1/2}q|^2 \leq 0.$$

Observe that we can write the middle portion on the left-hand side of (5.18) as

(5.19) $$(2\delta - 2\epsilon)|q_t|^2 + \frac{3}{2}\epsilon\alpha|A^{1/2}q|^2 + 2\epsilon\delta < q_t, q > +\epsilon(\beta + |A^{1/4}u|^2)|A^{1/4}q|^2$$

$$= \epsilon\{\left[\frac{2\delta - 2\epsilon}{\epsilon} - \frac{2\delta - 2\epsilon}{2\eta}\right]|q_t|^2 + \alpha|A^{1/2}q|^2 + 2\epsilon < q_t, q >$$

$$+(\beta + |A^{1/4}u|^2)|A^{1/4}q|^2\} + \epsilon[\frac{\alpha}{2}|A^{1/2}q|^2 - \frac{\eta}{2}(2\delta - 2\epsilon)|q|^2].$$

By taking the values of the undetermined constants $\eta > 0$ and $\epsilon > 0$ to be

$$(5.20) \qquad \eta = \frac{\alpha \mu_1^4}{2\delta},$$

$$(5.21) \qquad \epsilon = \min\{\frac{1}{2}, 2\delta \left[3 + \frac{2\delta^2}{\alpha \mu_1^4}\right]^{-1}\}$$

we obtain from (5.19) that

$$(5.22) \qquad (2\delta - 2\epsilon)|q_t|^2 + \frac{3}{2}\epsilon\alpha|A^{1/2}q|^2 + 2\epsilon\delta < q_t, q > +\epsilon(\beta + |A^{1/4}u|^2)|A^{1/4}q|^2$$

$$\geq \epsilon\left[|q_t|^2 + \alpha|A^{1/2}q|^2 + 2\epsilon < q_t, q > +(\beta + |A^{1/4}u|^2)|A^{1/4}q|^2\right],$$

because

$$\frac{2\delta - 2\epsilon}{\epsilon} - \frac{2\delta - 2\epsilon}{2\eta} \geq 1$$

and

$$\frac{\alpha}{2}|A^{1/2}q|^2 - \frac{\eta}{2}(2\delta - 2\epsilon)|q|^2 \geq 0.$$

Let

$$(5.23) \qquad L(t) = |q_t|^2 + \alpha|A^{1/2}q|^2 + 2\epsilon < q_t, q > +(\beta + |A^{1/4}u|^2)|A^{1/4}q|^2.$$

Let the integer $m$ be such that

$$(5.24) \qquad \mu_{m+1}^2 \geq \max\{\frac{4d}{\alpha}, \frac{8\rho_o^2}{\epsilon\alpha}\}$$

where $\epsilon > 0$ is given by (5.21). Then it follows that

$$\frac{\epsilon\alpha}{2} - \frac{\epsilon d}{\mu_{m+1}^2} - \frac{2\rho_0^2}{\mu_{m+1}^2} \geq \frac{\epsilon\alpha}{4} - \frac{2\rho_0^2}{\mu_{m+1}^2} \geq 0. \tag{5.25}$$

From (5.18), (5.22), (5.23), and (5.25) we obtain

$$\frac{d}{dt}L(t) + \epsilon L(t) \leq 0, t \geq 0,$$

so that

$$L(t) \leq L(0)e^{-\epsilon t}, t \geq 0. \tag{5.26}$$

Note that

$$L(t) \geq \frac{1}{2}|q_t|^2 + \frac{\alpha}{2}|A^{1/2}q|^2 + (\frac{1}{2} - \epsilon)|q_t|^2 +$$

$$+ \left[\frac{1}{2}\alpha - \frac{\epsilon + |\beta|}{\mu_{m+1}^2}\right] |A^{1/2}q|^2 \tag{5.27}$$

$$\geq \frac{1}{2}|q_t|^2 + \frac{\alpha}{2}|A^{1/2}q|^2,$$

provided that $m$ is large enough to satisfy

$$\mu_{m+1}^2 \geq \frac{2(|\beta| + \epsilon)}{\alpha}. \tag{5.28}$$

Therefore, we end up with

$$\frac{1}{2}\left[|q_t|^2 + \alpha|A^{1/2}q|^2\right] \le L(t) \le L(0)e^{-\epsilon t}$$

