ON THE POINCARE–WHITNEY CIRCUITSPACE AND OTHER PROPERTIES OF AN INCIDENCE MATRIX FOR BINARY TREES

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ON THE POINCARE-WHITNEY CIRCUITSPACE AND OTHER PROPERTIES OF AN INCIDENCE MATRIX FOR BINARY TREES*

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Abstract : The leaf-nonleaf incidence matrix, H of a binary tree T(N) is defined and shown to have interesting properties. The first result is an extension or analogue of the Poincare-Veblen-Alexander theorem on graph cycles to trees. The second result reported here is based on the notion of a S-pair (G, M) which is defined. It states that the H-matrix of a tree, H(T(N)) yields an example of a nontrivial S-pair. This construction solves a problem in celestial mechanics, that is, it gives a combinatorial characterization (and thus an algorithm) for a special class of symplectic matrices.

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On the Poincare-Whitney circuitspace and other properties of an incidence matrix for binary trees.

By Chjan C. Lim

The Poincare-Veblen-Alexander theorem [1, 2, 5, 9, 12]:

*Let G be an antisymmetric digraph and **S** its arc-node incidence matrix. The necessary and sufficient condition that a vector* $\boldsymbol{c} = (c_1, \ldots, c_m)$ *represent a cycle or a sum of cycles is that* $\boldsymbol{S}\,\boldsymbol{c} = \boldsymbol{o}$,

has been generalized by Whitney [14] and Tutte [11]. In the language of matroid theory [3, 4, 11, 13], the arc-node incidence matrix of a digraph gives a representation of the corresponding cycle matroid; its kernel represents the Poincare-Whitney circuitspace of the matroid. Our first result extends the above theorem to full binary trees in a nontrival way. In other words, we construct a binary matroid with a non-empty circuitspace for trees, which differs from other known matroids for graphs [13].

In the place of graph cycles, and the arc-nodes incidence matrix, we define internal sub-trees and a leaf-node incidence matrix, induced by the partial order relation of a rooted tree. For each full binary tree with N leaves, T(N), we define a N x (N-1) matrix, H as follows. *Let the row number of **H** represent (label) the N leaves of T(N), and the column numbers, the N-1 remaining nodes which are not leaves. We put:*

H_{ij} = 1 (-1) if leaf i is lower than node j and is connected to it via the right (left) branch.

H_{ij} = 0 if i is not lower than j in T(N).

The notion of lower than, sometimes denoted by the symbol < is a partial order relation [2]. It is clear that an oriented full binary tree can be reconstructed from its H-matrix.

Let us now extend the above definition to unoriented full binary trees: if we replace all the -1 entries in H by 1, leaving all other entries unchanged, then the resulting matrix is the incidence matrix H′ of the corresponding unoriented full binary tree T(N). The following properties of H and H′ are not difficult to prove:

(i) *H' has the column inclusion property which is a partial order relation on its columns (cf. Berge [2] pg. 144).*

(ii) *H' is totally unimodular, i.e. the determinant of every square submatrix of H' is 0 or ± 1. This follows from Heller's theorem (cf. Berge [2], pg. 144).*

(iii) *The kernel of H' is spanned by $(N-1) - N+p = p-1$ linearly independent vectors $\mathbf{x}_j$ consisting of 0 and ± 1. Here p is such that the largest sub-matrix of H' with non-zero determinant has dimension N-p. This result follows from a direct application of Cramer's Rule and (ii).*

(iv) *The number $p \geq 1$ is exactly the number of nodes in T(N) that have descendents which are all leaves.*

(v) *The i-th row of H' and H represent the path of nodes between leave i and the root of T(N). This follows directly from our construction of H and H'.*

In order to state our extension of the Poincare-Veblen-Alexander theorem, we will now define *internal sub-trees*:

Let us denote by T'(N) the tree which results if we cut all the leaves in T(N).

