EXPONENTIALLY GROWING SOLUTIONS FOR INVERSE PROBLEMS IN PDE

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Abstract

To study fixed energy inverse problems in potential scattering for a general class of partial differential operators, one needs solutions $\phi(x,\zeta)$ of $(P_0(D)+q)\phi = \lambda\phi$ which behave like $e^{ix\cdot\zeta}$ with $\zeta \in \mathbf{C}^n$ and $P_0(\zeta) = \lambda$. The construction of such solutions requires a generalized limiting absorption estimate for $P_0(D,\zeta) = P_0(D+\zeta) - \lambda$. In this paper, we give such estimates for a class of partial differential operators $P_0(D)$.

1 Introduction

The direct scattering theory for operators of the form $P_0(D) + V(x,D)$, where $P_0(D)$ is a partial differential operator in $\mathbf{R}^n$ with real constant coefficients which is simply characteristic (see Definition 1.1 below) and V is a short-range perturbation, has been developed successfully by Agmon and

Hörmander (see [H]) and many other authors. One of the key ingredients of such a theory is an estimate for the resolvent $(P_0(D)-z)^{-1}$ which remains valid as z approaches the real axis. This clearly can't happen on the space L^2 when z approaches the spectrum of $P_0(D)$. But if $R_0(z)=(P_0(D)-z)^{-1}$ is viewed as an operator from a suitable space X smaller than L^2 to another space Y larger than L^2, then its norm can be shown to remain bounded, independent of the distance from z to the real axis (if z stays in a bounded subset K contained in the complex plane $\mathbf{C}$),

$$\|(P_0(D)-z)^{-1}f\|_Y \leq C\|f\|_X. \tag{1.1}$$

Definition 1.1 *Let $P(\xi)$ be a real valued polynomial of degree m in $\xi \in \boldsymbol{R}^n$ such that*

$$\Lambda(P_0)=\{\eta \in \boldsymbol{R}^n : P_0(\xi+\eta)\equiv P_0(\xi)\}=\{0\}.$$

P_0 will be called simply characteristic if

$$\tilde{P}_0(\xi) \leq C(\sum_{|\alpha|\leq 1} |P_0^{(\alpha)}(\xi)|+1), \quad \xi \in \boldsymbol{R}^n, \tag{1.2}$$

where

$$\tilde{P}_0(\xi)=\sum_{0\leq|\alpha|\leq m} |P_0^{(\alpha)}(\xi)| \ .$$

The result of the form (1.1) is known as the "limiting absorption principle". It turns out that weighted-L^2 spaces L^2_δ and $L^2_{-\delta}$ with $\delta > 1/2$ are approriate spaces for X and Y, respectively, which were first proved by Agmon in [Ag]. Later Agmon and Hörmander showed that the following class of spaces B_s and their duals $B_s^*(-\infty < s < \infty)$:

$$B_s=\{v \in L^2_{loc}(\mathbf{R}^n) : \quad \sum_{j=1}^{\infty} R_j^s(\int_{\Omega_j} |v|^2 dx)^{1/2} < \infty\} \tag{1.3}$$

$$B_s^* = \{u \in L^2_{loc}(\mathbf{R}^n) : \quad \sup_{j \geq 1} R_j^{-s} \Big(\int_{\Omega_j} |u|^2 dx\Big)^{1/2} < \infty\} \tag{1.4}$$

where

$$R_0 = 0, \quad R_j = 2^{j-1}, \quad j = 1, 2, \cdots$$

$$\Omega_j = \{x \in \mathbf{R}^n : R_{j-1} < |x| < R_j\}, \quad j = 1, 2, \cdots$$

capture quite precisely the behavior of the resolvent operator at infinity.

To study fixed energy inverse problems in potential scattering for a general class of differential operators $P_0(D)$, we want solutions $\phi(x, \zeta)$ of $(P_0(D) + q)\phi = \lambda\phi$ which behave like $e^{ix\cdot\zeta}$ with $\zeta \in \mathbf{C}^n$, $P_0(\zeta) = \lambda$. The construction of such solutions requires a generalized limiting absorption estimate for $P_0(D, \zeta) = P_0(D + \zeta) - \lambda$. The first such estimate was obtained by Sylvester and Uhlmann ([S-U]) for the Laplacian $-\Delta$ at zero energy ($\zeta^2 = 0$):

$$\|(-\Delta - 2i\zeta \cdot \nabla)^{-1} f\|_{L^2_{-\delta}} \leq \frac{C}{|\zeta|} \|f\|_{L^2_{1-\delta}}, \quad 0 < \delta < 1. \tag{1.5}$$

Motivated by inverse scattering and the construction of exponentially growing solutins, an analogue of limitimg absorption principle for a general class of complex polynomials was developed in paper [Liu]. Based on those estimates, we are able to give estimates similar to (1.5), which are given in the following theorem, for a class of partial differential operators $P(D)$. We will denote by $\mathcal{F}$ or $\hat{}$ the Fourier transform and $\mathcal{F}^{-1}$ or $\vee$ its inverse.

Theorem 1.1 *If $P(\xi)$ is real-valued elliptic with $P(\xi, \zeta)$* ***uniformly simply characteristic*** *on M_ζ (see the definition 1.2 below) and let K be a compact subset of $\mathbf{C}$ containing no* ***critical value*** *of P in the sense given in the* *definition 1.3 below, then for any real number s satisfying $0 \leq s \leq 1$,*

$$\|R(z,\zeta)f\|_{B^*_{1-s}} \leq C(s,c_P) \sup_{\xi\in\boldsymbol{R}^n} \frac{1}{\tilde{P}(\xi,\zeta)} \|f\|_{B_s}, \quad f \in B_s, \quad z \in K,$$

where $R(z,\zeta) = \mathcal{F}^{-1}((P(\cdot+\zeta)-z)^{-1}\hat{f})$, with $C(s,c_P)$ independent of ζ, and $\sup_{\xi\in\boldsymbol{R}^n} \frac{1}{\tilde{P}(\xi,\zeta)} \to 0$ as $\zeta \to \infty$ in M_ζ.

Definition 1.2 *Let $P(\xi) = P_1(\xi) + iP_2(\xi)$, $\xi \in \boldsymbol{R}^n$ be an mth order polynomial with complex coefficients. We define a* ***simply characteristic*** *polynomial P to be one which satisfies*

$$\tilde{P}(\xi) \leq c_P(|P(\xi) - z_0| + |\nabla P(\xi)|) \tag{1.6}$$

for all $\xi \in \boldsymbol{R}^n$ and some $z_0 \in \boldsymbol{C}$ where

$$\tilde{P}(\xi) = \sum_{|\alpha|\leq m} |P^{(\alpha)}(\xi)|$$

and

$$\sum_{|\alpha|=1} |P^{(\alpha)}(\xi)| \overset{def.}{=} |\nabla P(\xi)| \overset{def.}{=} \left[\sum_{i\neq j, 1\leq i,j\leq m} \left| \det \begin{pmatrix} \frac{\partial P_1}{\partial \xi_i} & \frac{\partial P_1}{\partial \xi_j} \\ \frac{\partial P_2}{\partial \xi_i} & \frac{\partial P_2}{\partial \xi_j} \end{pmatrix} \right|^2 \right]^{1/4}.$$

The family $P(\xi,\zeta)$, where $P(\xi,\zeta) = P(\xi+\zeta) - z$, $z \in \boldsymbol{C}$ and $\zeta \in M_\zeta = \{\zeta \in \boldsymbol{C}^n : P(\zeta) = z\}$ is ***uniformly simply characteristic*** *if (1.6) holds with c_P independent of ζ.*

Definition 1.3 *If $\nabla P_1(\xi)$ and $\nabla P_2(\xi)$ are not linearly independent at some point $\xi \in \{\xi \in \boldsymbol{R}^n : P(\xi) - z = 0\}$, we say that the value z is a* ***critical value*** *of P.*

The next result gives a general class of elliptic polynomials which are **uniformly simply characteristic.**