(5.29)
$$\le \left[2(1+\alpha)\|\begin{pmatrix} q(0) \\ q_t(0) \end{pmatrix}\|_E^2 + d|A^{1/4}q(0)|^2\right] e^{-\epsilon t}$$

$$\le [2(1+\alpha)+d]\|\begin{pmatrix} q(0) \\ q_t(0) \end{pmatrix}\|_E^2 e^{-\epsilon t}, t \ge 0,$$

where $m > 0$ is sufficiently large so that

(5.30)
$$\mu_{m+1}^2 \ge \max\{\frac{4d}{\alpha}, \frac{8\rho_0^2}{\epsilon\alpha}, \frac{2|\beta|+1}{\alpha}\}$$

with $d, \rho_0$, and $\epsilon$ given by (5.8), (3.19), and (5.21) respectively. By the denseness of $\mathcal{D}(\mathcal{A})$ in E, inequality (5.29) remains valid for any initial data $\binom{u_0}{u_1} \in E$, under assumption (5.13).

Finally, by combining the inequalities (3.17) and (3.21) as the first stage and inequality (5.29) as the second stage, we conclude that there exists a uniform constant rate,

(5.31)
$$\nu = \min\left[\frac{1}{2}, \frac{1}{2}\alpha\mu_1^{\frac{1}{4}}, 2\delta\left[3 + \frac{2\delta^2}{\alpha\mu_1^{\frac{1}{4}}}\right]^{-1}\right] > 0$$

(cf. (3.2) and (5.21)), such that for any initial data $\binom{u_0}{u_1}$ in $E$, there is a constant $\rho(u_0, u_1) >$ 0 such that the solution of the original system (2.8) - (2.9) satisfies

(5.32)
$$\text{dist}_E(S(t)\begin{pmatrix} u_0 \\ u_1 \end{pmatrix}, M_m) \le \rho(u_0, u_1)e^{-\nu t}, t \ge 0.$$

Moreover, from (3.21) and (5.29) it can be seen that for $\binom{u_0}{u_1}$ in any given bounded subset $B$ of $E$, the constant $\rho(u_0, u_1)$ can be made uniform and depends only on $B$. Thus the exponential attractivity of $M_m$ is proved. The entire proof is completed. ■

We can estimate the dimension $m$ of the inertial manifold more explicitly as follows:

**Corollary 5.2** If $i$ is a nonnegative integer satisfying

$$(5.33) \qquad (2i-\frac{1}{2})^2\pi^2 \le \frac{1}{\alpha}\max\left[4|\beta|+1+\frac{8\rho_0^2}{\mu_1^2}, 8\rho_0^2\max\left[2,\frac{3}{2\delta}+\frac{\delta}{\alpha\mu_1^4}\right]\right]$$

$$< (2i+\frac{1}{2})^2\pi^2,$$

then (2.8) - (2.9) has an inertial manifold $M_m$ given by (5.15) of dimension $m = 2i$.

If $i$ is a nonnegative integer satisfying

$$(5.34) \qquad (2i+\frac{1}{2})^2\pi^2 \le \frac{1}{\alpha}\max\left[4|\beta|+1+\frac{8\rho_0^2}{\mu_1^2}, 8\rho_0^2\max\left[2,\frac{3}{2\delta}+\frac{\delta}{\alpha\mu_1^4}\right]\right]$$

$$< (2i+\frac{3}{2})^2\pi^2,$$

then there exists an inertial manifold $M_m$ given by (5.15) of dimension $m = 2i+1$.