Definition 1: *An internal sub-tree, μ of T(N) is a sub-tree of T'(N) with the following limitations: its root must not be adjacent to a leaf in T(N), its internal nodes (that is, those nodes if any, which are neither the root nor leaf in μ), must not be adjacent to a leaf in T(N), only the leaves of μ maybe adjacent to a leaf in T(N).*

An internal sub-tree of T(N), μ can be represented by a (N-1) - dimensional vector $\mathbf{b} = (b_1, \ldots, b_{n-1})$ in the following way: *if the root of μ is node k in T(N), then we put $b_k = -1$, and if the leaves of μ are labelled by the set of numbers, α then $b_i = 1$ for each $i \in \alpha$; all other entries $b_j = 0$.*

Definition 2a: A fundamental internal sub-tree, ν of T(N) is an internal sub-tree which has no internal nodes, that is, ν consists of a root and all its direct descendents which become the leaves of ν.

Definition 2b: A (N-1) dimensional vector **b** has special form if it has only 3 non-zero entries, $b_i = b_j = 1$, and $b_k = -1$.

Each fundamental internal sub-tree, ν of T(N) can be represented by a special form vector in the above manner.

Theorem 1: *Let T(N) be an unoriented full binary tree and H' its incidence matrix. Then*

(a) Each special form solution of $H'\mathbf{x} = 0$, $b^{(k)}$ represents a fundamental internal sub-tree of T(N).

(b) The special form representation of any fundamental internal sub-tree in T(N) is a solution of $H'\mathbf{x} = 0$.

(c) There are exactly p −1 special form solutions of $H'\mathbf{x} = 0$, and they are linearly-independent. Here p is the number of leaves in the tree T'(N), (cf. also property (iv) of the H-matrix.)

The following corollary gives the desired extension of the Poincare-Veblen-Alexander theorem to trees:

Corollary 1: *Given the incidence matrix, H' of an unoriented full binary tree, T(N), the necessary and sufficient condition that a vector $\mathbf{b} = (b_1, \ldots, b_{n-1})$ represents an internal sub-tree or a sum of such, is that $H'b = 0$.*

An immediate and useful corollary of theorem 1 is:

Corollary 2: *For any unoriented full binary tree T(N), the kernal of its H-matrix has a special basis which consists of p -1 linearly independent special form vectors.*

Another way to state theorem 1 and corollary 1 is to work in the framework of GF(2). In this case, the vector representation of a internal sub-tree μ, consists of 0 and 1 only, with $b_j = 1$ if node j in T(N) is either the root or a leave of μ. This representation of μ yields the elementary chains of a chain-group over GF(2). The supports of these elementary chains are, by definition, the circuits of a binary matroid, **M**(T) on the nodes of the tree T'(N), i.e. the non-leaf nodes of T(N). Clearly, the rank of **M**(T) equals the number of nodes, N-1 minus the dimension of its circuitspace, p-1, i.e. (N-1) - (p-1) = N-p, which is the rank of the H'-matrix of T(N). It turns out that the distinct paths between a leaf and the root of T(N) yields a (N-p)-dimensional basis of the cocircuitspace of **M**(T).

Remark: Theorem 1 can be generalized to arbitrary rooted trees. In general, it is possible that the kernel of $\mathbf{H}'$ is empty. If not, the dimension of the kernel is in general not directly related to the number p. The full proof of theorem 1 is given elsewhere [7].

The incidence-matrix **H** for T(N) has other interesting properties, including theorem 2, which appears to be the solution of a difficult combinatorial problem. It also plays a role in the symplectic transformations of celestial mechanics [6, 10].

In order to state the second theorem, we will need the following definitions:

Definition 3: A m x n matrix with real entries, **A** is the graph-matrix of a graph G if, after replacing the positive (negative) entries by 1 (-1) and leaving the zero elements unchanged, the resulting matrix completely defines the graph G, eg the resulting matrix is a incidence or adjacency matrix of G.

Definition 4: The pair (G, **A**) has the S-property if **A** is a graph-matrix of a certain type for the graph G, and the matrix $\mathbf{D} = (\mathbf{A}^t)^{-1}$ is a graph-matrix of the same type for G.

Consider any N x N orthogonal matrix, **O**, and let G(o) be its digraph, i.e. if $O_{ij} > 0$ (<o) then an arc links node i to node j (node j to node i). It is clear that the pair (G(o), **O**) has the S-property, for $(\mathbf{O}^t)^{-1} = \mathbf{O}$. These examples are trival; we must look beyond pairs (G,**A**) where the graph matrix **A**(G) is of the adjacency type, for nontrivial examples.

