Cones of hyperplane arrangements

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Galen Anna Dorpalen-Barry

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Abstract

Hyperplane arrangements dissect $\mathbb{R}^n$ into connected components called chambers, and a well-known theorem of Zaslavsky counts chambers as a sum of nonnegative integers called Whitney numbers of the first kind. His theorem generalizes to count chambers within any cone defined as the intersection of a collection of halfspaces from the arrangement, leading to a notion of Whitney numbers for each cone. This thesis concerns cones of hyperplane arrangement in two ways.

First we consider cones within the braid arrangement, consisting of the reflecting hyperplanes $x_i = x_j$ inside $\mathbb{R}^n$ for the symmetric group, thought of as the type A_{n-1} reflection group. Here,

- cones correspond to posets,
- chambers within the cone correspond to linear extensions of the poset,
- the Whitney numbers of the cone interestingly refine the number of linear extensions of the poset.

We interpret this refinement explicitly for two families of posets: width two posets, and disjoint unions of chains. In the latter case, this gives a geometric re-interpretation to Foata's theory of cycle decomposition for multiset permutations, and leads to a simple generating function compiling these Whitney numbers.

Second, we give an interpretation of the coefficients the Poincaré polynomial of a cone of an arbitrary arrangement via the Varchenko-Gel'fand ring, which is the ring of functions from the chambers of the arrangement to the integers with pointwise addition and multiplication. Varchenko and Gel'fand gave a simple presentation for this ring for an arbitrary arrangement, along with a filtration and associated graded ring whose Hilbert series is the Poincaré polynomial. We generalize these results to cones and prove a novel result for the Varchenko-Gel'fand ring of an arrangement: when the arrangement is supersolvable the associated graded ring of the arrangement is Koszul.

Acknowledgements

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There are many other people who deserve acknowledgement, but if I were to list them all, this thesis would be twice as long!

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1 Introduction

In this thesis, I will collect discrete data that tells me about a continuous setting. I will begin with a question about hyperplane arrangements, but the mathematics of the two central chapters (Chapters 3 and 4) will focus on the computation of a certain polynomial called the *Poincaré polynomial*.

This project, which was initially proposed to me by my advisor Victor Reiner, was largely inspired by the work of Regina Gente. In her thesis, she extended *Varchenko's determinant theorem* to *cones*, i.e. intersections of open halfspaces defined by some of the hyperplanes of an *arrangement* [30]. Cones of arrangements have been studied by various authors including Aguiar-Mahajan [2], Alexanderson-Wetzel (using the more general context of "convex bodies") [3], Brown [16], Zaslavsky [65], and, in Type A, the author together with Kim and Reiner [22]. Gente did not invent cones nor is her thesis the first place where cones arise. Her work, however, spawned the following question:

Question 1.1. What theorems about hyperplane arrangements extend to cones?

In this thesis, I will answer that question for a couple of theorems and pursue a number of adjacent questions. In the remainder of this chapter, I will give a more technical overview of the results in this thesis. For this reader's convenience, the chapter consists of two sections. Section 1.1 gives some context for our results, using a theorem of Zaslavsky, and Section 1.2 briefly summarizes the main results of this thesis.

1.1 Hyperplane Arrangements and a Theorem of Zaslavsky

This thesis concerns *arrangements* $\mathcal{A} = \{H_1, \dots, H_m\}$ of hyperplanes H_i, which are affine-linear codimension one subspaces of $V = \mathbb{R}^n$. Each such arrangement dissects V into the connected components of its complement $V \setminus \bigcup_{i=1}^m H_i$, called *chambers*. Let $\mathcal{C}(\mathcal{A})$ denote the collection of all such chambers.

The theory of hyperplane arrangements is rich and well-explored, with connections to reflection groups, braid groups, random walks, card-shuffling, and discrete geometry of polytopes and oriented matroids; see [40, 56]. In particular, the number of chambers $\#\mathcal{C}(\mathcal{A})$ has a famous formula due to Zaslavsky [64], expressed in terms of the *intersection poset* $\mathcal{L}(\mathcal{A})$, which consists of all *intersection subspaces* $X = H_{i_1} \cap H_{i_2} \cap \cdots \cap H_{i_k}$, ordered via reverse inclusion. This poset is known to have the property that every *lower interval*

$$[V, X] := \{Y \in \mathcal{L}(\mathcal{A}) : V \leq Y \leq X\}$$

from its unique bottom element V to any intersection space X forms a *geometric lattice*. In particular, each such $[V, X]$ is a *ranked* poset, where the rank of X is $\operatorname{codim}(X) := n - \dim(X)$. Zaslavsky's result says that

$$\#\mathcal{C}(\mathcal{A}) = \sum_{X \in \mathcal{L}(\mathcal{A})} |\mu(V, X)| = \sum_{k=0}^{n} c_k(\mathcal{A}) \; = [\operatorname{Poin}(\mathcal{A}, t)]_{t=1}, \tag{1.1}$$

where $\mu(-,-)$ denotes the *Möbius function* of $\mathcal{L}(\mathcal{A})$, while the nonnegative integers

$$c_k(\mathcal{A}) \; := \sum_{\substack{X \in \mathcal{L}(\mathcal{A}): \\ \operatorname{codim}(X) = k}} |\mu(V, X)|,$$

are often called the *(signless) Whitney numbers of the first kind* for $\mathcal{A}$, and their generating function

$$\operatorname{Poin}(\mathcal{A}, t) := \sum_{k \geq 0} c_k(\mathcal{A})\, t^k$$

is called the *Poincaré polynomial*[1].

Our starting point is a less widely-known generalization of equation (1.1), also due to Zaslavsky [65]. More generally, it allows us to to count the chambers of $\mathcal{A}$ that lie within a *cone* $\mathcal{K}$, defined to be the intersection of any collection of open halfspaces for hyperplanes of $\mathcal{A}$; said differently, a cone $\mathcal{K}$ of $\mathcal{A}$ is a chamber in $\mathcal{C}(\mathcal{A}')$ for some subarrangement $\mathcal{A}' \subseteq \mathcal{A}$. Results on the set $\mathcal{C}(\mathcal{K})$ of all chambers of $\mathcal{A}$ inside a cone $\mathcal{K}$ have appeared more recently in work of Brown on random walks [16], and in work of Gente on Varchenko determinants [30, Section 2.4], and independently in work of Aguiar and Mahajan [2, Theorem 8.22]. Define the *poset of interior intersections for* $\mathcal{K}$ to be the following order ideal in $\mathcal{L}(\mathcal{A})$:

$$\mathcal{L}^{\text{int}}(\mathcal{K}) = \{X \in \mathcal{L}(\mathcal{A}) \mid X \cap \mathcal{K} \neq \emptyset\}.$$

Zaslavsky observed in [65, Example A, p. 275] that (1.1) generalizes to cones $\mathcal{K}$, asserting

$$\#\mathcal{C}(\mathcal{K}) = \sum_{X \in \mathcal{L}^{\text{int}}(\mathcal{K})} |\mu(V, X)| = \sum_{k=0}^{n} c_k(\mathcal{K}) \; = [\operatorname{Poin}(\mathcal{K}, t)]_{t=1}\,. \tag{1.2}$$

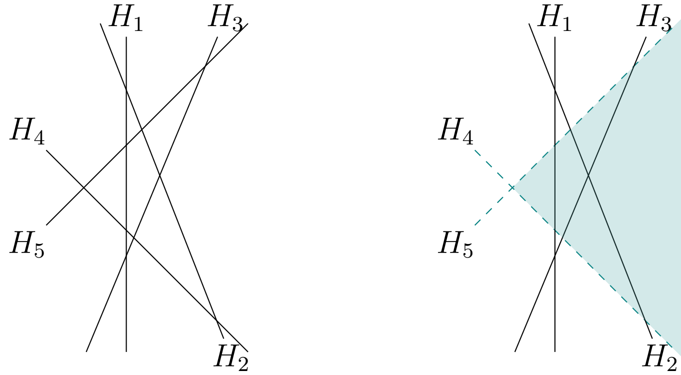
Here we again define nonnegative integers, the *(signless) Whitney numbers of the first kind* for the cone $\mathcal{K}$

$$c_k(\mathcal{K}) \; := \sum_{\substack{X \in \mathcal{L}^{\text{int}}(\mathcal{K}): \\ \operatorname{codim}(X) = k}} |\mu(V, X)|,$$

[1] This is because it is the generating function for the Betti numbers of the complexified complement $\mathbb{C}^n \setminus \mathcal{A}$; see [40, Chap. 5].

with generating function $\mathrm{Poin}(\mathcal{K},t) := \sum_k c_k(\mathcal{K})\, t^k$, which we call the *Poincaré polynomial* of $\mathcal{K}$.

For example, inside $\mathcal{A} = \{H_1, H_2, H_3, H_4, H_5\}$ in $\mathbb{R}^2$ shown below on the left, we have shaded one of four possible cones $\mathcal{K}$ defined by the subarrangement $\mathcal{A}' = \{H_4, H_5\}$, containing $\#\mathcal{C}(\mathcal{K}) = 5$ chambers of $\mathcal{A}$:



Zaslavsky's formula (1.2) computes this as follows. The poset of interior intersections $\mathcal{L}^{\mathrm{int}}(\mathcal{K})$ has Hasse diagram:

$$\begin{tikzcd}
& H_2 \cap H_3 \arrow[d, dash] \arrow[dr, dash] & \\
H_1 \arrow[dr, dash] & H_2 \arrow[d, dash] & H_3 \arrow[dl, dash] \\
& V &
\end{tikzcd}$$

Here $\mu(V,X) = (-1)^{\mathrm{codim}(X)}$ for all X, so that $(c_0(\mathcal{K}), c_1(\mathcal{K}), c_2(\mathcal{K})) = (1,3,1)$, and

$$\begin{aligned}
\mathrm{Poin}(\mathcal{K},t) &= c_0(\mathcal{K}) + c_1(\mathcal{K})t + c_2(\mathcal{K})t^2 = 1 + 3t + t^2, \\
\#\mathcal{C}(\mathcal{K}) = [\mathrm{Poin}(\mathcal{K},t)]_{t=1} &= c_0(\mathcal{K}) + c_1(\mathcal{K}) + c_2(\mathcal{K}) = 5.
\end{aligned}$$

In Chapter 3, we give several interpretations of the Poincaré polynomial of a cone of a special family of arrangements called *Type A reflection arrangements* (also called the braid arrangements). In Chapter 4, we interpret the Poincaré polynomial of a cone of an arbitrary arrangement as the *Hilbert series* of an *associated graded* of the *Varchenko-Gel'fand Ring*.

1.2 Summary of Results

The remainder of this thesis is divided into four chapters. Now we give a brief summary of the main results of each of these four chapters.

2. **Background**. In Chapter 2, we review some general background regarding hyperplanes, cones, posets, oriented matroids.

3. **Reflection Arrangements**. In Chapter 3 we give a combinatorial description for the coefficients of the Poincaré polynomial of the the Type A reflection arrangement (Theorems 3.1 and 3.2), as well as a number of interesting interpretations of the coefficients of the Poincaré polynomial for special cones of the Type A reflection arrangement (Theorems 3.3, 3.4, and 3.5). We also include some details regarding a general reflection arrangements heuristic. The results from this chapter are also contained in the author's paper with Kim and Reiner [22].

4. **Varchenko-Gel'fand Ring**. Chapter 4 consists of results pertaining to both the Varchenko-Gel'fand ring and associated graded of the Varchenko-Gel'fand ring with the degree filtration of a cone. The two main theorems of this chapter are (1) a *Gröbner basis-like* presentation of the Varchenko-Gel'fand ring and its associated graded ring (for the degree filtration), given in Theorem 4.1, and (2) a proof that every *supersolvable arrangement* admits a *Koszul* associated graded algebra, see Theorem 4.4. The material in this chapter appears in the author's recent paper [21].

5. **Questions and Remarks**. The final chapter, Chapter 5, is a compilation of a number of questions and results. We first collect some of the questions from previous chapters. Then we explore two other families of questions: (i) two

questions regarding the broken circuit complex of a cone (and the fact that it may not be shellable), and (ii) two questions regarding Shi arrangements.

In particular, the **main theorems** of this thesis are

- Theorem 3.1
- Theorem 3.2
- Theorem 3.3
- Theorem 3.4
- Theorem 3.5
- Theorem 4.1
- Theorem 4.4

2 Background

In this chapter, we will go over some of the background needed for the main results. The first part reviews hyperplane arrangements and cones. The second part concerns oriented matroids.

2.1 Hyperplane Arrangements and Cones

We begin with some background regarding hyperplanes, cones, and linear algebra then give a gentle introduction to oriented matroids. We will use the braid arrangement as our running example, but state a number of more general results. Good references include [40], [56], [57, §3.11], [43, §3.3].

Definition 2.1. A *hyperplane* in $V = \mathbb{R}^n$ is an affine linear subspace of codimension one. An *arrangement* of hyperplanes in $\mathbb{R}^n$ is a finite collection $\mathcal{A} = \{H_1, \ldots, H_m\}$ of distinct hyperplanes. A *chamber* of $\mathcal{A}$ is an open, connected component of $\mathbb{R}^n \setminus \bigcup_{H \in \mathcal{A}} H$. The set of all chambers of $\mathcal{A}$ is denoted by $\mathcal{C}(\mathcal{A})$.

To each hyperplane H we can associated a (not unique) normal vector v. When H passes through the origin, this vector satisfies $v \cdot x = 0$ for all $x \in H$. An analogous definition holds for affine hyperplanes. We omit that definition here, but note that we will be useful later to specify H by a choice of normal vector.

Example 2.2. The *type A reflection arrangement*, A_{n-1}, also called the braid arrangement, consists of the $\binom{n}{2}$ hyperplanes of the form

$$H_{ij} = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_i - x_j = 0\}$$

for integers $1 \leq i < j \leq n$. There are $n!$ chambers $\mathcal{K}_\sigma = \{x \in \mathbb{R}^n : x_{\sigma_1} < \cdots < x_{\sigma_n}\}$ of A_{n-1}, naturally indexed by the permutations $\sigma = [\sigma_1, \sigma_2, \dots, \sigma_n]$ of $[n]$, that give the strict inequalities ordering the coordinates within the chamber, as in (3.1). For example, when $n = 4$,

$$\mathcal{K}_{1243} = \{x \in \mathbb{R}^4 : x_1 < x_2 < x_4 < x_3\},$$
$$\mathcal{K}_{4213} = \{x \in \mathbb{R}^4 : x_4 < x_2 < x_1 < x_3\}$$

are two out of the $4! = 24$ chambers of $\mathcal{C}(A_{4-1})$.

Definition 2.3. Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{R}^n$. An *intersection* of $\mathcal{A}$ is a nonempty subspace of the form $X = H_{i_1} \cap H_{i_2} \cap \cdots \cap H_{i_k}$ where $\{H_{i_1}, H_{i_2}, \dots, H_{i_k}\} \subseteq \mathcal{A}$. Here the ambient vector space $V = \mathbb{R}^n$ is considered to be the intersection $\bigcap_{H \in \emptyset} H$ of the empty set of hyperplanes. We denote the set of intersections of $\mathcal{A}$ by $\mathcal{L}(\mathcal{A})$.

Example 2.4. The intersections of A_{n-1} are described by equalities between the variables.

- For all $n \geq 1$, the line $x_1 = x_2 = \cdots = x_n$ is the intersection of all the hyperplanes of A_{n-1}.
- When $n = 4$ the intersection of H_{12} and H_{34} in the subspace of $\mathbb{R}^4$ in which $x_1 = x_2$ and $x_3 = x_4$. On the other hand, the intersection of H_{12} and H_{13} is the subspace of $\mathbb{R}^4$ in which $x_1 = x_2 = x_3$.

More generally, there is a bijection $\pi \mapsto X_\pi$ between the collection Π_n of all *set partitions* $\pi = \{B_1, \ldots, B_k\}$ of $[n] = \{1, 2, \ldots, n\}$ and the set of all intersections of A_{n-1}. The bijection sends the set partition π to the subspace X_π where one has equal coordinates $x_i = x_j$ whenever i, j lie in a common block B_k of π. We sometimes denote the set partition $\pi = \{B_1, \ldots, B_k\}$ with the notation $\pi = B_1|B_2| \cdots |B_k$, and may or may not include commas and set braces around the elements of each block B_i. For example, $1 \mid 23 \mid 456$ and $\{\{1\}, \{2, 3\}, \{4, 5, 6\}\}$ represent the same set partition of [6].

- For example, the set partition $1|2| \cdots |n$ in which all elements appear as singletons corresponds to $X_{1|2|\cdots|n} = V = \mathbb{R}^n$, the empty intersection, which is the ambient space.
- For all $n \geq 1$, the set partition $123 \cdots n$ having all the elements in the same block corresponds to the line $X_{123\cdots n}$ defined by $x_1 = x_2 = \cdots = x_n$.
- When $n = 4$, one has $X_{12|34} = H_{12} \cap H_{34}$ and $X_{123|4} = H_{12} \cap H_{13}$.

The collection $\mathcal{L}(\mathcal{A})$ of all intersections of an arrangement $\mathcal{A}$ will be partially ordered by reverse inclusion, and called the *intersection poset* of $\mathcal{A}$. It has a unique minimal element, namely the intersection

$$\bigcap_{H \in \emptyset} H = V = \mathbb{R}^n.$$

For any hyperplane arrangement $\mathcal{A}$, each of the *lower intervals*

$$[V, X] := \{Y \in \mathcal{L}(\mathcal{A}) : V \leq Y \leq X\}$$

forms a *geometric lattice* [57, Prop. 3.11.2], meaning that:

- (Upper Semi-Modularity) For all $X, Y \in \mathcal{L}(\mathcal{A})$, the codimensions of X and Y satisfy

$$\operatorname{codim}(X) + \operatorname{codim}(Y) \geq \operatorname{codim}(X \vee Y) + \operatorname{codim}(X \wedge Y)$$

  where $X \vee Y$ is the intersection $X \cap Y$, and $X \wedge Y$ denotes the lowest-dimensional subspace $Z \in \mathcal{L}(\mathcal{A})$ containing both X and Y.

- (Atomicity) Every $X \in \mathcal{L}(\mathcal{A})$ is an intersection of some of the hyperplanes of $\mathcal{A}$.

In particular, this implies that each such lower interval is a *ranked poset*, with rank function given by the *codimension* $\operatorname{codim}(X) = \dim(V) - \dim(X)$. Furthermore this implies that its *Möbius function* values $\mu(V, X)$, defined recursively by $\mu(V, V) := 1$ and $\mu(V, X) := -\sum_{Y: V \leq Y < X} \mu(V, Y)$, will *alternate in sign* in the sense that $(-1)^{\operatorname{codim}(X)} \mu(V, X) \geq 0$.

Definition 2.5. Let $\mathcal{A}$ be an arrangement of hyperplanes in $\mathbb{R}^n$. For $0 \leq k \leq n$, the *kth signless Whitney number* of $\mathcal{L}(\mathcal{A})$ of the first kind is

$$c_k(\mathcal{A}) = \sum_{\substack{X \in \mathcal{L}(\mathcal{A}): \\ \operatorname{codim}(X) = k}} |\mu(V, X)| = (-1)^k \sum_{\substack{X \in \mathcal{L}(\mathcal{A}): \\ \operatorname{codim}(X) = k}} \mu(V, X).$$

Henceforth, we call $c_k(\mathcal{A})$, $0 \leq k \leq n$, the *Whitney numbers of* $\mathcal{A}$. One of the standard ways to compile them into a generating function is their *Poincaré polynomial* $\operatorname{Poin}(\mathcal{A}, t) := \sum_{k=0}^{n} c_k(\mathcal{A})\, t^k$; see [40, §2.3].

We aim to understand the chambers, intersections, and Whitney numbers for cones in $\mathcal{A}$; the chambers, intersections, and Whitney numbers for $\mathcal{A}$ are a special case.

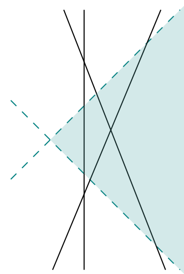


Figure 2.1: An arrangement of hyperplanes in in $\mathbb{R}^2$. A cone defined by two of the hyperplanes is shaded. Alternatively, we can view this as an affine slice of a central arrangement of hyperplanes in $\mathbb{R}^3$.

Definition 2.6. Let $\mathcal{A}$ be an arrangement of hyperplanes in $V = \mathbb{R}^n$. A *cone*[1] $\mathcal{K}$ of an arrangement $\mathcal{A}$ is an intersection of halfspaces defined by some of the hyperplanes of $\mathcal{A}$.

Unless otherwise stated we will assume that $\mathcal{A}$ is oriented so that $\mathcal{K}$ is an intersection of positive halfspaces and thus a cone $\mathcal{K}$ can be specified by a (potentially redundant) set of *walls* $W \subseteq [n]$ where

$$\mathcal{K} = \bigcap_{i \in W} H_i^+.$$

For example, in the arrangement given in Figure 2.1, there are four cones defined by the dashed hyperplanes and one such cone is shaded. When dealing with the cone in Figure 2.1, we will assume that $\mathcal{A}$ is oriented so that $\mathcal{K}$ is an intersection of two positive halfspaces.

Each cone $\mathcal{K}$ of $\mathcal{A}$ has its collection of *chambers*, namely those chambers in $\mathcal{C}(\mathcal{A})$ that lie inside $\mathcal{K}$:

$$\mathcal{C}(\mathcal{K}) = \{C \in \mathcal{C}(\mathcal{A}) : C \subset \mathcal{K}\}.$$

[1] Aguiar and Mahajan [2] call these objects *top-cones*.

The poset of *interior intersections* of the cone $\mathcal{K}$ is the following order ideal within the poset $\mathcal{L}(\mathcal{A})$:

$$\mathcal{L}^{\mathrm{int}}(\mathcal{K}) = \mathcal{L}^{\mathrm{int}}_{\mathcal{A}}(\mathcal{K}) = \{X \in \mathcal{L}(\mathcal{A}) \mid X \cap \mathcal{K} \neq \emptyset\}.$$

For each X in $\mathcal{L}^{\mathrm{int}}_{\mathcal{A}}(\mathcal{K})$, its lower interval $[V, X]$ is still a geometric lattice, with same rank function $\mathrm{codim}(X)$, so that one can define the k^{th} *(signless) Whitney number of* $\mathcal{K}$ by

$$c_k(\mathcal{K}) \;=\; \sum_{\substack{X \in \mathcal{L}^{\mathrm{int}}(\mathcal{K}): \\ \mathrm{codim}(X)=k}} |\mu(V, X)| = (-1)^k \sum_{\substack{X \in \mathcal{L}^{\mathrm{int}}(\mathcal{K}): \\ \mathrm{codim}(X)=k}} \mu(V, X),$$

along with their generating function $\mathrm{Poin}(\mathcal{K}, t) := \sum_{k=0}^{n} c_k(\mathcal{K})\, t^k$, the *Poincaré polynomial* for $\mathcal{K}$.

Chapter 4 concerns another interpretation of the Poincaré polynomial, this time in terms of certain data associated to the oriented matroid of $\mathcal{A}$ (the *no broken circuit sets*). We will introduce these sets in Section 2.2, where we also give the NBC-interpretation of the Poincaré polynomial.

For now, we big our study with the following result of Zaslavsky [65], which counts the number $\#\mathcal{C}(\mathcal{K})$ of chambers of an arrangement $\mathcal{A}$ lying inside one of its cones $\mathcal{K}$.