**Proof** Note that we have shown that the positive roots of the transcendental equation (2.3) are distributed in such a way that successive pairs lie in the intervals

$$\left((2i+\frac{1}{2})\pi, (2i+\frac{3}{2})\pi\right), i = 0, 1, 2, \cdots.$$

Then (5.13) and (5.14) lead to the two possibilities for the dimension of the inertial manifold. ∎

We remark that the estimate of the dimension of the inertial manifold in Corollary 5.2 has been made directly in terms of the physical parameters $\alpha, \beta$ and $\delta$.

**Corollary 5.3** For the inertial manifold $M_m = H_m \times H_m$, the inertial form of the system (2.8) - (2.9) is

$$(5.35) \qquad \frac{d^2p}{dt^2} + \alpha Ap + \delta\frac{dp}{dt} + (\beta + |A^{1/4}p|^2)A^{1/2}p = 0, t \ge 0, \text{ in } H_m,$$

$$p(0) = p_0 \in H_m, p_t(0) = p_1 \in H_m.$$

Let $p(t)$ be denoted by $p(t) =$ col $(p_1(t)), \cdots, p_m(t))$ in the sense that

$$p(t) = \sum_{i=1}^{m} p_i(t) w_i,$$

then (5.35) is an ordinary differential system with the matrix $A =$ diag$(\lambda_1, \cdots, \lambda_m)$.

**Proof.** Note that on the inertial manifold $M_m = H_m \times H_m$, the $Q_m H$-component $q(t) \equiv 0$, and that the solutions $u(t)$ of (2.8) are always strong solutions, we thus obtain the inertial form as shown by (5.35). ∎

## §6. Application: Exponential Stabilization by Finite Dimensional Control

In this section we will give an application of the result on the existence of inertial manifolds for the system (2.8) or (2.4) to a control problem.

As we mentioned in §1 and §3 (Remark 3.1), for a negative $\beta$ with suitably large magnitude, the asymptotic behavior of solutions for the equation (2.4) was not known. In this connection, the possibility of stabilizing (2.4) by a feedback control was an open issue. Specifically, consider the evolution equation (2.4) with a control $h(t) \in L_{loc}([0, \infty); H)$, i.e.

$$\frac{d^2u}{dt^2} + \alpha A u + \delta \frac{du}{dt} + (\beta + |A^{1/4}u|^2) A^{1/2} u = h(t), t \geq 0 \tag{6.1}$$

$$u(0) = u_0, u_t(0) = u_1$$

The stabilization problem can be stated as follows: if there is a linear or nonlinear feedback operator $J : E \to H$, such that the feedback control

$$h = J(u, u_t) \tag{6.2}$$

makes the closed-loop system stable and all the solutions $(u, u_t)$ of the closed-loop system

$$\frac{d^2u}{dt^2} + \alpha A u + \delta \frac{du}{dt} + (\beta + |A^{1/4}u|^2) A^{1/2} u = J(u, u_t), t \geq 0, \tag{6.3}$$

converge to zero in E as $t \to \infty$ then the controlled system (6.1) is said to be strongly stabilized by the feedback (6.2). If, in addition, the convergence of the closed-loop solution $(u, u_t)$ occurs at a uniform exponential rate, then the controlled system (6.1) is said to be exponentially stabilized by the feedback (6.2).

Based on the existence of inertial manifolds shown above, we shall prove the exponential stabilizability of the controlled system (6.1) for all values of the parameter $\beta$ to provide an affirmative answer to this open problem.

**Theorem 6.1** The controlled system (6.1) can be exponentially stabilized by a finite dimensional linear feedback control

$$h(t) = \beta A^{1/2} P_m u(t), t \geq 0, \tag{6.4}$$

where $P_m : H \to H_m$ is the orthogonal projection, and $M_m = H_m \times H_m$ is an inertial manifold for the system (2.4) or (2.8).