Theorem 1.2 *Let* $P(\xi) = \sum_{|\alpha| \leq m} a_\alpha \xi^\alpha$ *be an m-th order real-valued elliptic polynomial and* V, $\hat{V}$, U *and* $\hat{U}$ *be defined by*

$$V = \{\zeta \in \boldsymbol{C}^n : P(\zeta) = \lambda\}$$

$$\hat{V} = \overline{\{\hat{\zeta} = \zeta/|\zeta| : \zeta \in V, |\zeta| \neq 0\}}$$

$$U = \{\zeta \in \boldsymbol{C}^n : P_m(\zeta) = \sum_{|\alpha|=m} a_\alpha \zeta^\alpha = 0\}$$

$$\hat{U} = \{\hat{\zeta} = \zeta/|\zeta| : \zeta \in U, |\zeta| \neq 0\}.$$

Let $P_m(\eta, \hat{\zeta}) = \sum_{|\alpha|=m} a_\alpha (\eta + \hat{\zeta})^\alpha$ *and suppose that zero is not a* ***critical value*** *of* $P_m(\eta, \hat{\zeta})$ *for all* $\hat{\zeta} \in \hat{U}$. *Then zero is not a* ***critical value*** *of* $P(\xi, \zeta) = P(\xi + \zeta) - \lambda$ *for fixed* $\lambda \in \boldsymbol{R}$ *and all* $\zeta \in V$, $|\zeta| \geq R$ *for some constant* $R > 0$ *and* P *is* ***uniformly simply characteristic.*** *For* $0 \leq s \leq 1$ *we then have*

$$\|\mathcal{F}^{-1}((P(\cdot + \zeta) - \lambda)^{-1} \hat{f})\|_{B^*_{1-s}} \leq C(s, c_P) \sup_{\xi \in \boldsymbol{R}^n} \frac{1}{\tilde{P}(\xi, \zeta)} \|f\|_{B_s}, \quad f \in B_s.$$

Moreover, $\sup_{\xi \in \boldsymbol{R}^n} \frac{1}{\tilde{P}(\xi,\zeta)} \to 0$ *as* $|\zeta| \to \infty$ *in* $\{\zeta \in \boldsymbol{C}^n : P(\zeta) = \lambda\}$.

Section 2 and Section 3 are devoted to the proofs of Theorem 1.1 and Theorem 1.2, respectively. In Section 4, we'll discuss some examples for which generalized limiting absorption estimates hold. Exponentially growing solutions for the differential equation

$$(P_0(D) + q - \lambda)u = 0 \quad ,$$

where $P_0(\xi)$ is an elliptic polynomial, will be constructed in Section 5.

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2 The Proof of Theorem 1.1

Let

$$P(\xi) = \sum_{|\alpha| \le m} a_\alpha \xi^\alpha$$

be an m-th order elliptic polynomial with constant coefficients and z be a complex number in C. Recall that $P(\xi, \zeta)$ is defined as:

$$P(\xi, \zeta) = \sum_{|\alpha| \le m} a_\alpha (\xi + \zeta)^\alpha - z.$$

In paper [Liu], we obtained an analogue of limitimg absorption principle for simply characteristic complex polynomials. Since we'll need it to prove our theorems, we state it here without proof.

Theorem 2.1 *Assume that P is* ***simply characteristic*** *and let K be a compact subset of $\boldsymbol{C}$ containing no* ***critical value*** *of P in the sense given in the definition 1.3 in Introduction. If $f \in B_s$, $0 \le s \le 1$, it follows that $R(z)f = \mathcal{F}^{-1}((P(\cdot) - z)^{-1}\hat{f})$ belongs to B^*_{1-s}, for $z \in K$ and we have the bound:*

$$\|R(z)f\|_{B^*_{1-s}} \le C(s, c_P) \sup_{\xi \in \boldsymbol{R}^n} \frac{1}{\tilde{P}(\xi)} \|f\|_{B_s}, \quad z \in K \quad ,$$

where c_P is in condition (1.6) in the Definition 1.2.

As we remarked in paper [Liu], if $\{P(\xi,\zeta)\}$ is a family of polynomials of ξ depending on a parameter ζ in $M_\zeta \subset \mathbf{C}^n$ and constant c_P in condition (1.6) is independent of $\zeta \in M_\zeta$, then

$$|(R(z,\zeta)f,g)| \leq C(s,c_P) \sup_{\xi\in\mathbf{R}^n} \frac{1}{\tilde{P}(\xi,\zeta)} \|f\|_{B_s}\|g\|_{B_{1-s}} \tag{2.7}$$

for all $f \in B_s$ and $g \in B_{1-s}$ with $0 \leq s \leq 1$.

Now we begin to prove Theorem 1.1. Since c_P is independent of ζ, the first part of the theorem is implied by inequality (2.7). So it remains to prove the second part.

Since $P(\xi) = \sum_{|\alpha|\leq m} a_\alpha \xi^\alpha$ is elliptic, for each $1 \leq j \leq n$, there is $|\alpha| = m$ with $a_\alpha \neq 0$ and $\alpha_j \neq 0$, denoting this α by $\alpha^{(j)}$. Then for $\beta^{(j)} = (\alpha_1^{(j)}, \alpha_2^{(j)}, \cdots, \alpha_j^{(j)} - 1, \cdots, \alpha_n^{(j)})$

$$\frac{\partial^{\beta^{(j)}}}{\partial\xi^{\beta^{(j)}}} P(\xi,\zeta) = m! a_{\alpha^{(j)}}(\xi_j + \zeta_j) + c^{(j)}$$

(note: $|\beta^{(j)}| = m - 1$) where

$$c^{(j)} = \frac{\partial^{\beta^{(j)}}}{\partial\xi^{\beta^{(j)}}} \Big(\sum_{|\alpha|\leq m-1} a_\alpha(\xi+\zeta)^\alpha - \lambda \Big),$$

which is a constant. Then

$$\begin{aligned} \tilde{P}(\xi,\zeta) &\geq \sum_{j=1}^{n} |m! a_{\alpha^{(j)}}(\xi_j + \zeta_j) + c^{(j)}| \\ &\geq \sum_{j=1}^{n} [m! |a_{\alpha^{(j)}}(\xi_j + \zeta_j)| - |c^{(j)}|] \\ &= m! \sum_{j=1}^{n} |a_{\alpha^{(j)}}| |(\xi_j + \zeta_j)| - \sum_{j=1}^{n} |c^{(j)}|. \end{aligned}$$

Define vectors $\mathbf{a}$ and $\mathbf{b}(\xi,\zeta)$ in $\mathbf{R}^n$ by

$$\mathbf{a} = (|a_{\alpha(1)}|, |a_{\alpha(2)}|, \cdots, |a_{\alpha(n)}|)$$

$$\mathbf{b}(\xi,\zeta) = (|\xi_1+\zeta_1|, |\xi_2+\zeta_2|, \cdots, |\xi_n+\zeta_n|).$$

Then

$$\sum_{j=1}^{n} |a_{\alpha(j)}||(\xi_j+\zeta_j)| = (\mathbf{a},\mathbf{b})$$

where $(\mathbf{a},\mathbf{b})$ means inner product of vectors $\mathbf{a}$ and $\mathbf{b}$. Since each component of $\mathbf{a}$ is positive constant and each component of $\mathbf{b}$ is nonnegative, there is a constant $C_0 > 0$ such that $(\mathbf{a},\mathbf{b}) \geq C_0|\mathbf{a}||\mathbf{b}|$ for all $\xi \in \mathbf{R}^n$ and $\zeta \in \mathbf{C}^n$. So if $\zeta \in \{\zeta \in \mathbf{C}^n : P(\zeta) = \lambda\}$ with $|\zeta|$ sufficiently large (note that $|\zeta_I| \to \infty$ in $M_\zeta = \{\zeta \in \mathbf{C}^n : \quad P(\zeta) = z\}$)

$$\begin{aligned}
\tilde{P}(\xi,\zeta) &\geq m!\sum_{j=1}^{n} |a_{\alpha(j)}||(\xi_j+\zeta_j)| - \sum_{j=1}^{n}|c^{(j)}| \\
&= m!(\mathbf{a},\mathbf{b}) - \sum_{j=1}^{n}|c^{(j)}| \\
&\qquad (\text{ note: } |\mathbf{b}| = \sqrt{\sum_{j=1}^{n}(\xi_j+\zeta_{R,j})^2 + \sum_{j=1}^{n}\zeta_{I,j}^2} \ ) \\
&\geq m!|\mathbf{a}||\zeta_I|C_0 - \sum_{j=1}^{n}|c^{(j)}| \\
&\geq \frac{m!}{2}|\mathbf{a}||\zeta_I|C_0
\end{aligned}$$

Thus

$$\sup_{\xi\in\mathbf{R}^n} \frac{1}{\tilde{P}(\xi,\zeta)} \leq \frac{2}{m!|\mathbf{a}||\zeta_I|C_0} \to 0$$

as $|\zeta_I| \to \infty$, and the proof of Theorem 1.1 is finished. ♯

3 The Proof of Theorem 1.2

Recall that

$$V = \{\zeta \in \mathbf{C}^n : P(\zeta) = \lambda\} \tag{3.8}$$

$$\hat{V} = \overline{\{\hat{\zeta} = \zeta/|\zeta| : \zeta \in V, |\zeta| \neq 0\}} \tag{3.9}$$

$$U = \{\zeta \in \mathbf{C}^n : P_m(\zeta) = \sum_{|\alpha|=m} a_\alpha \zeta^\alpha = 0\} \tag{3.10}$$

$$\hat{U} = \{\hat{\zeta} = \zeta/|\zeta| : \zeta \in U, |\zeta| \neq 0\} \tag{3.11}$$

To prove Theorem 1.2, we first prove that if $\hat{\zeta}_0 \in \hat{V}$ and $\hat{\zeta}_0 \notin \{\hat{\zeta} : \zeta \in V\}$, then $\hat{\zeta}_0 \in \hat{U}$.