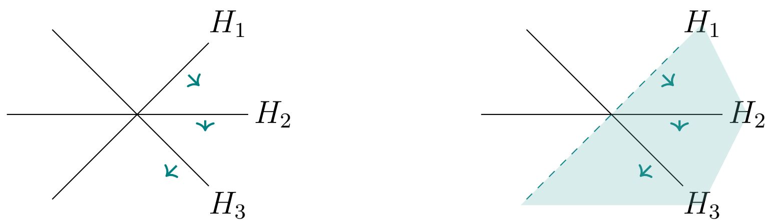
Theorem 2.7 ([65, Example A, p. 275])**.** *Let* $\mathcal{K}$ *be a cone of an arrangement* $\mathcal{A}$ *in* $V = \mathbb{R}^n$*. Then*

$$\#\mathcal{C}(\mathcal{K}) = \sum_{X \in \mathcal{L}^{\mathrm{int}}(\mathcal{K})} |\mu(V, K)| = \; c_0(\mathcal{K}) \;+\; c_1(\mathcal{K}) \;+ \cdots +\; c_n(\mathcal{K}) \;=\; [\mathrm{Poin}(\mathcal{K}, t)]_{t=1}\,.$$

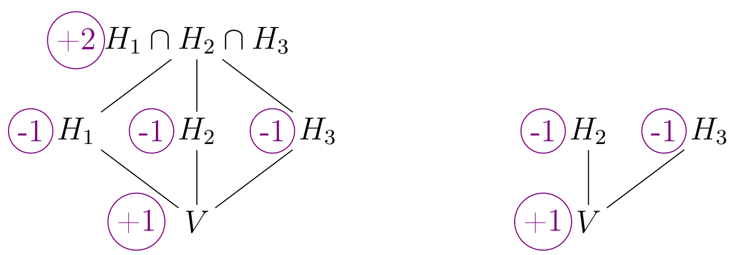
Zaslavsky proved in his doctoral thesis [64] the better-known special case of Theorem 2.7 for the full arrangement, that is, where $\mathcal{K} = V = \mathbb{R}^n$.

The following few examples illustrate Theorem 2.7 for various type A cones.

Example 2.8. Consider the arrangement in $\mathbb{R}^2$ below on the left with the given orientation. The cone $\mathcal{K} = H_1^+$ with $W = \{1\}$ is the shaded region below on the right.



Below we draw the Hasse diagrams of $\mathcal{L}(\mathcal{A})$ (left) and $\mathcal{L}^{\text{int}}(\mathcal{K})$ (right). The circled value next to X is $\mu(V, X)$.

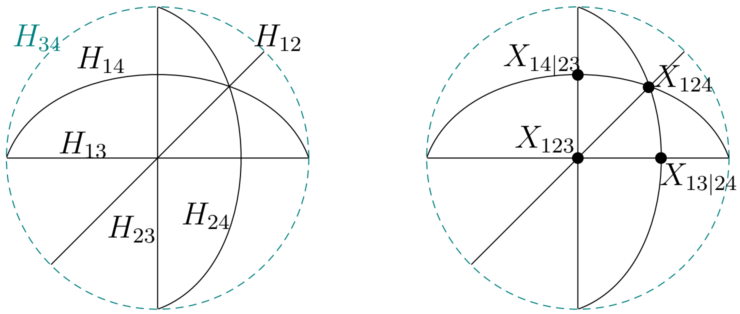


The associated Poincaré polynomials are

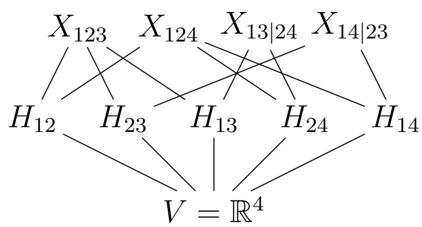
$$\mathsf{Poin}(\mathcal{A}, t) = 1 + 3t + 2t^2 \qquad \text{and} \qquad \mathsf{Poin}(\mathcal{K}, t) = 1 + 2t,$$

respectively. It is easy to see from the pictures in Example 2.8 that the arrangement has $1 + 3 + 2 = 6$ chambers and that the cone has $1 + 2 = 3$ chambers.

Example 2.9. Consider the arrangement $\mathcal{A} = A_{4-1} = \{H_{12}, H_{13}, H_{14}, H_{23}, H_{24}, H_{34}\}$ inside $V = \mathbb{R}^4$. On the left below we have drawn a linearly equivalent picture of its intersection with the hyperplane where $x_1 + x_2 + x_3 + x_4 = 0$, isomorphic to $\mathbb{R}^3$, and depicted the intersection of the hyperplanes with the unit 2-sphere in this 3-dimensional space. Here we pick the cone $\mathcal{K}$ to be the one defined by the halfspace $x_3 < x_4$ for the hyperplane H_{34}, and draw the intersection of H_{34} with the unit sphere as the equatorial circle, with the other five hyperplanes H_{ij} depicted as great circles intersecting the hemisphere where $x_3 < x_4$. On the right below the non-hyperplane interior intersection subspaces X_π are labeled.

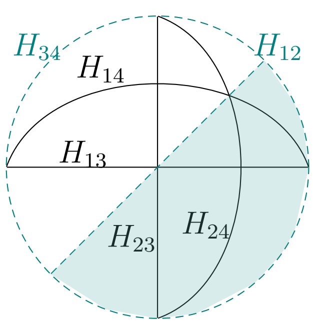


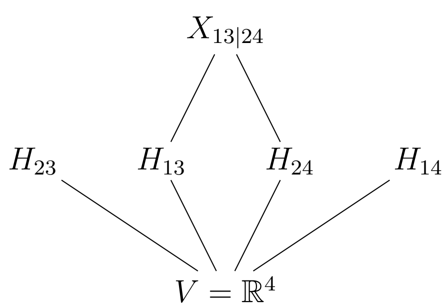
Therefore the intersection poset $\mathcal{L}^{\text{int}}(\mathcal{K})$ of this cone is



We have $(\, c_0(\mathcal{K})\,,\, c_1(\mathcal{K})\,,\, c_2(\mathcal{K})\,) = (1, 5, 6)$ and thus $\text{Poin}(\mathcal{K}, t) = 1 + 5t + 5t^2$. Summing the Whitney numbers gives $1+5+6 = 12$, and a quick visual verification assures that there are 12 chambers in this cone.

Example 2.10. Consider the cone $\mathcal{K}$ of A_3 in which $x_3 < x_4$ and $x_1 < x_2$. On the left below we have drawn the same picture as Example 2.9 with the cone corresponding to $\mathcal{K}$ shaded. We depict $\mathcal{L}^{\text{int}}(\mathcal{K})$ on the right.





We have $(\, c_0(\mathcal{K})\,,\, c_1(\mathcal{K})\,,\, c_3(\mathcal{K})\,) = (1,4,1)$, and thus $\mathrm{Poin}(\mathcal{K},t) = 1+4t+t^2$. Summing the Whitney numbers gives $1+4+1 = 6 = \#\mathcal{C}(\mathcal{K})$.

2.2 Oriented Matroids

The collection of normal vectors $\{v_1,\ldots,v_n\}$ of $\mathcal{A}$ naturally gives rise to an *oriented matroid*. The theory of (oriented) matroids arising from hyperplane arrangements is well-studied and there are many excellent sources on this topic including [9], [40, Section 2.1], and [56, Lecture 3]. We briefly review some basics but refer the reader to the preceding sources for a more detailed discussion.

Let E be a finite set. A *signed set* $D = (D^+, D^-)$ of E is a disjoint, ordered pair of subsets $D^+, D^- \subseteq E$. For a signed subset D of E and $e \in E$ and D, define

$$D_e := \begin{cases} + & \text{if } e \in D^+ \\ - & \text{if } e \in D^- \\ 0 & \text{else.} \end{cases}$$

The collection $\underline{D} := \{e \in E : D_e \neq 0\}$ is called the *(unsigned) support set* of E. Each signed set D has an “opposite" signed set $-D$ with the same (unsigned) support set but $(-D)_e = -(D_e)$ for all $e \in \underline{D}$. For an arbitrary pair C, D of signed sets, we use their *separating set* to keep track of the places where their signs are opposite, i.e. if C, D are signed sets of E then the separating set of C and D is

$$S(C,D) := \{e \in E \mid C_e = -D_e \neq 0\}.$$

Its easy to see that the separating set of D and $-D$ is $\underline{D}$.

Finally we define the *composition product* of two signed sets C, D with the same ground set E. The composition of C with D is the signed set $C \circ D$ where

$$(C \circ D)_e = \begin{cases} C_e & \text{if } C_e \neq 0 \\ D_e & \text{else} \end{cases} \qquad \text{for } e \in E.$$

Definition 2.11. Let E be a finite set and $\mathfrak{D}$ a collection of signed subsets of E. We say that $\mathfrak{D}$ is the set of *dependencies*[2] of an *oriented matroid* on E if $\mathfrak{D}$ satisfies the *dependence axioms*:

V0. $0 \in \mathfrak{D}$, where 0 denotes the signed set with $(0)_e = 0$ for all $e \in E$,

V1. If $D \in \mathfrak{D}$, then $-D \in \mathfrak{D}$,

V2. If $C, D \in \mathfrak{D}$, then $C \circ D \in \mathfrak{D}$,

V3. If $C, D \in \mathfrak{D}$ and $e \in S(C, D)$ then there exists $F \in \mathfrak{D}$ with

$$\begin{aligned} F_e &= 0 \\ F_f &= (C \circ D)_f \qquad \text{for } f \in E \backslash S(C, D). \end{aligned}$$

The proof of the main result of Chapter 4 concerns the connection between an oriented matroid and its *dual*. For an oriented matroid $M = (E, \mathfrak{D})$, there is a unique oriented matroid $M^* = (E, \mathfrak{D}^*)$ with vectors

$$\mathfrak{D}^* = \{(F^+, F^-) \mid D \perp F \text{ for all } D \in \mathfrak{D}\}$$

where $F \perp D$ if F and D are *orthogonal*. By orthogonal, we mean that if $\cdot$ is our

[2]In the study of oriented matroids, these are often referred to as "vectors." Here we use the nonstandard terminology of "dependencies" to avoid confusion.

usual multiplication on $\{+,-,0\}$ with

$$a \cdot b = \begin{cases} + & \text{if } a = b \neq 0 \\ - & \text{if } a, b \neq 0 \text{ and } a \neq b \\ 0 & \text{if } a = 0 \text{ or } b = 0 \end{cases}$$

then $\{F_e \cdot D_e \mid e \in E\}$ either equals $\{0\}$ or contains $\{+,-\}$ [32, §6.2.5]. We call M^* the dual matroid to M and call $\mathfrak{D}^*$ the *covectors* of M. One can show that the covectors of M also satisfy the dependence axioms so that the dual oriented matroid is in fact an oriented matroid.

We will be concerned with oriented matroids defined by sets of vectors $\{v_1, \ldots, v_n\}$ in $\mathbb{R}^d$, which naturally come equipped with *signed dependencies* given by linear combinations. Whenever $\sum_{v \in \underline{D}} \lambda_v v = 0$, one has a signed dependency $D = (D^+, D^-)$ where

$$D_v = \begin{cases} + & \text{if } \lambda_v > 0 \\ - & \text{if } \lambda_v < 0 \\ 0 & \text{else.} \end{cases}$$

In this context, one can think of the composition product of as a sum of dependencies where the second dependency is multiplied by a small, positive number.

That is, suppose C, D yield dependencies

$$\sum_{v \in \underline{C}} \lambda_v v = 0 \qquad \text{and} \qquad \sum_{v \in \underline{D}} \gamma_v v = 0.$$

Then we can choose some $\varepsilon > 0$ so that $\lambda_v + \varepsilon \gamma_v$ is greater than zero if $v \in C^+$ and less than zero if $v \in C^-$. Thus the signed dependency given by

$$\sum_{v \in \underline{C}} \lambda_v v + \varepsilon \left(\sum_{v \in \underline{D}} \gamma_v v \right) = 0.$$

satisfies the definition of $C \circ D$ given above.

Proposition 2.12. *For $E = \{v_1, \ldots, v_n\}$, the set of signed dependencies satisfies the dependence axioms.*

Proof. Let $\mathfrak{D}$ be the set of signed dependencies of E. Axioms zero and one are trivially satisfied from the definition of a signed dependency. The second follows from the definition of the circle product and the fact that $0 + 0 = 0$. For the final axiom, we use the same procedure as the setup for the circle product construction, but we engineer some cancellation by taking $\varepsilon = -\frac{\lambda_e}{\gamma_e}$. Since $e \in S(C, D)$, it follows that λ_e and γ_e have opposite signs and thus $\varepsilon > 0$. We take F to be the signed dependency given by the coefficients of

$$\sum_{v \in \underline{C}} \lambda_v v + \left(-\frac{\lambda_e}{\gamma_e}\right) \sum_{v \in \underline{D}} \gamma_v v = 0.$$

By our construction, we guarantee that the coefficient of v_e will be zero. Note: since $-\frac{\lambda_e}{\gamma_e} > 0$, the only possible cancellation comes from $g \in S(C, D)$. $\square$

The covectors of this oriented matroid also have a well-known geometric interpretation via nonempty intersections of halfspaces defined by some of the hyperplanes of $\mathcal{A}$, see [2, Section 1.1.3] for example. We will use the fact that if an intersection $\bigcap_i H_i^{\varepsilon_i}$ is nonempty for $\{\varepsilon_i\}_i$, then there is a covector $F \in \mathfrak{D}^*$ such that $F_i = \varepsilon_i$ for all $i \in \underline{F}$.

The inclusion-minimal (inclusion of the underlying sets) , nonempty signed dependencies of an oriented matroid are called *signed circuits* and we denote the set of all signed circuits of $\mathfrak{D}$ by $\mathfrak{C}$, i.e.

$$\mathfrak{C} := \{C \in \mathfrak{D} \mid \underline{C} \text{ is minimal under inclusion}, C \neq 0\}.$$

A theorem of Bland-Las Vergnas and Edmonds-Mandel [9, Theorem 3.7.5] says that every vector $D \in \mathfrak{D}$ is a composition of signed circuits, i.e. there is some

$k \in \mathbb{Z}$ and collection $C^{(1)}, \ldots, C^{(k)} \in \mathfrak{C}$ such that

$$D = C^{(1)} \circ C^{(2)} \circ \cdots \circ C^{(k)}.$$

Furthermore, one can select the circuits of this composition so that they *conform* to D, meaning that for all $i = 1, \ldots, k$ and $e \in \underline{D}$: if $C_e^{(i)}$ is nonzero then $C_e^{(i)} = D_e$ [9, Proposition 3.7.2]. We will use a weaker version in the proof of our main theorem: every signed dependence D can be written as a composition of circuits $D = C^{(1)} \circ C^{(2)} \circ \cdots \circ C^{(k)}$ and it is easy to see that the first circuit $C^{(1)}$ always conforms to D.

In order to simplify notation, we will hereafter conflate a vector v_i with its index i. We take $E = [n]$, so that $\mathfrak{C}$ is a collection of signed subsets of $[n]$. For $C \in \mathfrak{C}$, we say that $\underline{C} - \{i\}$ is a *broken circuit* if i is the smallest index (under the usual order on $[n]$ in which $1 < 2 < 3 < \cdots < n$) such that $i \in C$. We will also consider the *no broken circuit sets* of $\mathcal{A}$, denoted $NBC(\mathcal{A})$, which are the subsets of $[n]$ containing no broken circuits. A no broken circuit set $N \in NBC(\mathcal{A})$ of $\mathcal{A}$ is a $\mathcal{K}$ *no broken circuit set* or $\mathcal{K}$-NBC set if

$$\bigcap_{i \in N} H_i^0 \in \mathcal{L}^{\text{int}}(\mathcal{K}).$$

If we take our cone to be the intersection of no halfspaces, i.e. the wall set W is empty , then we recover the full arrangement, and the $\mathcal{K}$-NBC sets are precisely the usual NBC sets of the arrangement. We will denote the set of $\mathcal{K}$-NBC sets by $NBC(\mathcal{K})$.

These sets naturally arise when one studies the Poincaré polynomial of the cone. Since every lower interval $[V, X]$ of $\mathcal{L}^{\text{int}}(\mathcal{K})$ is isomorphic to the corresponding lower interval $[V, X]$ in $\mathcal{L}(\mathcal{A})$, a theorem of Rota [50, Section 7] (see also [51, Theorem 1.1]) allows us to compute the Möbius function of the interval $[V, X]$ in $\mathcal{L}^{\text{int}}(\mathcal{K})$ via the $\mathcal{K}$-NBC sets, i.e., for all $X \in \mathcal{L}^{\mathrm{int}}(\mathcal{K})$

$$\mu(V, X) = (-1)^{\operatorname{codim}(X)} \cdot \# \left\{ N \in NBC(\mathcal{A}) \;\middle|\; \bigcap_{i \in N} H_i = X \right\}.$$

As a result, the Poincaré polynomial of a cone also has an expression in terms of $\mathcal{K}$-NBC sets:

$$\operatorname{Poin}(\mathcal{K}, t) = \sum_{N \in NBC(\mathcal{K})} t^{\# N}$$

Taking $t = 1$, gives

$$\#\mathcal{C}(\mathcal{K}) = \# NBC(\mathcal{K}). \tag{2.1}$$

This equality will be useful for the results in Chapter 4.

3 Type A Reflection Arrangements

In this section we discuss results for reflection arrangements. The results for Type A and the aside on reflection arrangements were discussed in the author's collaborative paper with Jang Soo Kim and Victor Reiner [22].

3.1 The Results

The object of this section is to understand the distribution of the signless Whitney numbers as a refinement of $\#\mathcal{C}(\mathcal{K})$ as in equation (1.2), for cones $\mathcal{K}$ in the *braid arrangement*. The braid arrangement $A_{n-1} = \{H_{ij}\}_{1\leq i<j\leq n}$ is the set of $\binom{n}{2}$ reflecting hyperplanes

$$H_{ij} = \{(x_1, \ldots, x_n) \in V = \mathbb{R}^n \mid x_i - x_j = 0\}$$

for the symmetric group $\mathfrak{S}_n$ on n letters, thought of as the reflection group of type A_{n-1}. There is a well-known and easy bijection between the chambers $\mathcal{C}(A_{n-1})$ and the permutations $\sigma = [\sigma_1, \sigma_2, \ldots, \sigma_n]$ in $\mathfrak{S}_n$, sending σ to the chamber:

$$\mathcal{K}_\sigma := \{x \in V = \mathbb{R}^n : x_{\sigma_1} < x_{\sigma_2} < \cdots < x_{\sigma_n}\}. \tag{3.1}$$

More generally, one has an easy bijection, reviewed in Section 3.2, between posets on the set $[n] := \{1, 2, \ldots, n\}$ and cones in the braid arrangement A_{n-1}, sending a poset P to the cone

$$\mathcal{K}_P := \{x \in V = \mathbb{R}^n : x_i < x_j \text{ for } i <_P j\}.$$

It is readily checked that the chamber $\mathcal{K}_\sigma$ lies in the cone $\mathcal{C}(\mathcal{K}_P)$ if and only if σ is a *linear extension* of P, meaning that $i <_P j$ implies $i <_\sigma j$, regarding σ as a total order $\sigma_1 < \sigma_2 < \cdots < \sigma_n$ on $[n]$. Letting $\mathrm{LinExt}(P)$ denote the set of all linear extensions of P, this shows that $\#\mathcal{C}(\mathcal{K}_P) = \#\,\mathrm{LinExt}(P)$, and hence equation (1.2) becomes

$$\#\,\mathrm{LinExt}(P) = \sum_{k \geq 0} c_k(P) \; = [\mathrm{Poin}(P,t)]_{t=1} \tag{3.2}$$

abbreviating $c_k(P) := c_k(\mathcal{K}_P)$ and $\mathrm{Poin}(P,t) := \mathrm{Poin}(\mathcal{K}_P, t)$ from now on.

A motivating special case occurs when $P = \mathsf{Antichain}_n$ is the *antichain poset* on $[n]$ that has no order relations. In this case $\mathrm{LinExt}(\mathsf{Antichain}_n) = \mathfrak{S}_n$ itself, and the signless Whitney number $c_{n-k}(\mathsf{Antichain}_n)$ of the first kind is well-known [57, Prop. 1.3.7] to be the *signless Stirling number of the first kind* $c(n,k)$ that counts the permutations in $\mathfrak{S}_n$ having k cycles. Consequently, (3.2) becomes the easy summation formula

$$n! = \#\mathfrak{S}_n = \sum_{k \geq 0} c(n,k). \tag{3.3}$$

The **main results** of this chapter generalize three well-known expressions for $\mathrm{Poin}(\mathsf{Antichain}_n, t)$, explained below:

$$\begin{aligned} \mathrm{Poin}(\mathsf{Antichain}_n, t) = \sum_{k \geq 0} c(n,k) t^{n-k} &= \sum_{\sigma \in \mathfrak{S}_n} t^{n - \mathsf{cyc}(\sigma)} && (3.4) \\ &= \sum_{\sigma \in \mathfrak{S}_n} t^{n - \mathsf{LRmax}(\sigma)} && (3.5) \\ &= 1(1+t)(1+2t)\cdots(1+(n-1)\,t). && (3.6) \end{aligned}$$

Equation (3.4) comes from the definition of $c(n,k)$ mentioned above, where $\mathsf{cyc}(\sigma)$ is the number of cycles of σ. This interpretation of the Poincaré polynomial will be generalized to all posets in Theorem 3.1, and generalized in a different way to posets which are *disjoint unions of chains* in Theorem 3.3.

Equation (3.5) arises from a well-known bijection $\tau \mapsto \sigma$ from $\mathfrak{S}_n$ to itself, called the *fundamental bijection*[1], such that $\mathsf{cyc}(\tau) = \mathsf{LRmax}(\sigma)$, see [57, Proposition 1.3.1]; here $\mathsf{LRmax}(\sigma)$ is the number of *left-to-right maxima* of $\sigma = \sigma_1\sigma_2\ldots\sigma_n$, i.e., the number of integers $1 \leq j \leq n$ such that $\sigma_j = \max\{\sigma_1, \sigma_2, \ldots, \sigma_j\}$. This bijection and interpretation of the Poincaré polynomial will be generalized to all posets in Theorem 3.2.

Equation (3.6) is well-known [57, Proposition 1.3.7], and will be generalized to a generating function compiling the Poincaré polynomials of all posets which are disjoint unions of chains in Theorem 3.4 below.

The remainder of this paper is organized as follows.

Section 3.2 gives preliminaries on the intersection lattice and cones in braid arrangements. In particular, for a poset P on $[n]$, it gives an explicit combinatorial description of the interior intersections $\mathcal{L}^{\text{int}}(\mathcal{K}_P)$ for a poset cone $\mathcal{K}_P$, as the subset $\Pi^{\pitchfork}(P)$ of *P-transverse partitions* inside the lattice Π_n of set partitions of $[n]$; roughly speaking, these are the set partitions π for which the quotient preposet P/π does not collapse any strict order relations $i <_P j$ into equalities. With this in hand, it defines the subset $\mathfrak{S}^{\pitchfork}(P)$ of *P-transverse permutations* inside $\mathfrak{S}_n$ as those permutations σ whose cycles form a P-transverse partition, in order to prove the following generalization of equation (3.4).

[1] This bijection, often called *Foata's first fundamental transformation*, seems to have appeared also in work of Rényi [49, §4].

Theorem 3.1. *For any poset P on $[n]$, one has*

$$\mathrm{Poin}(P,t) = \sum_{\sigma \in \mathfrak{S}^{\pitchfork}(P)} t^{n-\mathsf{cyc}(\sigma)}.$$

In particular, $\#\,\mathrm{LinExt}(P) = \#\mathfrak{S}^{\pitchfork}(P)$.