**Proof.** Using this feedback control, for the given $H_m$ determined such that $M_m = H_m \times H_m$ is an inertial manifold for (2.4) or (2.8), we have a decomposition of the closed-loop equation as follows:

$$\frac{d^2p}{dt^2} + \alpha Ap + \delta\frac{dp}{dt} + |A^{1/4}u|^2 A^{1/2}p = 0,$$
$$\frac{d^2q}{dt^2} + \alpha Aq + \delta\frac{dq}{st} + (\beta + |A^{1/4}u|^2) A^{1/2}q = 0,$$

where $u = p + q$ is the closed-loop state function.

Since we have a cancellation of $h(t)$ with $\beta A^{1/2} P_m u(t)$ on the left-hand side of the equation (2.11), an easy adaptation in the proof of Lemma 3.1 and Theorem 3.2 shows that the absorbing property remains valid and that the same $B_0$ given in (3.20) remains an absorbing set for the closed-loop system (6.5).

After truncation as in §5, by the same argument we know that Theorem 5.1 holds with the same condition (5.13) - (5.14) and the same constant $d$ defined by (5.8). More precisely, it holds that for any $t_0 \geq 0$,

$$\|\begin{pmatrix} q(t) \\ q_t(t) \end{pmatrix}\|_E^2 \leq c_q(u_0, u_1) e^{-\epsilon(t-t_0)}, t \geq t_0, \tag{6.6}$$

where $q(\cdot)$ is the mild solution of the second equation in (6.5), and $c_q(u_0, u_1)$ is a positive constant depending on the initial data and $\epsilon > 0$ is chosen in the same manner as in Theorem 5.1.

On the other hand, due to the finite dimensional feedback control (6.4), the mild solution of the first equation in (6.5),

$$\frac{d^2p}{dt^2} + \alpha Ap + \delta\frac{dp}{dt} + |A^{1/4}u|^2 A^{1/2}p = 0, \tag{6.7}$$

$$p(0) = p_0 \in H_m, p_t(0) = p_1 \in H_m$$

also decays exponentially. In fact, by taking the inner product of the equation (6.7) with $2p_t + \tilde{\epsilon}p$, in $H_m$, where $\tilde{\epsilon} > 0$ can be arbitrarily chosen, we get

$$\begin{aligned}&\frac{d}{dt}\{|p_t|^2 + \alpha|A^{1/2}p|^2 + \tilde{\epsilon} < p_t, p > + \frac{1}{2}|A^{1/4}p|^4 + \\ &+ |A^{1/4}q|^2|A^{1/4}p|^2\} + \{(2\delta - \tilde{\epsilon})|p_t|^2 + \tilde{\epsilon}\alpha|A^{1/2}p|^2 + \tilde{\epsilon}\delta < p_t, p > \\ &+ \tilde{\epsilon}|A^{1/2}u|^2|A^{1/4}p|^2\} \leq 2|A^{1/4}p|^2| < A^{1/2}q, q_t > |\end{aligned} \tag{6.8}$$

where an integration by parts is used in the term $2|A^{1/4}q|^2 < A^{1/2}p, p_t >$. Choose $\tilde{\epsilon} > 0$ satisfying (3.2), so that one can apply a similar argument as in Lemma 3.1. It turns out that

$$\begin{aligned}&\frac{d}{dt}\{|p_t|^2 + \alpha|A^{1/2}p|^2 + \epsilon < p_t, P > + \\ &+ \frac{1}{2}|A^{1/4}p|^4 + |A^{1/4}q|^2|A^{1/4}p|^2\} + \\ &+ \frac{\epsilon}{2}\{|p_t|^2 + \alpha|A^{1/2}p|^2 + \epsilon < p_t, p > + \frac{1}{2}|A^{1/4}p|^4 + \\ &+ |A^{1/4}q|^2|A^{1/4}p|^2\} \\ &\leq 2d|A^{1/2}q| \quad |q_t| \leq d\|\begin{pmatrix} q \\ q_t \end{pmatrix}\|_E^2 \leq dc_q(u(t_0), u_t(t_0))e^{-1\epsilon}(t - t_0), \;\; for\ t \geq t_0,\end{aligned} \tag{6.9}$$

