ON THE POWER OF THE NONCENTRAL F- AND WILKS' U-TESTS: CHOOSING VARIATES FOR INCREASING THE POWER OF HOTELLING'S T^2-TEST*

by

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Summary

It is shown that the power of the noncentral F-test with m and n degrees of freedom is increasing in n and decreasing in m. For the problems of testing the means of one or two multivariate normal populations, this implies that increasing the dimension of the observed vectors may decrease the power of Hotelling's T^2-test, unless the additional "distance" provided by the extra variates is sufficiently large. Procedures are given which are designed to test, on the basis of a preliminary sample, whether the additional distance is sufficiently large to justify the inclusion of the extra variates. These procedures are more appropriate than the usual "test for additional information," but require the use of noncentral F tables to determine the critical points. Power properties of Wilks' U-test for MANOVA are also discussed.

Some Key Words

Noncentral F distribution; power of ANOVA test; additional variates for Hotelling's T^2-test; test for additional information; MANOVA; power of Wilks' U-test.

1. Introduction.

It is widely accepted that the power of the non-central F-test is increasing in the number of denominator degrees of freedom and decreasing in the number of numerator d.f. (see Seber (1966) p. 34). However, no proofs of these facts appear in the literature, although the former follows from the optimality property of the ANOVA test (see Remark 2.2). Attempts at an analytic proof based on explicit representations of the power function (see (2.1)) are unsuccessful, but a proof can be obtained by suitably applying the statistical notions of monotone likelihood ratio and the Neyman-Pearson lemma. This argument is given in section 2; other applications of this method occur in sections 5 and 6.

Our motivation for studying these properties of the non-central F-distribution stems from the question of whether or not the inclusion of additional variates provides increased power when testing the means of one or two multivariate normal populations by Hotelling's T^2-test. This problem is discussed in detail in section 3. In section 4 we present new procedures for testing, on the basis of a preliminary sample, whether to include the additional variates, and we compare these procedures to the usual "test for additional information." In the process we discuss the estimation of a noncentrality parameter (section 4) and study certain functions g which determine the amount of additional "distance" necessary for the inclusion of the extra variates to increase the power (section 5). Properties of the power function of Wilks' U-test for MANOVA, when the non-centrality matrix has rank 1, are discussed in section 6.

2. Properties of the Power Function of the Noncentral F-test.

We use the following notation for the noncentral F (not normalized) and beta distributions. Let U and V be independent χ^2 variates,

$U \sim \chi^2_m(\lambda)$ (noncentral), $V \sim \chi^2_n$ (central), and define $f = U/V$ and $b = V/(U+V)$. We denote the random variables f and b, and also their distributions, by

$$f(\lambda;\ m,\ n) \quad \text{and} \quad b(\lambda;\ \tfrac{1}{2}n,\ \tfrac{1}{2}m)$$

respectively. For $0 < \alpha < 1$ let $f^{\alpha}(\lambda;\ m,\ n)$ be defined by

$$\alpha = P\{f(\lambda;\ m,\ n) > f^{\alpha}(\lambda;\ m,\ n)\}$$

and write $f^{\alpha}(m,\ n) = f^{\alpha}(0;\ m,\ n)$. The power function of the size-α F-test with m and n degrees of freedom and noncentrality parameter λ is

$$\pi_{\alpha}(\lambda;\ m,\ n) \equiv P\{f(\lambda;\ m,\ n) > f^{\alpha}(m,\ n)\}.$$

The main result of this section is stated as

Theorem 2.1.

(a) For fixed α, m, and $\lambda > 0$, $\pi_{\alpha}(\lambda;\ m,\ n)$ strictly increases with n.

(b) For fixed α, n, and $\lambda > 0$, $\pi_{\alpha}(\lambda;\ m,\ n)$ strictly decreases with m.

To prove this theorem we require the following lemmas.

Lemma 2.1.

Let X and Y be distributed independently as $f(0;\ m+\theta,\ n+s)$ and $1/b(0;\tfrac{1}{2}n,\tfrac{1}{2}s)$ respectively. Then

$$XY \sim f(0;\ m+\theta,\ n).$$

Proof:

Let $U \sim \chi^2_{m+\theta}$, $V \sim \chi^2_n$, $W \sim \chi^2_s$, with U, V, and W independent. Define

$$X = U/(V+W),\ Y = (V+W)/V.$$

Then X and Y are independent (since $V + W$ and W/V are independent),

$$X \sim f(0;\ m+\theta,\ n+s),\ Y \sim 1/b(0;\ n,\ s),$$

and

$$XY = U/V \sim f(0;\ m+\theta,\ n).$$

Lemma 2.2.

$f(0;\ m,\ n)$ has the strict monotone likelihood ratio property with m as the parameter.

Proof:

The density function of $f(0;\ m,\ n)$ is given by

$$\varphi(f;\ m,\ n) = c(m,\ n)f^{\frac{1}{2}m-1}/(1+f)^{\frac{1}{2}(m+n)}.$$

For $m^* > m$, the likelihood ratio is

$$\varphi(f;\ m^*,\ n)/\varphi(f;\ m,\ n) = k(m,\ m^*,\ n)\{f/(1+f)\}^{\frac{1}{2}(m^*-m)},$$

which is a strictly increasing function of f.

Lemma 2.3.

Let X and Y be distributed independently as $f(0;\ m+\theta,\ n)$ and $f(0;\ s,\ m+\theta+n)$ respectively. Then

$$Y + (1+Y)X \sim f(0;\ m+\theta+s,\ n).$$

Proof:

Let U, V, and W be defined as in Lemma 2.1. Define

$$X = U/V,\ Y = W/(U+V).$$

Clearly, X and Y are independent,

$$X \sim f(0;\ m+\theta,\ n),\ Y \sim f(0;\ s,\ m+\theta+n),$$

and

$$Y + (1+Y)X = X + (1+X)Y = (U+W)/V \sim f(0;\ m+\theta+s,\ n).$$

Proof of Theorem 2.1(a).

Note that

$$\pi_\alpha(\lambda;\ m,n) = e^{-\lambda/2} \sum_{k=0}^{\infty} (\lambda/2)^k (1/k!) P\{f(0;\ m+2k,n) > f^\alpha(m,n)\} \tag{2.1}$$

(Rao (1965), p. 175). We shall prove a result stronger than (a), namely

$$P\{f(0;\ m+\theta,\ n+s) > f^\alpha(m,n+s)\} > P\{f(0;\ m+\theta,n) > f^\alpha(m,n)\} \tag{2.2}$$

for all $\theta > 0$ and $s > 0$. The result (a) follows by taking $\theta = 2k$, $k = 0, 1, 2,\ldots$ in (2.2). Define X and Y as in Lemma 2.1. Consider the problem of testing $\theta = 0$ against $\theta > 0$ based on X and Y. It follows from Lemma 2.2 and the Neyman -Pearson lemma that the unique UMP level-α test is given by the critical region

$$X > f^\alpha(m,\ n+s).$$

From Lemma 2.1 we note that the test given by the critical region

$$XY > f^\alpha(m,\ n)$$

is also a level-α test, which implies (2.2).