This result even generalizes to a statement (Theorem 3.64) about cones in any *real reflection arrangement*.

In light of Theorem 3.1, one expects bijections between $\mathfrak{S}^{\pitchfork}(P)$ and $\mathrm{LinExt}(P)$. One such bijection is the goal of Section 3.3, which defines a notion of *P-left-to-right maximum* for a linear extension of P, and then generalizes the fundamental bijection $\tau \mapsto \sigma$ and equation (3.5) as follows.

Theorem 3.2. *For any poset P on $[n]$, one has a bijection*

$$\begin{array}{ccc} \mathfrak{S}^{\pitchfork}(P) & \xrightarrow{\Phi} & \mathrm{LinExt}(P) \\ \tau & \longmapsto & \sigma \end{array}$$

such that $\mathsf{cyc}(\tau) = \mathsf{LRmax}_P(\sigma)$, *where* $\mathsf{LRmax}_P(\sigma)$ *is the number of P-left-to-right maxima of σ. Therefore,*

$$\mathrm{Poin}(P,t) = \sum_{\sigma \in \mathrm{LinExt}(P)} t^{n-\mathsf{LRmax}_P(\sigma)}.$$

Section 3.4 examines posets P which are disjoint unions of chains, and produces a second interesting bijection $\mathfrak{S}^{\pitchfork}(P) \to \mathrm{LinExt}(P)$. Given any *composition* $\overline{a} = (a_1, \ldots, a_\ell)$ of n, meaning that $\overline{a} \in \{0,1,2,\ldots\}^\ell$ and $|\overline{a}| := \sum_{i=1}^{\ell} a_i = n$, let $\mathsf{a_i}$ denote a chain (totally ordered) poset on a_i elements, and then

$$P_{\overline{a}} := \mathsf{a}_1 \sqcup \mathsf{a}_2 \sqcup \cdots \sqcup \mathsf{a}_\ell$$

is a disjoint union of incomparable chains of sizes $a_1, a_2, \ldots, a_\ell$. Here one can identify elements in $\mathrm{LinExt}(P_{\overline{a}})$ with *multiset permutations* of $\{1^{a_1}, 2^{a_2}, \ldots, \ell^{a_\ell}\}$.

Section 3.4 reviews the beautiful theory of *prime cycle decompositions* for such multiset permutations due to Foata [24], and then reinterprets it as giving a bijection $\mathrm{LinExt}(P_{\overline{a}}) \to \mathfrak{S}^{\pitchfork}(P_{\overline{a}})$, and the following (second) generalization of equation (3.4).

Theorem 3.3. *For any composition $\overline{a}$ of n, the disjoint union $P_{\overline{a}}$ of chains has a bijection*

$$\begin{array}{rcl} \mathrm{LinExt}(P_{\overline{a}}) & \longrightarrow & \mathfrak{S}^{\pitchfork}(P_{\overline{a}}) \\ \sigma & \longmapsto & \tau \end{array}$$

with $\mathsf{cyc}(\tau) = \mathsf{pcyc}(\sigma)$, the number of prime cycles in Foata's unique decomposition for σ. Thus

$$\mathrm{Poin}(P_{\overline{a}}, t) = \sum_{\sigma \in \mathrm{LinExt}(P_{\overline{a}})} t^{n - \mathsf{pcyc}(\sigma)}.$$

Foata's theory is then used to prove the following generating function, generalizing equation (3.6).

Theorem 3.4. *For $\ell = 1, 2, \ldots,$ one has*

$$\sum_{\overline{a} \in \{1,2,\ldots\}^{\ell}} \mathrm{Poin}(P_{\overline{a}}, t) \cdot \mathbf{x}^{\overline{a}} = \frac{1}{1 - \sum_{j=1}^{\ell} e_j(\mathbf{x}) \cdot (t-1)(2t-1)\cdots((j-1)t-1)},$$

where $\mathbf{x}^{\overline{a}} := x_1^{a_1} \cdots x_\ell^{a_\ell}$ and $e_j(\mathbf{x}) := \sum_{1 \le i_1 < \cdots < i_j \le \ell} x_{i_1} \cdots x_{i_j}$ is the j^{th} elementary symmetric function.

Section 3.6 then examines posets of *width two*, that is, posets P decomposable as $P = P_1 \cup P_2$ where the subposets P_1, P_2 are *chains* (i.e. totally ordered subsets) inside P. Here the Whitney numbers $c_k(P)$ are interpreted by a descent-like statistic on σ in $\mathrm{LinExt}(P)$:

$$\mathsf{des}_{P_1,P_2}(\sigma) := \#\{i \in [n-1] : \sigma_i \in P_2,\ \sigma_{i+1} \in P_1,\ \text{with } \sigma_i, \sigma_{i+1} \text{ incomparable in } P\}.$$

Theorem 3.5. *For a width two poset decomposed into two chains as $P = P_1 \cup P_2$, one has*

$$\mathrm{Poin}(P,t) = \sum_{\sigma \in \mathrm{LinExt}(P)} t^{\mathsf{des}_{P_1,P_2}(\sigma)}.$$

Theorem 3.5 will be used to show (Corollary 3.57) that, for certain very special width two posets P, the Poincaré polynomial $\mathrm{Poin}(P,t)$ coincides with the P-*Eulerian polynomial*,

$$\sum_{\sigma \in \mathrm{LinExt}(P)} t^{\mathsf{des}(\sigma)} \tag{3.7}$$

which counts linear extensions σ of P according to their number of (usual) *descents*

$$\mathsf{des}(\sigma) := \#\{i \in [n-1] : \sigma_i > \sigma_{i+1}\},$$

assuming that P has been *naturally labeled* in the sense that the identity permutation $\sigma = [1, 2, \ldots, n]$ lies in $\mathrm{LinExt}(P)$. In particular, Example 3.58 uses this to deduce that for $P = 2 \times \mathsf{n}$, the *Cartesian product* of chains having sizes 2 and n, the Whitney numbers $c_k(2 \times \mathsf{n})$ are *Narayana numbers*, counting $2 \times n$ standard tableaux according to their number of descents. This P-Eulerian polynomial has many interpretations, e.g., as the *h-polynomial* of the order complex for the *distributive lattice* $J(P)$ of order ideals in P, or of the P*-partition triangulation* of the *order polytope* for P; see [48, Proposition 2.1, Proposition 2.2] and [57, Sections 3.4, 3.8, 3.13] for more on this. Unfortunately, in general the P-Eulerian polynomial differs from the Poincaré polynomial $\mathrm{Poin}(P,t)$ considered here. For example, when P is an antichain with three elements, the P-Eulerian polynomial is $1 + 4t + t^2$, while $\mathrm{Poin}(P,t) = 1 + 3t + 2t^2$.

Lastly, we note that the cone $\mathcal{K}_P$ associated to a poset P has appeared in many places in the literature, for example, implicitly in Stanley's theory of P-*partitions* [57, §3.15] and related work of Gessel on *quasisymmetric functions* [31], more explicitly in work of Björner and Wachs [10–12], and as the *ranking COMs* of Bandelt, Chepoi and Knauer [7]. A pointed version of this cone, called an *order cone* is discussed in Beck and Sanyal [8, Chapter 6]. When the underlying Hasse diagram of P is a tree, results on the number of chambers $\#\mathcal{C}(\mathcal{K}_P) = \#\operatorname{LinExt}(P)$ in these cones appear in work of K. Saito [52].

3.2 Intersections, Set Partitions, and Proof of Theorem 3.1

In this section, we review this correspondence and some useful results regarding the structure of the lattice of set partitions.

Let Π_n denote the lattice of set partitions on $[n]$, ordered via refinement: $\pi_1 \leq \pi_2$ if π_1 refines π_2. For the braid arrangement A_{n-1}, the intersection poset $\mathcal{L}(A_{n-1})$ is isomorphic to Π_n. This isomorphism will be useful for our work in Chapter 3, so we include the formal statement below, as a proposition:

Proposition 3.6 ([56, pp. 26-27])**.** *The map $\pi \mapsto X_\pi$ in Example 2.4 is a poset isomorphism $\Pi_n \cong \mathcal{L}(A_{n-1})$.*

In the case of the braid arrangement, these Möbius function values have a simple expression.

Proposition 3.7 ([57, Example 3.10.4])**.** *For any set partition $\pi = \{B_1, \ldots, B_k\}$ in Π_n, one has*

$$\mu(V, X_\pi) = (-1)^{n-k} \prod_{i=1}^{k} (\#B_i - 1)! \quad (= \mu(\ 1|2|\cdots|n\ ,\ \pi\ )\ )$$

with the convention $0! := 1$. Here $\mu(V, X_\pi), \mu(1|2|\cdots|n, \pi)$ are $\mu(-,-)$ values in $\mathcal{L}(A_{n-1}), \Pi_n$, respectively.

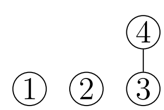
For the sake of our later discussion, we point out a re-interpretation of this formula involving permutations. Given a permutation σ in $\mathfrak{S}_n$, when considered as acting on $V = \mathbb{R}^n$, its *fixed space* $V^\sigma := \{\mathbf{x} \in \mathbb{R}^n : \sigma(\mathbf{x}) = \mathbf{x}\}$ will be the intersection subspace $V^\sigma = X_\pi$, where $\pi = \{B_1, B_2, \ldots, B_k\}$ is the set partition in Π_n given by the *cycles* of σ. Consequently, one can re-interpret

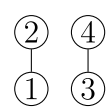
$$\begin{aligned} |\mu(V, X_\pi)| &= \prod_{i=1}^{k} (\#B_i - 1)! \\ &= \#\{\sigma \in \mathfrak{S}_n \text{ with cycle partition } \pi\} \\ &= \#\{\sigma \in \mathfrak{S}_n : V^\sigma = X_\pi\}, \end{aligned} \tag{3.8}$$

since each block B_i of π has $(\#B_i - 1)!$ choices of a cyclic orientation.

It is well-known (see [43, §3.3], for example) and easy to see that such cones correspond bijectively with posets P on $[n]$ via this rule: one has $x_i < x_j$ for all points in the cone $\mathcal{K}$ if and only if $i <_P j$. We will denote the cone associated to P by $\mathcal{K}_P$, and abbreviate $c_k(P) := c_k(\mathcal{K}_P)$ and $\mathrm{Poin}(P, t) := \mathrm{Poin}(\mathcal{K}_P, t)$.

Example 3.8. The cone inside A_3 in Example 2.9 given by the inequality $x_3 < x_4$ on $V = \mathbb{R}^4$ has defining poset P_1 with order relation $3 <_{P_1} 4$ on $[4] = \{1, 2, 3, 4\}$, while the cone in Example 2.10 given by the inequalities $x_1 < x_2$ and $x_3 < x_4$ has defining poset P_2 with order relations $1 <_{P_2} 2$ and $3 <_{P_2} 4$. These posets P_1, P_2 are shown here:

$P_1 =$ 

$P_2 =$ 

3.2.1 A cone-preposet dictionary and the interior intersections of a poset cone

By Theorem 3.6, the intersection poset $\mathcal{L}(A_{n-1})$ is isomorphic to the set partition lattice Π_n, and hence for each cone $\mathcal{K}_P$ in $\mathcal{A}_{n-1}$, one should be able to identify the interior intersection poset $\mathcal{L}^{\text{int}}(\mathcal{K}_P)$ as some order ideal inside Π_n. This is our next goal, which will be aided by recalling some facts about preposets, posets, binary relations, and cones.

Definition 3.9. Recall that a *preposet* Q on $[n]$ is a binary relation $Q \subseteq [n] \times [n]$ which is both

– *reflexive*, meaning $(i,i) \in Q$ for all i, and

– *transitive*, meaning $(i,j),(j,k) \in Q$ implies $(i,k) \in Q$.

If in addition Q is *antisymmetric*, meaning $(i,j),(j,i) \in Q$ implies $i = j$, then Q is called a *poset* on $[n]$; in this case, we sometimes write $i \leq_Q j$ when $(i,j) \in Q$.

A set partition $\pi \in \Pi_n$ is identified with an *equivalence relation* $\pi \subseteq [n] \times [n]$ having $(i,j) \in \pi$ when i,j appear in the same block of π. That is, π is reflexive, transitive, and *symmetric*, meaning $(i,j) \in \pi$ implies $(j,i) \in \pi$. We will sometimes write this binary relation as $i \equiv_\pi j$ when $(i,j) \in \pi$.

The *union* $Q_1 \cup Q_2 \subseteq [n] \times [n]$ of two preposets will be reflexive, but possibly not transitive, so not always a preposet. However, the transitive closure operation $Q \mapsto \overline{Q}$ lets one complete it to a preposet $\overline{Q_1 \cup Q_2}$.

We will use a slight rephrasing of the folklore *cone-preposet dictionary*, as discussed by Postnikov, Reiner, and Williams in [43, Section 3.3]. This dictionary is a bijection between preposets Q on $[n]$ and *closed* cones of any dimension that are intersections in $V = \mathbb{R}^n$ of closed halfspaces of the form $\{x_i \leq x_j\}$. Under this bijection, any such closed cone C corresponds to a preposet Q_C via

$$C \mapsto Q_C := \{(i,j) \mid x_i \leq x_j \text{ for all } \mathbf{x} \in C\}.$$

Conversely, any preposet Q on $[n]$ corresponds to a closed cone C_Q via

$$Q \mapsto C_Q := \bigcap_{(i,j)\in Q} \{x_i \leq x_j\} = \{\mathbf{x} \in \mathbb{R}^n \mid x_i \leq x_j \text{ for all } (i,j) \in Q\}.$$

For a subset $A \subseteq \mathbb{R}^n$, denote its *interior* and *relative interior* by $\mathsf{int}(A), \mathsf{relint}A$. Then for a preposet Q,

$$\mathsf{relint} C_Q = \left\{ \mathbf{x} \in \mathbb{R}^n : \begin{array}{c} x_i < x_j \text{ if } (i,j) \in Q \text{ but } (j,i) \notin Q, \\ x_i = x_j \text{ if both } (i,j),(j,i) \in Q \end{array} \right\}. \tag{3.9}$$

Also, one has the following assertions, using the notation of this dictionary:

- for π in Π_n, the subspace denoted X_π is the (non-pointed) cone C_π, regarding π as a preposet, and
- for any poset P on $[n]$, the open n-dimensional cone denoted $\mathcal{K}_P$ earlier is $\mathsf{relint} C_P$ $(= \mathsf{int}(C_P))$.

We will need one further dictionary fact from the Type A case.

Proposition 3.10 ([43, Proposition 3.5]). *For preposets Q, Q', one has*

$$C_Q \cap C_{Q'} = C_{\overline{Q \cup Q'}}.$$

The following definition will help to characterize the set partitions π having X_π in $\mathcal{L}^{\mathrm{int}}(\mathcal{K}_P)$.

Definition 3.11. Given a poset P on $[n]$ and a set partition $\pi = \{B_1, \ldots, B_k\}$ in Π_n, define a preposet P/π on the set $\{B_1, \ldots, B_k\}$ as the transitive closure of the (reflexive) binary relation having $(B_i, B_j) \in P/\pi$ whenever there exist $p \in B_i$ and $q \in B_j$ with $p \leq_P q$.

Proposition 3.12. *For P a poset on $[n]$ and $\pi = \{B_1, \ldots, B_k\}$ a set partition in Π_n, the following are equivalent.*

(i) $X_\pi \in \mathcal{L}^{\text{int}}(\mathcal{K}_P)$*, that is, one has a nonempty intersection* $X_\pi \cap \mathcal{K}_P \neq \emptyset$*.*

(ii) If $i <_P j$, meaning that $i \leq_P j$ and $i \neq j$, then $(j, i) \notin \overline{P \cup \pi}$.

(iii) Every block $B_i \in \pi$ is an antichain of P, and the preposet P/π is actually a poset.

We give a proof of Proposition 3.12 toward the end of this section, after some discussion and examples.

Definition 3.13. Let P be a poset on $[n]$. A set partition π of P is called a *P-transverse partition* if it satisfies one of the equivalent conditions in Proposition 3.12. We denote by $\Pi^{\pitchfork}(P)$ the induced subset of Π_n consisting of P-transverse partitions.

Remark 3.14. Aguiar and Mahajan [2, p.230] have a similar concept, which they call a *prelinear extension* of P. A prelinear extension of P is equivalent to a P-transverse partition π together with a linear ordering on the blocks of π that extends the partial order P/π from Proposition 3.12(iii).

Proposition 3.12 and Corollary 3.7 immediately imply the following corollary.

Corollary 3.15. *Let P be a poset on $[n]$. Then $\Pi^{\pitchfork}(P)$ and $\mathcal{L}^{\text{int}}(\mathcal{K}_P)$ are isomorphic as posets. Consequently,*

$$\mathrm{Poin}(P, t) = \sum_{\pi \in \Pi^{\pitchfork}(P)} |\mu(V, X_\pi)| \cdot t^{n - \#\mathsf{blocks}(\pi)} = \sum_{\substack{\pi = \{B_1, \ldots, B_k\} \\ \text{in } \Pi^{\pitchfork}(P)}} \prod_{i=1}^{k} (\#B_i - 1)! \cdot t^{n-k}.$$

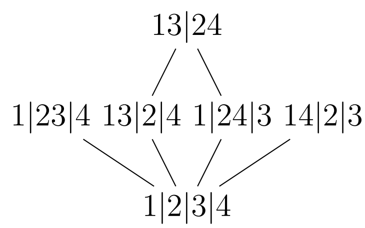
Example 3.16. Let $P := P_2$ be the second poset from Example 3.8, with $x_1 <_P x_2$ and $x_3 <_P x_4$. Then

- $\pi = 13|24$ is P-transverse.

- $\pi = 12|3|4$ is *not* P-transverse as it fails condition (ii): $1 <_P 2$, but $(2,1) \in \pi \subset \overline{P \cup \pi}$.

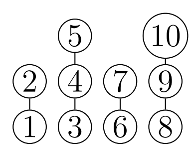
- $\pi = 14|23$ is *not* P-transverse, failing condition (ii): $1 <_P 2$, but $(2,1) \in \overline{P \cup \pi}$, though $(2,1) \notin P \cup \pi$.

The six P-transverse partitions give this subposet $\Pi^{\pitchfork}(P)$ of Π_4 isomorphic to $\mathcal{L}^{\text{int}}(\mathcal{K}_P)$, as in Example 2.10:

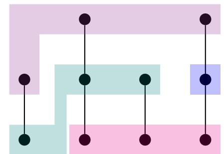


It happens that here $|\mu(V, X_\pi)| = 1$ for all π in $\Pi^{\pitchfork}(P)$, so that $\text{Poin}(P,t) = 1 + 4t + t^2$.

Example 3.17. Let P be the following poset:



Then $\pi = \{\{1,4,7\},\{2,5,10\},\{3,6,8\},\{9\}\}$ in Π_{10} is P-transverse, represented here by shading the blocks:



Viewed in this way, Proposition 3.12(iii), roughly speaking, states that π is P-transverse if and only if its blocks are antichains that can be "stacked without crossings" with respect to the Hasse diagram for P.

Proof of Proposition 3.12. We will show a cycle of implications: (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i).

(i) implies (ii):

Assume (i), so that there exists some $\mathbf{x}$ in $\mathbb{R}^n$ lying in the nonempty set

$$X_\pi \cap \mathcal{K}_P = X_\pi \cap \mathsf{int}(C_P) = \mathsf{relint} X_\pi \cap C_P = \mathsf{relint} C_{\overline{P \cup \pi}}$$
$$= \left\{ \mathbf{x} \in \mathbb{R}^n : \begin{array}{c} x_i < x_j \text{ if } (i,j) \in \overline{P \cup \pi} \text{ but } (j,i) \notin \overline{P \cup \pi}, \\ x_i = x_j \text{ if both } (i,j),(j,i) \in \overline{P \cup \pi} \end{array} \right\},$$

where the first equality comes from the definition of $\mathcal{K}_P$ and C_P, the second from the fact that $\mathcal{K}_P, C_P$ are full n-dimensional, the third from Proposition 3.10, and the fourth from equation (3.9) above. Now to see that (ii) holds, given any pair i, j with $i <_P j$, then $x_i < x_j$ since $\mathbf{x} \in \mathcal{K}_P$, but then since $(i,j) \in P \subseteq \overline{P \cup \pi}$, the conditions above imply $(j,i) \notin \overline{P \cup \pi}$, as desired for (ii).

(ii) implies (iii):

Assume (ii) holds. Then every block B of π must be an antichain in P, else there exists $i \neq j$ in B with $i <_P j$, and then $(j,i) \in \pi \subseteq \overline{P \cup \pi}$, contradicting (ii).

Now suppose for the sake of contradiction that P/π is not a poset. Since P/π is a preposet, it can only fail to be antisymmetric, that is, there are blocks $B \neq B'$

of π having both $(B, B'), (B', B)$ in P/π. Since both P, π are transitive binary relations, this means there must exist a (periodic) sequence of elements of the form

$$\cdots \equiv_\pi p_1 <_P p_2 \equiv_\pi p_3 <_P p_4 \equiv_\pi \cdots <_P p_{m-2} \equiv_\pi p_{m-1} <_P p_m \equiv_\pi p_1 <_P p_2 \equiv \cdots$$

alternating relations (p_i, p_{i+1}) lying in P and in π. Then $p_1 <_P p_2$ and $(p_2, p_1) \in \overline{P \cup \pi}$, contradicting (ii).

(iii) implies (i):
Assume (iii), that is, the blocks of π are antichains of P, and P/π is a poset. One can then reindex the blocks of π such that $(B_1, B_2, \ldots, B_k)$ is a linear extension of P/π. Use this indexing to define a point $\mathbf{x} \in \mathbb{R}^n$ whose p^{th} coordinate $x_p = i$ if p lies in block B_i of π.

We claim $\mathbf{x}$ lies in $X_\pi \cap \mathcal{K}_P$, verifying (i). By construction $\mathbf{x}$ lies in X_π, since its coordinates are constant within the blocks of π. To verify $\mathbf{x} \in \mathcal{K}_P$, given $p <_P q$, one must check that $x_p < x_q$. Assume that p, q lie in blocks B_i, B_j of π, so that $x_p = i$ and $x_q = j$. Since the blocks of π are antichains in P and $p <_P q$, one has $i \neq j$, and since $(B_1, B_2, \ldots, B_k)$ is a linear extension of P/π, one must have $i < j$, that is, $x_p < x_q$. $\square$

It will help later in identifying P-transverse partitions to also have the following *recursive* characterization.

Proposition 3.18. *Let P be a poset on $[n]$, and π a set partition of $[n]$. Then π lies in $\Pi^{\pitchfork}(P)$ if and only if it contains a block B with these two properties:*

(a) B is a subset of the minimal elements of P, and

(b) if $\hat{\pi} := \pi \backslash \{B\}$ and $\hat{P}$ is the poset on $[n] \backslash B$ obtained from P by removing the elements in B, then $\hat{\pi} \in \Pi^{\pitchfork}(\hat{P})$.

Proof. For the forward implication, assume $\pi \in \Pi^{\pitchfork}(P)$ and $B \in \pi$ is a minimal block. We use Proposition 3.12(iii) to show that (a) and (b) hold. For (a), assume there exists some $x \in B$ which is *not* minimal in P, i.e., there is some $y \in P$ with $x >_P y$. The block B is an antichain, so necessarily $y \notin B$ and so y lies in a block $B' \neq B$ of π. But then $B >_{P/\pi} B'$ contradicts the minimality of B in P/π.