It is easy to see that $U = \{r\hat{U} : r \geq 0\}$ and $\hat{U}$ is closed in the unit sphere S^{2n-1}. In fact, if $\hat{\zeta}_0 = \lim_{l\to\infty} \hat{\zeta}_l$ and $\hat{\zeta}_l \in \hat{U}$, then

$$\sum_{|\alpha|=m} a_\alpha \zeta_0^\alpha = \lim_{l\to\infty} \sum_{|\alpha|=m} a_\alpha \zeta_l^\alpha = 0.$$

$\hat{\zeta}_0 \in \hat{V}$ implies there is a sequence $\{\zeta_l\} \subset V$ such that $\hat{\zeta}_0 = \lim_{l\to\infty} \frac{\zeta_l}{|\zeta_l|}$ with $\zeta_l \neq 0$.

<u>Claim:</u> $|\zeta_l| \to \infty$ as $l \to \infty$.

If $\{|\zeta_l|\}$ is bounded, there is a subsequence $\{|\zeta_{lk}|\}$ and $r_0 > 0$ such that $r_0 = \lim_{k\to\infty} |\zeta_{lk}|$. Then

$$P(r_0\hat{\zeta}_0) - \lambda = \lim_{k\to\infty} P(|\zeta_{lk}|\hat{\zeta}_{lk}) - \lambda = 0 \quad .$$

This implies $\zeta_0 = r_0\hat{\zeta}_0 \in V$ and $\hat{\zeta}_0 \in \{\hat{\zeta} : \zeta \in V\}$ which is a contradiction.

The ellipticity of P implies that $|\zeta| \geq |Im\zeta| \to \infty$ in V. Since $|\zeta_l| \to \infty$ and $P(\zeta_l) - \lambda = 0$,

$$\sum_{|\alpha|=m} a_\alpha(|\zeta_l|\hat{\zeta}_l)^\alpha + \sum_{|\alpha|\leq m-1} a_\alpha(|\zeta_l|\hat{\zeta}_l)^\alpha - \lambda = 0$$

implies

$$\sum_{|\alpha|=m} a_\alpha\hat{\zeta}_l^\alpha + \frac{1}{|\zeta_l|}\sum_{|\alpha|=m-1} a_\alpha\hat{\zeta}_l^\alpha + \cdots + \frac{a_0-\lambda}{|\zeta_l|^m} = 0.$$

Letting $l \to \infty$, we have

$$\sum_{|\alpha|=m} a_\alpha\hat{\zeta}_0^\alpha = -\lim_{l\to\infty}[\frac{1}{|\zeta_l|}\sum_{|\alpha|=m-1} a_\alpha\hat{\zeta}_l^\alpha + \cdots + \frac{a_0-\lambda}{|\zeta_l|^m}] = 0.$$

So $\hat{\zeta}_0 \in \hat{U}$ and we have

$$\hat{V} \subset \{\hat{\zeta} : \zeta \in V\} \cup \hat{U}.$$

Write $\xi = |\zeta|\eta$ and $\zeta = |\zeta|\hat{\zeta}$. Then

$$\begin{aligned} P(\xi,\zeta) &= \sum_{|\alpha|=m} a_\alpha(\xi+\zeta)^\alpha + \cdots + a_0 - \lambda \qquad (3.12) \\ &\quad (\xi = |\zeta|\eta, \zeta = |\zeta|\hat{\zeta}) \\ &= |\zeta|^m \sum_{|\alpha|=m} a_\alpha(\eta+\hat{\zeta})^\alpha + |\zeta|^{m-1}\sum_{|\alpha|=m-1} a_\alpha(\eta+\hat{\zeta})^\alpha + \cdots + a_0 - \lambda \\ &= |\zeta|^m[\sum_{|\alpha|=m} a_\alpha(\eta+\hat{\zeta})^\alpha + \frac{1}{|\zeta|}\sum_{|\alpha|=m-1} a_\alpha(\eta+\hat{\zeta})^\alpha + \cdots + \frac{a_0-\lambda}{|\zeta|^m}] \end{aligned}$$

For each $r > 0$ and $\hat{\zeta} \in S^{2n-1}$, define

$$\begin{aligned} Q(\eta,\hat{\zeta},r) &= \sum_{|\alpha|=m} a_\alpha(\eta+\hat{\zeta})^\alpha + \cdots + \frac{a_0-\lambda}{r^m} \\ P_m(\eta,\hat{\zeta}) &= \sum_{|\alpha|=m} a_\alpha(\eta+\hat{\zeta})^\alpha. \end{aligned}$$

Since $\hat{\zeta} \in \hat{U}$ iff $P_m(\hat{\zeta}) = \sum_{|\alpha|=m} a_\alpha \hat{\zeta}^\alpha = 0$ and $\zeta \in V$ iff

$$\sum_{|\alpha|=m} a_\alpha \hat{\zeta}^\alpha + \frac{1}{|\zeta|} \sum_{|\alpha|=m-1} a_\alpha \hat{\zeta}^\alpha + \cdots + \frac{a_0 - \lambda}{|\zeta|^m} = 0,$$

$\hat{V}$ is very close to $\hat{U}$ in $S^{2n-1} = \{|\hat{\zeta}| = 1 : \zeta \in \mathbf{C}^n\}$ when $|\zeta|$ is sufficiently large.

Now we consider $Q(\eta, \hat{\zeta}, r)$ and $P_m(\eta, \hat{\zeta})$. Since $P_m(\eta, \hat{\zeta})$ is elliptic as function of η and $\hat{U}$ is a bounded closed set,

$$W_1 = \{(\eta, \hat{\zeta}) \in \mathbf{R}^n \times \mathbf{C}^n : P_m(\eta, \hat{\zeta}) = 0, \hat{\zeta} \in \hat{U}\}$$

is a bounded closed subset in $\mathbf{R}^n \times \mathbf{C}^n$ (the coefficients of $P_m(\eta, \hat{\zeta})$ are bounded for all $\hat{\zeta} \in \hat{U}$).

Suppose

$$\begin{aligned} P_m(\eta, \hat{\zeta}) &= \sum_{|\alpha|=m} a_\alpha (\eta + \hat{\zeta})^\alpha \\ &= P_{m1}(\eta, \hat{\zeta}) + i P_{m2}(\eta, \hat{\zeta}) \end{aligned}$$

and

$$\begin{aligned} |\nabla_\eta P_m(\eta, \hat{\zeta})| &= \left[\textstyle\sum_{j \neq l} \left| \begin{array}{cc} \frac{\partial P_{m1}(\eta,\hat{\zeta})}{\partial \eta_j} & \frac{\partial P_{m1}(\eta,\hat{\zeta})}{\partial \eta_l} \\ \frac{\partial P_{m2}(\eta,\hat{\zeta})}{\partial \eta_j} & \frac{\partial P_{m2}(\eta,\hat{\zeta})}{\partial \eta_l} \end{array} \right|^2 \right]^{1/4} \\ &\neq 0 \end{aligned} \tag{3.13}$$

on $\{\eta \in \mathbf{R}^n : P_m(\eta, \hat{\zeta}) = 0\}$ for all $\hat{\zeta} \in \hat{U}$. Since W_1 is closed and bounded, we can find a bounded closed neighborhood K_1 of W_1 and a number $\epsilon > 0$ such that

$$|\nabla_\eta P_m(\eta, \hat{\zeta})| \geq \epsilon > 0$$

on K_1 and (note: $|P_m(\eta, \hat{\zeta})| \to \infty$ as $|\eta| \to \infty$)

$$|P_m(\eta, \hat{\zeta})| \geq \epsilon$$

on $K_2 \overset{def}{=} \{(\eta, \hat{\zeta}) \in \mathbf{R}^n \times \mathbf{C}^n : (\eta, \hat{\zeta}) \in (\mathbf{R}^n \times \mathbf{C}^n) \backslash K_1^0 \quad \text{and} \quad \hat{\zeta} \in K_1|_{\mathbf{C}^n}\}$ where K_1^0 means the interior of K_1 and $K_1|_{\mathbf{C}^n}$ means the projection of K_1 onto $\mathbf{C}^n$ (see Figure 1).