Proof of Theorem 2.1(b).

Here again we prove a result stronger than (b), namely,

$$P\{f(0;\ m+s+\theta,n) > f^\alpha(m+s,n)\} < P\{f(0;\ m+\theta,n) > f^\alpha(m,n)\} \tag{2.3}$$

for all $\theta > 0$, $s > 0$. The result (b) follows from (2.1) and (2.3).

Let X and Y be as defined in Lemma 2.3. Let Y^* be distributed independently of X as $f(0;\ s,m+n)$. Consider the problem of testing $\theta = 0$ against $\theta > 0$ based on X and Y^*. By Lemma 2.2 and the Neyman - Pearson lemma, the unique UMP level-α test is given by the critical region

$$X > f^\alpha(m,\ n).$$

By Lemma 2.3, the test with critical region

$$Y^* + (1+Y^*)X > f^\alpha(m+s,n)$$

is also a level-α test. Hence for all $\theta > 0$

$$P\{f(0;\ m+\theta,n) > f^\alpha(m,n)\} = P\{X > f^\alpha(m,n)\}$$
$$> P\{Y^* + (1+Y^*)X > f^\alpha(m+s,n)\}.$$

Since Y^* is stochastically larger than Y,

$$P\{Y^* + (1+Y^*)X > f^\alpha(m+s,n)\} > P\{Y + (1+Y)X > f^\alpha(m+s,n)\}$$
$$= P\{f(0;\ m+\theta+s,n) > f^\alpha(m+s,n)\}$$

which proves (2.3).

The following corollary is an immediate consequence of Theorem 1.1(a) and (b).

<u>Corollary 2.1</u>.

If $\lambda > 0$ and $q > 0$,

$$P\{f(\lambda;\ m,n) > f^\alpha(m,n)\} > P\{f(\lambda;\ m+q,n-q) > f^\alpha(m+q,n-q)\}.$$

A similar result holds for the power function of the noncentral chi-square test. This power function is defined as

$$\pi_\alpha(\lambda;\ m) = P\{\chi^2_m(\lambda) > \chi^2_{m,\alpha}\} \tag{2.4}$$

where $\alpha = P\{\chi^2_m > \chi^2_{m,\alpha}\}$.

<u>Theorem 2.2</u>.

For fixed α and $\lambda > 0$, $\pi_\alpha(\lambda;\ m)$ is a strictly decreasing function of m.

Proof:

Since

$$\pi_\alpha(\lambda;\ m) = e^{-\lambda/2} \sum_{k=0}^{\infty} (\lambda/2)^k(1/k!)P\{\chi^2_{m+2k} > \chi^2_{m,\alpha}\},$$

it suffices to prove that

$$P\{\chi^2_{m+\theta} > \chi^2_{m,\alpha}\} \tag{2.5}$$

is a strictly decreasing function of m, for each $\theta > 0$. Let X and Y be distributed independently as $\chi^2_{m+\theta}$ and χ^2_s, respectively. It is easy to see that the distribution of $\chi^2_{m+\theta}$ has the strict monotone likelihood ratio property with θ as the parameter. Consider the problem of testing $\theta = 0$ vs. $\theta > 0$ based on X and Y. The unique UMP level-α test is given by the critical region $X > \chi^2_{m,\alpha}$, while $X + Y > \chi^2_{m+s,\alpha}$ is also a critical region of level-α. Hence for all $\theta > 0$,

$$P\{\chi^2_{m+\theta} > \chi^2_{m,\alpha}\} = P\{X > \chi^2_{m,\alpha}\} > P\{X + Y > \chi^2_{m+s,\alpha}\}$$

$$= P\{\chi^2_{m+\theta+s} > \chi^2_{m+s,\alpha}\},$$

which proves (2.5).

Remark 2.1:

The above results remain valid for positive non-integral degrees of freedom.

Remark 2.2:

Theorem 2.1(a) and Corollary 2.1 (but not Theorem 2.1(b)) can be obtained by an argument based on invariance. Let U, V, W be mutually independent random variables with $U \sim \sigma^2\chi^2_m(\lambda)$, $V \sim \sigma^2\chi^2_n$, $W \sim \sigma^2\chi^2_s$, and consider the problem of testing $\lambda = 0$ vs. $\lambda > 0$. This problem is invariant under the group of scale transformations, and the pair $\{U/(V+W), V/W\}$ is a maximal invariant. These are independent statistics and the distribution of V/W does not involve λ so from the Neyman-Pearson lemma it follows that the F-test based on $U/(V+W)$ is the unique UMP invariant test. Two invariant competitors are the F-tests based on U/V and $(U+W)/V$. That these tests are less powerful than the test based on $U/(V+W)$ implies Theorem 2.1(a) and Corollary 2.1, respectively. This argument could be shortened by appealing to Theorem 5 of Lehmann (1959), p. 228; this is essentially done by Lehmann on pp. 267-268.

3. Choosing Variates for Increased Power.

Let the (p+q)-dimensional column vectors $X_i = \begin{pmatrix} Y_i \\ Z_i \end{pmatrix}$, $i = 1,\ldots, N$, be independent observations from a multivariate normal population with mean $\mu = \begin{pmatrix} \eta \\ \xi \end{pmatrix}$ and nonsingular covariance matrix Σ, where each Y_i is $p \times 1$ and Z_i is $q \times 1$. Consider the problem of testing

$$H: \mu = 0 \text{ vs. } K: \mu \neq 0.$$

We assume throughout that Σ is unknown (if Σ is known a similar but simpler discussion applies--see the Remark following Theorem 5.1). If it is difficult or expensive to obtain the observations Z_i on the last q variates, one may wish to determine whether the inclusion of these variates will increase the power of the usual D^2-test (Hotelling's T^2-test) for H vs. K. Let the $(p+q) \times (p+q)$ Wishart matrix $S \equiv \sum_{i=1}^{N} (X_i - \bar{X})(X_i - \bar{X})'$ with $N - 1$ degrees of freedom be partitioned as

$$S = \begin{pmatrix} S_{YY} & S_{YZ} \\ S_{ZY} & S_{ZZ} \end{pmatrix}, \; S_{YY} : p \times p, \; S_{ZZ} : q \times q,$$

and partition Σ accordingly. If only the first p variates $Y_1, \ldots, Y_N$ are observed, the size-α D^2-test for testing $\eta = 0$ vs. $\eta \neq 0$ rejects $\eta = 0$ if

$$D_p^2 \equiv N\bar{Y}'S_{YY}^{-1}\bar{Y} > f^{\alpha}(p, N-p),$$

while the size-α D^2-test based on all $p + q$ variates $X_1, \ldots, X_N$ for testing H vs. K rejects H if

$$D_{p+q}^2 \equiv N\bar{X}'S^{-1}\bar{X} > f^{\alpha}(p+q, N-p-q).$$

(To insure that S^{-1} exists we assume $N \geq p + q + 1$.) The power functions of these two tests are

$$\pi_{\alpha}(N\Delta_p^2; \; p, \; N-p) \quad \text{and} \quad \pi_{\alpha}(N\Delta_{p+q}^2; \; p+q, \; N-p-q)$$

respectively, where

$$\Delta_p^2 \equiv \eta'\Sigma_{YY}^{-1}\eta \quad \text{and} \quad \Delta_{p+q}^2 \equiv \mu'\Sigma^{-1}\mu$$

are the (Mahalanobis) population distances based on p and $p + q$ variates.