where $t_0$ is the time for the trajectory to enter the absorbing ball $B_o$ and $c_q$ is a constant depending on $u(t_0)$ and $u_t(t_0)$. Let

$$R(t) = |p_t|^2 + \alpha|A^{1/2}p|^2 + \epsilon < p_t, p > + \tag{6.10}$$

$$+\frac{1}{2}|A^{1/4}p|^4 + |A^{1/4}q|^2|A^{1/4}p|^2.$$

Then (6.9) can be written as

$$\frac{d}{dt}R(t) + \frac{\epsilon}{2}R(t) \le dc_q(u(t_0), u_t(t_0))e^{-\epsilon(t-t_0)}, \text{ for } t \ge t_0. \tag{6.11}$$

Solve this differential inequality to obtain

$$R(t) \le e^{-\epsilon(t-t_0)/2}R(t_0) + \frac{2d}{\epsilon}c_q(u(t_0), u_t(t_0))e^{-\epsilon(t-t_0)/2}, \tag{6.12}$$

Note that

$$R(t) \ge \frac{\min\{1,\alpha\}}{2}\|\begin{pmatrix} p(t) \\ p_t(t) \end{pmatrix}\|_E^2, \tag{6.13}$$

and

$$R(t_0) \le 2(1+\alpha)\|\begin{pmatrix} u(t_0) \\ u_t(t_0) \end{pmatrix}\|^2 + \frac{1}{\mu_1^4}\|\begin{pmatrix} u(t_0) \\ u_t(t_0) \end{pmatrix}\|^4. \tag{6.14}$$

If we restrict $\binom{u(t_0)}{u_t(t_0)}$ to the absorbing ball $B_0$, then

$$R(t_0) \le c_1(\rho_0), \tag{6.15}$$

so that we have

$$\|\begin{pmatrix} p(t) \\ p_t(t) \end{pmatrix}\|_E^2 \le \frac{2}{\min\{1,\alpha\}}[c_1(\rho_0) + c_2(\rho_0)]e^{-\epsilon(t-t_0)/2}, \tag{6.16}$$

where $t \geq t_0$, and

$$c_2(\rho_0) = \frac{2d}{\epsilon} \sup_{(u_0,u_1) \in B_0} c_q(u_o, u_1). \tag{6.17}$$

Finally, by combining the absorbing property with (6.6) and (6.16), we conclude that (6.4) is a finite dimensional feedback control which exponentially stabilizes the original system (2.4) or (2.8). The proof is completed. ∎

A new feature in this theorem is that it indicates how to stabilize a nonlinear damped beam system by a finite dimensional feedback controller by first establishing the existence of an inertial manifold. This approach can also be useful in other stabilization problems, and we will discuss such applications more extensively elsewhere. This result shows a direct connection between two very interesting developments in the theory of infinite dimensional dynamical systems: inertial manifolds and the stabilization of control systems via finite dimensional controllers.

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## References

[1] J.M. Ball, Initial-boundary value problems for an extensible beam, J. Math. Anal. Appl., **42** (1973), 671-90.

[2] A.M. Bloch, Analysis of a rotating flexible system, Acta Appl. Mathematicae, **15** (1989), 211-234.

[3] A.M. Bloch and E.S. Titi, On the dynamics of rotating elastic beams, to appear in Proceedings of the Conference on New Trends in Systems Theory, 1990, Genoa, Italy, G. Conti et al, edit., Birkhauser.

[4] P. Biler, Remarks on the decay for damped string and beam equations, Nonlinear Analysis, **10** (1986), 839-842.

[5] E.H. De Brito, Nonlinear initial-boundary value problems, Nonlinear Analysis, **11** (1987), 124-137.

[6] P. Constantin, C. Foias, B. Nicolaenko and R. Temam, Integral Manifolds and Inertial Manifolds for Dissipative Partial Differential Equations, Springer-Verlag, New York, 1989.