So we can choose $R > 0$ large enough such that for all $r \geq R$

$$|Q(\eta, \hat{\zeta}, r)| \geq \epsilon/2 \quad \text{on} \quad K_2$$

and

$$|\nabla_\eta Q(\eta, \hat{\zeta}, r)| \geq \epsilon/2 \quad \text{on} \quad K_1.$$

Of course we can choose R large enough such that $\hat{V} \subset K_1$.Therefore, $\nabla_\eta Re(Q(\eta, \hat{\zeta}, r))$ and $\nabla_\eta Im(Q(\eta, \hat{\zeta}, r))$ are linearly independent on $\{\eta \in \mathbf{R}^n : Q(\eta, \hat{\zeta}, r) = 0\}$ for all $\hat{\zeta} \in \hat{V}$ and $r \geq R$. Since $\hat{V}$ is closed and bounded and $Q(\eta, \hat{\zeta}, r)$ is elliptic with uniformly bounded coefficients for all $\hat{\zeta} \in \hat{V}$ and $r \geq R$, we can find $C > 0$ which is independent of $\hat{\zeta}$ and $r \geq R$ such that

$$\tilde{Q}(\eta, \hat{\zeta}, r) \leq C(|Q(\eta, \hat{\zeta}, r)| + |\nabla_\eta Q(\eta, \hat{\zeta}, r)|)$$

for all $\eta \in \mathbf{R}^n$ where

$$\tilde{Q}(\eta, \hat{\zeta}, r) = |Q(\eta, \hat{\zeta}, r)| + |\nabla_\eta Q(\eta, \hat{\zeta}, r)| + \sum_{2 \leq |\alpha| \leq m} |Q^{(\alpha)}(\eta, \hat{\zeta}, r)| \ .$$

So $Q(\eta, \hat{\zeta}, r)$ is uniformly **simply characteristic** for all $\hat{\zeta} \in \hat{V}$ and $r \geq R$. This implies for all $2 \leq |\beta| \leq m$,

$$|Q^{(\beta)}(\eta, \hat{\zeta}, r)| \quad = \quad |\frac{\partial}{\partial \eta^\beta} Q(\eta, \hat{\zeta}, r)|$$

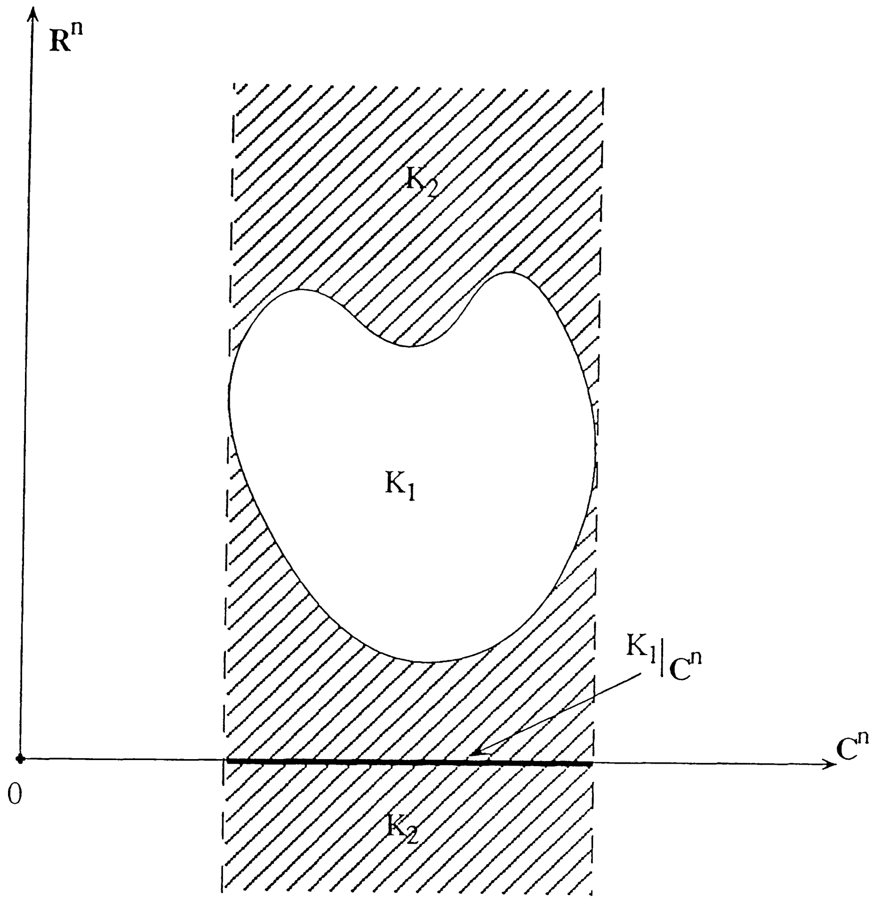

Figure 1: The graph of K_1 and K_2

$$
\begin{aligned}
&= \left|\frac{\partial}{\partial\eta^{\beta}}\Big[\sum_{|\alpha|=m} a_{\alpha}(\eta+\hat{\zeta})^{\alpha} + \frac{1}{r}\sum_{|\alpha|=m-1} a_{\alpha}(\eta+\hat{\zeta})^{\alpha} + \cdots + \frac{a_0-\lambda}{r^m}\Big]\right| \\
&\leq C(|Q(\eta,\hat{\zeta},r)| + |\nabla_{\eta}Q(\eta,\hat{\zeta},r)|),
\end{aligned}
$$

and then

$$
|Q^{(\beta)}(\frac{\xi}{r},\hat{\zeta},r)| \leq C(|Q(\frac{\xi}{r},\hat{\zeta},r)| + |\nabla_{\eta}Q(\frac{\xi}{r},\hat{\zeta},r)|). \tag{3.14}
$$

For any $r \geq R$, there exists $\hat{\zeta} \in \hat{V}$ such that $\zeta = r\hat{\zeta}$ satisfies $P(\zeta) = \lambda$. Since

$$
P(\xi,\zeta) = |\zeta|^m Q(\eta,\hat{\zeta},r), \quad \eta = \frac{\xi}{|\zeta|},
$$

$P(\xi,\zeta) = 0$ iff $Q(\eta,\hat{\zeta},|\zeta|) = 0$ and for each β, $1 \leq |\beta| \leq m$,

$$
\begin{aligned}
\frac{\partial P}{\partial\xi^{\beta}}(\xi,\zeta) &= |\zeta|^m \frac{\partial Q}{\partial\eta^{\beta}}(\eta,\hat{\zeta},|\zeta|)\frac{1}{|\zeta|^{|\beta|}} \\
&= |\zeta|^{m-|\beta|}\frac{\partial Q}{\partial\eta^{\beta}}(\eta,\hat{\zeta},|\zeta|)\ .
\end{aligned}
$$

In particular

$$
|\nabla_{\xi}P(\xi,\zeta)| = |\zeta|^{m-1}|\nabla_{\eta}Q(\eta,\hat{\zeta},|\zeta|)|\ .
$$

Inequality (3.14) implies that for each $2 \leq |\beta| \leq m$

$$
\frac{1}{|\zeta|^{m-|\beta|}}\left|\frac{\partial P}{\partial\xi^{\beta}}(\xi,\zeta)\right| \leq C(\frac{1}{|\zeta|^m}|P(\xi,\zeta)| + \frac{1}{|\zeta|^{m-1}}|\nabla_{\xi}P(\xi,\zeta)|).
$$

This implies

$$
\left|\frac{\partial P}{\partial\xi^{\beta}}(\xi,\zeta)\right| \leq C(\frac{1}{|\zeta|^{|\beta|}}|P(\xi,\zeta)| + \frac{1}{|\zeta|^{|\beta|-1}}|\nabla_{\xi}P(\xi,\zeta)|).
$$

Therefore,

$$
\begin{aligned}
\tilde{P}(\xi,\zeta) &= |P(\xi,\zeta)| + |\nabla P(\xi,\zeta)| + \sum_{|\beta|\geq 2}|P^{(\beta)}(\xi,\zeta)| \\
&\leq C(|P(\xi,\zeta)| + |\nabla P(\xi,\zeta)| \\
&\qquad + \sum_{|\beta|\geq 2}[\frac{1}{|\zeta|^{|\beta|}}|P(\xi,\zeta)| + \frac{1}{|\zeta|^{|\beta|-1}}|\nabla_{\xi}P(\xi,\zeta)|]) \\
&\leq C^*(|P(\xi,\zeta)| + |\nabla P(\xi,\zeta)|)
\end{aligned}
$$

for all $\xi \in \mathbf{R}^n$, and $\zeta \in \{\zeta \in \mathbf{C}^n : P(\zeta) = \lambda\}$ with $|\zeta| \geq R$; i.e., $P(\xi, \zeta)$ is **simply characteristic** with constant C^* independent of $\zeta \in \{\zeta \in \mathbf{C}^n : P(\zeta) = \lambda, |\zeta| \geq R\}$.