Numerical studies of these power functions in earlier papers (Rao (1949), (1966)) have indicated, and Corollary 2.1 now verifies, that if the additional distance $N\Delta^2_{p+q} - N\Delta^2_p$ provided by the last q variates is not large then

$$\pi_\alpha(N\Delta^2_{p+q}\ ;\ p+q,\ N-p-q) < \pi_\alpha(N\Delta^2_p,\ p,\ N-p),$$

so the power may actually be decreased by including additional variates. Since $\pi_\alpha(N\Delta^2_{p+q};\ p+q,\ N-p-q)$ increases strictly and continuously to one as $\Delta^2_{p+q} \to \infty$, for fixed $\Delta^2_p > 0$ there is a unique value

$$g(N\Delta^2_p) \equiv g(N\Delta^2_p;\ p,\ N-p,\ q,\ \alpha) > 0$$

such that

$$\pi_\alpha(N\Delta^2_p + g(N\Delta^2_p);\ p+q,\ N-p-q) = \pi_\alpha(N\Delta^2_p;\ p,\ N-p). \tag{3.1}$$

The power of the D^2_{p+q}-test will exceed that of the D^2_p-test if and only if $N\Delta^2_{p+q} - N\Delta^2_p > g(N\Delta^2_p)$. Therefore, the problem of testing whether the additional q variates provide increased power is properly posed as follows: test

$$H_1:\ N\Delta^2_{p+q} - N\Delta^2_p \leq g(N\Delta^2_p) \text{ vs. } K_1:\ N\Delta^2_{p+q} - N\Delta^2_p > g(N\Delta^2_p).$$

We call this the problem of "testing for increased power." This differs from the usual formulation of the problem of "testing for additional information" : test

$$H_2:\ N\Delta^2_{p+q} - N\Delta^2_p = 0 \text{ vs. } K_2:\ N\Delta^2_{p+q} - N\Delta^2_p > 0,$$

which has been considered by Rao (1948), (1949), (1965), (1966), (1970), Giri (1964), (1965), Das Gupta (1968), and others. Even if K_2 is true, the inclusion of the extra q variates may decrease the power of the D^2-test, so H_1 vs. K_1 is a more accurate formulation of our problem.

4. Testing for Increased Power on the Basis of a Preliminary Sample.

In this section we discuss testing for increased power, based on a preliminary sample $x_1, \ldots, x_n$. (We may now think of $X_1, \ldots, X_N$ as representing future observations, with N substantially larger than n.) We adopt the convention of using small letters for statistics based on the preliminary sample, so write

$$s = \sum_{i=1}^{n} (x_i - \bar{x})(x_i - \bar{x})', \quad d^2_{p+q} = n\bar{x}' s^{-1} \bar{x} \ , \quad d^2_p = n\bar{y} s^{-1}_{yy} \bar{y} \ ,$$

and assume that $n \geq p + q + 1$. It is well-known that

$$d^2_p \sim f(n\Delta^2_p \ ; \ p, \ n-p),$$

while conditional on d^2_p ,

$$(d^2_{p+q} - d^2_p)(1+d^2_p)^{-1} \sim f\left((n\Delta^2_{p+q} - n\Delta^2_p)(1+d^2_p)^{-1} \ ; \ q, \ n-p-q\right). \qquad (4.1)$$

The usual size-β "test for additional information" for testing H_2 vs. K_2 (Rao (1965), p. 482) rejects H_2 if

$$(d^2_{p+q} - d^2_p)(1+d^2_p)^{-1} > f^{\beta}(q, \ n-p-q). \qquad (4.2)$$

Giri (1964) has shown that among invariant tests, (4.2) is the UMP similar level-β test for H_2 vs. K_2. However one can object to the test (4.2) on the grounds that it is designed to test H_2 vs. K_2 rather than the more relevant problem H_1 vs. K_1.

Procedures which seem more appropriate for testing H_1 vs. K_1 can be based on the conditional distribution (4.1) according to the following idea. If $\hat{\Delta}^2_p \equiv \hat{\Delta}^2_p(d^2_p)$ is an estimate of Δ^2_p based on d^2_p, we can estimate $g(N\Delta^2_p)$ by $g(N\hat{\Delta}^2_p)$ and then approximate the problem H_1 vs. K_1 by the problem of testing

No method for the computation of this solution is given, however, nor are any properties of the MLE studied. When $n > p + 2$ we content ourselves with a reasonable alternative based on the linear unbiased estimate $n^{-1}\{(n-p-2)d_p^2 - p\}$. Since this may assume negative values we consider instead

$$\hat{\Delta}_{p,1}^2 \equiv n^{-1}\{(n-p-2)d_p^2 - p\}^+$$

(where $a^+ = (a+|a|)/2$) which is not unbiased but satisfies $E(\hat{\Delta}_{p,1}^2) > \Delta_p^2$. Since g is increasing and approximately linear, we have roughly that

$$Eg(N\hat{\Delta}_{p,1}^2) \approx g(NE(\hat{\Delta}_{p,1}^2)) > g(N\Delta_p^2)$$

so $g(N\hat{\Delta}_{p,1}^2)$ tends to overestimate $g(N\Delta_p^2)$. Since $H_1' \supset H_1$ whenever $g(N\hat{\Delta}_{p,1}^2) > g(N\Delta_p^2)$, the test $T(\hat{\Delta}_{p,1}^2;\, \beta)$ tends to be conservative as a test for H_1 vs. K_1 in the sense that the conditional level will usually be less than β, i.e., one will be led to include the additional q variates in the D^2-test with a probability less than β when H_1 is true.

Due to the approximations involved, the above discussion does not establish that $T(\hat{\Delta}_{p,1}^2;\, \beta)$ is a level-β test for H_1 vs. K_1 (although this is probably true). We can construct a conservative level-β test for this problem as follows. For $0 < \epsilon < 1$ define

$$\hat{\Delta}_{p,\epsilon}^2 = \inf\{\lambda \geq 0 \mid d_p^2 \leq f^{1-\epsilon}(\lambda;\, p,n-p)\}.$$

Then $\hat{\Delta}_{p,\epsilon}^2$ is an upper $(1-\epsilon)$-confidence bound for Δ_p^2; in fact

$$P_{\Delta_p^2}\{\hat{\Delta}_{p,\epsilon}^2 \geq \Delta_p^2\} = \begin{cases} 1 & \text{if } \Delta_p^2 = 0 \\ 1-\epsilon & \text{if } \Delta_p^2 > 0 \end{cases} .$$