[7] R.W. Dickey, Free vibrations and dynamic buckling of extensible beams, J. Math. Anal. Appl., **29**(1970), 443-454.

[8] J.G. Eisley, Nonlinear vibrations of beams and rectangular plates, Z. Angew. Math, Phys., **15** (1964), 167-175.

[9] C. Foias, G.R. Sell and R. Temam, Inertial Manifolds for nonlinear evolutionary equations, J.Diff. Eqns., **73** (1988), 309-353.

[10] C. Foias, G.R. Sell and E.S. Titi, Exponential tracking and approximation of inertial manifolds for dissipative equations, J. Dynamics and Diff. Eqns. **1** (1989), 199-224.

[11] J. Hale, Asymptotic Behavior of Dissipative Systems, Math. Surveys and Monographs, Vol. 25, AMS, Providence, RI, 1988.

[12] D.B. Henry, Geometric Theory of Semilinear Parabolic Equations, Lect. Notes in Math., No. 840, Springer-Verlag, New York, 1981.

[13] L.A. Medeiros, On a new class of nonlinear wave equations, J. Math. Anal. Appl., **69** (1979), 252-262.

[14] G.P. Menzala, On global classical solutions of a nonlinear wave equation, Applicable Analysis, **10** (1980), 179-195.

[15] X. Mora, Finite-dimensional attracting invariant manifolds for damped semilinear wave equations, in Contributions to Nonlinear Partial Differential Equations, Vol. II, J.I. Diaz and P.L. Lions, eds. Pitman Research Notes on Math, Vol. 155, (1983), 172-183.

[16] B. Nicolaenko, Inertial manifolds for models of compressible gas dynamics, Cont. Math., AMS 99, (1987), 165-179.

[17] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Springer-Verlag, New York, 1983.

[18] D.C. Pereira, Existence, uniqueness, and asymptotic behavior for solutions of the nonlinear beam equation, Nonlinear Analysis, **14** (1990), 613-623.

[19] P.H. Rivera, On a nonlinear hyperbolic equation in a Hilbert space, Anais. Acad. Bras. Ciências, **50** (1978), 133-135.

[20] E.L. Reiss and B.J. Matkowsky, Nonlinear dynamic buckling of a compressed elastic column, Quart. Appl. Math., **29** (1971), 245-260.

[21] Y. Shibata and M. Kikuchi, On the mixed problem for some quasi-linear hyperbolic systems with fully nonlinear boundary conditions, J. Diff. Eqns., **80** (1989), 154-197.

[22] R. Temam, Infinite Dimensional Dynamical Systems in Mechanics and Physics, Springer-Verlag, New York, 1988.

[23] W. Woinowsky-Krieger, The effect of axial force on the vibration of hinged bars, J. Appl. Mech., **17** (1950), 35-56.

[24] I.D. Chueshov, Inertial manifold in the problem of nonlinear vibrations of an infinite panel, Ukrainian Math. J. **42**, 9, (1990), 1291-1293.

[25] J.M. Ghidaglia and A. Marzocchi, Long time behavior of strongly damped wave equations, global attractors and their dimensions, to appear in SIAM J. Math. Analysis.

[26] D. Sevcovic, Existence and limiting behavior for damped nonlinear evolution equations with nonlocal terms, Comm. Math. Univ. Carolinae 31 (1990), 283-293.