For (b), note that the blocks of $\hat{\pi}$ are a subset of the blocks of π, so they are antichains in P and necessarily also antichains in $\hat{P}$. Furthermore, the preposet $\hat{P}/\hat{\pi}$ on the blocks of $\hat{\pi}$ must be a poset, else if B', B'' were two blocks of $\hat{\pi}$ having $B' \geq_{\hat{P}/\hat{\pi}} B''$ and $B'' \geq_{\hat{P}/\hat{\pi}} B'$, then these same two blocks B', B'' in π and would have have $B' \geq_{P/\pi} B''$ and $B'' \geq_{P/\pi} B'$, a contradiction.

For the backward implication, assume π in Π_n has a block B satisfying properties (a), (b). We use Proposition 3.12(i) to show that $\pi \in \Pi^{\pitchfork}(P)$. Since $\hat{\pi} \in \Pi^{\pitchfork}(\hat{P})$, there is a point $\hat{\mathbf{x}} \in \mathbb{R}^{[n]\setminus B}$ in the (nonempty) set $X_{\hat{\pi}} \cap \mathcal{K}_{\hat{P}}$, i.e., the coordinates of $\mathbf{x}$ are constant within each block of $\hat{\pi}$, and $\hat{\mathbf{x}}_p < \hat{\mathbf{x}}_q$ whenever $p <_{\hat{P}} q$. Let $\mathfrak{m} := \min_{\mathbb{R}}\{\hat{\mathbf{x}}_p : p \in [n] \setminus B\}$ be the smallest coordinate of $\hat{\mathbf{x}}$, and then extend $\hat{\mathbf{x}} \in \mathbb{R}^{[n]\setminus B}$ to a point $\mathbf{x} \in \mathbb{R}^n$ by assigning all of the new coordinates x_p for $p \in B$ to have the same value, but strictly smaller than $\mathfrak{m}$, e.g., $x_p := \mathfrak{m} - 1$ for all $p \in B$. One then checks that this $\mathbf{x}$ lies in $X_\pi \cap \mathcal{K}_P$: it lies in $\mathcal{K}_p$ due to the fact that B is a subset of the minimal elements of P, and it lies in X_π because it is constant on the new block B of π not already in $\hat{\pi}$, as well as constant on the blocks of $\hat{\pi}$. $\square$

3.2.2 More examples of $\Pi^{\pitchfork}(P)$

Example 3.19. Given a poset P on $[n]$, its *dual* or *opposite* poset P^{opp} has the same underlying set $[n]$, but with opposite order relation: $i \leq_P j$ if and only if $j \leq_{P^{\mathsf{opp}}} i$. One can readily check that conditions (ii) and (iii) in Proposition 3.12 are self-dual in the sense that π in Π_n is P-transverse if and only if it is P^{opp}-transverse.

Consequently, $\Pi^{\pitchfork}(P^{\mathsf{opp}}) = \Pi^{\pitchfork}(P)$, and hence $\mathrm{Poin}(P^{\mathsf{opp}}, t) = \mathrm{Poin}(P, t)$.

Example 3.20. Given posets P_1, P_2, respectively, their *ordinal sum* $P_1 \oplus P_2$ is the poset whose underlying set is the disjoint union $P_1 \sqcup P_2$, and having order relations $x \leq_{P_1 \oplus P_2} y$ if either

- $x, y \in P_i$ and $x_{\leq_{P_i}} y$ for some $i = 1, 2$, or
- $x \in P_1$ and $y \in P_2$.

If the underlying sets for P_1, P_2 are $[n_1], [n_2]$, one can readily check from either of Proposition 3.12 (ii) or (iii) that a partition π of $[n_1] \sqcup [n_2]$ is $P_1 \oplus P_2$-transverse if and only if it is of form $\pi = \{A_1, \ldots, A_k, B_1, \ldots, B_\ell\}$ where $\pi_1 = \{A_i\}_{i=1}^k$ and $\pi_2 = \{B_j\}_{j=1}^\ell$ are P_1-transverse and P_2-transverse partitions of $[n_1]$ and $[n_2]$, respectively. Bearing in mind that $V = \mathbb{R}^{n_1+n_2} = V_1 \oplus V_2$ where $V_i = \mathbb{R}^{n_i}$ for $i = 1, 2$, this gives isomorphisms

$$\Pi^{\pitchfork}(P \oplus P_2) \cong \Pi^{\pitchfork}(P_1) \times \Pi^{\pitchfork}(P_2)$$
$$[V, X_\pi] \cong [V_1, X_{\pi_1}] \times [V_2, X_{\pi_2}]$$

and therefore also

$$\mathrm{Poin}(P_1 \oplus P_2, t) = \mathrm{Poin}(P_1, t) \cdot \mathrm{Poin}(P_2, t).$$

3.2.3 Proof of Theorem 3.1

We recall here the bijection between the chambers of braid arrangement A_{n-1} inside a cone $\mathcal{K}_P$ and the linear extensions of P. We then define P-transverse permutations and use them to combinatorially re-interpret $\mathrm{Poin}(P, t)$.

Definition 3.21. Given two posets P, Q on $[n]$, say that Q *extends* P if $i \leq_P j$ implies $i \leq_Q j$, that is, $P \subseteq Q$ as binary relations on $[n]$, or equivalently, their cones satisfy $\mathcal{K}_Q \subseteq \mathcal{K}_P$. When Q is a total or linear order $\sigma_1 < \cdots < \sigma_n$ on $[n]$, we identify it with a permutation $\sigma = \sigma_1 \ldots \sigma_n$, and call σ a *linear extension* of P. Let $\mathrm{LinExt}(P)$ denote the set of all linear extensions of P.

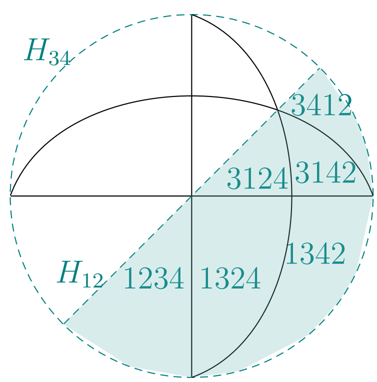
Example 2.2 noted that chambers of the braid arrangement A_{n-1} are of the form $\mathcal{K}_\sigma$ for permutations σ. Then $\mathcal{K}_\sigma$ is a chamber lying in the cone $\mathcal{C}(\mathcal{K}_P)$ if and only if σ lies in $\mathrm{LinExt}(P)$, giving a bijection

$$\begin{array}{rcl} \mathrm{LinExt}(P) & \longrightarrow & \mathcal{C}(\mathcal{K}_P) \\ \sigma & \longmapsto & \mathcal{K}_\sigma. \end{array}$$

See also [56, Example 1.3]. Consequently, as noted in (3.2) one has

$$\#\,\mathrm{LinExt}(P) = \sum_{k \geq 0} c_k(P) \ = [\mathrm{Poin}(P,t)]_{t=1}\,.$$

Example 3.22. The poset P defined by $1 <_P 2$ and $3 <_P 4$ from Example 2.10 has six linear extensions, shown here labeling the chambers in $\mathcal{C}(\mathcal{K}_P)$:



Recall our goal of finding combinatorial interpretations for $\mathrm{Poin}(P,t)$. Comparing the expression for $\mathrm{Poin}(P,t)$ given by Corollary 3.15 in terms of P-transverse partitions π, and the interpretation of $\mu(V, X_\pi)$ in terms of permutations given in (3.8) motivates the following definition.

Definition 3.23. Given a poset P on $[n]$, a *P-transverse permutation* is a permutation σ in $\mathfrak{S}_n$ for which the set partition π given by the cycles of σ is a P-transverse partition. Denote by $\mathfrak{S}^{\pitchfork}(P)$ the set of all P-transverse permutations.

Corollary 3.15 and equation (3.8) then immediately imply this interpretation for $\mathrm{Poin}(P,t)$.

Thereom 3.1. *For any poset P on $[n]$, one has*

$$\mathrm{Poin}(P,t) = \sum_{\sigma \in \mathfrak{S}^{\pitchfork}(P)} t^{n-\mathsf{cyc}(\sigma)}.$$

In particular, setting $t = 1$, one has $\#\,\mathrm{LinExt}(P) = \#\mathfrak{S}^{\pitchfork}(P)$.

3.3 Proof of Theorem 3.2

The goal in the next few subsections is to define mutually inverse bijections

$$\begin{array}{rcl} \mathfrak{S}^{\pitchfork}(P) & \xrightarrow{\Phi} & \mathrm{LinExt}(P) \\ \mathrm{LinExt}(P) & \xrightarrow{\Psi} & \mathfrak{S}^{\pitchfork}(P) \end{array}$$

along with the notion of *P-left-to-right maxima* for σ in $\mathrm{LinExt}(P)$, to prove this result from the Introduction.

Theorem 3.2. *For any poset P on $[n]$, one has a bijection $\Phi : \mathfrak{S}^{\pitchfork}(P) \to \mathrm{LinExt}(P)$ sending the number of cycles to the number of P-left-to-right-maxima. Therefore,*

$$\mathrm{Poin}(P,t) = \sum_{\sigma \in \mathrm{LinExt}(P)} t^{n-\mathsf{LRmax}_P(\sigma)}$$

where $\mathsf{LRmax}_P(\sigma)$ denotes the number of P-left-to-right maxima of σ.

To this end, we first recall the special case the antichain poset $P = \mathsf{Antichain}_n$ on $[n]$, where Φ is known as the *fundamental bijection* [57, Proposition 1.3.1]. A permutation τ in $\mathfrak{S}_n$ sending $i \mapsto \tau(i)$ may be written

– in a one-line notation as $\tau = [\tau(1), \tau(2), \ldots, \tau(n)]$, or

– in a two-line notation $\tau = \begin{pmatrix} 1 & 2 & \cdots & n \\ \tau(1) & \tau(2) & \cdots & \tau(n) \end{pmatrix}$, or

– in various cycle notations that list the τ-orbits on $[n]$, called its *cycles*, in some arbitrarily chosen order, with each cycle listed as $(j, \tau(j), \tau^2(j), \ldots)$ for some arbitrary choice of the first element j.

One way to make the choices non-arbitrary and put the cycle notation in *standard form* insists that the first element j listed within each cycle $\tau^{(i)}$ is the maximum element of the cycle, and then insists that the cycles $\tau^{(1)}, \tau^{(2)}, \ldots$ are listed with their maximum elements in increasing order as integers, that is, $j_1 <_{\mathbb{Z}} j_2 <_{\mathbb{Z}} \cdots$. The fundamental bijection $\mathfrak{S}_n \to \mathfrak{S}_n$ sends τ to $\Phi(\tau) := \sigma = [\sigma_1, \sigma_2, \ldots, \sigma_n]$ by erasing the parentheses around the standard form cycle notation for τ.

Example 3.24. The permutation $\tau = [7, 5, 9, 4, 2, 8, 3, 6, 1]$ in $\mathfrak{S}_9$ in one-line notation can also be written in two-line notation and factored according to its τ-orbits or cycles

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 7 & 5 & 9 & 4 & 2 & 8 & 3 & 6 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 7 & 9 \\ 7 & 9 & 3 & 1 \end{pmatrix} \begin{pmatrix} 2 & 5 \\ 5 & 2 \end{pmatrix} \begin{pmatrix} 4 \\ 4 \end{pmatrix} \begin{pmatrix} 6 & 8 \\ 8 & 6 \end{pmatrix}.$$

Its cycle notation in standard form and image $\sigma = \Phi(\tau)$ are then

$$\begin{aligned} \tau &= (4)(5,2)(8,6)(9,1,7,3) \\ \Phi(\tau) = \sigma &= [4,5,2,8,6,9,1,7,3]. \end{aligned}$$

The inverse map Ψ starts with $\sigma = [\sigma_1, \ldots, \sigma_n]$ in one-line notation, and determines where to re-insert the parenthesis pairs in the sequence to obtain the standard form for the cycles of τ. One only needs to know the locations of the left parentheses, since then the right parenthesis locations are determined. There will be one left parenthesis just to the left of each σ_j which is a *left-to-right maximum* (or *LR-maximum* for short) in σ, meaning $\sigma_j = \max_{\mathbb{Z}}\{\sigma_1, \sigma_2, \ldots, \sigma_j\}$. It is not hard to check that Φ, Ψ are mutual inverses, and if $\sigma = \Phi(\tau)$, one has $\mathsf{cyc}(\tau) = \mathsf{LRmax}(\sigma)$, the number of LR-maxima of σ.

3.3.1 The map $\Phi : \mathfrak{S}^{\pitchfork}(P) \to \mathrm{LinExt}(P)$

To define the map $\Phi : \mathfrak{S}^{\pitchfork}(P) \to \mathrm{LinExt}(P)$ on a P-transverse permutation τ, we will first use the P-transverse partition π whose blocks are the cycles of τ to separate the blocks of π and the elements of P into *levels*, and then define a notion of *essential* elements. Recall that because σ lies in $\mathfrak{S}^{\pitchfork}(P)$, meaning π lies in $\Pi^{\pitchfork}(P)$, the quotient preposet P/π on the blocks of π is actually a poset. This leads to the following definition:

Definition 3.25. Say that a block of π which is minimal in P/π is of *Level* 1. For $k \geq 2$, the blocks of π of *Level* k are the minimal ones in the poset obtained from P/π by removing all blocks of Level less than k.

In other words, a block B of π is of Level k if and only if

$$k = \max\{\ell : \text{ there exists a chain } B =: B_1 >_{P/\pi} B_2 >_{P/\pi} \cdots >_{P/\pi} B_\ell\},$$

or even more concretely, k is the maximum among integers ℓ with the property that there exist blocks $B =: B_1, B_2, \ldots, B_\ell$ of π and elements $x_i >_P y_{i+1}$ with $x_i \in B_i, y_{i+1} \in B_{i+1}$ for each $i = 1, 2, \ldots, \ell - 1$.

For an element x in $[n]$, define the *Level* of x to be the Level of the unique block of π containing x.

Definition 3.26. An element x is *essential* if x has Level k and there exists some y of Level $k - 1$ with $x >_P y$; by convention, all Level 1 elements x are essential.

In order to define the map $\Phi : \mathfrak{S}^{\pitchfork}(P) \to \mathrm{LinExt}(P)$, we first introduce a *standard form* for a P-transverse permutation τ. Let τ have cycle partition $\pi = \{B_1, \ldots, B_m\}$, lying in $\Pi^{\pitchfork}(P)$. List the cycles of τ in order $\tau^{(1)}, \tau^{(2)}, \ldots, \tau^{(m)}$, with the block B_i of π corresponding to the cycle $\tau^{(i)}$, and write the cycle $\tau^{(i)}$ as $\tau^{(i)} = (x_i, \tau(x_i), \tau^2(x_i), \ldots)$ for some x_i in B_i. Then this listing is the *standard form* of τ if and only if it has the following properties:

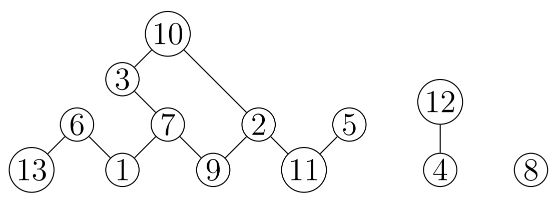
– If blocks B_i, B_j have Levels $k, k+1$ in π, respectively, then their indices satisfy $i \leq_{\mathbb{Z}} j$.

– For each i, the first element x_i listed in the cycle $\tau^{(i)} = (x_i, \tau(x_i), \tau^2(x_i), \ldots)$ is the maximal essential element of B_i; by Lemma 3.28(b), below, every block B_i contains an essential element.

– If B_i, B_j are blocks of Level k with $i <_{\mathbb{Z}} j$ then $x_i <_{\mathbb{Z}} x_j$.

Following the fundamental bijection, the map $\Phi : \mathfrak{S}^{\pitchfork}(P) \to \mathrm{LinExt}(P)$ is defined by putting $\tau \in \mathfrak{S}^{\pitchfork}(P)$ into standard form and erasing parenthesis. The following example illustrates this process.

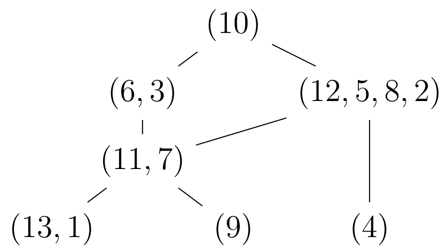
Example 3.27. Let P be the following poset on $[13]$:



Let

$$\begin{aligned} \tau &= (4)(6,3)(9)(10)(11,7)(12,5,8,2)(13,1) && \in \mathfrak{S}^{\pitchfork}(P), \text{ so that} \\ \pi &= \{\{4\},\{3,6\},\{9\},\{7,10\},\{2,5,8,12\},\{1,13\}\} && \in \Pi^{\pitchfork}(P), \end{aligned}$$

and the poset P/π, drawn as a poset on the cycles of τ, looks as follows:



One can check that

- the Level 1 cycles are $(13,1),(9),(4)$, with essential elements $1,4,9,13$,
- there is one Level 2 cycle $(11,7)$, with one essential element 7,
- the Level 3 cycles are $(6,3),(12,5,8,2)$, with essential elements $2,3,5$,
- there is one Level 4 cycle (10), with one essential element 10.

Here is τ in standard form, with essential elements overlined, and Levels separated by bars:

$$\tau = (\overline{4})(\overline{9})(\overline{13},\overline{1}) \mid (\overline{7},11) \mid (\overline{3},6)(\overline{5},8,\overline{2},12) \mid (\overline{10}),$$

Removing the parentheses (and bars), one obtains its image under Φ:

$$\Phi(\tau) = \sigma = [4,9,13,1,7,11,3,6,5,8,2,12,10] \in \mathrm{LinExt}(P).$$

The next lemma is used to prove the image of Φ lies in $\mathrm{LinExt}(P)$, and $\mathfrak{S}^{\pitchfork}(P) \xrightarrow{\Phi} \mathrm{LinExt}(P)$ is bijective.

Lemma 3.28. *Let P be a poset on $[n]$ and $\tau \in \mathfrak{S}^{\pitchfork}(P)$. Then the following properties hold:*

(a) For each $k \geq 1$, the Level k elements of $[n]$ form an antichain in P.

(b) Every cycle contains at least one essential element.

(c) The image $\Phi(\tau)$ of τ is a linear extension of P.

Proof. For (a), assume that there were two comparable elements $x <_P y$ with x, y both of Level k. Either x, y lie in the *same* block of π, contradicting P-transversality, or they lie in *different* blocks of π, which would be comparable in P/π, contradicting both blocks being of Level k.

For (b), note that our most concrete description of a block B having Level k shows that there exist blocks $B =: B_1, B_2, \ldots, B_k$ of π and elements $x_i >_P y_{i+1}$ with $x_i \in B_i, y_{i+1} \in B_{i+1}$ for each $i = 1, 2, \ldots, k-1$. But then this shows that the block B_2 must be of Level $k-1$, and $x_1 >_P y_2$ shows x_1 is essential in $B_1 (= B)$.

For (c), one can show via induction on k that the restriction of $\Phi(\tau)$ to the order ideal of elements of P having Level at most k is a linear extension. In both the base case $k = 1$, and in the inductive step, one notes that one can add in the elements of Level k in any order, because they form an antichain by part (a). $\square$

3.3.2 The inverse map Ψ

To define the inverse map $\Psi = \Phi^{-1}$ on a linear extension σ, we proceed similarly to the previous subsection. We first cut σ into consecutive strings, suggestively called *Levels*, define a notion of *essential* elements of σ, and a notion of *P-left-to-right-maximum.*

Definition 3.29. Given $\sigma = [\sigma_1, \ldots, \sigma_n]$ in $\mathrm{LinExt}(P)$, we recursively break σ into disjoint contiguous sequences $[\sigma_i, \sigma_{i+1}, \ldots, \sigma_{i+j}]$, each forming an antichain of P, and each maximal in the sense that the slightly longer sequence $[\sigma_i, \sigma_{i+1}, \ldots, \sigma_{i+j}, \sigma_{i+j+1}]$ is *not* an antichain of P:

- Let $[\sigma_1, \sigma_2, \ldots, \sigma_r]$ be the longest initial segment of σ whose underlying set $\{\sigma_1, \sigma_2, \ldots, \sigma_r\}$ is an antichain of P; call $\{\sigma_1, \sigma_2, \ldots, \sigma_r\}$ the *Level 1* elements of σ.

– The *Level 2* elements of σ are $\{\sigma_{r+1}, \sigma_{r+2}, \ldots, \sigma_s\}$, where $[\sigma_{r+1}, \sigma_{r+2}, \ldots, \sigma_s]$ is the longest initial segment of $[\sigma_{r+1}, \sigma_{r+2}, \ldots, \sigma_n]$ that forms an antichain in P.

– Similarly, for $k \geq 3$, the Level k elements of σ are defined as follows: if the union of all elements of Levels $1, 2, \ldots, k-1$ are $\{\sigma_1, \sigma_2, \ldots, \sigma_t\}$, then the set of Level k elements is the underlying set of the longest initial segment in $[\sigma_{t+1}, \sigma_{t+2}, \ldots, \sigma_n]$ that forms an antichain in P.

Definition 3.30. As in the previous section, say x in σ is *essential* if x has Level k in σ and there exists an element y of Level $k-1$ in σ with $x >_P y$; again by convention, Level 1 elements of σ are all essential.

Definition 3.31. Say that an element x is a *P-left-to-right-maximum* of σ, or *P-LR-maximum* for short, if x is essential in σ, and x appears as a LR-maximum in the usual sense among the subsequence of essential elements of σ having the same Level as x. In other words, if x has Level k, it is a P-LR-maximum if the subsequence of essential Level k elements is $[\sigma_{i_1}, \sigma_{i_2}, \ldots, \sigma_{i_r}]$ for some indices $i_1 < i_2 < \cdots < i_r$, and there is some j with $1 \leq j \leq r$ for which $\sigma_{i_j} = x = \max\{\sigma_{i_1}, \sigma_{i_2}, \ldots, \sigma_{i_j}\}$. We denote the number of P-LR maxima of σ by $\mathsf{LRmax}_P(\sigma)$.

A map $\Psi : \mathrm{LinExt}(P) \to \mathfrak{S}_n$ can now be defined in much the same way as for the fundamental bijection: starting with $\sigma = [\sigma_1, \ldots, \sigma_n]$ in $\mathrm{LinExt}(P)$, one must determine where to re-insert the parenthesis pairs in the sequence to recover the cycles of τ. In fact, one only needs to know the locations of the left parentheses, since then the right parenthesis locations are determined, and there will be one left parenthesis just to the left of each $x = \sigma_j$ which is a P-LR maximum.

Example 3.32. Let P be the poset in Example 3.27 and let

$$\sigma = [4, 9, 13, 1, 7, 11, 3, 6, 5, 8, 2, 12, 10] \in \mathrm{LinExt}(P),$$

which is $\Phi(\tau)$ from Example 3.27. The Level decomposition of σ, with essential elements overlined, looks like

$$\overline{4}, \overline{9}, \overline{13}, \overline{1} \mid \overline{7}, 11 \mid \overline{3}, 6, \overline{5}, 8, \overline{2}, 12 \mid \overline{10}.$$

Now create cycles by placing left parentheses just before each P-LR maximum:

$$(\overline{4})(\overline{9})(\overline{13}, \overline{1}) \mid (\overline{7}, 11) \mid (\overline{3}, 6)(\overline{5}, 8, \overline{2}, 12) \mid (\overline{10}).$$

The resulting cycle structure gives the P-transverse permutation $\Psi(\sigma)$:

$$\Psi(\sigma) = (4)(9)(13, 1)(7, 11)(3, 6)(5, 8, 2, 12)(10),$$

which is the P-transverse permutation τ in Example 3.27.