(Since the distributions $f(\lambda;\, p,n-p)$ have a MLR it follows from Corollary 3 of Lehmann (1959, p. 80) that $\hat{\Delta}_{p,\epsilon}^2$ is the uniformly most accurate upper $(1-\varepsilon)$-confidence bound for Δ_p^2 based on d_p^2.) If for a given value of β we choose $\varepsilon = \varepsilon(\beta)$ and $\delta = \delta(\beta)$ in the interval $(0, 1)$ such that $(1-\delta)(1-\varepsilon) = 1 - \beta$, then the test $T \equiv T(\hat{\Delta}_{p,\varepsilon}^2; \delta)$ is of level-β for H_1 vs. K_1. To see this, define

$$\psi = \psi(d_p^2) = f^{\delta}\Big((n/N)g(N\hat{\Delta}_{p,\varepsilon}^2)(1+d_p^2)^{-1};\ q,\ n-p-q\Big).$$

Then

$$P_{H_1}\{T \text{ accepts } H_1\} = E_{H_1}[P_{H_1}\{(d_{p+q}^2 - d_p^2)(1+d_p^2)^{-1} \le \psi | d_p^2\}]$$

$$\ge E_{\Delta_p^2}[P\{f\Big((n/N)g(N\Delta_p^2)(1+d_p^2)^{-1};\ q,n-p-q\Big) \le \psi | d_p^2\}]$$

$$\ge \int_{\{d_p^2 | \hat{\Delta}_{p,\varepsilon}^2 \ge \Delta_p^2\}} P\{f\Big((n/N)g(N\hat{\Delta}_{p,\varepsilon}^2)(1+d_p^2)^{-1};q,n-p-q\Big) \le \psi | d_p^2\} dG$$

$$\ge (1-\delta)(1-\varepsilon) = 1 - \beta,$$

where G denotes the distribution function of d_p^2. Here the first two inequalities follow from the facts that $f(\lambda;\ q,n-p-q)$ is stochastically increasing in λ and that $g(\lambda)$ is increasing in λ.

Compared to the usual test (4.2), the proposed tests $T(\hat{\Delta}_p^2; \beta)$ for H_1 vs. K_1 have the disadvantage that they require knowledge of the percentage points of the noncentral F distribution to obtain the critical values $f^{\beta}(\lambda;\ q,n-p-q)$, the values of the functions g, and the values of the upper confidence bound $\hat{\Delta}_{p,\varepsilon}^2$, but as already pointed out, the test (4.2) is designed for a different testing problem. The computation of these values is discussed in section 5.

The idea of approximating the testing problem H_1 vs. K_1 by the (conditional) problem H_1' vs. K_1' described above (4.3) can be used to obtain suitable tests for any testing problem having the same form but with g replaced by other functions of interest. For example, instead of asking if the inclusion of the extra q variates will merely increase the power (equivalently, decrease the Type II error probability) of the D^2-test, one may ask if the Type II error probability will be decreased by a certain fraction γ, $0 < \gamma < 1$. If we define

$$g_\gamma(N\Delta_p^2) \equiv g_\gamma(N\Delta_p^2;\ p,N-p,q,\alpha) > g(N\Delta_p^2) > 0$$

to be the unique value satisfying

$$1 - \pi_\alpha(N\Delta_p^2 + g_\gamma(N\Delta_p^2);\ p+q\ \ ,N-p-q) = \gamma\{1-\pi_\alpha(N\Delta_p^2;\ p,N-p)\},$$

then this question can be stated as the problem of testing

$$N\Delta_{p+q}^2 - N\Delta_p^2 \le g_\gamma(N\Delta_p^2) \text{ vs. } N\Delta_{p+q}^2 - N\Delta_p^2 > g_\gamma(N\Delta_p^2).$$

This testing problem has the same form as H_1 vs. K_1 except that g is replaced by g_γ, so by simply substituting g_γ for g in (4.3) we obtain appropriate tests.

Finally, consider the case of two populations. The entire preceding discussion applies with only notational changes. Suppose $X_i^{(j)} = \begin{pmatrix} Y^{(j)} \\ Z_i^{(j)} \end{pmatrix}$, $i = 1,\ldots, N$, $j = 1, 2$, are independent observations from two $(p+q)$-dimensional normal populations with means $\mu^{(j)} = \begin{pmatrix} \eta^{(j)} \\ \xi^{(j)} \end{pmatrix}$ and common covariance matrix Σ. Consider the problem of testing

$$\tilde{H}:\ \mu^{(1)} = \mu^{(2)} \text{ vs. } \tilde{K}:\ \mu^{(1)} \neq \mu^{(2)}.$$

Define $\tilde{S} = \sum_{j=1}^{2} \sum_{i=1}^{N_j} (X_i^{(j)} - \bar{X}^{(j)})(X_i^{(j)} - \bar{X}^{(j)})'$, which has a Wishart distribution with $N_1 + N_2 - 2$ degrees of freedom (assume $N_1 + N_2 \geq p + q + 2$), and partition $\tilde{S}$ and Σ as before. Setting $\tilde{N} = (N_1^{-1} + N_2^{-1})^{-1}$, the size-$\alpha$ D^2-test for testing $\eta^{(1)} = \eta^{(2)}$ vs. $\eta^{(1)} \neq \eta^{(2)}$ based on the first p variates $Y_i^{(j)}$ rejects the null hypothesis if

$$\tilde{D}_p^2 \equiv \tilde{N}(\overline{Y}^{(1)} - \overline{Y}^{(2)})'\tilde{S}_{YY}^{-1}(\overline{Y}^{(1)} - \overline{Y}^{(2)}) > f^{\alpha}(p, N_1 + N_2 - 1 - p),$$

while the size-α D^2-test based on all $p + q$ variates $X_i^{(j)}$ rejects $\tilde{H}$ if

$$\tilde{D}_{p+q}^2 \equiv \tilde{N}(\overline{X}^{(1)} - \overline{X}^{(2)})'\tilde{S}^{-1}(\overline{X}^{(1)} - \overline{X}^{(2)}) > f^{\alpha}(p+q, N_1 + N_2 - 1 - p - q).$$

The power of these tests are

$$\pi_{\alpha}(\tilde{N}\tilde{\Delta}_p^2; p, N_1 + N_2 - 1 - p), \quad \pi_{\alpha}(\tilde{N}\tilde{\Delta}_{p+q}^2; p+q, N_1 + N_2 - 1 - p - q)$$

respectively, where

$$\tilde{\Delta}_p^2 = (\eta^{(1)} - \eta^{(2)})'\Sigma_{YY}^{-1}(\eta^{(1)} - \eta^{(2)}), \quad \tilde{\Delta}_{p+q}^2 = (\mu^{(1)} - \mu^{(2)})'\Sigma^{-1}(\mu^{(1)} - \mu^{(2)}).$$

Therefore the problem of testing whether the additional q variates provide increased power is the following: test

$$\tilde{H}_1 : \tilde{N}\tilde{\Delta}_{p+q}^2 - \tilde{N}\tilde{\Delta}_p^2 \leq \tilde{g}(\tilde{N}\tilde{\Delta}_p^2) \text{ vs. } \tilde{K}_1 : \tilde{N}\tilde{\Delta}_{p+q}^2 - \tilde{N}\tilde{\Delta}_p^2 > \tilde{g}(\tilde{N}\tilde{\Delta}_p^2),$$

where

$$\tilde{g}(\tilde{N}\tilde{\Delta}_p^2) = g(\tilde{N}\tilde{\Delta}_p^2; p, N_1 + N_2 - 1 - p, q, \alpha).$$

The problem $\tilde{H}_1$ vs. $\tilde{K}_1$ is of exactly the same form as the problem H_1 vs. K_1 except that N and $N - p$ are replaced by $\tilde{N}$ and $N_1 + N_2 - 1 - p$ respectively.