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| 791 | **Chao–Nien Chen**, On the uniqueness of solutions of some second order differential equations |
| 792 | **Xinfu Chen and Avner Friedman**, The thermistor problem for conductivity which vanishes at large temperature |
| 793 | **Xinfu Chen and Avner Friedman**, The thermistor problem with one-zero conductivity |
| 794 | **E.G. Kalnins and W. Miller, Jr.**, Separation of variables for the Dirac equation in Kerr Newman space time |
| 795 | **E. Knobloch, M.R.E. Proctor and N.O. Weiss**, Finite-dimensional description of doubly diffusive convection |
| 796 | **V.V. Pukhnachov**, Mathematical model of natural convection under low gravity |
| 797 | **M.C. Knaap**, Existence and non-existence for quasi-linear elliptic equations with the p-laplacian involving critical Sobolev exponents |
| 798 | **Stathis Filippas and Wenxiong Liu**, On the blowup of multidimensional semilinear heat equations |
| 799 | **A.M. Meirmanov**, The Stefan problem with surface tension in the three dimensional case with spherical symmetry: non-existence of the classical solution |
| 800 | **Bo Guan and Joel Spruck**, Interior gradient estimates for solutions of prescribed curvature equations of parabolic type |
| 801 | **Hi Jun Choe**, Regularity for solutions of nonlinear variational inequalities with gradient constraints |
| 802 | **Peter Shi and Yongzhi Xu**, Quasistatic linear thermoelasticity on the unit disk |
| 803 | **Satyanad Kichenassamy and Peter J. Olver**, Existence and non-existence of solitary wave solutions to higher order model evolution equations |
| 804 | **Dening Li**, Regularity of solutions for a two-phase degenerate Stefan Problem |
| 805 | **Marek Fila, Bernhard Kawohl and Howard A. Levine**, Quenching for quasilinear equations |
| 806 | **Yoshikazu Giga, Shun'ichi Goto and Hitoshi Ishii**, Global existence of weak solutions for interface equations coupled with diffusion equations |
| 807 | **Mark J. Friedman and Eusebius J. Doedel**, Computational methods for global analysis of homoclinic and hetero clinic orbits: a case study |
| 808 | **Mark J. Friedman**, Numerical analysis and accurate computation of heteroclinic orbits in the case of center manifolds |
| 809 | **Peter W. Bates and Songmu Zheng**, Inertial manifolds and inertial sets for the phase-field equations |
| 810 | **J. López Gómez, V. Márquez and N. Wolanski**, Global behavior of positive solutions to a semilinear equation with a nonlinear flux condition |
| 811 | **Xinfu Chen and Fahuai Yi**, Regularity of the free boundary of a continuous casting problem |
| 812 | **Eden, A., Foias, C., Nicolaenko, B. and Temam, R.**, Inertial sets for dissipative evolution equations Part I: Construction and applications |
| 813 | **Jose–Francisco Rodrigues and Boris Zaltzman**, On classical solutions of the two-phase steady-state Stefan problem in strips |
| 814 | **Viorel Barbu and Srdjan Stojanovic**, Controlling the free boundary of elliptic variational inequalities on a variable domain |
| 815 | **Viorel Barbu and Srdjan Stojanovic**, A variational approach to a free boundary problem arising in electro-photography |
| 816 | **B.H. Gilding and R. Kersner**, Diffusion-convection-reaction, free boundaries, and an integral equation |