Note that it is not yet clear that the image of Ψ lies in $\mathfrak{S}^{\pitchfork}(P)$, but it will follow from the proof of Theorem 3.2. First we need a technical lemma.

Lemma 3.33. *Let $\sigma \in \mathrm{LinExt}(P)$ and $\tau \in \mathfrak{S}^{\pitchfork}(P)$ such that $\sigma = \Phi(\tau)$. Then the set of Level k elements of τ is precisely the set of Level k elements of σ.*

Proof. We prove this by induction on k. For the base case ($k = 1$), note that the Level 1 elements of τ will form an initial segment $\{\sigma_1, \sigma_2, \ldots, \sigma_\ell\}$ of $\sigma = [\sigma_1, \ldots, \sigma_n]$, by the definition of Φ. These Level 1 elements of τ will also form an antichain of P by Lemma 3.28(a). On the other hand, we claim that the longer initial segment $\{\sigma_1, \sigma_2, \ldots, \sigma_\ell, \sigma_{\ell+1}\}$ *cannot* form an antichain in P, because $\sigma_{\ell+1}$ is of Level 2 in τ by definition of Φ, and it is also essential in τ because it is leftmost in its cycle in the standard form for τ, and all such elements are essential. Thus $\sigma_{\ell+1} >_P \sigma_i$

for some $i = 1, 2, \ldots, \ell$, showing that the longer segment is not an antichain of P. By definition of Ψ, this means that the Level 1 elements of σ will be those in the shorter segment $\{\sigma_1, \sigma_2, \ldots, \sigma_\ell\}$.

For the inductive step ($k \geq 2$), we perform the same argument as in the base case, but replace P, τ, σ with their counterparts $\hat{P}, \hat{\tau}, \hat{\sigma}$ in which the elements of Level $1, 2, \ldots, k-1$ have been removed. One must check that $\hat{\tau}$ lies in $\mathfrak{S}^{\pitchfork}(\hat{P})$, but this is straightforward, because if $\sigma, \hat{\sigma}$ have cycle partitions $\pi, \hat{\pi}$, then $\hat{P}/\hat{\pi}$ is obtained from P/π by removing its minimal blocks. $\square$

Proof of Theorem 3.2. Note $\#\mathfrak{S}^{\pitchfork}(P) = \#\,\mathrm{LinExt}(P)$ by Theorem 3.1, and Φ maps $\mathfrak{S}^{\pitchfork}(P) \to \mathrm{LinExt}(P)$ by Lemma 3.28(c). We therefore claim that it suffices to check $\Psi(\Phi(\tau)) = \tau$ for all τ in $\mathfrak{S}^{\pitchfork}(P)$. This will imply that Φ is injective, hence bijective, with Ψ its inverse bijection, and thus the image of Ψ is $\mathfrak{S}^{\pitchfork}(P)$. Note that by construction, if $\Psi(\sigma) = \tau$, then $\mathsf{LRmax}_P(\sigma) = \mathsf{cyc}(\tau)$, so Theorem 3.2 would follow.

By Lemma 3.33, we have if $\sigma = \Phi(\tau)$, then the sets of Level k elements of σ and τ coincide. It follows immediately from Lemma 3.33 that the *essential* elements in σ and τ coincide, since in each case, their definition uses only the order P and the partition by Levels. Therefore, we can focus our attention on each Level k separately, where the definition of Φ and Ψ coincides almost exactly with their definition in the fundamental bijection, ignoring the non-essential elements carried along in each Level. It then follows that $\Psi(\Phi(\tau)) = \tau$ via the same argument as for the fundamental bijection. $\square$

3.4 Proofs of Theorems 3.3 and 3.4

The goal of this section is to understand $\mathrm{Poin}(P,t)$ for a poset P that is a disjoint union of chains. Although Theorem 3.2 applies to *any* poset, when P is a disjoint union of chains, there turns out to be another elegant expression for $\mathrm{Poin}(P,t)$ stemming from Foata's theory of multiset permutations, generalizing equation (3.6) for the antichain poset P.

In Subsection 3.4.1 we review Foata's theory of multiset permutations, in particular his work with the *intercalation product* and *prime cycle decompositions*. Subsection 3.4.2 reviews its relation to partial commutation monoids. Subsection 3.4.3 shows how the results in Section 3.3 can be rephrased in terms of multiset permutations when P is a disjoint union of chains. Theorem 3.3 is also proved in this subsection. Finally Subsection 3.5 employs Foata's theory to give a generalization of MacMahon's Master Theorem which specializes to Theorem 3.4, a generating function compiling the Poincaré polynomials for disjoint unions of chains.

3.4.1 Multiset Permutations

This subsection gives background on the theory of multiset permutations as introduced by Foata in his PhD thesis [24, Section 3.2], and extended in later publications [25, Chapters 3-5]. It also appears in Knuth [35, Section 5.1.2].

Definition 3.34. Recall that a *(weak) composition* $\overline{a} = (a_1, \ldots, a_\ell)$ *of* n is a sequence of nonnegative integers having sum $|a| := \sum_i a_i = n$. We will regard $\overline{a}$ as specifying the multiplicities in a *multiset* $M(\overline{a}) := \{1^{a_1}, 2^{a_2}, \ldots, \ell^{a_\ell}\}$, that is, a set with repetitions

$$M(\overline{a}) = \{\underbrace{1,1,\ldots,1}_{a_1 \text{ times}}, \underbrace{2,2,\ldots,2}_{a_2 \text{ times}}, \ldots, \underbrace{\ell,\ell,\ldots,\ell}_{a_\ell \text{ times}}\}.$$

A *multiset permutation* $\sigma = [\sigma_1, \ldots, \sigma_n]$ is a rearrangement of the elements of $M(\overline{a})$, which we will often write in a two-line notation that generalizes that of permutations:

$$\sigma = \begin{pmatrix} 1 & \cdots & 1 & 2 & \cdots & 2 & \cdots & \ell & \cdots & \ell \\ \sigma_1 & \cdots & \sigma_{a_1} & \sigma_{a_1+1} & \cdots & \sigma_{a_1+a_2} & \cdots & \sigma_{a_1+\cdots+a_{\ell-1}+1} & \cdots & \sigma_n \end{pmatrix}.$$

We denote the set of all multiset permutations of $M(\overline{a})$ by $\mathfrak{S}_{M(\overline{a})}$. For any $\sigma \in \mathfrak{S}_{M(\overline{a})}$, we call $M(\overline{a})$ the *support* of σ, and write $M(\overline{a}) = \mathsf{supp}(\sigma)$.

Example 3.35. The composition $\overline{a} = (2, 3, 2, 3)$ gives the multiplicities of the multiset

$$M(\overline{a}) = \{1^2, 2^3, 3^2, 4^3\} = \{1, 1, 2, 2, 2, 3, 3, 4, 4, 4\}.$$

Then the following multiset permutation σ is an element of $\mathfrak{S}_{M(\overline{a})}$:

$$\sigma = \begin{pmatrix} 1 & 1 & 2 & 2 & 2 & 3 & 3 & 4 & 4 & 4 \\ 2 & 4 & 4 & 3 & 1 & 2 & 1 & 3 & 4 & 2 \end{pmatrix}.$$

Foata [24, §3.2] defined an associative *intercalation product* operation on multiset permutations $(\sigma, \rho) \mapsto \sigma \intercal \rho$. Knuth [35, §5.1.2] describes it algorithmically: think of σ, ρ in two-line notation as sequences of columns $\binom{i}{j}$, and juxtapose these sequences of columns. Then perform swaps to *sort* the columns according to their top entries, never swapping two with the *same* top entry. For example,

$$\begin{pmatrix} 2 & 3 & 4 \\ 4 & 2 & 3 \end{pmatrix} \intercal \begin{pmatrix} 1 & 1 & 2 & 2 & 3 & 4 & 4 \\ 2 & 4 & 3 & 1 & 1 & 4 & 2 \end{pmatrix} = \left(\begin{array}{ccc|ccccccc} 2 & 3 & 4 & 1 & 1 & 2 & 2 & 3 & 4 & 4 \\ 4 & 2 & 3 & 2 & 4 & 3 & 1 & 1 & 4 & 2 \end{array}\right)$$
$$= \begin{pmatrix} 1 & 1 & 2 & 2 & 2 & 3 & 3 & 4 & 4 & 4 \\ 2 & 4 & 4 & 3 & 1 & 2 & 1 & 3 & 4 & 2 \end{pmatrix}.$$

Definition 3.36. For each ℓ, the *intercalation monoid* Int_ℓ is the submonoid of all multiset permutations σ whose support $M = \{1^{a_1}, 2^{a_2}, \ldots, \ell^{a_\ell}\}$ involves only the letters in $\{1, 2, \ldots, \ell\}$. The empty permutation $()$ is the identity element for $\top$, since $() \top \sigma = \sigma = \sigma \top ()$.

Note that, just as permutations in the symmetric group $\mathfrak{S}_n$ do not commute in general, the monoid Int_ℓ is not commutative. For example

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \top \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 2 & 3 \\ 2 & 3 & 1 & 1 \end{pmatrix} \neq \begin{pmatrix} 1 & 1 & 2 & 3 \\ 3 & 2 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} \top \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

However, one can check that $\sigma \top \rho = \rho \top \sigma$ when σ, ρ are *disjoint*, that is, $\mathsf{supp}(\sigma) \cap \mathsf{supp}(\rho) = \emptyset$.

Definition 3.37. Say σ in Int_ℓ is *prime* if the only factorizations $\sigma = \rho \top \tau$ have either $\rho = ()$ or $\tau = ()$.

Example 3.38. The permutation $\begin{pmatrix} 2 & 4 & 5 & 7 \\ 5 & 7 & 4 & 2 \end{pmatrix}$ is prime. However, $\begin{pmatrix} 1 & 1 & 2 & 3 \\ 2 & 3 & 1 & 1 \end{pmatrix}$ is not prime, since

$$\begin{pmatrix} 1 & 1 & 2 & 3 \\ 2 & 3 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \top \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}.$$

On the other hand $\begin{pmatrix} 2 & 4 & 5 & 7 \\ 5 & 7 & 2 & 4 \end{pmatrix}$ is not prime, even though its support is multiplicity free, since

$$\begin{pmatrix} 2 & 4 & 5 & 7 \\ 5 & 7 & 2 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 5 \\ 5 & 2 \end{pmatrix} \top \begin{pmatrix} 4 & 7 \\ 7 & 4 \end{pmatrix} = \begin{pmatrix} 4 & 7 \\ 7 & 4 \end{pmatrix} \top \begin{pmatrix} 2 & 5 \\ 5 & 2 \end{pmatrix}.$$

It is not obvious, but turns out to be true that σ is prime if and only if both

- $\mathsf{supp}(\sigma) = M$ is multiplicity free, that is, M is a set not a multiset, and

– σ has only one cycle when considered as an ordinary permutation of the set M.

We therefore call *prime* elements *prime cycles.* More generally, one has the following.

Theorem 3.39 (Foata, 1969 [25, 35])**.** *Let σ be a multiset permutation. Then σ has a decomposition into a product of prime cycles. That is, there exist $k \geq 0$ prime cycles $\sigma^{(1)}, \ldots, \sigma^{(k)}$ such that*

$$\sigma = \sigma^{(1)} \mathsf{T}\, \sigma^{(2)} \mathsf{T} \cdots \mathsf{T}\, \sigma^{(k)}. \tag{3.10}$$

Further, this cycle decomposition of σ is unique up to successively interchanging pairs of adjacent prime cycles with disjoint support. In particular k is unique.

Definition 3.40. Call $\mathsf{pcyc}(\sigma) := k$ the number of prime cycles in the decomposition of σ from Theorem 3.39.

Example 3.41. The element σ from Example 3.35 has $\mathsf{pcyc}(\sigma) = 4$ and two prime cycle decompositions

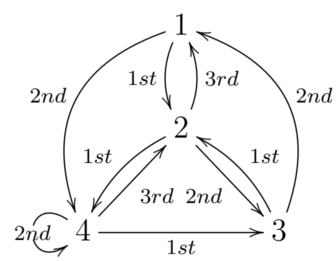
$$\begin{aligned}\begin{pmatrix}1&1&2&2&2&3&3&4&4&4\\2&4&4&3&1&2&1&3&4&2\end{pmatrix} &= \begin{pmatrix}2&3&4\\4&2&3\end{pmatrix} \mathsf{T} \begin{pmatrix}1&2&3\\2&3&1\end{pmatrix} \mathsf{T} \begin{pmatrix}4\\4\end{pmatrix} \mathsf{T} \begin{pmatrix}1&2&4\\4&1&2\end{pmatrix}\\ &= \begin{pmatrix}2&3&4\\4&2&3\end{pmatrix} \mathsf{T} \begin{pmatrix}4\\4\end{pmatrix} \mathsf{T} \begin{pmatrix}1&2&3\\2&3&1\end{pmatrix} \mathsf{T} \begin{pmatrix}1&2&4\\4&1&2\end{pmatrix}.\end{aligned}$$

We describe here an algorithm to find a prime cycle decomposition of a multiset permutation, which can be deduced from [35, §5.1.2], and illustrate how it produces the first of the two decompositions in Example 3.41. Encode a multiset permutation σ in $\mathfrak{S}_{M(\overline{a})}$ as two pieces of data:

– a directed graph D_σ on vertex set $\{1, 2, \ldots, \ell\}$ having one copy of the directed arc $i \to j$ for each occurrence of the column $\binom{i}{j}$ in its two-line notation, along with

- specification for each vertex x in $\{1, 2, \ldots, \ell\}$ the linear ordering of the arcs $x \to y$ emanating from x, indicating the left-to-right ordering of the corresponding columns in the two-line notation.

The resulting digraphs are those with the outdegree equal to the indegree equal to a_i for each i. E.g., the σ from Example 3.41 has this directed graph D_σ, with linear orderings indicated on the arcs out of each vertex:



With this identification, one factors σ recursively. First produce a prime cycle $\sigma^{(1)}$ for which

$$\sigma = \sigma^{(1)} \top \hat{\sigma} \tag{3.11}$$

via the following algorithm that takes a directed walk in D_σ.

- Start at the smallest vertex i_0 in $\{1, 2, \ldots, \ell\}$ with outdegree $a_{i_0} \geq 1$, and follow its first outward arc $i_0 \to i_1$. Then follow i_1's first outward arc $i_1 \to i_2$, follow i_2's first outward arc $i_2 \to i_3$, etc.

- Repeat until first arriving at a previously-visited[2] vertex i_s, say $i_s = i_r$ with $r < s$; possibly $r = s - 1$.

[2] During the process, when one enters a new vertex i_j along an arc $i_{j_1} \to i_j$, there will always be at least one outward arc $i_j \to i_{j+1}$ leaving i_j, because each vertex started with its indegree matching its outdegree.

– The directed *circuit* C of arcs $i_r \to i_{r+1} \to i_{r+2} \to \cdots \to i_{s-1} \to i_s (= i_r)$ corresponds to a prime cycle $\sigma^{(1)}$ that one can factor out to the left as in (3.11): by construction, each of its corresponding columns $\binom{i_t}{j_t}$ occurs as the *leftmost* column of σ having i_t as its top element.

– Complete the factorization recursively, replacing σ by $\hat{\sigma}$, removing the arcs C from D_σ to give $D_{\hat{\sigma}}$.

Example 3.42. The multiset permutation

$$\sigma = \begin{pmatrix} 1 & 1 & 2 & 2 & 2 & 3 & 3 & 4 & 4 & 4 \\ 2 & 4 & 4 & 3 & 1 & 2 & 1 & 3 & 4 & 2 \end{pmatrix}$$

factors as follows. A graphical depiction of that factorization process is given in Figure 3.1, where dotted arrows showing the directed walks in D_σ.

$$\begin{aligned} &\begin{pmatrix} 1 & 1 & 2 & 2 & 2 & 3 & 3 & 4 & 4 & 4 \\ 2 & 4 & 4 & 3 & 1 & 2 & 1 & 3 & 4 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 3 & 4 \\ 4 & 2 & 3 \end{pmatrix} \mathsf{T} \begin{pmatrix} 1 & 1 & 2 & 2 & 3 & 4 & 4 \\ 2 & 4 & 3 & 1 & 1 & 4 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 3 & 4 \\ 4 & 2 & 3 \end{pmatrix} \mathsf{T} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \mathsf{T} \begin{pmatrix} 1 & 2 & 4 & 4 \\ 4 & 1 & 4 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 3 & 4 \\ 4 & 2 & 3 \end{pmatrix} \mathsf{T} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \mathsf{T} \begin{pmatrix} 4 \\ 4 \end{pmatrix} \mathsf{T} \begin{pmatrix} 1 & 2 & 4 \\ 4 & 1 & 2 \end{pmatrix}. \end{aligned}$$

3.4.2 Partial Commutation Monoids

It will be helpful to view the intercalation monoid Int_ℓ as a partial commutation monoid. We briefly review some relevant facts about partial commutation monoids.

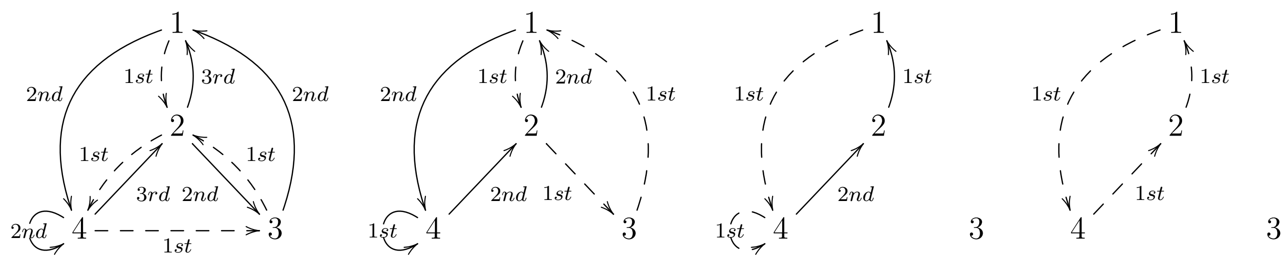


Figure 3.1: An illustration of the decomposition algorithm for multiset permutations for σ in Example 3.42. Dotted arrows represent the directed walks in D_σ.

Definition 3.43. Given a set $\mathbb{A}$, which we call an *alphabet* and a subset of its pairs $C \subseteq \binom{\mathbb{A}}{2}$, the associated *partial commutation monoid* $\mathcal{M}$ is defined to be the set of equivalence classes on words $\alpha_1\alpha_2\ldots\alpha_k$ in the alphabet $\mathbb{A}$ under the equivalence relation

$$\alpha_1\alpha_2\ldots\alpha_i\alpha_{i+1}\ldots\alpha_k \equiv \alpha_1\alpha_2\ldots\alpha_{i+1}\alpha_i\ldots\alpha_k \tag{3.12}$$

if $\{\alpha_i, \alpha_{i+1}\} \in C$.

From this perspective, Foata's Theorem 3.39 asserts that Int_ℓ is a partial commutation monoid, whose associated alphabet $\mathbb{A}$ is the set of all prime cycles, and C the pairs of prime cycles with disjoint supports.

For later use, we point out the following (nontrivial) proposition, see [35, §5.1.2, Exercise 11] and [57, Exercise 3.123]. Given a factorization of an element $\alpha = \alpha_1\alpha_2\ldots\alpha_k$ in $\mathcal{M}$ a partial commutation monoid, define a poset $\mathcal{P}_\alpha$ on $[k]$ as the transitive closure of the binary relation containing $(i,j) \in \mathcal{P}_\alpha$ when $i <_{\mathbb{Z}} j$ and either $\alpha_i = \alpha_j$ or $\alpha_i\alpha_j \not\equiv \alpha_j\alpha_i$.

Proposition 3.44. *Given a factorization of $\alpha = \alpha_1\alpha_2\ldots\alpha_k$ in $\mathcal{M}$ a partial commutation monoid,*

1. *$\mathcal{P}_\alpha$ does not depend on the choice of factorization of α, and*

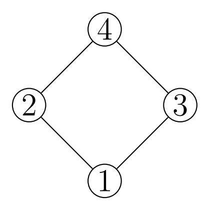
2. *there is a bijection between* $\mathrm{LinExt}(\mathcal{P}_\alpha)$ *and the factorizations of α given by*

$$(i_1, \ldots, i_k) \mapsto \alpha_{i_1} \ldots \alpha_{i_k}.$$

Example 3.45. The multiset permutation σ from Example 3.41 had two prime cycle factorizations

$$\begin{aligned} \sigma &= \sigma^{(1)} \mathsf{T} \sigma^{(2)} \mathsf{T} \sigma^{(3)} \mathsf{T} \sigma^{(4)} \\ &= \sigma^{(1)} \mathsf{T} \sigma^{(3)} \mathsf{T} \sigma^{(2)} \mathsf{T} \sigma^{(4)} \end{aligned}$$

corresponding to the two linear extensions of the poset $\mathcal{P}_\sigma$ on $[4]$ with this Hasse diagram:

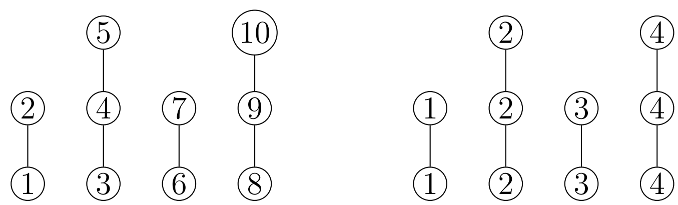


3.4.3 Connection with linear extensions and P-transverse permutations

We wish to use Foata's prime cycle decomposition to define a bijection $\mathrm{LinExt}(P_{\overline{a}}) \to \mathfrak{S}^{\pitchfork}(P_{\overline{a}})$, and use this to prove Theorem 3.3.

We begin with an easy identification of $\mathrm{LinExt}(P_{\overline{a}})$ with $\mathfrak{S}_{M(\overline{a})}$. For this purpose, given any weak composition $\overline{a} = (a_1, \ldots, a_\ell)$ of n, consider two labelings of $P_{\overline{a}}$, one by the elements $\{1, 2, \ldots, n\}$ which we will call the *standardized labeling*, and the second by the elements of the multiset $M(\overline{a})$, which we will call the *multiset* *labeling*. The standardized labeling labels the first chain a_1 by $1, 2, \ldots, a_1$ from bottom-to-top, then the second chain a_2 by $a_1 + 1, \ldots, a_1 + a_2$ from bottom-to-top, and so on. The multiset labeling labels the elements in the first chain a_1 all by 1, the second chain a_2 all by 2, etc.

Example 3.46. For $n = 10$ and $\overline{a} = (2, 3, 2, 3)$, the standardized and multiset labelings of $P_{\overline{a}}$ are



With this in hand, the following proposition is a straightforward observation.

Proposition 3.47. *For any weak composition $\overline{a}$ of n, one has a bijection*

$$\begin{array}{rcl} \mathrm{LinExt}(P_{\overline{a}}) & \longrightarrow & \mathfrak{S}_{M(\overline{a})} \\ \lambda = [\lambda_1, \ldots, \lambda_n] & \longmapsto & \sigma = [\sigma_1, \ldots, \sigma_n] \end{array}$$

replacing λ_i by its corresponding multiset label σ_i, that is, if λ_i lies on the j^{th} chain a_j in $P_{\overline{a}}$, then $\sigma_i := j$.