This finishes the proof of the first part. The second part follows from Theorem 1.1. ♯

4 Examples

Let $P(\xi)$ be an mth order polynomial in $\xi \in \mathbf{R}^n$ and $\zeta \in M_\zeta = \{\zeta \in \mathbf{C}^n : P(\zeta) = \lambda\}$.

If $P(\xi + \zeta)$ is **simply characteristic** and λ is not a **critical value** for all ζ in M_ζ, then Theorem 2.1 implies

$$\|(P(D+\zeta) - \lambda)^{-1} f\|_{B^*_{1-s}} \leq C_s(\zeta) \sup_{\xi \in \mathbf{R}^n} \frac{1}{\tilde{P}(\xi, \zeta)} \|f\|_{B_s}$$

for $0 \leq s \leq 1$, where

$$P(\xi, \zeta) = P(\xi + \zeta) - \lambda$$

and

$$\tilde{P}(\xi, \zeta) = \sum_{|\alpha| \leq m, |\alpha| \neq 1} |P^{(\alpha)}(\xi, \zeta)| + |\nabla_\xi P(\xi, \zeta)| \quad .$$

Thus if $C_s(\zeta)$ is independent of $\zeta \in M_\zeta$ and

$$\sup_{\xi \in \mathbf{R}^n} \frac{1}{\tilde{P}(\xi, \zeta)} \to 0 \tag{4.15}$$

as $|\zeta| \to \infty$ in M_ζ, we have a control on large ζ of solution u to the differential equation:

$$(P(D+\zeta) - \lambda) u = f \in B_s, \quad 0 \leq s \leq 1 \quad .$$

Of course not every polynomial $P(\xi)$ has property (4.15). Theorem 1.1 and Theorem 1.2 give a class of elliptic polynomials which have the property (4.15). But there are many examples of non-elliptic polynomials which have the property (4.15). We are going to calculate $\sup_{\xi\in\mathbf{R}^n}\frac{1}{P(\xi,\zeta)}$ on sevaral examples and show that (4.15) holds at least for certain direction ζ in M_ζ (this turns out to be sufficient for the application to inverse problems).

Example 1: The Elliptic Polynomial $P(\xi) = \xi^2$

For this polynomial, the control on $\zeta \in M_\zeta = \{\zeta \in \mathbf{C}^n : \zeta^2 = k^2 \geq 0\}$ for large ζ was obtained by Sylvester and Uhlmann ([S-U]), considered $\zeta^2 = 0$. More generally, it can be easily shown that the control on large ζ holds for all $|\zeta_I|^2 \geq \rho$ and $|\zeta_R|^2 \geq \rho$, $\rho > 0$ and this fact will be used in Example 4.

Let ρ be a positive number and suppose $\zeta = \zeta_R + i\zeta_I \in \mathbf{C}^n$ is such that $|\zeta_I|^2 \geq \rho$, $|\zeta_R|^2 \geq \rho$. Let $\lambda > 0$ and

$$\begin{aligned} P(\xi,\zeta) &= P(\xi+\zeta) - \lambda \\ &= (\xi+\zeta)^2 - \lambda \\ &= (\xi+\zeta_R)^2 - \zeta_I^2 - \lambda + 2i(\xi+\zeta_R)\cdot\zeta_I \end{aligned}$$

$$\begin{aligned} P_1(\xi,\zeta) &= Re(P(\xi,\zeta)) = (\xi+\zeta_R)^2 - \zeta_I^2 - \lambda \\ P_2(\xi,\zeta) &= Im(P(\xi,\zeta)) = 2\zeta_I\cdot(\xi+\zeta_R) \\ \nabla P_1(\xi,\zeta) &= 2(\xi+\zeta_R) \\ \nabla P_2(\xi,\zeta) &= 2\zeta_I. \end{aligned}$$

A straight forward calculation shows that

$$\begin{aligned} \|R(\lambda,\zeta)f\|_{B^*_{1-s}} &\leq C_s \sup_{\xi\in\mathbf{R}^n} \frac{1}{\tilde{P}(\xi,\zeta)}\|f\|_{B_s} \\ &\leq \frac{C_s}{C(\rho,\zeta)}\|f\|_{B_s} \end{aligned}$$

for $0 \leq s \leq 1$, where

$$R(\lambda,\zeta)f = \mathcal{F}^{-1}(\frac{1}{(\cdot+\zeta)^2-\lambda}\hat{f}) \quad ,$$

$$\begin{aligned} \tilde{P}(\xi,\zeta) &= |P(\xi,\zeta)| + |\nabla P(\xi,\zeta)| + \sum_{|\alpha|=2} |P^{(\alpha)}(\xi,\zeta)| \\ &= |(\xi+\zeta_R)^2 - (\zeta_I^2+\lambda)| + 2s|\xi_2+\zeta_{R,2}| + 2\sqrt{s}(\sum_{j\neq 2}(\xi_j+\zeta_{R,j})^2)^{1/4} + 2n. \end{aligned}$$

and $C(\rho,\zeta)$ is defined by

$$C(\rho,\zeta) = \min\{(1-\epsilon)|\zeta_R|^2, 2(\epsilon-\tilde{\epsilon})^{1/4}|\zeta_I|^{1/2}|\zeta_R|^{1/2}, 2\sqrt{\tilde{\epsilon}}|\zeta_I||\zeta_R|\} \tag{4.16}$$

with ϵ and $\tilde{\epsilon}$ satisfying $0 < \tilde{\epsilon} < \epsilon < 1$.

Example 2: Second Order Elliptic Polynomials

Let

$$P(\xi) = \sum_{|\alpha|\leq 2} a_\alpha \xi^\alpha$$

be an elliptic polynomial of degree 2. Then the principal part of $P(\xi)$ is the positive definite quadratic form,

$$P_2(\xi) = \sum_{|\alpha|=2} a_\alpha \xi^\alpha$$

$$= \quad (\xi_1, \cdots, \xi_n) \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix},$$

where $A = (a_{ij})$ may be taken to be symmetric. Then there is a non-degenerate $n \times n$ matrix B such that $B^*AB = I$. So if $\xi = B\eta$,

$$P_2(\xi) = \eta^* B^* A B \eta = \eta_1^2 + \eta_2^2 + \cdots + \eta_n^2.$$

From Example 1 we know that zero is not a **critical value** of

$$P_2(\eta, \hat{\zeta}) = \sum_{i=1}^{n} (\eta + \hat{\zeta})^2$$

for all

$$\hat{\zeta} \in \hat{U} = \{\hat{\zeta} \in C^n : \ |\hat{\zeta}| = 1 \ \text{ and } \ \hat{\zeta}_1^2 + \cdots + \hat{\zeta}_n^2 = 0\}$$

So zero is not a **critical value** of

$$P_2(\xi, \zeta) = \sum_{|\alpha| \leq 2} a_\alpha (\xi + \zeta)^\alpha$$

for all

$$\hat{\zeta} \in \hat{U} = \{\hat{\zeta} \in C^n : \ |\hat{\zeta}| = 1 \ \text{ and } \ \sum_{|\alpha|=2} a_\alpha \hat{\zeta}^\alpha = 0\}.$$

Then by Theorem 1.2, for any $\lambda \in \mathbf{R}$, there is a number $R > 0$ such that λ is not a **critical value** of

$$P(\xi, \zeta) = \sum_{|\alpha| \leq 2} a_\alpha (\xi + \zeta)^\alpha$$

for all

$$\zeta \in V = \{\zeta \in C^n : \ \sum_{|\alpha| \leq 2} a_\alpha \zeta^\alpha = \lambda, \ \ |\zeta| \geq R\},$$

and

$$\sup_{\xi \in \mathbf{R}^n} \frac{1}{\tilde{P}(\xi, \zeta)} \to 0 \ \text{ as } \ |\zeta| \to \infty \ \text{ in } \ V.$$

Example 3: The Parabolic Polynomial $P(\xi) = \xi'^2 + i\xi_{n+1}$

Let

$$P(\xi) = \xi'^2 + i\xi_{n+1}$$

where $\xi = (\xi', \xi_{n+1}) \in \mathbf{R}^{n+1}$. Then

$$\begin{aligned} P(\zeta) &= \zeta'^2 + i\zeta_{n+1} = (\zeta'_R + i\zeta'_I)^2 + i(\zeta_{R,n+1} + i\zeta_{I,n+1}) \\ &= \zeta'^2_R - \zeta'^2_I - \zeta_{I,n+1} + i(2\zeta'_R \cdot \zeta'_I + \zeta_{R,n+1}) \ . \end{aligned}$$

For the simplicity, we can take $\lambda = 0$. Then $P(\zeta) = 0$ if and only if

$$\begin{cases} \zeta'^2_R - \zeta'^2_I - \zeta_{I,n+1} = 0 \\ 2\zeta'_R \cdot \zeta'_I + \zeta_{R,n+1} = \zeta_R \cdot (2\zeta'_I, 1) = 0 \end{cases} \tag{4.17}$$