Suppose we have available preliminary samples $x_i^{(j)}$, $i = 1, \ldots, n_j$, $j = 1, 2$ $(n_1 + n_2 \geq p+q+2)$ with which to test $\tilde{H}_1$ vs. $\tilde{K}_1$. Define $\tilde{n}$, $\tilde{s}$, $\tilde{d}_p^2$, $\tilde{d}_{p+q}^2$ in the same way as $\tilde{N}$, $\tilde{S}$, $\tilde{D}_p^2$, $\tilde{D}_{p+q}^2$. Then the joint distribution of $\tilde{d}_p^2$ and $(\tilde{d}_{p+q}^2 - \tilde{d}_p^2)(1+\tilde{d}_p^2)^{-1}$ is the same as that of d_p^2 and $(d_{p+q}^2 - d_p^2)(1+d_p^2)^{-1}$ except that n, $n-p$, $n-p-q$, Δ_p^2, and Δ_{p+q}^2 must be replaced by $\tilde{n}$, $n_1 + n_2 - 1 - p$, $n_1 + n_2 - 1 - p - q$, $\tilde{\Delta}_p^2$, and $\tilde{\Delta}_{p+q}^2$, respectively.

If $\hat{\tilde{\Delta}}^2_p$ is an estimate of $\tilde{\Delta}^2_p$ based on $\tilde{d}^2_p$, let $\tilde{T}(\hat{\tilde{\Delta}}^2_p;\ \beta)$ be the test which rejects $\tilde{H}_1$ if

$$(\tilde{d}^2_{p+q} - \tilde{d}^2_p)(1+\tilde{d}^2_p)^{-1} > f^{\beta}\Big((\tilde{n}/\tilde{N})\tilde{g}(\tilde{N}\hat{\tilde{\Delta}}^2_p)(1+\tilde{d}^2_p)^{-1};\ q, n_1+ n_2-1-p-q\Big).$$

If $\hat{\tilde{\Delta}}^2_{p,1}$ and $\hat{\tilde{\Delta}}^2_{p,\epsilon}$ are defined by

$$\hat{\tilde{\Delta}}^2_{p,1} = \tilde{n}^{-1}\{(n_1+ n_2-1-p)\tilde{d}^2_p - p\}^+,$$

$$\hat{\tilde{\Delta}}^2_{p,\epsilon} = \inf\{\lambda \geq 0 \,|\, \tilde{d}^2_p \leq f^{1-\epsilon}(\lambda;\ p, n_1+ n_2-1-p)\},$$

then $\tilde{T}(\hat{\Delta}^2_{p,1};\ \beta)$ is approximately level-β and $\tilde{T}(\hat{\Delta}^2_{p,\epsilon};\ \delta)$ is level-β for testing $\tilde{H}_1$ vs. $\tilde{K}_1$, where $(1-\delta)(1-\epsilon) = 1 - \beta$.

5. <u>Properties and Computation of the Functions g</u>.

Write the defining relation (3.1) for the function $g(\lambda) = g(\lambda;\ p, N-p, q, \alpha)$ in the abbreviated form

$$\pi_2(\lambda + g(\lambda)) = \pi_1(\lambda), \tag{5.1}$$

where $\pi_1(\lambda) = \pi_\alpha(\lambda;\ p,\ N-p)$, $\pi_2(\lambda) = \pi_\alpha(\lambda;\ p+q,\ N-p-q)$.
The power functions π_1 and π_2 are strictly increasing, analytic (see (2.1)) functions with $\pi_1(0) = \pi_2(0) = \alpha$, so we can write

$$g(\lambda) = \pi_2^{-1}\Big(\pi_1(\lambda)\Big) - \lambda, \tag{5.2}$$

from which it is seen that g is arbitrarily smooth and $g(0) = 0$. The problem of showing that g is strictly increasing again yields not to analytic methods (e.g., differentiating (5.2)) but to a statistical argument.

<u>Theorem 5.1</u>.

The functions $g(\lambda) = g(\lambda;\ p, N-p, q, \alpha)$ are strictly increasing in λ.

<u>Proof</u>:

It suffices to show that

$$\pi_2(\lambda + g(\lambda_0)) < \pi_1(\lambda) \tag{5.3}$$

for all $0 \leq \lambda_0 < \lambda$, since this implies that

$$\lambda + g(\lambda_0) < \pi_2^{-1}\left(\pi_1(\lambda)\right) = \lambda + g(\lambda).$$

Let the pair of random variables X, Y have the following joint distribution:

$$Y \sim f(\lambda;\, p,\, N\text{-}p),$$

while conditional on Y,

$$X \sim f\left(g(\lambda_0)(1+Y)^{-1};\, q,\, N\text{-}p\text{-}q\right).$$

Since X and Y have the same joint distribution as $(D^2_{p+q} - D^2_p)(1+D^2_p)^{-1}$ and D^2_p (with $N\Delta^2_p = \lambda$ and $N\Delta^2_{p+q} = \lambda + g(\lambda_0)$),

$$Y + (1+Y)X \sim D^2_{p+q} \sim f(\lambda + g(\lambda_0);\, p+q,\, N\text{-}p\text{-}q) \qquad (5.4).$$

Consider the problem of testing $\lambda = \lambda_0$ vs. $\lambda > \lambda_0$ on the basis of X and Y. The joint density of X and Y is of the form $\varphi_{\lambda_0}(x|y)\varphi_\lambda(y)$, so by the MLR property of $f(\lambda;\, p,\, N\text{-}p)$, the test with critical region

$$Y > f^\alpha(p,\, N\text{-}p)$$

is the unique UMP level-α^* test, where

$$\alpha^* \equiv P_{\lambda=\lambda_0}\{Y > f^\alpha(p,\, N\text{-}p)\} = \pi_1(\lambda_0).$$

However,

$$Y + (1+Y)X > f^\alpha(p+q,\, N\text{-}p\text{-}q)$$

is also a level-α^* critical region, since (5.1) and (5.4) imply that

$$\begin{aligned} P_{\lambda=\lambda_0}\{Y + (1+Y)X > f^\alpha(p+q, N\text{-}p\text{-}q)\} &= P\{f(\lambda_0 + g(\lambda_0)\,;\, p+q, N\text{-}p\text{-}q) \\ &\qquad > f^\alpha(p+q, N\text{-}p\text{-}q)\} \\ &= \pi_2(\lambda_0 + g(\lambda_0)) = \pi_1(\lambda_0) = \alpha^*. \end{aligned}$$

Therefore for $\lambda > \lambda_0$,

$$\pi_2(\lambda + g(\lambda_0)) = P_\lambda\{Y + (1+Y)X > f^\alpha(p+q,\ N-p-q)\} < P_\lambda\{Y > f^\alpha(p,\ N-p)\}$$

$$= \pi_1(\lambda),$$

which proves (5.3).