817 **Shoshana Kamin, Lambertus A. Peletier and Juan Luis Vazquez**, On the Barenblatt equation of elasto-plastic filtration
818 **Avner Friedman and Bei Hu**, The Stefan problem with kinetic condition at the free boundary
819 **M.A. Grinfeld**, The stress driven instabilities in crystals: mathematical models and physical manifestations
820 **Bei Hu and Lihe Wang**, A free boundary problem arising in electrophotography: solutions with connected toner region
821 **Yongzhi Xu, T. Craig Poling, and Trent Brundage**, Direct and inverse scattering of time harmonic acoustic waves in an inhomogeneous shallow ocean
822 **Steven J. Altschuler**, Singularities of the curve shrinking flow for space curves
823 **Steven J. Altschuler and Matthew A. Grayson**, Shortening space curves and flow through singularities
824 **Tong Li**, On the Riemann problem of a combustion model
825 **L.A. Peletier & W.C. Troy**, Self-similar solutions for diffusion in semiconductors
826 **C.J. van Duijn, L.A. Peletier & R.J. Schotting**, On the analysis of brine transport in porous media
827 **Minkyu Kwak**, Finite dimensional description of convective reaction-diffusion equations
828 **Minkyu Kwak**, Finite dimensional inertial forms for the 2D Navier–Stokes equations
829 **Victor A. Galaktionov and Sergey A. Posashkov**, On some monotonicity in time properties for a quasilinear parabolic equation with source
830 **Victor A. Galaktionov**, Remark on the fast diffusion equation in a ball
831 **Hi Jun Choe and Lihe Wang**, A regularity theory for degenerate vector valued variational inequalities
832 **Vladimir I. Oliker and Nina N. Uraltseva**, Evolution of nonparametric surfaces with speed depending on curvature, II. The mean curvature case.
833 **S. Kamin and W. Liu**, Large time behavior of a nonlinear diffusion equation with a source
834 **Shoshana Kamin and Juan Luis Vazquez**, Singular solutions of some nonlinear parabolic equations
835 **Bernhard Kawohl and Robert Kersner**, On degenerate diffusion with very strong absorption
836 **Avner Friedman and Fernandor Reitich**, Parameter identification in reaction-diffusion models
837 **E.G. Kalnins, H.L. Manocha and Willard Miller, Jr.**, Models of $q$-algebra representations I. Tensor products of special unitary and oscillator algebras
838 **Robert J. Sacker and George R. Sell**, Dichotomies for linear evolutionary equations in Banach spaces
839 **Oscar P. Bruno and Fernando Reitich**, Numerical solution of diffraction problems: a method of variation of boundaries
840 **Oscar P. Bruno and Fernando Reitich**, Solution of a boundary value problem for Helmholtz equation via variation of the boundary into the complex domain
841 **Victor A. Galaktionov and Juan L. Vazquez**, Asymptotic behaviour for an equation of superslow diffusion. The Cauchy problem
842 **Josephus Hulshof and Juan Luis Vazquez**, The Dipole solution for the porous medium equation in several space dimensions
843 **Shoshana Kamin and Juan Luis Vazquez**, The propagation of turbulent bursts
844 **Miguel Escobedo, Juan Luis Vazquez and Enrike Zuazua**, Source-type solutions and asymptotic behaviour for a diffusion-convection equation
845 **Marco Biroli and Umberto Mosco**, Discontinuous media and Dirichlet forms of diffusion type
846 **Stathis Filippas and Jong-Shenq Guo**, Quenching profiles for one-dimensional semilinear heat equations
847 **H. Scott Dumas**, A Nekhoroshev-like theory of classical particle channeling in perfect crystals
848 **R. Natalini and A. Tesei**, On a class of perturbed conservation laws
849 **Paul K. Newton and Shinya Watanabe**, The geometry of nonlinear Schrödinger standing waves
850 **S.S. Sritharan**, On the nonsmooth verification technique for the dynamic programming of viscous flow
851 **Mario Taboada and Yuncheng You**, Global attractor, inertial manifolds and stabilization of nonlinear damped beam equations
852 **Shigeru Sakaguchi**, Critical points of solutions to the obstacle problem in the plane
853 **F. Abergel, D. Hilhorst and F. Issard-Roch**, On a dissolution-growth problem with surface tension in the neighborhood of a stationary solution
854 **Erasmus Langer**, Numerical simulation of MOS transistors
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856 **Johannes C.C. Nitsche**, Boundary value problems for variational integrals involving surface curvatures
857 **Chao–Nien Chen**, Multiple solutions for a semilinear elliptic equation on $\mathbb{R}^N$ with nonlinear dependence on the gradient
858 **D. Brochet, X. Chen and D. Hilhorst**, Finite dimensional exponential atttractor for the phase field model
859 **Joseph D. Fehribach**, Mullins-Sekerka stability analysis for melting-freezing waves in helium-4
860 **Walter Schempp**, Quantum holography and neurocomputer architectures
861 **D.V. Anosov**, An introduction to Hilbert's 21st problem
862 **Herbert E Huppert and M Grae Worster**, Vigorous motions in magma chambers and lava lakes