Note that $P(\zeta) = 0$ implies $\zeta'^2_I + \zeta_{I,n+1} \geq 0$. Due to the ξ'-rotational invariance of P we may assume that

$$\zeta_I = (0, \zeta_{I,2}, 0 \cdots, \zeta_{I,n+1}) = (0, s, \cdots, 0, \zeta_{I,n+1})$$

and $s > 0$. Then when $P(\zeta) = \zeta'^2 + i\zeta_{n+1} = 0$,

$$\begin{aligned} P(\xi, \zeta) &= (\xi + \zeta)'^2 + i(\xi_{n+1} + \zeta_{n+1}) \\ &= \xi'^2 + 2\xi' \cdot \zeta' + \zeta'^2 + i\xi_{n+1} + i\zeta_{n+1} \\ &= \xi'^2 + 2\xi' \cdot \zeta' + i\xi_{n+1} \\ &= (\xi' + \zeta'_R)^2 - \zeta'^2_R + i(2s\xi_2 + \xi_{n+1}) \end{aligned}$$

$$\begin{aligned} P_1(\xi, \zeta) &= Re(P(\xi, \zeta)) = (\xi' + \zeta'_R)^2 - \zeta'^2_R \\ P_2(\xi, \zeta) &= Im(P(\xi, \zeta)) = 2s\xi_2 + \xi_{n+1}. \end{aligned}$$

$$\begin{aligned}\tilde{P}(\xi,\zeta) &= |P(\xi,\zeta)| + |\nabla P(\xi,\zeta)| + \sum_{|\alpha|=2} |P^{(\alpha)}(\xi,\zeta)| \\ &= |r'^2 - \zeta_R'^2| + |2s\xi_2 + \xi_{n+1}| + [4s^2(r'^2 - (\xi_2 + \zeta_{R,2})^2) + r'^2]^{1/4} + 2n\end{aligned}$$

It is not hard to prove that if $|\zeta_I'^2 + \zeta_{I,n+1}|$ is sufficiently large,

$$\begin{cases} \tilde{P}(\xi,\zeta) \leq 2(|P(\xi,\zeta)| + |\nabla P(\xi,\zeta)|) \\ \frac{1}{\tilde{P}(\xi,\zeta)} \leq \frac{C}{(\zeta_I'^2 + \zeta_{I,n+1})^{1/4}} \end{cases} \tag{4.18}$$

for all $\xi \in \mathbf{R}^{n+1}$. Then

$$\|\mathcal{F}^{-1}(\frac{1}{P(\cdot + \zeta) - 0}\hat{f})\|_{B^*_{1-s}} \leq \frac{C_s}{(\zeta_I'^2 + \zeta_{I,n+1})^{1/4}}\|f\|_{B_s}$$

for $0 \leq s \leq 1$ for large $(\zeta_I'^2 + \zeta_{I,n+1})$ in $M_\zeta = \{\zeta \in \mathbf{C}^2 : \ P(\zeta) = 0\}$.

Example 4: The Ultra-Hyperbolic Polynomial $P(\xi) = \xi'^2 - \xi''^2$

Let

$$P(\xi) = \xi'^2 - \xi''^2$$

where $\xi' \in \mathbf{R}^n$, $\xi'' \in \mathbf{R}^m$, $n \geq 2$ and $m \geq 1$. Suppose $\zeta = (\zeta', \zeta'')$, $\zeta' \in \mathbf{C}^n$, $\zeta'' \in \mathbf{C}^m$ and $\lambda = 0$. Then

$$\begin{aligned} P(\zeta) &= \zeta'^2 - \zeta''^2 = (\zeta_R' + i\zeta_I')^2 - (\zeta_R'' + i\zeta_I'')^2 \\ &= \zeta_R'^2 - \zeta_R''^2 - (\zeta_I'^2 - \zeta_I''^2) + 2i(\zeta_R' \cdot \zeta_I' - \zeta_R'' \cdot \zeta_I'') \\ &= 0 \end{aligned}$$

if and only if

$$\begin{cases} \zeta_R'^2 - \zeta_R''^2 = (\zeta_I'^2 - \zeta_I''^2) \\ \zeta_R \cdot (\zeta_I', -\zeta_I'') = \zeta_R' \cdot \zeta_I' - \zeta_R'' \cdot \zeta_I'' = 0 \end{cases} \quad . \tag{4.19}$$

If $\zeta_I'^2 - \zeta_I''^2 < 0$, then no point (ζ_R', ζ_R'') satisfies (4.19). So $P(\zeta) = 0$ if and only if

$$\begin{cases} \zeta_R'^2 - \zeta_R''^2 = (\zeta_I'^2 - \zeta_I''^2) \geq 0 \\ \zeta_R \cdot (\zeta_I', -\zeta_I'') = \zeta_R' \cdot \zeta_I' - \zeta_R'' \cdot \zeta_I'' = 0 \end{cases} \quad . \tag{4.20}$$

$$\begin{aligned} P(\xi,\zeta) &= (\xi' + \zeta')^2 - (\xi'' + \zeta'')^2 \\ &= \xi'^2 + 2\xi' \cdot \zeta' + \zeta'^2 - (\xi''^2 + 2\xi'' \cdot \zeta'' + \zeta''^2) \\ & \text{(since } P(\zeta) = 0) \\ &= \xi'^2 - \xi''^2 + 2\xi' \cdot \zeta' - 2\xi'' \cdot \zeta'' \\ &= (\xi' + \zeta_R')^2 - \zeta_R'^2 - (\xi'' + \zeta_R'')^2 + \zeta_R''^2 + 2i[\xi' \cdot \zeta_I' - \xi'' \cdot \zeta_I''] \\ P_1(\xi,\zeta) &= (\xi' + \zeta_R')^2 - \zeta_R'^2 - (\xi'' + \zeta_R'')^2 + \zeta_R''^2 \\ P_2(\xi,\zeta) &= 2[\xi' \cdot \zeta_I' - \xi'' \cdot \zeta_I''] \ . \end{aligned}$$

$$\begin{aligned} \tilde{P}(\xi,\zeta) &= |P(\xi,\zeta)| + |\nabla P(\xi,\zeta)| + \sum_{|\alpha|=2} |P^{(\alpha)}(\xi,\zeta)| \\ &= |(\xi' + \zeta_R')^2 - \zeta_R'^2 - (\xi'' + \zeta_R'')^2 + \zeta_R''^2| + 2|\xi \cdot (\zeta_I', -\zeta_I'')| \\ &= 2[(s^2+t^2) \sum_{j \neq 2, n+1} (\xi_j + \zeta_{R,j})^2 + |t(\xi_2 + \zeta_{R,2}) - s(\xi_{n+1} + \zeta_{R,n+1})|^2]^{1/4} + 2(n+m) \ . \end{aligned}$$

We now assume $\zeta_I = (0, s, 0, \cdots, 0)$; i.e.,

$$\zeta_I' = (0, s, 0, \cdots, 0) \text{ and } \zeta_I'' = (0, \cdots, 0).$$

Then $P(\zeta) = 0$ if and only if $\zeta_I'^2 - \zeta_I''^2 = s^2$ and $\zeta_R \cdot (\zeta_I', -\zeta_I'') = 0$. This implies $s\zeta_{R,2} = 0$ and therefore $\zeta_{R,2} = 0$. So we have

$$\begin{aligned} \tilde{P}(\xi,\zeta) &= |P(\xi,\zeta)| + |\nabla P(\xi,\zeta)| + \sum_{|\alpha|=2} |P^{(\alpha)}(\xi,\zeta)| \\ &= |(\xi' + \zeta_R')^2 - \zeta_R'^2 - (\xi'' + \zeta_R'')^2 + \zeta_R''^2| + 2|\xi_2|s \\ &\quad +2\sqrt{s}[\sum_{j\neq 2}(\xi_j + \zeta_{R,j})^2]^{1/4} + 2(n+m). \end{aligned}$$

If we take $0 < \epsilon < 1/2$, we can easily show that

$$\tilde{P}(\xi,\zeta) \leq C_1(|P(\xi,\zeta)| + |\nabla P(\xi,\zeta)|)$$

for all $\xi \in \mathbf{R}^{n+m}$, and

$$\tilde{P}(\xi,\zeta) \geq C_2(\zeta_R'^2 - \zeta_R''^2)^{1/2}$$

for all $\xi \in \mathbf{R}^{n+m}$ with C_1 and C_2 are independent of

$$\zeta \in \{\zeta \in \mathbf{C}^{n+m} : P(\zeta) = 0, \zeta_I = (0, s, 0, \cdots, 0)\} \ .$$

Therefore

$$\|\mathcal{F}^{-1}(\frac{1}{P(\cdot + \zeta) - 0}\hat{f})\|_{B_{1-s}^*} \leq \frac{C_s}{(\zeta_R'^2 - \zeta_R''^2)^{1/2}}\|f\|_{B_s}$$

for $0 \leq s \leq 1$ on $\{\zeta \in \mathbf{C}^{n+m} : \quad P(\zeta) = 0, \quad \zeta_I = (0, s, 0, \cdots, 0)\} \subset M_\zeta$.