Proof. The inverse map recovers λ from σ by labeling the a_j occurrences of the value j within σ from left-to-right with the integers in the interval $[a_1 + a_2 + \cdots + a_{j-1} + 1, a_1 + a_2 + \cdots + a_{j-1} + a_j]$. $\square$

Example 3.48. For $\overline{a} = (2, 3, 2, 3)$, this bijection maps $\lambda = [3, 8, 9, 6, 1, 4, 2, 7, 10, 5]$ in $\mathrm{LinExt}(P_{\overline{a}})$ to

$$\sigma = [2, 4, 4, 3, 1, 2, 1, 3, 4, 2] = \begin{pmatrix} 1 & 1 & 2 & 2 & 2 & 3 & 3 & 4 & 4 & 4 \\ 2 & 4 & 4 & 3 & 1 & 2 & 1 & 3 & 4 & 2 \end{pmatrix}.$$

We can now define a map $\varphi : \mathrm{LinExt}(P_{\overline{a}}) \to \mathfrak{S}_n$, which will turn out to be a bijection onto $\mathfrak{S}^{\pitchfork}(P_{\overline{a}})$.

Definition 3.49. Fix a weak composition $\overline{a}$ of n. Given λ in $\mathrm{LinExt}(P_{\overline{a}})$,

- let $\sigma \in \mathfrak{S}_{M(\overline{a})}$ be its corresponding multiset permutation from Proposition 3.47,
- label the entries in the top line of σ's two-line notation with subscripts $1, 2, \ldots, n$ from left-to-right,
- use Foata's Theorem 3.39 to decompose $\sigma = \sigma^{(1)} \intercal \cdots \intercal \sigma^{(\ell)}$ into prime cycles $\sigma^{(i)}$, carrying along the subscripts in the top line, and finally
- replace each prime cycle $\sigma^{(i)}$ with the cyclic permutation $\tau^{(i)}$ of the subscripts of its top line.

Then $\varphi(\lambda) := \tau = \tau^{(1)} \cdots \tau^{(\ell)}$ in $\mathfrak{S}_n$.

Example 3.50. We continue Example 3.48. Let $\overline{a} = (2, 3, 2, 3)$ and $\lambda = [3, 8, 9, 6, 1, 4, 2, 7, 10, 5]$ in $\mathrm{LinExt}(P_{\overline{a}})$. Subscript the top line of its corresponding σ in $\mathfrak{S}_{M(\overline{a})}$, and factor as in Theorem 3.39, carrying along subscripts:

$$\begin{aligned}
\sigma &= \begin{pmatrix} 1_1 & 1_2 & 2_3 & 2_4 & 2_5 & 3_6 & 3_7 & 4_8 & 4_9 & 4_{10} \\ 2 & 4 & 4 & 3 & 1 & 2 & 1 & 3 & 4 & 2 \end{pmatrix} \\
&= \begin{pmatrix} 2_3 & 3_6 & 4_8 \\ 4 & 2 & 3 \end{pmatrix} \intercal \begin{pmatrix} 1_1 & 2_4 & 3_7 \\ 2 & 3 & 1 \end{pmatrix} \intercal \begin{pmatrix} 4_9 \\ 4 \end{pmatrix} \intercal \begin{pmatrix} 1_2 & 2_5 & 4_{10} \\ 4 & 1 & 2 \end{pmatrix} \\
&= (2_3, 4_8, 3_6) \intercal (1_1, 2_4, 3_7) \intercal (4_9) \intercal (1_2, 4_{10}, 2_5) \\
&= \sigma^{(1)} \intercal \sigma^{(2)} \intercal \sigma^{(3)} \intercal \sigma^{(4)}
\end{aligned}$$

Replacing each prime cycle $\sigma^{(i)}$ with the cycle $\tau^{(i)}$ on its subscripts gives $\varphi(\lambda) = \tau \in \mathfrak{S}_{10}$:

$$\varphi(\lambda) = \tau = \tau^{(1)}\tau^{(2)}\tau^{(3)}\tau^{(4)} = (3,8,6)(1,4,7)(9)(2,10,5).$$

We can now prove Theorem 3.3, whose statement we recall here.

Theorem 3.3. *For any composition $\overline{a}$ of n, the disjoint union $P_{\overline{a}}$ of chains has a bijection*

$$\begin{array}{rcl} \mathrm{LinExt}(P_{\overline{a}}) & \longrightarrow & \mathfrak{S}^{\pitchfork}(P_{\overline{a}}) \\ \sigma & \longmapsto & \tau \end{array}$$

with $\mathsf{cyc}(\tau) = \mathsf{pcyc}(\sigma)$*, the number of prime cycles in Foata's unique decomposition for σ. Thus*

$$\mathrm{Poin}(P_{\overline{a}}, t) = \sum_{\sigma \in \mathrm{LinExt}(P_{\overline{a}})} t^{n-\mathsf{pcyc}(\sigma)}.$$

Proof. We claim that the above map $\varphi : \mathrm{LinExt}(P_{\overline{a}}) \to \mathfrak{S}_n$ is the desired bijection. Since Theorem 3.1 showed $\#\,\mathrm{LinExt}(P_{\overline{a}}) = \#\mathfrak{S}^{\pitchfork}(P_{\overline{a}})$, it suffices to show that the image of φ lies in $\mathfrak{S}^{\pitchfork}(P_{\overline{a}})$, and that φ is injective.

To see that every λ in $\mathrm{LinExt}(P_{\overline{a}})$ has $\varphi(\lambda) = \tau$ lying in $\mathfrak{S}^{\pitchfork}(P_{\overline{a}})$, we will induct on the number ℓ of cycles in $\tau = \tau^{(1)} \cdots \tau^{(\ell)}$, which is also the number of prime cycles in the decomposition $\sigma = \sigma^{(1)} \mathsf{T} \cdots \mathsf{T}\, \sigma^{(\ell)}$. By definition of $\mathfrak{S}^{\pitchfork}(P_{\overline{a}})$, we must check that the cycle partition $\pi = \{B_1, \ldots, B_\ell\}$ of τ lies in $\Pi^{\pitchfork}(P_{\overline{a}})$, where B_i is the set underlying the cycle $\tau^{(i)}$. To this end, assume that we produce the factorization $\sigma = \sigma^{(1)} \mathsf{T} \cdots \mathsf{T}\, \sigma^{(\ell)}$ according to the algorithm presented in Subsection 3.4.1, and let us check that the block $B = B_1$ underlying $\tau^{(1)}$ satisfies the two properties (a), (b) in the recursive characterization of $\Pi^{\pitchfork}(P_{\overline{a}})$ from Proposition 3.18:

– For (a), the elements of B are all minimal in $P_{\overline{a}}$ because, in the initial factorization step $\sigma = \sigma^{(1)} \intercal \hat{\sigma}$, each column $\binom{i}{j}$ in the two-line notation of $\sigma^{(1)}$ is the *leftmost* column of σ having i as its top element, so it corresponds to the *bottom* element in the i^{th} chain $\mathsf{a_i}$ of $P_{\overline{a}}$.

– For (b), note that after that initial factorization step, the poset $\hat{P}_{\overline{a}}$ and partition $\hat{\pi} = \{B_2, \ldots, B_\ell\}$ will correspond to $\hat{\sigma}$ in the above factorization, coming from a $\hat{\lambda}$ in $\mathrm{LinExt}(\hat{P}_{\overline{a}})$ with $\varphi(\hat{\lambda}) = \hat{\sigma}$ for which the result holds by induction on ℓ.

To show that φ is injective, we must give an algorithm to recover λ from $\tau = \varphi(\lambda)$. It would be equivalent to recover τ's multiset labeled image σ in $\mathfrak{S}_{M(\overline{a})}$ from the bijection in Proposition 3.47. Factoring τ lets us recover its *unordered* set of cycles $\{\tau^{(1)}, \ldots, \tau^{(\ell)}\}$, and hence also the *unordered* set of prime cycles $\{\sigma^{(1)}, \ldots, \sigma^{(\ell)}\}$ that will appear in an intercalation factorization of σ. We would like to know how to properly index $\{\sigma^{(i)}\}_{i=1,\ldots,\ell}$, up to interchanging commuting elements, so that we could recover σ as their intercalation product $\sigma = \sigma^{(1)} \intercal \cdots \intercal \sigma^{(\ell)}$. We claim that this (partial) ordering is already contained in the information of the unordered set of cycles $\{\tau^{(i)}\}_{i=1,\ldots,\ell}$ as follows. When two prime cycles $\sigma^{(r)}, \sigma^{(s)}$ do not commute, it is because they share a common element i, so there must exist two elements x, y in $[n]$ that come from the i^{th} chain a_i in the standardized labeling of $P_{\overline{a}}$, with $x \in \tau^{(r)}, y \in \tau^{(s)}$. If $x <_{\mathbb{Z}} y$, then $\sigma^{(r)}$ must occur to the left of $\sigma^{(s)}$ in the intercalation product. $\square$

3.5 Proof of Theorem 3.4

Our goal here is to find a generating function compiling the Poincaré polynomials $\mathrm{Poin}(P_{\overline{a}}, t)$ for all compositions $\overline{a}$ of length ℓ. This uses more of Foata's theory for the intercalation monoid Int_ℓ, similar to his deduction of MacMahon's *Master Theorem*.

Since each multiset permutation σ has only finitely many intercalation factorizations $\sigma = \rho \intercal \tau$, one can define a convolution algebra on the set of functions $\phi : \mathrm{Int}_\ell \to \mathbb{Z}$ with pointwise addition:

$$(\phi_1 * \phi_2)(\sigma) := \sum_{\rho \intercal \tau = \sigma} \phi_1(\rho) \cdot \phi_2(\tau).$$

Let $\zeta : \mathrm{Int}_\ell \to \mathbb{Z}$ denote the *zeta function* defined by $\zeta(\sigma) = 1$ for all σ in Int_ℓ. The zeta function has a unique convolutional inverse μ, called the *Möbius function*. Foata proved that the Möbius function can be expressed by the following explicit formula

$$\mu(\sigma) = \begin{cases} (-1)^{\mathsf{pcyc}(\sigma)} & \text{if } \sigma \text{ is simple,} \\ 0 & \text{else,} \end{cases}$$

where $\sigma \in \mathrm{Int}_\ell$ is *simple* if all the letters of σ are distinct, that is, $\mathsf{supp}(\sigma)$ is a set, not a multiset. This may be formulated as an identity in a completion $\mathbb{Z}[[\mathrm{Int}_\ell]] := \left\{ \sum_{\sigma \in \mathrm{Int}_\ell} z_\sigma \sigma : z_\sigma \in \mathbb{Z} \right\}$ of the monoid algebra $\mathbb{Z}[\mathrm{Int}_\ell]$, allowing infinite $\mathbb{Z}$-linear combinations of elements of Int_ℓ (see [25, Théorème 2.4]):

$$1 = \left(\sum_{\sigma \in \mathrm{Int}_\ell} \sigma \right) \left(\sum_{\sigma \in \mathrm{Int}_\ell} \mu(\sigma)\, \sigma \right) = \left(\sum_{\sigma \in \mathrm{Int}_\ell} \sigma \right) \left(\sum_{\text{simple } \sigma \in \mathrm{Int}_\ell} (-1)^{\mathsf{pcyc}(\sigma)} \sigma \right). \quad (3.13)$$

Now introduce an $\ell \times \ell$ matrix $B := (b_{ij})_{i,j=1,2,\ldots,\ell}$ of indeterminates, and let $\mathbb{Z}[[b_{ij}, t]]$ be the (usual, commutative) power series ring in $\{b_{ij}\}_{i,j=1}^{\ell}$ along with one further indeterminate t. One can then define a ring homomorphism

$$\begin{array}{rcl} \mathbb{Z}[[\mathrm{Int}_\ell]] & \xrightarrow{u_t} & \mathbb{Z}[[b_{ij}, t]] \\ \sigma & \longmapsto & t^{\mathsf{pcyc}(\sigma)} \cdot \underline{b}_\sigma \end{array}$$

where if $\sigma = \begin{pmatrix} i_1 & i_2 & \cdots & i_n \\ \sigma_1 & \sigma_2 & \cdots & \sigma_n \end{pmatrix}$ then $\underline{b}_\sigma := \prod_{k=1}^n b_{i_k \sigma_k}$.

Applying the homomorphism u_t to both sides of (3.13) gives a t-version of MacMahon's Master Theorem.

Theorem 3.51. *In $\mathbb{Z}[[b_{ij}, t]]$ one has the identity*

$$\sum_{\sigma \in \mathrm{Int}_\ell} t^{\mathsf{pcyc}(\sigma)} \underline{b}_\sigma = \left(\sum_{simple\ \sigma \in \mathrm{Int}_\ell} (-t)^{\mathsf{pcyc}(\sigma)} \underline{b}_\sigma \right)^{-1} = \left(\sum_{H \subseteq [\ell]} \sum_{\sigma \in \mathfrak{S}_H} (-t)^{\mathsf{pcyc}(\sigma)} \underline{b}_\sigma \right)^{-1}.$$

Remark 3.52. Setting $t = 1$ in Theorem 3.51 gives an identity in $\mathbb{Z}[[b_{ij}]]$:

$$\sum_{\sigma \in \mathrm{Int}_\ell} \underline{b}_\sigma = \left(\sum_{H \subseteq [\ell]} \sum_{\sigma \in \mathfrak{S}_H} (-1)^{\mathsf{pcyc}(\sigma)} \underline{b}_\sigma \right)^{-1}, \tag{3.14}$$

which is equivalent to an identity in Foata's proof of the (commutative) MacMahon Master Theorem, as we recall here. Introduce two sets of ℓ variables $\mathbf{x} = (x_1, \ldots, x_\ell), \mathbf{y} = (y_1, \ldots, y_\ell)$ related by the matrix B of indeterminates as follows: $\mathbf{y} = B\mathbf{x}$, that is, $y_i = \sum_j b_{ij} x_j$. Then MacMahon's Master Theorem is this identity in $\mathbb{Z}[[b_{ij}]]$:

$$\sum_{\overline{a} \in \{0,1,2,\ldots\}^\ell} \big(\text{coefficient of } \mathbf{x}^{\overline{a}} \text{ in } \mathbf{y}^{\overline{a}}\big) = \det(I_\ell - B)^{-1}, \tag{3.15}$$

where $\mathbf{x}^{\overline{a}} := x_1^{a_1} \cdots x_\ell^{a_\ell}$. It is not hard to check that the left sides and right sides of (3.15) and (3.14) are the same: the left side of (3.14) needs to be grouped according to the multiplicity vector $\overline{a}$ giving the support $\mathsf{supp}(\sigma)$, and the right side must be reinterpreted in terms of the permutation expansion of a determinant.

Remark 3.53. Theorem 3.51 is similar in spirit to Garoufalidis-Lê-Zeilberger's *quantum MacMahon Master Theorem* [28, Theorem 1] (see also Konvalinka-Pak [36, Theorem 1.2]). Their quantum version inserts a $(-q)^{-\mathrm{inv}(\sigma)}$ in order to produce a q-determinant, but $\mathrm{inv}(\sigma) \neq \mathsf{pcyc}(\sigma)$.

We now specialize $b_{ij} = x_j$ in Theorem 3.51 to deduce Theorem 3.4, whose statement we recall here.

Theorem 3.4. *For $\ell = 1, 2, \ldots$, one has*

$$\sum_{\overline{a} \in \{1,2,\ldots\}^\ell} \mathrm{Poin}(P_{\overline{a}}, t) \cdot \mathbf{x}^{\overline{a}} = \frac{1}{1 - \sum_{j=1}^{\ell} e_j(\mathbf{x}) \cdot (t-1)(2t-1)\cdots((j-1)t-1)},$$

where $e_j(\mathbf{x}) := \sum_{1 \le i_1 < \cdots < i_j \le \ell} x_{i_1} \cdots x_{i_j}$ is the j^{th} elementary symmetric function.

Proof. Setting $b_{ij} = x_j$ in Theorem 3.51 gives

$$\sum_{\sigma \in \mathrm{Int}_\ell} t^{\mathsf{pcyc}(\sigma)} \prod_k x_{\sigma_k} = \left(\sum_{H \subseteq [\ell]} \sum_{\sigma \in \mathfrak{S}_H} (-t)^{\mathsf{pcyc}(\sigma)} \prod_{k \in H} x_k \right)^{-1}. \tag{3.16}$$

Let us manipulate both sides of equation (3.16). On the left side, grouping terms according to $\mathsf{supp}(\sigma)$ gives

$$\sum_{\overline{a} \in \{0,1,2,\ldots\}^\ell} \mathbf{x}^{\overline{a}} \sum_{\sigma \in \mathfrak{S}_{M(\overline{a})}} t^{\mathsf{pcyc}(\sigma)}.$$

On the right side of (3.16), any subset $H \subseteq [\ell]$ of cardinality $j \geq 1$ satisfies

$$\sum_{\sigma \in \mathfrak{S}_H} (-t)^{\mathsf{pcyc}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_j} (-t)^{\mathsf{cyc}(\sigma)} = (-t)(1-t)(2-t)\cdots(j-1-t)$$

by (3.6). Therefore grouping according to $j = \#H$, and noting $\sum_{\substack{H \subseteq [\ell]: \\ \#H = j}} \prod_{k \in H} x_k = e_j(\mathbf{x})$ lets one rewrite the sum inside the parentheses on the right side of (3.16) as this:

$$1 + \sum_{j=1}^{\ell} (-t)(1-t)(2-t)\cdots(j-1-t) \cdot e_j(\mathbf{x}).$$

So far this gives

$$\sum_{\overline{a}\in\{0,1,2,\ldots\}^\ell} \mathbf{x}^{\overline{a}} \sum_{\sigma\in\mathfrak{S}_{M(\overline{a})}} t^{\mathsf{pcyc}(\sigma)} = \left(1+\sum_{j=1}^{\ell}(-t)(1-t)(2-t)\cdots(j-1-t)\cdot e_j(\mathbf{x})\right)^{-1}.$$

Now perform two more substitutions: first replace t by t^{-1}, giving this

$$\sum_{\overline{a}\in\{0,1,2,\ldots\}^\ell} \mathbf{x}^{\overline{a}} \sum_{\sigma\in\mathfrak{S}_{M(\overline{a})}} t^{-\mathsf{pcyc}(\sigma)} = \left(1+\sum_{j=1}^{\ell}(-t^{-1})(1-t^{-1})(2-t^{-1})\cdots(j-1-t^{-1})\cdot e_j(\mathbf{x})\right)^{-1},$$

and then replace x_i by tx_i for $i=1,2,\ldots,\ell$, so that $\mathbf{x}^{\overline{a}} \mapsto t^{|\overline{a}|}\mathbf{x}^{\overline{a}}$ and $e_j(\mathbf{x}) \mapsto t^j e_j(\mathbf{x})$, giving this

$$\sum_{\overline{a}\in\{0,1,2,\ldots\}^\ell} \mathbf{x}^{\overline{a}} \sum_{\sigma\in\mathfrak{S}_{M(\overline{a})}} t^{|\overline{a}|-\mathsf{pcyc}(\sigma)} = \left(1-\sum_{j=1}^{\ell}(t-1)(2t-1)\cdots((j-1)t-1)\cdot e_j(\mathbf{x})\right)^{-1}.$$

Comparison of the left side with Theorem 3.3 shows that this last equation is Theorem 3.4. □

Remark 3.54. We justify here the claim from the Introduction that Theorem 3.4 generalizes the formula (3.6):

$$\mathrm{Poin}(\mathsf{Antichain}_\ell, t) = 1(1+t)(1+2t)\cdots(1+(\ell-1)t).$$

Since $\mathsf{Antichain}_\ell = P_{\overline{a}}$ where $\overline{a}=(1,1,\ldots,1)$, we seek to explain why the coefficient of $x_1\ldots x_\ell$ in the power series on the right side in Theorem 3.4 should be $1(1+t)(1+2t)\cdots(1+(\ell-1)t)$. Introducing the abbreviation $\langle t\rangle_j := (t-1)(2t-1)\cdots((j-1)t-1)$, the right side in Theorem 3.4 can be rewritten and expanded as

$$\frac{1}{1-\sum_{j=1}^{\ell} e_j(\mathbf{x})\langle t\rangle_j} = \sum_{n\geq 0}\left(\sum_{j=1}^{\ell} e_j(\mathbf{x})\langle t\rangle_j\right)^n. \tag{3.17}$$

If we let A_ℓ denote the coefficient of $x_1\cdots x_\ell$ in this series, then it suffices to explain why

$$A_{\ell+1} = (1+\ell t)\cdot A_\ell. \tag{3.18}$$

For this coefficient extraction, it is safe to replace each $e_j(\mathbf{x}) = e_j(x_1, x_2, \ldots, x_\ell)$ in (3.17) with an infinite variable version $e_j(\mathbf{x}) = e_j(x_1, x_2, \ldots)$. Extracting the coefficient of $x_1 \cdots x_\ell$ on the right side in (3.17) shows

$$A_\ell = \sum_{\substack{\text{ordered set partitions} \\ \pi=(B_1,\ldots,B_n) \text{ of } [\ell]}} w(\pi), \qquad \text{where } w(\pi) := \prod_{B_i \in \pi} \langle t \rangle_{|B_i|}.$$

To explain (3.18), note that each ordered set partition $\hat{\pi}$ of $[\ell+1]$ can be obtained from a unique ordered set partition $\pi = (B_1, \ldots, B_n)$ of $[\ell]$ as follows: either $\hat{\pi}$ has added $\ell+1$ into one of the preexisting blocks B_i of π, or $\hat{\pi}$ has a singleton block $\{\ell+1\}$, inserted into one of the $n+1$ locations in the sequence $(B_1, \ldots, B_n)$. Thus having fixed an ordered set partition π of $[\ell]$, the sum of $w(\hat{\pi})$ over $\hat{\pi}$ which correspond to π is this sum:

$$\begin{aligned} &w(\pi)\cdot(|B_1|t-1)+\cdots+w(\pi)\cdot(|B_n|t-1)+\underbrace{w(\pi)+w(\pi)+\cdots+w(\pi)}_{n+1 \text{ times}} \\ &= w(\pi)\left(\sum_{i=1}^{n}(|B_i|t-1)+n+1\right) = w(\pi)\,(\ell t - n + n + 1) = w(\pi)\cdot(1+\ell t). \end{aligned}$$

Summing this over all possible π gives (3.18).

3.6 Proof of Theorem 3.5

The *width* of a poset P is the maximum size of an antichain in P. A famous result of Dilworth from 1950 (see [57, Ch 3, Exer 77(d)]), asserts that the width d of P is the minimum number of chains required in a *chain decomposition* $P = P_1 \cup P_2 \cup \cdots \cup P_d$, that is, where each P_i is a totally ordered subset $P_i \subseteq P$.

Consequently, a poset P of width two can be decomposed into two chains $P = P_1 \cup P_2$, possibly with some order relations between elements of P_1 and P_2.

Recall that in the Introduction we defined a descent-like statistic on $\sigma = [\sigma_1, \ldots, \sigma_n]$ in $\mathrm{LinExt}(P)$, as the cardinality $\mathsf{des}_{P_1,P_2}(\sigma) := \#\mathsf{Des}_{P_1,P_2}(\sigma)$ of this set,

$$\mathsf{Des}_{P_1,P_2}(\sigma) := \{i \in [n-1] : \sigma_i \in P_2,\ \sigma_{i+1} \in P_1,\ \text{with } \sigma_i, \sigma_{i+1} \text{ incomparable in } P\}, \tag{3.19}$$

in order to state the following result.

Theorem 3.5. *For a width two poset decomposed into two chains as $P = P_1 \cup P_2$, one has*

$$\mathrm{Poin}(P,t) = \sum_{\sigma \in \mathrm{LinExt}(P)} t^{\mathsf{des}_{P_1,P_2}(\sigma)}.$$

To prove this, we start with the following observation.