Remark 1.

Theorem 5.1 implies that $g(\lambda) > g(0) = 0$ for $\lambda > 0$, and therefore includes Corollary 2.1 as a special case. Also, the proof of Theorem 5.1 can be modified to prove Theorem 2.1 directly, without using the expansion (2.1).

Remark 2.

If $q^* > q$ then Corollary 2.1 implies that

$$\pi_3(\lambda + g(\lambda)) < \pi_2(\lambda + g(\lambda)) = \pi_1(\lambda),$$

where $\pi_3(\lambda) = \pi_\alpha(\lambda;\ p+q^*,\ N-p-q^*)$. This in turn yields

$$\lambda + g(\lambda) < \pi_3^{-1}(\pi_1(\lambda)) = \lambda + g^*(\lambda),$$

where $g^*(\lambda) = g(\lambda;\ p,\ N-p,\ q^*,\ \alpha)$. Hence $g(\lambda;\ p,\ N-p,\ q,\ \alpha)$ is a strictly increasing function of q, which contradicts a statement of Rao (1966, p. 92).

Remark 3.

If we define $h(\lambda) \equiv h(\lambda;\ p,\ q,\ \alpha) > 0$ to be the unique value satisfying

$$\pi_\alpha(\lambda + h(\lambda);\ p+q) = \pi_\alpha(\lambda;\ p),$$

where $\pi_\alpha(\lambda;\ m)$ is the power function of the noncentral χ^2-test (see Theorem 2.2), then a proof similar to that of Theorem 5.1 shows that $h(\lambda)$ is strictly increasing in λ. If in section 3 we had assumed that the covariance matrix Σ were known and replaced S by Σ in the

D^2-statistics, then the function h would replace g in determining whether the additional distance provided by the extra q variates leads to increased power.

To apply the procedures discussed in section 4, one needs to obtain numerical values of the percentage points f^β, the power functions π, and the functions g. A list of tables and approximations available for f^β and π is given by Johnson and Kotz (1970, Chapter 30); the computer programs of Bargmann and Ghosh (1964) seem especially useful. The value of $g(\lambda)$ can be approximated simply by comparing the power functions π_1 and π_2 (c.f. (3.1) or (5.1)). If one has available a computer program for calculating π_1 and π_2, an accurate value for $g(\lambda)$ can be obtained by an iterative procedure now described.

Rewrite the defining equation (5.1) for $g(\lambda)$ in the form

$$g(\lambda) = 2 \log[e^{\frac{1}{2}g(\lambda)}\{1 - \pi_2\Big(\lambda + g(\lambda)\Big)\}/\{1 - \pi_1(\lambda)\}]. \qquad (5.5)$$

Let $g_0(\lambda) = 0$ and define recursively

$$g_{n+1}(\lambda) = 2 \log[e^{\frac{1}{2}g_n(\lambda)}\{1 - \pi_2\Big(\lambda + g_n(\lambda)\Big)\}/\{1 - \pi_1(\lambda)\}]. \qquad (5.6)$$

Then $\{g_n(\lambda)\}$ is a strictly increasing sequence of lower bounds for $g(\lambda)$, and $g_n(\lambda)$ converges to $g(\lambda)$ at a geometric rate. (That $g_1(\lambda) > g_0(\lambda)$ is seen immediately from Corollary 2.1).

To prove this, note that (c.f. (2.1)) for $i = 1,2$,

$$1 - \pi_i(\lambda) = e^{-\frac{1}{2}\lambda}R_i(\lambda) \qquad (5.7)$$

where

$$R_1(\lambda) = \sum_{k=0}^{\infty} (\tfrac{1}{2}\lambda)^k\{1 - \pi_\alpha(0;\ p+2k,\ N-p)\}/k!$$

$$R_2(\lambda) = \sum_{k=0}^{\infty} (\tfrac{1}{2}\lambda)^k\{1 - \pi_\alpha(0;\ p+q+2k,\ N-p-q)\}/k!\ .$$

Hence (5.5) and (5.6) can be rewritten as

$$g(\lambda) = 2\log\{R_2(\lambda + g(\lambda))/R_1(\lambda)\} \tag{5.8}$$

$$g_{n+1}(\lambda) = 2\log\{R_2(\lambda + g_n(\lambda))/R_1(\lambda)\} . \tag{5.9}$$

Since $R_2(\lambda)$ is strictly increasing we can use equation (5.9), the initial inequalities $g_0(\lambda) < g_1(\lambda)$ and $g_0(\lambda) < g(\lambda)$, and an induction argument to show that $g_n(\lambda) < g_{n+1}(\lambda)$ and $g_n(\lambda) < g(\lambda)$ for all n. Next, for fixed λ and $x \geq 0$ define

$$\varphi_\lambda(x) = 2\log\{R_2(\lambda+x)/R_1(\lambda)\} = x + 2\log[\{1 - \pi_2(\lambda+x)\}/\{1 - \pi_1(\lambda)\}].$$

The function φ_λ is strictly increasing, $\varphi_\lambda(0) > 0$, and the equation $x = \varphi_\lambda(x)$ has a unique solution, namely $g(\lambda)$. Furthermore φ_λ has derivative

$$\varphi_\lambda'(x) = 1 - 2\pi_2'(\lambda+x)\{1 - \pi_2(\lambda+x)\}^{-1} < 1.$$

From these facts, it follows by the usual functional iteration argument that $g_n(\lambda)$ converges to $g(\lambda)$ at the geometric rate $\{\varphi_\lambda'\left(g(\lambda)\right)\}^n$.

When q and $N - p$ are even integers, the above computations may be facilitated by means of the finite expansion of the noncentral F power function (Johnson and Kotz (1970), p. 192). With $x_1 = \{1 + f^\alpha(p, N-p)\}^{-1}$, $x_2 = \{1 + f^\alpha(p+q, N-p-q)\}^{-1}$, $a_1 = \frac{1}{2}p$, $a_2 = \frac{1}{2}(p+q)$, $b_1 = \frac{1}{2}(N-p)$, $b_2 = \frac{1}{2}(N-p-q)$, one has

$$1 - \pi_i(\lambda) = e^{-\frac{1}{2}\lambda x_i} Q_i(\lambda) \tag{5.10}$$

for $i = 1, 2$, where Q_1 and Q_2 are polynomials defined by

$$Q_i(\lambda) = \sum_{k=0}^{b_i-1} (\tfrac{1}{2}\lambda x_i)^k I_{1-x_i}(a_i+k, b_i-k)/k!$$

and $I_z(a, b)$ is the incomplete beta function ratio. Therefore (5.5) and (5.6) can be rewritten as

$$g(\lambda) = (\frac{x_1}{x_2} - 1)\lambda + \frac{2}{x_2} \log\{Q_2\Big(\lambda + g(\lambda)\Big)/Q_1(\lambda)\} \tag{5.11}$$

$$g_{n+1}(\lambda) = (\frac{x_1}{x_2} - 1)\lambda + \frac{2}{x_2} \log\{Q_2\Big(\lambda + g_n(\lambda)\Big)/Q_1(\lambda)\}, \tag{5.12}$$

and a program for (5.12) can easily be written. Incidentally, (5.11) can be used to show that $g(\lambda)$ is approximately linear for large or small λ, while the inequality

$$g(\lambda) > g_1(\lambda) = (\frac{x_1}{x_2} - 1)\lambda + \frac{2}{x_2} \log\{Q_2(\lambda)/Q_1(\lambda)\},$$

obtained from (5.12), shows that $\lim_{\lambda\to\infty} g(\lambda) = \infty$ as stated earlier, since $x_1 > x_2$ and $\log Q_i(\lambda)$ is dominated by λ.