Remark: In paper [I] Isakov proved that for any $\eta \in \mathbf{R}^n$, there are two sequences $\{\zeta_j^{(k)}\} \subset \{\zeta \in \mathbf{C}^n : P(\zeta) = 0, \zeta_I = (0, s, 0, \cdots, 0)\}$ such that $|\zeta_j^{(k)}| \to \infty$ as $j \to \infty$ and

$$\eta = \zeta_j^{(1)} + \zeta_j^{(2)}$$

for each $j \geq 1$, which shows that the control for $\mathcal{F}^{-1}(\frac{1}{P(\cdot+\zeta)-0}\hat{f})$ on large ζ in $\{\zeta \in \mathbf{C}^n : P(\zeta) = 0, \zeta_I = (0, s, 0, \cdots, 0)\}$ is enough in appication.

Example 5: The Elliptic Polynomial $P(\xi) = \xi^4$

Let

$$P(\xi) = \xi^4, \quad \xi \in \mathbf{R}^n$$

and fix $k \in \mathbf{R}^n$ with $|k| > 0$. Then

$$P(\zeta) - |k|^2 = \zeta^4 - k^2 = (\zeta^2 - |k|)(\zeta^2 + |k|) = 0$$

if and only if

$$(\zeta^2 - |k|) = 0 \quad \text{or} \quad (\zeta^2 + |k|) = 0$$

if and only if

$$\begin{cases} \zeta_R^2 - \zeta_I^2 = |k| \\ \zeta_R \cdot \zeta_I = 0 \end{cases} \text{or} \quad \begin{cases} \zeta_R^2 - \zeta_I^2 = -|k| \\ \zeta_R \cdot \zeta_I = 0 \end{cases}.$$

$$\begin{aligned} P(\xi, \zeta) &= P(\xi + \zeta) - P(\zeta) = (\xi + \zeta)^4 - |k|^2 \\ &= [(\xi + \zeta)^2 - |k|)][(\xi + \zeta)^2 + |k|)] \\ &= 0 \end{aligned}$$

if and only if

$$((\xi + \zeta)^2 - |k|) = 0 \quad \text{or} \quad ((\xi + \zeta)^2 + |k|) = 0$$

if and only if

$$\begin{cases} (\xi+\zeta_R)^2 = \zeta_I^2 - |k| \\ \xi \cdot \zeta_I = 0 \end{cases} \text{or} \quad \begin{cases} (\xi+\zeta_R)^2 = \zeta_I^2 + |k| \\ \xi \cdot \zeta_I = 0 \end{cases} .$$

Now we assume $\zeta_R^2 = \zeta_I^2 + |k|$. The graph of $P(\xi,\zeta) = 0$ is given in Figure 2.

The zero set of $P(\xi,\zeta)$ consists of two concentric circles with center $-\zeta_R$ and radii $\sqrt{\zeta_I^2 + |k|}$ and $\sqrt{\zeta_I^2 - |k|}$, respectively. Note that, as $|\zeta_I| \to \infty$, the distance between these two circles tends to zero:

$$\sqrt{\zeta_I^2 + |k|} - \sqrt{\zeta_I^2 - |k|} = \frac{2|k|}{\sqrt{\zeta_I^2 + |k|} + \sqrt{\zeta_I^2 - |k|}} \to 0.$$

Since

$$\begin{aligned} P(\xi,\zeta) &= (\xi+\zeta)^4 - k^2 \\ &= [(\xi+\zeta)^2 - |k|)][(\xi+\zeta)^2 + |k|)] \quad , \end{aligned}$$

$$\begin{aligned} \nabla_\xi P(\xi,\zeta) &= 4(\xi+\zeta)^2(\xi+\zeta) \\ &= 4[(\xi+\zeta)^2 - \zeta_I^2 + i2\xi\cdot\zeta_I][\xi+\zeta_R+i\zeta_I] \quad , \end{aligned}$$

and

$$\begin{aligned} \frac{\partial^2 P(\xi,\zeta)}{\partial \xi_j^2} &= 4(\xi+\zeta)^2 + 8(\xi_j+\zeta_j)^2 \\ &= 4[(\xi+\zeta_R)^2 - \zeta_I^2 + i2\xi\cdot\zeta_I] + 8(\xi_j+\zeta_j)^2 \quad . \end{aligned}$$

If $\xi \in \{\xi \in \mathbf{R}^n : (\xi+\zeta_R)^2 - \zeta_I^2 = 0, \xi\cdot\zeta_I = 0\}$; i.e., $(\xi+\zeta)^2 = 0$, then

$$|P(\xi,\zeta)| = k^2$$

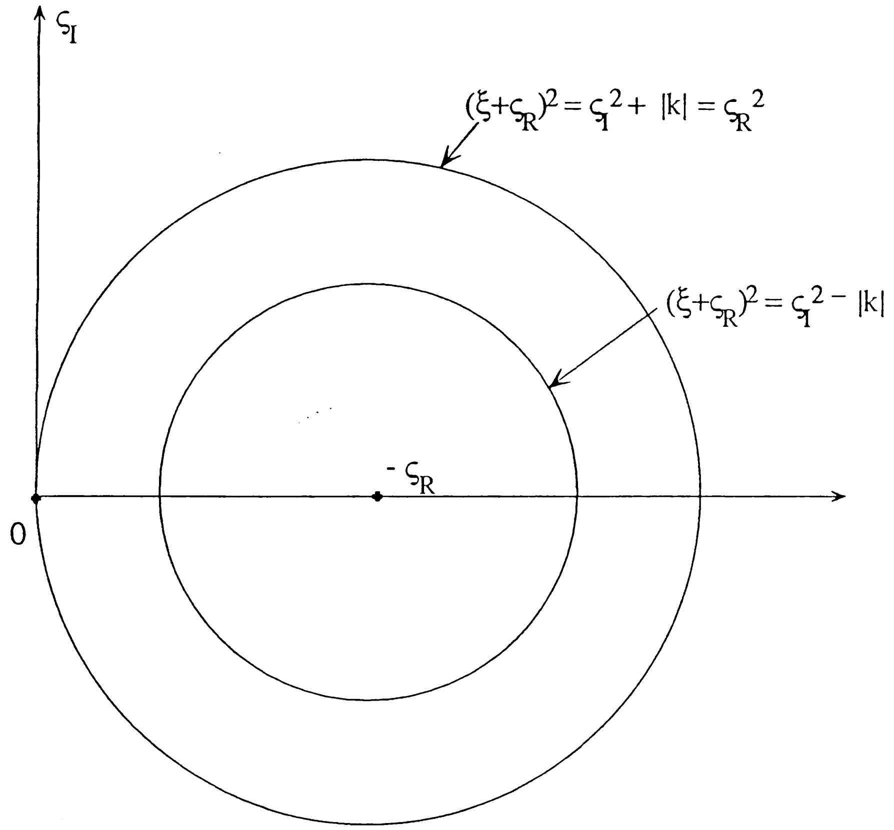

Figure 2: The graph of $P(\xi, \zeta) = 0$

and

$$\nabla_\xi P(\xi, \zeta) = 0.$$

But

$$\begin{aligned}
\sum_{|\alpha|=2} |P^{(\alpha)}(\xi,\zeta)| &\geq \sum_{j=1}^{n} |\frac{\partial^2 P(\xi,\zeta)}{\partial \xi_j^2}| \\
&= 8\sum_{j=1}^{n} |(\xi_j+\zeta_j)^2| = 8\sum_{j=1}^{n} |\xi_j+\zeta_j|^2 \\
&= 8\sum_{j=1}^{n} [(\xi_j+\zeta_{R,j})^2 + \zeta_{I,j}^2] \\
&= 8(\zeta_I^2+\zeta_I^2) = 16\zeta_I^2
\end{aligned}$$

So as $|\zeta| \to \infty$, $\sum_{|\alpha|=2} |P^{(\alpha)}(\xi,\zeta)| \to \infty$ on $M_\xi = \{\xi \in \mathbf{R}^n : (\xi+\zeta)^2 = k^2\}$. This means we can not find a constant C such that

$$\tilde{P}(\xi,\zeta) \leq C(|P(\xi,\zeta)| + |\nabla P(\xi,\zeta)|)$$

for all $\xi \in \mathbf{R}^n$ and all $\zeta \in M_\zeta = \{\zeta \in \mathbf{C}^n : P(\zeta) = k^2\}$.