Corollary 3.55. *For posets P of width two, one has*

$$\mathrm{Poin}(P,t) = \sum_{\pi \in \Pi^{\pitchfork}(P)} t^{\mathsf{pairs}(\pi)}$$

where $\mathsf{pairs}(\pi)$ is the number of two-element blocks B_i in π. In particular, setting $t = 1$,

$$\#\,\mathrm{LinExt}(P) = \#\Pi^{\pitchfork}(P) (= \#\mathfrak{S}^{\pitchfork}(P)).$$

Proof. Antichains in P have at most two elements, so Proposition 3.12(iii) implies that P-transverse permutations have only 1-cycles and 2-cycles. But then this implies that the map $\mathfrak{S}^{\pitchfork}(P) \to \Pi^{\pitchfork}(P)$ sending a P-transverse permutation τ to the set partition π given by its cycles is a bijection, with $\mathsf{pairs}(\pi) = n - \mathsf{cyc}(\tau)$. The result then follows from Theorem 3.1. $\square$

Example 3.56. Let $P = \mathsf{a} \sqcup \mathsf{b}$ be a poset which is a disjoint union of two chains a, b having a, b elements respectively. One can check that a P-transverse partition having $\mathsf{pairs}(\pi) = k$ is completely determined by the choice of a k element subset $x_1 <_P \cdots <_P x_k$ from a and a k element subset $y_1 <_P \cdots <_P y_k$ from b to constitute the two-element blocks, as follows: $\{x_1, y_1\}, \ldots, \{x_k, y_k\}$. This implies

$$\mathrm{Poin}(\mathsf{a} \sqcup \mathsf{b}, t) = \sum_{k=0}^{\min(a,b)} \binom{a}{k}\binom{b}{k} t^k.$$

This is consistent with $\#\,\mathrm{LinExt}(\mathsf{a} \sqcup \mathsf{b}) = \binom{a+b}{a}$, since setting $t = 1$ in the equation above gives

$$\binom{a+b}{a} = \sum_{k=0}^{\min(a,b)} \binom{a}{k}\binom{b}{k}$$

which is an instance of the Chu-Vandermonde summation.

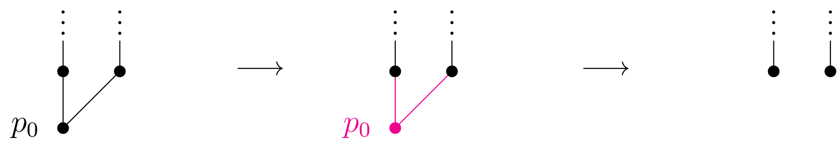
In light of Corollary 3.55, to prove Theorem 3.5, one would need a bijection from $\mathrm{LinExt}(P)$ to $\Pi^{\pitchfork}(P)$ (or $\mathfrak{S}^{\pitchfork}(P)$) that sends the statistic $\mathsf{des}_{(P_1,P_2)}(-)$ to the number of pairs or number of 2-cycles. Unfortunately, there does not seem to be a consistent labeling of a width two poset $P = P_1 \cup P_2$ to make the bijection Ψ from Section 3.3 play this role. Nevertheless, having fixed the chain decomposition[3] $P = P_1 \sqcup P_2$, we provide in the proof below such a bijection $\Omega : \mathrm{LinExt}(P) \to \Pi^{\pitchfork}(P)$.

Proof of Theorem 3.5. We describe Ω and Ω^{-1} recursively, via induction on $n := \#P$. There are two cases, based on whether P has one or two minimal elements.

Case 1. There is a unique minimum element $p_0 \in P$.

In this case, given $\sigma = [\sigma_1, \ldots, \sigma_n]$ in $\mathrm{LinExt}(P)$, we must have $\sigma_1 = p_0$, so that $\{p_0\}$ should be a singleton block of $\pi = \Omega(\sigma)$, and one produces the remaining blocks of π by applying Ω recursively to $[\sigma_2, \ldots, \sigma_n]$. This is depicted schematically here:

[3] So we assume here that $P_1 \cap P_2 = \varnothing$, but there may be order relations between elements of P_1 and P_2.



$(P, [\sigma_1, \sigma_2, \ldots, \sigma_n])$ $\qquad$ $(P, [p_0, \sigma_2 \ldots, \sigma_n])$ $\qquad$ $(P - \{p_0\}, [\sigma_2, \ldots, \sigma_n])$

For the inverse map Ω^{-1}, given a P-transverse partition π, since the blocks of π are antichains in P, the unique minimum element p_0 of P must lie in a singleton block $\{p_0\}$ in π. So make $\Omega^{-1}(\pi) = \sigma$ have $\sigma_1 = p_0$, and construct $[\sigma_2, \ldots, \sigma_n]$ by applying Ω^{-1} recursively to the $(P - \{p_0\})$-transverse partition obtained from π by removing the block $\{p_0\}$.

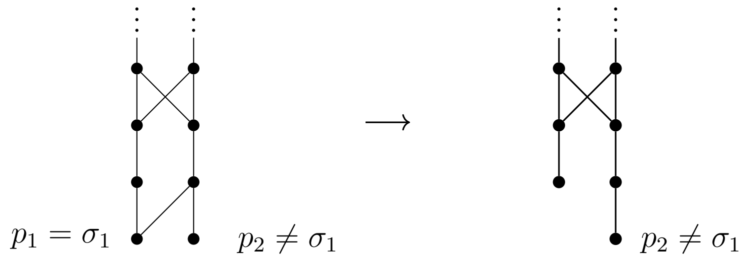
Case 2. There are two minimal elements of P.

Label these two minimal elements p_1, p_2 of P so that $p_i \in P_i$ for $i = 1, 2$. Note that this implies that every $\sigma = [\sigma_1, \sigma_2, \ldots, \sigma_n]$ in $\mathrm{LinExt}(P)$ has either $\sigma_1 = p_1$ or $\sigma_1 = p_2$. Note also that any P-transverse partition π only has blocks of cardinality 1 or 2, which yields two subcases for defining Ω and Ω^{-1}:

- The Subcase 2a for
  - defining Ω occurs when $\sigma_1 = p_1$,
  - defining Ω^{-1} occurs when $\{p_1\}$ appears as a singleton block within π.
- The Subcase 2b for
  - defining Ω occurs when $\sigma_1 = p_2$,
  - defining Ω^{-1} occurs when p_1 appears in a two-element block within π.

Subcase 2a.

When defining Ω, if $\sigma_1 = p_1$, then make $\{p_1\}$ a singleton block of $\pi = \Omega(\sigma)$, and produce the remaining blocks of π by applying Ω recursively to $[\sigma_2, \ldots, \sigma_n]$.

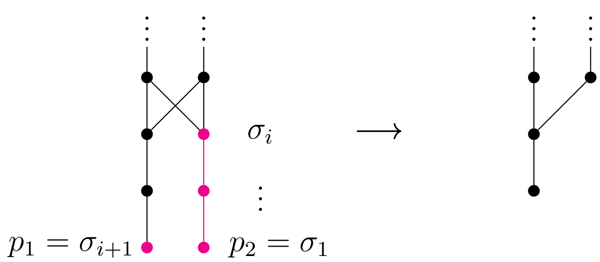


$(P, (\sigma_1, \sigma_2, \ldots, \sigma_n))$ $\qquad$ $(P - \{p_1\}, (\sigma_2, \ldots, \sigma_n))$

To define Ω^{-1}, if $\{p_1\}$ is a singleton block of π, make $\Omega^{-1}(\pi) = \sigma$ have $\sigma_1 = p_1$, and construct $[\sigma_2, \ldots, \sigma_n]$ by applying Ω^{-1} recursively to the $(P - \{p_1\})$-transverse partition obtained from π by removing the block $\{p_1\}$.

Subcase 2b.

When defining Ω, if $\sigma_1 = p_2$, then p_1 appears elsewhere in σ, say $p_1 = \sigma_{i+1}$ where $i \geq 1$. Because σ lies in $\mathrm{LinExt}(P)$ and $\sigma_{i+1} = p_1$ is the minimum element of P_1, this forces $\sigma_1, \sigma_2, \ldots, \sigma_i$ to all be elements of P_2. In this case, add to π the singleton blocks $\{\sigma_1\}, \{\sigma_2\}, \ldots, \{\sigma_{i-1}\}$ along with the two-element block $\{\sigma_i, \sigma_{i+1}\} = \{\sigma_i, p_1\}$, and compute the rest of $\Omega(\sigma) = \pi$ recursively by replacing (P, σ) with $(P - \{\sigma_1, \sigma_2 \ldots, \sigma_{i+1}\}, [\sigma_{i+2}, \sigma_{i+3}, \ldots, \sigma_n])$. Here is the schematic picture:



$(P, (\sigma_1, \sigma_2, \ldots, \sigma_n))$ $\qquad$ $(P - \{\sigma_1, \ldots, \sigma_i\}, (\sigma_{i+1}, \sigma_{i+2}, \ldots, \sigma_n))$

When defining $\Omega^{-1}(\pi)$, if p_1 appears in some two-element block of π, then it appears in some block $\{p_1, p_2'\}$ for some p_2' in P_2. We claim that π being P-

transverse then forces any elements $p <_P p'_2$ in P_2 to lie in singleton blocks $\{p\}$ of π. To see this claim, assume not, so that some such p lies in a two-element block of π, necessarily of the form $\{p'_1, p\}$ for some p'_1 in P_1 with $p_1 <_P p'_1$. This leads to a contradiction of Proposition 3.12(ii), since (p'_1, p_1) would then be a relation in $\overline{P \cup \pi}$ via this transitive chain of relations: $p'_1 \equiv_\pi p <_P p'_2 \equiv_\pi p_1$.

In this subcase, list the totally ordered (and possibly empty) collection of all elements p in P_2 with $p <_P p'_2$ at the beginning of σ as $\sigma_1, \sigma_2, \ldots, \sigma_{i-1}$, followed by $\sigma_i\sigma_{i+1} = p'_2 p_1$. Then compute the rest of $\Omega^{-1}(\pi) = \sigma$ recursively, by applying Ω^{-1} to the $(P - \{\sigma_1, \sigma_2, \ldots, \sigma_{i+1}\})$-transverse partition obtained from π by removing the singleton blocks $\{\sigma_1\}, \{\sigma_2\}, \ldots, \{\sigma_{i-1}\}$ and the two-element block $\{\sigma_i, \sigma_{i+1}\} = \{p'_2, p_1\}$.

It is not hard to check that the two maps Ω, Ω^{-1} defined recursively in this way are actually mutually inverse bijections. By construction, Ω has the property that the two-element blocks of $\pi = \Omega(\sigma)$ are exactly those containing P-incomparable pairs $\{\sigma_i, \sigma_{i+1}\}$ for which $\sigma_i \in P_1$ and $\sigma_{i+1} \in P_2$, as claimed. $\square$

Recall from the Introduction that the number of usual descents of a permuation σ is defined as $\mathsf{des}(\sigma) = \#\mathsf{Des}(\sigma)$ where

$$\mathsf{Des}(\sigma) := \{i \in [n-1] \mid \sigma_i > \sigma_{i+1}\}.$$

This was used to define the P-Eulerian polynomial in equation (3.7) as $\sum_{\sigma \in \mathrm{LinExt}(P)} t^{\mathsf{des}(\sigma)}$, assuming that P is naturally labeled, that is, $\mathrm{LinExt}(P)$ contains the identity permutation $\sigma = [1, 2, \ldots, n]$.

Corollary 3.57. *When P is a width two poset having a chain decomposition $P_1 \cup P_2$ with P_1 an order ideal of P, then the Poincaré polynomial for P coincides with the*

P-Eulerian polynomial:

$$\operatorname{Poin}(P,t) = \sum_{\sigma \in \operatorname{LinExt}(P)} t^{\mathsf{des}(\sigma)}.$$

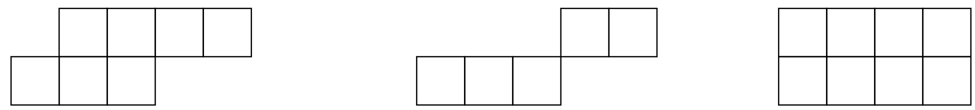
Proof. Let $\#P_i = n_i$ for $i = 1, 2$, so that $n = \#P = n_1 + n_2$. One can then choose a natural labeling for P by $[n]$ such that the elements of the order ideal P_1 are labeled by the initial segment $[n_1] = \{1, 2, \ldots, n_1\}$, and P_2 is labeled by $\{n_1 + 1, n_1 + 2, \ldots, n\}$. We claim that with this natural labeling, one has $\mathsf{Des}_{(P_1,P_2)}(w) = \mathsf{Des}(w)$. This is because the labeling renders one of the conditions in the definition (3.19) of $\mathsf{Des}_{(P_1,P_2)}(\sigma)$ superfluous: assuming that $\sigma_i \in P_2$ and $\sigma_{i+1} \in P_1$, then σ_i, σ_{i+1} must already be incomparable in P, because otherwise $\sigma_i <_P \sigma_{i+1}$ (since σ was a linear extension of P) and then P_1 being an order ideal would force $\sigma_i \in P_1$, a contradiction. Now since P_1, P_2 are totally ordered in P, and σ lies in $\operatorname{LinExt}(P)$, one has $\sigma_i \in P_2$ and σ_{i+1} in P_1 if and only if $\sigma_i >_{\mathbb{Z}} \sigma_{i+1}$. That is $\mathsf{Des}_{(P_1,P_2)}(w) = \mathsf{Des}(w)$. $\square$

Example 3.58. An interesting family of posets to which Corollary 3.57 applies are the posets $P(\lambda/\mu)$ associated with two-row skew Ferrers diagrams λ/μ. A *Ferrers diagram* associated to a partition (of a number) $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ has λ_i square cells drawn left-justified in row i. A *skew Ferrers diagram* λ/μ for two partitions λ, μ having $\lambda_i \geq \mu_i$ is the diagram for λ with the cells occupied by the diagram for μ removed. There is a poset structure $P(\lambda/\mu)$ on the cells of λ/μ in which a cell (i, j) in row i and column j has $(i, j) \leq_{P(\lambda/\mu)} (i', j')$ if both $i \leq i'$ and $j \leq j'$.

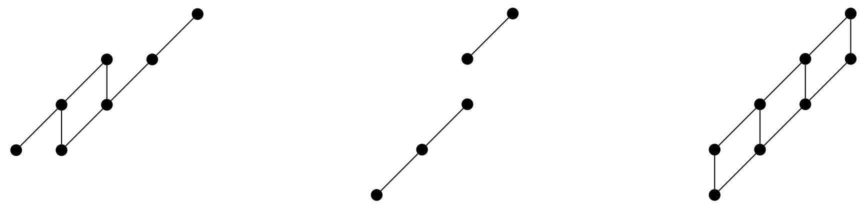
When λ/μ has only two parts, we will call it a *two-row skew Ferrers diagram.* Three examples of such λ/μ and their associated $P(\lambda/\mu)$ are shown below.

λ/μ:	$(5,3)/(1,0)$	$(5,3)/(3,0)$	$(4,4)/(0,0)$

Diagram:



$P(\lambda/\mu)$:



The decomposition $P(\lambda/\mu) = P_1 \cup P_2$ where P_i correspond to the cells in row i of λ/μ shows that $P(\lambda/\mu)$ has width two, and furthermore P_1 forms an order ideal. Therefore Corollary 3.57 implies that for any two-row skew Ferrers diagram λ/μ one has

$$\mathrm{Poin}(P(\lambda/\mu), t) = \sum_{\sigma \in \mathrm{LinExt}(P(\lambda/\mu))} t^{\mathsf{des}(\sigma)}. \tag{3.20}$$

On the other hand, there is a well-known bijection between linear extensions σ of $P(\lambda/\mu)$ and the *standard Young tableaux* Q of shape λ/μ, which are (bijective) labelings of the cells of the diagram by $[n]$ where $n = \sum_i \lambda_i - \sum_i \mu_i$, with the numbers increasing left-to-right in rows and top-to-bottom in columns; see [55, §7.10]. There is also a notion of *descent set* $\mathsf{Des}(Q)$ for such tableaux, having $i \in \mathsf{Des}(Q)$ whenever $i+1$ labels a cell in a lower row of Q than i. However, in general when σ corresponds to Q, one does *not* have $\mathsf{des}(\sigma) = \mathsf{des}(Q)$, so that $\mathrm{Poin}(P(\lambda/\mu), t)$ differs from the generating function $\sum_Q t^{\mathsf{des}(Q)}$ of standard tableaux Q shape λ/μ by $\mathsf{des}(Q)$. For example, there are two standard tableaux of shape $\lambda/\mu = (2,1)/(0,0)$

$$Q_1 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1}\end{array} \qquad Q_2 = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \\ \cline{1-1}\end{array}$$

both having $\mathsf{des}(Q_i) = 1$, however $\mathrm{Poin}(P(\lambda/\mu), t) = 1 + t$.

In two special cases, however, they (essentially) coincide.

- When $P_{\lambda/\mu} = \mathsf{a} \sqcup \mathsf{b}$ is a disjoint union of two chains, as in Example 3.56, one can check that, if one (naturally) labels $\mathsf{a} \sqcup \mathsf{b}$ so that the elements of the order ideal b are labeled $1, 2, \ldots, b$ while a is labeled $b+1, b+2, \ldots, b+a$, then one *does* have $\mathsf{des}(\sigma) = \mathsf{des}(Q)$, and hence
$$\sum_Q t^{\mathsf{des}(Q)} = \mathrm{Poin}(P_{\mathsf{a}\sqcup\mathsf{b}}, t) = \sum_k \binom{a}{k}\binom{b}{k} t^k.$$
- When λ/μ is a $2 \times n$ rectangle, so that $P_{\lambda/\mu} = 2 \times \mathsf{n}$ is a Cartesian product poset, then σ in $\mathrm{LinExt}(P)$ and standard Young tableaux Q of shape $2 \times n$ can both be identified with *Dyck paths of semilength* n, that is, lattice paths from $(0,0)$ to $(2n, 0)$ in $\mathbb{Z}^2$ taking steps northeast or southeast and staying weakly above the x-axis. One can check that
  - $\mathsf{Des}(\sigma)$ corresponds to *valleys* (i.e. southwest steps followed by a northeast step), while
  - $\mathsf{Des}(Q)$ correspond to *peaks* (i.e. northeast steps followed by a southwest step).

  In general, such a Dyck path has one more peak than valley [55, Exercises 6.19(i, ww, aaa)]. Hence one has
$$\mathrm{Poin}(2 \times \mathsf{n}, t) = \frac{1}{t} \sum_Q t^{\mathsf{des}(Q)} = \sum_{k=0}^{n-1} \frac{1}{n} \binom{n}{k}\binom{n}{k+1} t^k,$$
  which is the generating function for the *Narayana numbers* $N(n,k) := \frac{1}{n}\binom{n}{k-1}\binom{n}{k}$ (see [15, p.2] and [55, Exer. 6.36(a)]). Upon setting $t = 1$, the Naryana numbers sum to the *Catalan number*
$$\#\,\mathrm{LinExt}(2 \times \mathsf{n}) = \frac{1}{n+1}\binom{2n}{n}.$$

n	$\mathrm{Poin}(3 \times n, t)$
2	$1 + 3t + t^2$
3	$1 + 9t + 19t^2 + 11t^3 + 2t^4$
4	$1 + 18t + 92t^2 + 174t^3 + 133t^4 + 40t^5 + 4t^6$
5	$1 + 30t + 280t^2 + 1091t^3 + 1987t^4 + 1746t^5 + 731t^6 + 132t^7 + 8t^8$
6	$1 + 45t + 665t^2 + 4383t^3 + 14603t^4 + 25957t^5 + 25064t^6 + 12965t^7 + 3413t^8 + 404t^9$
7	$1 + 63t + 1351t^2 + 13475t^3 + 71305t^4 + 213539t^5 + 373651t^6 + 385578t^7 + 232310t^8$ $+79023t^9 + 14174t^{10} + 1168t^{11} + 32t^{12}$
8	$1 + 84t + 2464t^2 + 34608t^3 + 266470t^4 + 1206826t^5 + 3343958t^6 + 5782699t^7$ $+6275503t^8 + 4240489t^9 + 1743730t^{10} + 417622t^{11} + 53884t^{12} + 3232t^{13} + 64t^{14}$

Table 3.1: Computations of $\mathrm{Poin}(3 \times \mathsf{n}, t)$ for $2 \leq n \leq 8$.

Note that for any (non-skew) partition λ, the celebrated *hook-length formula* of Frame, Robinson and Thrall [55, Corollary 7.21.6] gives a simple product formula for $\#\,\mathrm{LinExt}(P(\lambda)) = [\mathrm{Poin}(P(\lambda), t)]_{t=1}$.

Open Problem 3.59. *Combinatorially interpret* $\mathrm{Poin}(P(\lambda), t)$ *for other partitions* λ*, and in particular, for* $m \times n$ *rectangular partitions, where* $P(\lambda) = \mathsf{m} \times \mathsf{n}$ *is a Cartesian product of chains.*

In Table 3.1, we give $\mathrm{Poin}(3 \times \mathsf{n}, t)$ for $2 \leq n \leq 8$.

Remark 3.60. Since equation (3.6) shows that the Poincaré polynomial $\mathrm{Poin}(P, t)$ for the antichain poset $P = P_{(1^n)} = \mathsf{Antichain}_n$ has *only real roots*, one might wonder whether this holds for some more general class of posets. It does not hold for all posets, and not even for all disjoint unions of chains $P_{\overline{a}}$, since

$$\mathrm{Poin}(P_{(2,2,2)}, t) = 1 + 12t + 43t^2 + 30t^3 + 4t^4$$

has a pair of non-real complex roots. It can fail even for rectangular Ferrers posets, e.g., $\lambda = (3,3,3)$ has $\mathrm{Poin}(P(\lambda), t) = \mathrm{Poin}(3 \times 3, t) = 1 + 9t + 19t^2 + 11t^3 + 2t^4$ in the above table, with two non-real complex roots.

On the other hand, computations show that $\mathrm{Poin}(P, t)$ *is* real-rooted for all posets P of width two having at most 9 elements. This leads to the following question.

Question 3.61. Is $\mathrm{Poin}(P, t)$ real-rooted when the poset P has width two?

3.7 An aside on cones in reflection arrangements

We digress here to generalize Theorem 3.1 from posets P parametrizing cones in the type A_{n-1} reflection arrangement, to *any real reflection arrangement.*

We start first at the level of generality of a *complex reflection group* W acting on $V = \mathcal{C}^n$. This means that W is a finite subgroup of $GL(V) \cong GL_n(\mathcal{C})$ generated by *(complex, unitary, pseudo-) reflections*, which are elements w in W whose fixed space $V^w = H$ is a hyperplane, that is, a codimension one ($\mathcal{C}-$)linear subspace. Let $\mathcal{A}_W$ denote the arrangement of all such reflecting hyperplanes, and $\mathcal{L}(\mathcal{A}_W)$ its poset of intersections, as before. This generalizes the type A_{n-1} setting, where $W = \mathfrak{S}_n$ and $\mathcal{L}(\mathcal{A}_W) \cong \Pi_n$ is the poset of set partitions of $[n]$. There is also a well-known generalization of the map $\mathfrak{S}_n \to \Pi_n$ that sends a permutation σ to the set partition $\pi = \{B_1, B_2, \ldots, B_k\}$ whose blocks B_i are the cycles of σ, given by

$$\begin{array}{ccc} W & \longrightarrow & \mathcal{L}(\mathcal{A}_W) \\ w & \longmapsto & V^w. \end{array} \tag{3.21}$$

Orlik and Solomon proved [39, §4] the following facts about this map.