For small values of λ, the derivative $g'(0)$ is of interest. From (5.8) we obtain, after some manipulation of integrals,

$$g'(0) = \{a_2!\ (b_2-1)!\ x_1^{b_1}\ (1-x_1)^{a_1}\}\ \{a_1!\ (b_1-1)!\ x_2^{b_2}\ (1-x_2)^{a_2}\}^{-1} - 1,$$

valid for all q, $N - p$.

Finally, in the particular case where q and $N - p$ are even integers and $N - p - q = 2$, then Q_2 is just the constant $1 - \alpha$, so (5.11) yields an explicit expression for $g(\lambda)$:

$$g(\lambda) = (\frac{x_1}{x_2} - 1)\lambda - \frac{2}{x_2} \log\{Q_1(\lambda)\} + \frac{2}{x_2} \log(1-\alpha).$$

6. On Wilks' U-test for MANOVA.

The canonical form of the multivariate analysis of variance (MANOVA) problem can be stated as follows: Let $V_1, \ldots, V_m, W_1, \ldots, W_n$ $(n \geq p)$ be mutually independent $p \times 1$ random vectors having normal distributions with the same nonsingular covariance matrix Σ. It is known that $EW_\alpha = 0$, $\alpha = 1, \ldots, n$. The problem is to test the hypothesis

$$H: \; EV_\alpha \equiv \mu_\alpha = 0, \; \alpha = 1, \ldots, m \quad \text{vs.} \quad K: \text{not } H.$$

The technique described in Section 2 cannot be used generally to study the behavior of the power functions of different tests for the MANOVA when the error degrees of freedom n, or the hypothesis degrees of freedom m, or the number of components p change separately. The key to success of the above method is mainly the monotone likelihood-ratio property of the relevant statistic involved. For the multivariate case, even for the likelihood-ratio test (LRT) it is possible to obtain similar results only when the noncentrality matrix $\tau: m \times m$ given by

$$\tau = (\mu_1 \; \cdots \; \mu_m)'\Sigma^{-1}(\mu_1 \; \cdots \; \mu_m) \tag{6.1}$$

has rank 1. The LRT rejects H if

$$U_p = \left| \sum_{\alpha=1}^{n} W_\alpha W_\alpha' \right| \Big/ \left| \sum_{\alpha=1}^{n} W_\alpha W_\alpha' + \sum_{\alpha=1}^{m} V_\alpha V_\alpha' \right| \leq U_p^{(1-\alpha)}(n, m), \tag{6.2}$$

where $U_p^{(1-\alpha)}(n, m)$ is the upper $(1-\alpha)$-level point of the null distribution of U_p. We shall assume <u>throughout</u> that rank $(\tau) = 1$ and define $\lambda = tr(\tau)$. Without loss of generality (for studying the distribution of U_p), we shall assume that

$$\Sigma = I_p, \; \mu_1 = (\lambda^{\frac{1}{2}}, 0, \ldots, 0)', \; \mu_2 = \cdots = \mu_m = 0.$$

Let $U_p(\lambda; n, m)$ denote the distribution of U_p.

We need some results on decompositions of U_p. It follows from the results of Das Gupta (1971), or otherwise, that U_p can be decomposed as

$$U_p = \prod_{i=1}^{p} Z_i \ , \tag{6.3}$$

where Z_i's are mutually independent,

$$Z_1 \sim b(\lambda;\ \tfrac{1}{2}n,\ \tfrac{1}{2}m), \tag{6.4}$$

$$Z_i \sim b(0;\ \tfrac{1}{2}(n-i+1),\ \tfrac{1}{2}m), \qquad i = 2,\ldots,\ p.$$

Also note that

$$U_p = \prod_{i=1}^{m} \left| \sum_{\alpha=1}^{n} W_\alpha W'_\alpha + \sum_{\alpha=i+1}^{m} V_\alpha V'_\alpha \right| \Big/ \left| \sum_{\alpha=1}^{n} W_\alpha W'_\alpha + \sum_{\alpha=i}^{m} V_\alpha V'_\alpha \right| \tag{6.5}$$

$$= \prod_{i=1}^{m} Z_i^* \ , \text{ say.}$$

It can be seen that Z_i^*'s are mutually independent, and

$$Z_1^* \sim b(\lambda;\ \tfrac{1}{2}(n+m-p),\ \tfrac{1}{2}p), \tag{6.6}$$

$$Z_i^* \sim b(0;\ \tfrac{1}{2}(n+m-p-i+1),\ \tfrac{1}{2}p),\ i = 2,\ldots,\ p.$$

Combining (6.3) - (6.6) we get the following lemma:

Lemma 6.1.

$$U_p(\lambda;\ n,\ m) = U_m(\lambda;\ n+m-p,\ p).$$

This generalizes the corresponding result when $\lambda = 0$ (see Anderson (1958), Theorem 8.4.2).

The distribution of Z_1 in (6.4) can be expressed as $b(0;\ \tfrac{1}{2}n,\ \tfrac{1}{2}m + K)$, where K is distributed as Poisson with the mean $\lambda/2$. Let $Z^{(\theta)}$ be a random variable distributed as $b(0;\ \tfrac{1}{2}n,\ \tfrac{1}{2}m + \theta)$, independently of $Z_2,\ldots,\ Z_p$. Define

$$U^*_{p,\theta} = Z^{(\theta)} Z_2 \ldots Z_p \, , \tag{6.7}$$

and denote its distribution by $U^*_{p,\theta}(n, m)$. Note that for $\theta = 0$,

$$U^*_{p,\theta}(n, m) = U_p(0; n, m) \tag{6.8}$$

We need the following lemma:

Lemma 6.2.

The density of $U^*_{p,\theta}$ has the strict monotone likelihood ratio (MLR) property in $-\theta$.

Proof:

It follows from Lemma 2.2 and the relation between the f and b distributions that the density of $Z^{(\theta)} \sim b(0; \frac{1}{2}n, \frac{1}{2}m + \theta)$ has the strict MLR property in $-\theta$. It is sufficient to prove the strict MLR property for $Z^{(\theta)}Z_2$; the rest follows by induction. Let the density functions of $Z^{(\theta)}$ and Z_2 be $g(\cdot, \theta)$ and $h(\cdot)$, respectively. Then the density of $Y = Z^{(\theta)}Z_2$ is given by

$$p(y; \theta) = \int g(z; \theta) h(y/z)(dz/z).$$

Using the fact that h is a beta density it can be checked that $h(y/z)$, as a function of y and z, is strictly totally positive of order 2 (i.e., has the strict MLR property). It now follows from Lemma 5 of Karlin (1956) that $p(y; \theta)$ has the strict MLR property in $-\theta$.