However we note that for any $\xi \in \mathbf{R}^n$ with $P(\xi,\zeta) \neq 0$

$$\begin{aligned}
\frac{1}{P(\xi,\zeta)} &= \frac{1}{((\xi+\zeta)^2+|k|)((\xi+\zeta)^2-|k|)} \\
&= \frac{1}{2|k|}[\frac{1}{((\xi+\zeta)^2-|k|)} - \frac{1}{((\xi+\zeta)^2+|k|)}].
\end{aligned}$$

So as a distribution,

$$\frac{1}{P(\xi,\zeta)} = \frac{1}{2|k|}[\frac{1}{((\xi+\zeta)^2-|k|)} - \frac{1}{((\xi+\zeta)^2+|k|)}].$$

Since for each $\zeta \in \{P(\zeta) = |k|^2\}$, $\mathcal{F}^{-1}(\frac{1}{P(\cdot,\zeta)}\hat{f})$, $\mathcal{F}^{-1}(\frac{1}{(\cdot+\zeta)^2-|k|}\hat{f})$ and $\mathcal{F}^{-1}(\frac{1}{(\cdot+\zeta)^2+|k|}\hat{f})$ are bounded from B_s to B_{1-s}^*, we can write

$$\mathcal{F}^{-1}(\frac{1}{P(\cdot,\zeta)}\hat{f}) = \frac{1}{2|k|}[\mathcal{F}^{-1}(\frac{1}{(\cdot+\zeta)^2-|k|}\hat{f}) - \mathcal{F}^{-1}(\frac{1}{(\cdot+\zeta)^2+|k|}\hat{f})]$$

for all $f \in B_s$, for $0 \leq s \leq 1$. Then by Example 1, there is a constant C_s which is independent of $\zeta \in M_\zeta$ such that

$$\|\mathcal{F}^{-1}(\frac{1}{P(\cdot,\zeta)}\hat{f})\|_{B^*_{1-s}} \leq \frac{C_s}{|\zeta|}\|f\|_{B_s}$$

for any $f \in B_s$.

If $0 < s \leq 1$, it was proved in paper [Liu] that solution to the equation

$$(P(D+\zeta) - k^2)u = f \in B_s$$

in B^*_{1-s} is unique. Therefore we still obtained the control of the solution of above equation on large ζ

$$\|u\|_{B^*_{1-s}} = \|\mathcal{F}^{-1}(\frac{1}{P(\cdot,\zeta)}\hat{f})\|_{B^*_{1-s}} \leq \frac{C_s}{|\zeta|}\|f\|_{B_s}.$$

Remark 1: If $|k| = 0$, then on $\{\xi \in \mathbf{R}^n : P(\xi,\zeta) = 0\}$, $Re(\nabla P(\xi,\zeta))$ and $Im(\nabla P(\xi,\zeta))$ are both zero, so they are not linearly independent. Then Theorem 2.1 can not be applied. For this special case, the control on $|\zeta|$ for large $|\zeta|$ was studied by Ikehata ([Ik]).

Remark 2: If we let

$$P(\xi) = (\xi^2 - k_1)(\xi^2 - k_2)$$

with $k_1 > k_2 > 0$, then by the similar method discussed above we can obtain

$$\|\mathcal{F}^{-1}(\frac{1}{P(\cdot,\zeta)}\hat{f})\|_{B^*_{1-s}} \leq \frac{C_s}{|\zeta|}\|f\|_{B_s}$$

for any $f \in B_s$, $0 \leq s \leq 1$ on $M_\zeta = \{\zeta \in \mathbf{C}^n : \ \zeta^2 = k_1 \ \text{ or } \ \zeta^2 = k_2\}$.

5 Construction of Exponentially Growing Solutions

In this section we construct solutions of

$$(P_0(D)+q-\lambda)u=0$$

which for large ζ behave like the so-called inhomogeneous plane waves $exp(ix\cdot\zeta)$ with $P_0(\zeta)=\lambda$ in a standard method, see [S-U].

Theorem 5.1 *Let* $P_0(\xi)=\sum_{|\alpha|\le m}a_\alpha\xi^\alpha$ *be a real valued elliptic polynomial and* $\lambda\in\boldsymbol{R}$. *Let* $q\in L^\infty$, $|q(x)|\le C(1+|x|)^{-(n/2)-\epsilon}$ *(*$n\ge 2$*) with some* $\epsilon>0$. *Suppose for some complex subset* $\tilde{\Sigma}_0\subset M_\zeta=\{\zeta\in\boldsymbol{C}^n : P_0(\zeta)=\lambda\}$, λ *is not a* ***critical value*** *of*

$$P_0(\xi+\zeta)=\sum_{|\alpha|\le m}a_\alpha(\xi+\zeta)^\alpha$$

for all $\zeta\in\tilde{\Sigma}_0$ *and the estimate*

$$\|\mathcal{F}^{-1}((P_0(\cdot+\zeta)-\lambda)^{-1}\hat{f}\|_{B^*_{1-s}}\le C(\zeta)\|f\|_{B_s},\quad f\in B_s,\quad 0\le s\le 1,$$

holds with $C(\zeta)\to 0$ *as* $|\zeta|\to\infty$ *in* $\tilde{\Sigma}_0$. *Then for any* $0\le s<\epsilon$ *and* $s\le 1$ *there exists a constant* $\mu>0$, *such that for any* $\zeta\in\tilde{\Sigma}_0\subset M_\zeta\subset\boldsymbol{C}^n$ *with* $|\zeta|>\mu$, *there exists a solution* $\psi(x,\zeta)$ *to the equation* $(P_0(D)-\lambda+q)\psi=0$ *of the form* $\psi=e^{ix\cdot\zeta}(1+\rho(x,\zeta))$ *such that* $\rho(\cdot,\zeta)\in B^*_{1-s}$ *and* $\|\rho(\cdot,\zeta)\|_{B^*_{1-s}}<C(\zeta)$.

Proof: Substituting the expression $\psi=e^{ix\cdot\zeta}(1+\rho)$ into the equation $(P_0(D)-\lambda+q)\psi=0$, we get the following equation for ρ:

$$(P_0(D+\zeta)-\lambda)\rho+q\rho=-q \tag{5.21}$$

where $D = -i(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \cdots, \frac{\partial}{\partial x_n})$. Denote by $G_0(x,\zeta)$ the following fundamental solution of (5.21)

$$G_0(x,\zeta) = \frac{1}{(2\pi)^n} \int \frac{e^{ix\cdot\xi}}{P_0(\xi+\zeta)-\lambda} d\xi.$$

Assumptions on $P_0(\xi)$ imply that the operator $G_0(\zeta)$, given by

$$G_0(\zeta)f = \int G_0(x-y,\zeta)f(y)dy = \mathcal{F}^{-1}(\frac{1}{P_0(\cdot+\zeta)-\lambda}\hat{f})$$

satisfies the estimate

$$\|G_0(\zeta)\|_{\mathcal{B}(B_s,B^*_{1-s})} < C(\zeta), \quad 0 \le s \le 1. \tag{5.22}$$

for all $\zeta \in \tilde{\Sigma}_0$ and $|\zeta| \ge M > 0$. For $0 \le s < \epsilon$ we have $q \in B_s$, so $G_0(\zeta)q \in B^*_{1-s}$ if $s \le 1$ or $G_0(\zeta)q \in B^*_0$ if $1 < s < \epsilon$. Moreover, the multiplication by q is a bounded operator from B^*_{1-s} to B_s if $s \le 1$ or bounded from B^*_0 to B_s if $1 < s < \epsilon$. Estimate (5.22) implies that

$$\|G_0(\zeta)q\|_{\mathcal{B}(B^*_{1-s},B^*_{1-s})} < C_1 C_s(\zeta).$$

Since $C_s(\zeta) \to 0$ as $|\zeta| \to \infty$, there is $\mu > 0$ such that if $|\zeta| > \mu$, $(I + G_0(\zeta)q)$ is invertible in B^*_{1-s} and

$$\|(I+G_0(\zeta)q)^{-1}\|_{\mathcal{B}(B^*_{1-s},B^*_{1-s})} < 2.$$

So the function ρ, defined by

$$\rho = -(I+G_0(\zeta)q)^{-1}G_0(\zeta)q$$

is a well defined element in B^*_{1-s} for $0 \le s <$ min $(\epsilon, 1)$ and

$$\begin{aligned} \|\rho\|_{B^*_{1-s}} &\le \|(I+G_0(\zeta)q)^{-1}\|_{\mathcal{B}(B^*_{1-s},B^*_{1-s})}\|G_0(\zeta)\|_{\mathcal{B}(B_s,B^*_{1-s})}\|q\|_{B_s} \\ &\le C(\zeta), \end{aligned}$$

which completes the proof. ♯

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