The problem of finding nontrivial pairs appears to be difficult. Nevertheless the next property of the H-matrix is an important example.

Theorem 2: For any given full binary tree, T(N) and N positive real numbers, $\Gamma = (\Gamma_i, \ldots, \Gamma_N)$, there exists a N x (N-1) matrix $\mathbf{A}(\Gamma)$ such that (T(N), $\mathbf{A}(\Gamma)$) has the S-property and $\mathbf{A}(\Gamma)$ is a graph-matrix of the H-matrix type for T(N).

The proof is based on a construction of $\mathbf{A}(\Gamma)$. There appears to be at least two distinct constructions. One version gives a graph-matrix which has (N-1) pairwise orthogonal columns i.e. $\mathbf{A}^t\mathbf{A} = \mathbf{I}$. The second construction gives a complete proof of a result of Jacobi in celestial mechanics [6, 10].

Let $\mathbf{H}_+$ be the augmented H-matrix composed of **H** and the column of N 1's.

Let the matrix $\mathbf{M}(\Gamma) = \begin{pmatrix} \mathbf{A}_+(\Gamma) & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_+(\Gamma) \end{pmatrix}$ where the augmented graph-matrices $\mathbf{A}_+(\Gamma)$ and $\mathbf{D}_+(\Gamma)$ are related to $\mathbf{H}_+$ in the same way (cf. definition 3). Then for any given tree T(N) and Γ, $\mathbf{M}(\Gamma)$ is symplectic, i.e. $\mathbf{M}^t \, \mathbf{J} \, \mathbf{M} = \mathbf{J}$.

Appendix

Proof of Theorem 1.

(a) Suppose that $\mathbf{H}' \mathbf{b}_\nu = \mathbf{0}$ and $\mathbf{b}_\nu$ has special form, then $H'_{ij} + H'_{ik} - H'_{i\ell} = 0$ for each $i = 1, \ldots, N$. There are 3 possible cases: (i) all 3 terms are zero, (ii) $H'_{ij} = 0$, $H'_{ik} = H'_{i\ell} = 1$, and (iii) $H'_{ik} = 0$, $H'_{ij} = H'_{i\ell} = 1$. The first case occurs when the path from leaf i does not pass through nodes j, k, or ℓ. The other two cases must both occur in the same tree T(N): by property (v), case (ii) implies that $k < \ell$ or $\ell < k$, and case (iii) implies that $j < \ell$ or $\ell < j$. If $k < \ell$, then $j < \ell$ because it is not possible to have $k < \ell < j$. The case $\ell < k$ is not possible because it implies $j < \ell < k$ or $\ell < k$ and $\ell < j$ which are both excluded. Thus $k < \ell$ and $j < \ell$ and it remains to show that j, k are direct descendents of ℓ. Suppose not, then there is a path from leaf i', say that contains node ℓ but not nodes j nor k. Then $\sum_{j=1}^{N-1} H'_{i'j} b_\nu^j = 1 \neq 0$. Thus $\mathbf{b}_\nu$ represents a fundamental sub-tree.

(b) Let $\mathbf{b}_\nu$ be a special form vector that represents some fundamental sub-tree ν in T(N). Then for $i = 1, \ldots, N$, $\sum_{j=1}^{N-1} H'_{ij} b_\nu^j = H'_{ij} + H'_{ik} - H'_{i\ell}$. Since the rows of H' represents ascending paths (property (v)), either $H'_{ij} = 0$ or $H'_{ik} = 0$. In both cases the triple sums above are zero for each $i = 1, \ldots, N$. Thus $\mathbf{H}' \mathbf{b}_\nu = \mathbf{0}$.

(c) By property (iii) the null-space of $\mathbf{H}'$ has dimension p -1, and by property (iv) and a straight-forward calculation for full binary trees, there are p -1 fundamental sub-tree in T(N). Part (b) implies that there are p -1 distinct special form vectors in the null-space, and since they are linearly independent, there are exactly p -1 special form solutions of $\mathbf{H}' \mathbf{x} = 0$.

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