For $0 < \alpha < 1$, denote the power of the LRT by

$$\pi_\alpha(\lambda; m,n,p) = P\{U_p(\lambda; n,m) \leq U_p^{(1-\alpha)}(n,m)\} \, . \tag{6.9}$$

This has the series expansion

$$e^{-\lambda/2} \sum_{k=0}^{\infty} (\lambda/2)^k (1/k!) P\{U^*_{p,k}(n,m) \leq U_p^{(1-\alpha)}(n,m)\} \, . \tag{6.10}$$

The following is the main result of this Section.

Theorem 6.1.

(a) $\pi_\alpha(\lambda;\ m,\ n,\ p+q) < \pi_\alpha(\lambda;\ m,\ n,\ p)$ for $q > 0$.

(b) $\pi_\alpha(\lambda;\ m+s,\ n,\ p) < \pi_\alpha(\lambda;\ m,\ n,\ p)$ for $s > 0$.

Proof of Theorem 6.1(a).

We shall prove a stronger result than (a), namely

$$P\{U^*_{p+q,k}(n,\ m) \leq U^{(1-\alpha)}_{p+q}(n,\ m)\} < P\{U^*_{p,k}(n,\ m) \leq U^{(1-\alpha)}_{p}(n,\ m)\} \quad (6.11)$$

for any $k > 0$ and $q > 0$. By (6.7)

$$U^*_{p+q,\theta} = [Z^{(\theta)}Z_2 \ldots Z_p][Z_{p+1} \ldots Z_{p+q}], \quad (6.12)$$

and the distribution of $Z^{(\theta)}Z_2 \ldots Z_p$ is $U^*_{p,\theta}(n,\ m)$, independent of $Z_{p+1} \ldots Z_{p+q} \equiv Q$, say.

Consider two independent random variables X_1 and X_2 distributed as $U^*_{p,\theta}(n,\ m)$ and Q, respectively. By Neyman-Pearson lemma and lemma 6.2, the unique UMP size α test of $\theta = 0$ vs. $\theta > 0$ based on X_1 and X_2 has the critical region

$$X_1 \leq U^{(1-\alpha)}_{p}(n,\ m)\ , \quad (6.13)$$

since the distribution of X_1 is $U_p(0;\ n,\ m)$ when $\theta = 0$. In particular, this test is strictly more powerful than the test with the critical region

$$X_1X_2 \leq U^{(1-\alpha)}_{p+q}(n,\ m), \quad (6.14)$$

since the distribution of X_1X_2 is $U_{p+q}(0;\ n,m)$ when $\theta = 0$. Comparing powers of the critical regions (6.13) and (6.14) and setting $\theta = k$, we get (6.11)

To prove part (b) we need the following lemma which follows from Lemma 2.3.

Lemma 6.3.

Let X_1 and X_2 be distributed as the products of mutually independent beta variates as follows:

$$X_1 \sim b(0;\ \tfrac{1}{2}n,\ \tfrac{1}{2}m + \theta) \prod_{i=2}^{p} b(0;\ \tfrac{1}{2}(n-i+1),\ \tfrac{1}{2}m) = U^*_{p,\theta}(n,\ m)$$

$$X_2 \sim b(0;\ \tfrac{1}{2}(n+m) + \theta,\ \tfrac{1}{2}s) \prod_{i=2}^{p} b(0;\ \tfrac{1}{2}(n+m-i+1),\ \tfrac{1}{2}s).$$

(These 2p beta variates are taken to be mutually independent.) Then

$$X \equiv X_1X_2 \sim U^*_{p,\theta}\ (n,\ m+s).$$

Proof of Theorem 6.1(b).

Here again we shall prove a result stronger than (b), namely,

$$P\{U^*_{p,k}(n,\ m+s) \le U_p^{(1-\alpha)}(n,\ m+s)\} < P\{U^*_{p,k}(n,\ m) \le U_p^{(1-\alpha)}(n,m)\} \quad (6.15)$$

for any $k > 0$, $s > 0$. Define X_1, X_2 as in Lemma 6.3, and consider a random variable X_2^* distributed, independently of X_1, as the product of p mutually independent beta variates given by

$$X_2^* \sim \prod_{i=1}^{p} b(0;\ \tfrac{1}{2}(n+m-i+1),\ \tfrac{1}{2}s) = U_p(0;\ n+m,\ s). \qquad (6.16)$$

Consider the problem of testing $\theta = 0$ vs. $\theta > 0$ based on X_1 and X_2^*. By Neyman-Pearson lemma and Lemma 6.2, the unique UMP sized test has the critical region

$$X_1 \le U_p^{(1-\alpha)}(n,\ m)\ . \qquad (6.17)$$

This test is strictly more powerful than the test with the critical region

$$X_1X_2^* \le U_p^{(1-\alpha)}(n,\ m+s)\ , \qquad (6.18)$$

since the distribution of $X_1X_2^*$ is $U_p(0;\ n,\ m+s)$ when $\theta = 0$. Note that X_2 is stochastically larger than X_2^*. Hence the critical region (6.18) is more powerful than the size α critical region given by

$$X_1X_2 \le U_p^{(1-\alpha)}(n,\ m+s)\ . \qquad (6.19)$$

Comparing the power functions of the critical regions (6.17) and (6.19), and setting $\theta = k$, we get (6.15).

Notice that by Lemma 6.1, we have

$$\pi_\alpha(\lambda;\ m,\ n,\ p) = \pi_\alpha(\lambda;\ p,\ n+m-p,\ m)\ . \qquad (6.20)$$

This leads to the following corollary of Theorem 6.1.

Corollary 6.1.

(a) $\pi_\alpha(\lambda;\ p+q,\ n+m-p-q,\ m) < \pi_\alpha(\lambda;\ p,\ n+m-p,\ m)$ for $q > 0$.

(b) $\pi_\alpha(\lambda;\ p,\ n+m-p+s,\ m+s) < \pi_\alpha(\lambda;\ p,\ n+m-p,\ m)$ for $s > 0$.

Remark.

Since $U_1(\lambda;\ n,\ m) = b(\lambda;\ \frac{1}{2}n,\ \frac{1}{2}m)$, we have

$$\pi_\alpha(\lambda;\ m,\ n,\ 1) = \pi_\alpha(\lambda;\ m,\ n), \tag{6.21}$$

these two sides being defined in (6.9) and Section 2. Applying (6.20), we get also

$$\pi_\alpha(\lambda;\ 1,\ n,\ p) = \pi_\alpha(\lambda;\ p,\ n-p+1)\ . \tag{6.22}$$

From (6.21) and (6.22) it is seen that Theorem 6.1(a) and (b) are generalizations of Corollary 2.1 and Theorem 2.1(b), respectively. It is conjectured that Theorem 2.1(a) can also be generalized as

$$\pi_\alpha(\lambda;\ m,\ n,\ p) < \pi_\alpha(\lambda;\ m,\ n+s,\ p)$$

for $s > 0$, but the earlier method of proof fails.

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