Proposition 3.62. *For any finite complex reflection group W, the map defined in* (3.21) *has these properties:*

(a) *The map is well-defined:* V^w *is an intersection of reflecting hyperplanes, so it lies in* $\mathcal{L}(\mathcal{A}_W)$*.*

(b) *The map* **surjects** $W \twoheadrightarrow \mathcal{L}(\mathcal{A}_W)$*.*

(c) *The Möbius function values for lower intervals* $[V, X]$ *in* $\mathcal{L}(\mathcal{A}_W)$ *can be expressed via this map as*

$$\mu(V, X) = \sum_{w \in W : V^w = X} \det(w).$$

Proof. Parts (a) and (b) are rephrasings of [39, Lemma (4.4)] and part (c) is [39, Lemma (4.7)]. However, we include here a shorter proof, due to C. Athanasiadis[4]. Recall that the values $\mu(V, X)$ are uniquely determined by the identity $\delta_{X,V} = \sum_{Y : V \supseteq Y \supseteq X} \mu(V, Y)$ where $\delta_{X,V} = 1$ if $X = V$ and 0 otherwise. It therefore suffices to check these equalities:

$$\sum_{Y : V \supseteq Y \supseteq X} \left(\sum_{w \in W : V^w = Y} \det(w) \right) = \sum_{w \in W : V^w \supseteq X} \det(w) = \sum_{w \in W_X} \det(w) = \delta_{X,V}$$

where here W_X denotes the subgroup of W that fixes X pointwise. The last equality follows from Steinberg's Theorem [58, Thm. 1.5]: he showed W_X is generated by the reflections whose hyperplane contains X, so that $W_X = \{1\}$ when $X = V$ (implying $\sum_{w \in W_X} \det(w) = \det(1) = 1$), and otherwise if $X \neq V$, summing the (nontrivial) character $\det(-)$ over W_X yields $\sum_{w \in W_X} \det(w) = 0$. $\square$

For *real* reflection groups, part (c) above has the following reformulation, generalizing equation (3.8) above; see [2, Lemma 5.17], [37, §2, pp. 413-414], and [47, Prop. 7.2].

[4] A version of this proof for *real* reflection groups appears (in Greek) within the proof of Theorem 5.1 on pages 33-34 in the Masters Thesis of Athanasiadis' student C. Savvidou.

Corollary 3.63. *Let W be a finite real reflection group acting on $V = \mathbb{R}^n$. For any X in $\mathcal{L}(\mathcal{A}_W)$, one has*

$$\mu(V, X) = (-1)^{n-\dim(X)} \#\{w \in W : V^w = X\}.$$

Proof. Note W acts orthogonally. We claim that any w acting *orthogonally* on $\mathbb{R}^n$ has $\det(w) = (-1)^{n-\dim(V^w)}$; given this, Lemma 3.62(c) would finish the proof. To see this claim, note that the eigenvalues λ of w lie on the unit circle in $\mathcal{C}$, so $\lambda\overline{\lambda} = 1$. If the eigenvalue λ has multiplicity m_λ, then $m_{\overline{\lambda}} = m_\lambda$. Thus

$$\det(w) = \prod_{\lambda \in \mathcal{C}} \lambda^{m_\lambda} = (+1)^{m_1}(-1)^{m_{-1}} \prod_{\text{pairs}\{\lambda \neq \overline{\lambda}\}} (\lambda\overline{\lambda})^{m_\lambda} = (-1)^{m_{-1}}.$$

Modulo two, however, we have

$$m_{-1} = n - \sum_{\lambda \neq -1} m_\lambda = n - m_{+1} - \sum_{\lambda \neq \overline{\lambda}} 2m_\lambda = n - m_{+1} = n - \dim(V^w),$$

and so $\det(w) = (-1)^{n-\dim(V^w)}$. □

We specialize now to real reflection groups W. Here it is known (see, e.g., [34, Chapter 1]) that W permutes the chambers $\mathcal{C}(\mathcal{A}_W)$ simply transitively. Thus by fixing a choice of base chamber C_0, every other chamber wC_0 has a unique label by some w in W, giving a bijection $\mathcal{C}(\mathcal{A}_W) \leftrightarrow W$.

Cones $\mathcal{K}$ inside the reflection arrangement $\mathcal{A}_W$ correspond to what were called *parsets* by the third author [45, Chapter 3], or *Coxeter cones* by Stembridge [59], where they were studied as well-behaved generalizations of posets P on $[n]$. In particular, the set of chambers $\mathcal{C}(\mathcal{K})$ inside $\mathcal{K}$ generalizes the set $\mathrm{LinExt}(P)$ of linear extensions of P. For a cone $\mathcal{K}$ in $\mathcal{A}_W$, we consider as before the subposet $\mathcal{L}^{\text{int}}(\mathcal{K})$ of intersection subspaces interior to $\mathcal{K}$, playing the role of the P-transverse set partitions $\Pi^{\pitchfork}(P)$ in type A_{n-1}. Generalizing the P-transverse permutations $\mathfrak{S}^{\pitchfork}(P)$, define the subset

$$W^{\pitchfork}(\mathcal{K}) := \{w \in W : V^w \in \mathcal{L}^{\mathrm{int}}(\mathcal{K})\}.$$

The real reflection group generalization of Theorem 3.1 is the following.

Theorem 3.64. *Any cone $\mathcal{K}$ in the reflection arrangement $\mathcal{A}_W$ for a finite real reflection group W has*

$$\mathrm{Poin}(\mathcal{K}, t) = \sum_{w \in W^{\pitchfork}(\mathcal{K})} t^{n-\dim(V^w)}.$$

In particular, setting $t = 1$, one has $\#\mathcal{C}(\mathcal{K}) = \#W^{\pitchfork}(\mathcal{K})$.

Proof.

$$\begin{aligned}\mathrm{Poin}(\mathcal{K}, t) &= \sum_{X \in \mathcal{L}^{\mathrm{int}}(\mathcal{K})} |\mu(V, X)| \cdot t^{n-\dim(X)} \\ &= \sum_{X \in \mathcal{L}^{\mathrm{int}}(\mathcal{K})} \#\{w \in W : V^w = X\} \cdot t^{n-\dim(X)} = \sum_{w \in W^{\pitchfork}(\mathcal{K})} t^{n-\dim(V^w)}\end{aligned}$$

where the second equality used Corollary 3.63, and the last equality used the definition of $W^{\pitchfork}(\mathcal{K})$. $\square$

4 The Varchenko-Gel'fand Ring

This chapter concerns a ring called the Varchenko-Gel'fand[1] ring of $\mathcal{A}$, a ring consisting of maps $\mathcal{C}(\mathcal{A}) \to \mathbb{Z}$ under pointwise addition and multiplication. The material from this chapter is also discussed in the author's arXiv preprint "The Varchenko-Gel'fand ring of a cone" [21].

This ring, which we denote $VG(\mathcal{A})$, was first introduced by Gel'fand and Varchenko [61], who proved that it has a $\mathbb{Z}$-basis of monomials indexed by no broken circuit sets of the oriented matroid and showed that the *degree filtration* of $VG(\mathcal{A})$ yields an *associated graded ring* with Hilbert series completely determined by $\mathcal{L}(\mathcal{A})$. Since then, Cordovil [19], Gel'fand-Rybnikov [29], Moseley [38], Proudfoot [44], and others have studied the Varchenko-Gel'fand ring of an arrangement as well as generalizations of this structure.

One can show that the Varchenko-Gel'fand ring $VG(\mathcal{K})$ of a cone is generated (as a ring) by Heaviside functions associated to the hyperplanes of $\mathcal{A}$. This endows $VG(\mathcal{K})$ with a *degree filtration* $\mathcal{F} = \{F_d\}$, where F_d is the $\mathbb{Z}$-span of products of

[1] In Russian, Varchenko comes first (alphabetically speaking).

	$g \in \mathcal{G}$	$\mathrm{in}_{\deg}(g)$	$\mathrm{in}_{\prec}(g)$
Idempotent	For all $i \in [n]$, $e_i^2 - e_i$	e_i^2	e_i^2
Unit	For all $i \in W$, $e_i - 1$	e_i	e_i
Cone Circuit	For all circuits $C = (C^+, C^-)$ with $\emptyset \neq W \cap C^+ = W \cap \underline{C}$, $e_{C^+ \setminus W} \cdot \prod_{j \in C^-} (e_j - 1)$ (similarly if $\emptyset \neq W \cap C^- = W \cap \underline{C}$)	$e_{\underline{C} \setminus W}$	$e_{\underline{C} \setminus W}$
Circuit	For signed circuits $C = (C^+, C^-)$ with $\emptyset = W \cap \underline{C}$, $(e_{C^+}) \prod_{j \in C^-} (e_j - 1) - (e_{C^-}) \prod_{j \in C^+} (e_j - 1)$	$+ \sum_{i \in C^+} e_{\underline{C} \setminus \{i\}} - \sum_{j \in C^-} e_{\underline{C} \setminus \{j\}}$	$e_{\underline{C} \setminus \{i_0\}}$ where $i_0 := \min_{\prec}(\underline{C})$

Table 4.1: The relations g in $\mathcal{G}$, along with their degree-initial form $\mathrm{in}_{\deg}(g)$ and their initial term $\mathrm{in}_{\prec}(g)$ for any monomial order $\prec$ that satisfies $e_1 \prec \cdots \prec e_n$.

such Heaviside functions of degree at most d. The *associated graded ring*[2] is

$$\mathcal{V}(\mathcal{K}) := \mathfrak{gr}_{\mathcal{F}} VG(\mathcal{K}) = \bigoplus_{d \geq 0} F_d / F_{d-1}$$

and its Hilbert series is

$$\operatorname{Hilb}(\mathcal{V}(\mathcal{K}), t) := \sum_{d \geq 0} \operatorname{rank}_{\mathbb{Z}}(F_d/F_{d-1})\ t^d$$

where $\operatorname{rank}_{\mathbb{Z}}(F_d/F_{d-1})$ denotes the rank of F_d/F_{d-1} as a $\mathbb{Z}$-module. When $\mathcal{K}$ contains the full arrangement, the graded algebra $\mathcal{V}(\mathcal{A})$ is also called the *Cordovil algebra*. Varchenko and Gel'fand proved that when the cone $\mathcal{K}$ is the full arrangement, then $\mathcal{V}(\mathcal{A})$ is torsion-free and $\operatorname{Hilb}(\mathcal{V}(\mathcal{A}), t) = \operatorname{Poin}(\mathcal{A}, t)$ [61]. We will show the same equality for cones and, specializing to the full arrangement, obtain a novel proof of Varchenko and Gel'fand's original result.

Our proof comes from giving a generating set $\mathcal{G}$ of relations that plays the role of a Gröbner basis presentation for $VG(\mathcal{K})$ and $\mathcal{V}(\mathcal{K})$ as quotients of $\mathbb{Z}[e_1, \ldots, e_n]$; when working over a field instead of $\mathbb{Z}$, they are honest Gröbner bases. The relations in $\mathcal{G} \subseteq \mathbb{Z}[e_1, \ldots, e_n]$ are summarized in Table 4.1, where we have made the (harmless) assumption that $\mathcal{K}$ is an intersection of (open) positive halfspaces, i.e. $\mathcal{K} = \bigcap_{i \in W} H_i^+$ where $W \subseteq [n]$ and $H_i^+ := \{\mathbf{x} \in \mathbb{R}^\ell \mid v_i \cdot \mathbf{x} > 0\}$ for some choice of normal vector v_i to H_i. For a subset of $[n]$, we use the notation $e_S := \prod_{i \in S} e_i$ to describe a squarefree monomial indexed by S in the variables $e_1, \ldots, e_n$. Our main theorem will assert that the elements of $\mathcal{G}$ (given in the second column of Table 4.1) and $\{\operatorname{in}_{\deg}(g) \mid g \in \mathcal{G}\}$ (given in the third column) give presentations for $VG(\mathcal{K})$ and $\mathcal{V}(\mathcal{K})$, respectively.

[2]In the case of the full arrangement, this associated graded ring is sometimes also called the *Cordovil algebra* of the arrangement. In this thesis, we will occasionally call $\mathcal{V}(\mathcal{K})$ the Cordovil algebra of the cone $\mathcal{K}$.

The most interesting polynomials in this Gröbner basis $\mathcal{G}$ are defined by signed circuits (broken circuits were defined in Chapter 2). Our Gröbner basis presentations for $VG(\mathcal{K})$ and $\mathcal{V}(\mathcal{K})$ concerns signed circuits $C = (C^+, C^-)$, which are signed dependencies for which $\underline{C}$ is minimal under inclusion.

In the main theorem of this chapter, our presentations of $VG(\mathcal{K})$ and $\mathcal{V}(\mathcal{K})$ will give a $\mathbb{Z}$-basis for both rings in terms of a certain family of monomials indexed by $\mathcal{K}$-no broken circuit sets, which introduced in Chapter 2.

Theorem 4.1. *Let $\mathcal{K}$ be a cone of an arrangement $\mathcal{A}$, and choose a monomial order $\prec$ on $\mathbb{Z}[e_1, \ldots, e_n]$ which refines the ordering by degree. Then both $VG(\mathcal{K}), \mathcal{V}(\mathcal{K})$ have presentations*

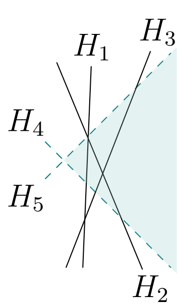
$$VG(\mathcal{K}) \cong \mathbb{Z}[e_1, \ldots, e_n]/(\mathcal{G}),$$

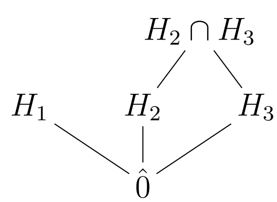
$$\mathcal{V}(\mathcal{K}) \cong \mathbb{Z}[e_1, \ldots, e_n]/(\mathrm{in}_{\deg}(\mathcal{G})),$$

and free $\mathbb{Z}$-modules bases given by the images of the $\mathcal{K}$-NBC monomials $\{e_N\}_{N \in NBC(\mathcal{K})}$. In particular,

$$\mathrm{Hilb}(\mathcal{V}(\mathcal{K}), t) = \sum_{N \in NBC(\mathcal{K})} t^{\#N} = \mathrm{Poin}(\mathcal{K}, t).$$

Example 4.2. Consider the cone $\mathcal{K}$ of a central arrangement in in $\mathbb{R}^3$ of which an affine slice is drawn below on the left. We can compute the Poincaré polynomial of the cone from $\mathcal{L}^{\mathrm{int}}(\mathcal{K})$ (below, on the right): $\mathrm{Poin}(\mathcal{K}, t) = 1 + 3t + t^2$.





For some choice of orientation of $\mathcal{A}$, the set of relations $\mathcal{G}$ is given in Table 4.2.

	$g \in \mathcal{G}$	$\mathrm{in}_{\deg}(g)$	$\mathrm{in}_{\prec}(g)$
(Idempotent)	For all $i \in [5]$ $e_i^2 - e_i$	e_i^2	e_i^2
(Unit)	$e_4 - 1$, $e_5 - 1$	e_4, e_5	e_4, e_5
(Circuit)	(none)	(none)	(none)
(Cone-Circuit)	$e_1e_2 - e_2$, $e_1e_2e_3 - e_2e_3$, $e_1e_3 - e_3$	e_1e_2, $e_1e_2e_3$, e_1e_3	e_1e_2, $e_1e_2e_3$, e_1e_3

Table 4.2: The elements of $\mathcal{G}$ for the cone in Example 4.2. To save space, we omit the the redundant Cone-Circuit relations for which the opposite orientation is already given.

Our main theorem says that $VG(\mathcal{K}) \cong \mathbb{Z}[e_1, e_2, e_3, e_4, e_5]/(\mathcal{G})$ and the associated graded ring has presentation $\mathcal{V}(\mathcal{K}) \cong \mathbb{Z}[e_1, e_2, e_3, e_4, e_5]/(\mathrm{in}_{\deg}\mathcal{G})$. Furthermore

$$\mathcal{V}(\mathcal{K}) \cong \mathbb{Z} \cdot \{1\} \oplus \mathbb{Z} \cdot \{x_1, x_2, x_3\} \oplus \mathbb{Z} \cdot \{x_2 x_3\}$$

which makes it very easy to see that $\mathrm{Hilb}(\mathcal{V}(\mathcal{K}), t) = 1 + 3t + t^2$.

Corollary 4.3. *$VG(\mathcal{K}) \cong \mathbb{Z}[e_1, \ldots e_n]/I_{\mathcal{K}}$ where $I_{\mathcal{K}}$ is generated by*

- (Idempotent) *$e_i^2 - e_i$ for $i \in [n]$,*
- (Unit) *$e_i - 1$ for $i \in [n]$ such that H_i is a wall of $\mathcal{K}$,*
- (Circuit) *$(e_{C^+}) \prod_{j \in C^-} (e_j - 1) - (e_{C^-}) \prod_{i \in C^+} (e_i - 1)$ for signed circuits $C = (C^+, C^-)$.*

Theorem 4.1 has an algebraic interesting consequence when $\mathcal{A}$ is a *supersolvable arrangement*: we obtain a Varchenko-Gel'fand ring analogue of a result on *Koszul algebras*[3] proven by Peeva for the *Orlik-Solomon algebra* of $\mathcal{A}$. In order to state this result, we work with coefficients in a field $\mathbb{F}$, and consider the ring $VG_{\mathbb{F}}(\mathcal{A})$ of maps from the chambers of $\mathcal{A}$ to a field $\mathbb{F}$, denoting the associated graded ring by $\mathcal{V}_{\mathbb{F}}(\mathcal{A})$.

Theorem 4.4. *If $\mathcal{A}$ is a supersolvable arrangement, then $\mathcal{V}_{\mathbb{F}}(\mathcal{A})$ is a Koszul algebra.*

Unfortunately, not every cone of a supersolvable arrangement yields a Koszul $\mathcal{V}_{\mathbb{F}}(\mathcal{K})$. In Section 4.4, we provide an example of a cone $\mathcal{K}$ of a supersolvable arrangement whose $\mathcal{V}_{\mathbb{F}}(\mathcal{K})$ is not Koszul.

The remainder of this chapter is devoted to proving Theorems 4.1 and 4.4. In Section 4.2, we give some background on commutative algebra. In Section 4.3, we

[3] See Section 4.4 for a definition of a Koszul algebra.

show that the set $\mathcal{G}$ lies in the kernel of φ and prove that the $NBC(\mathcal{K})$ monomials are the $\text{in}_{\prec}(\mathcal{G})$-standard monomials, which gives Theorem 4.1 as a consequence. Finally, in Section 4.4 we review Koszulity and prove Theorem 4.4.

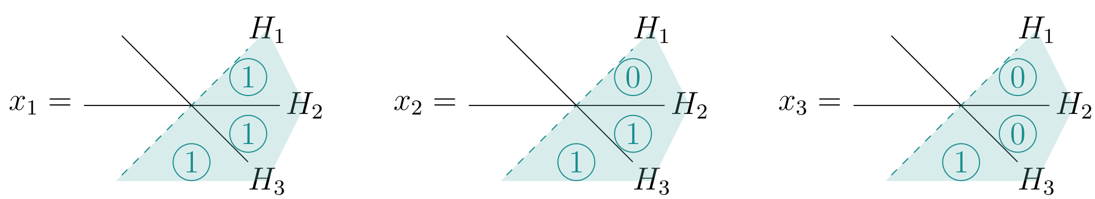
4.1 Definition of the Varchenko-Gel'fand Ring

The *Varchenko-Gel'fand ring* of an arrangement $\mathcal{A}$ is the ring of maps $f : \mathcal{C}(\mathcal{A}) \to \mathbb{Z}$ under pointwise addition and multiplication [61]. Similarly, we define the *Varchenko-Gel'fand ring of a cone* $\mathcal{K}$ to be the ring with underlying set

$$VG(\mathcal{K}) = \{f : \mathcal{C}(\mathcal{K}) \to \mathbb{Z}\}$$

under pointwise addition and multiplication. We can represent elements of $VG(\mathcal{K})$ as a labelling of the chambers of $\mathcal{K}$ with integers.

Example 4.5. Consider the cone from Example 2.8. Below are several elements of $VG(\mathcal{K})$:



In this example, the elements are suggestively labelled x_1, x_2, and x_3 to represent *Heaviside functions* (defined below) given by some orientation of the hyperplanes H_1, H_2, and H_3. The Heaviside functions associated to H_4 and H_5 are not included as they would both be 1 on every chamber of the cone. In Varchenko and Gel'fand's original paper [61], they observe that $VG(\mathcal{A})$ is generated as a $\mathbb{Z}$-algebra by

Heaviside functions

$$x_i(C) = \begin{cases} 1 & \text{if } C \subseteq H_i^+ \\ 0 & \text{else} \end{cases} \qquad \text{for } C \in \mathcal{C}(\mathcal{A}).$$

for each hyperplane $H_i \in \mathcal{L}(\mathcal{A})$. It suffices to check that $f : \mathcal{C}(\mathcal{A}) \to \mathbb{Z}$ is

$$f = \sum_{C \in \mathcal{C}(\mathcal{A})} f(C) \prod_{\substack{i \in W_C \\ C \subseteq H_i^+}} x_i \prod_{\substack{j \in W_C \\ C \subseteq H_j^-}} (1 - x_i)$$

where W_C is the set of indices of hyperplanes which are the linear spans of the codimension 1 faces of C. Their proof extends without modification to the cone case, where $x_i(C)$ is 1 when both $C \subseteq H_i^+$ and $C \in \mathcal{C}(\mathcal{K})$, and 0 otherwise. In particular this means that $x_i \equiv 1$ for each $H_i \in W$.

Remark 4.6. We will usually view the Varchenko-Gel'fand ring of a cone as a ring of functions $\mathcal{C}(\mathcal{K}) \to \mathbb{Z}$. However, it is also a quotient of $VG(\mathcal{A})$: one has a surjective *restriction map* $\mathsf{res} : VG(\mathcal{A}) \twoheadrightarrow VG(\mathcal{K})$ sending $f \mapsto f|_{\mathcal{K}}$ defined by $f|_{\mathcal{K}}(C) := f(C)$ for $C \in \mathcal{C}(\mathcal{K})$.

In the previous section, we introduced oriented matroids and defined a family of sets called the $\mathcal{K}$-NBC sets. By the definition of the Varchenko-Gel'fand ring, we have $VG(\mathcal{K}) \cong \mathbb{Z}^{\#\mathcal{C}(\mathcal{K})}$. Combining this isomorphism with Equation (2.1) implies

$$VG(\mathcal{K}) \cong \mathbb{Z}^{\#\mathcal{C}(\mathcal{K})} = \mathbb{Z}^{\#NBC(\mathcal{K})}.$$

This chain of equivalences will be crucial in the proof of the main theorem, which provides an explicit basis for $VG(\mathcal{K})$ in terms of the $\mathcal{K}$*-NBC monomials* $e_N = \prod_{i \in N} e_i$ for $N \in NBC(\mathcal{K})$.

4.2 Some Commutative Algebra

This section reviews some commutative algebra material on polynomial rings over $\mathbb{Z}$ and and their quotients. For more details, see [1, 6, 23].

4.2.1 Monomial orders

A polynomial in $\mathbb{Z}[e_1, \ldots, e_n]$ is a sum

$$f = \sum_{a=(a_1,\ldots,a_n)} c_a \, e_1^{a_1} \cdots e_n^{a_n}$$

where $c_a \in \mathbb{Z}$. When $c_a \neq 0$, one calls $c_a \, e_1^{a_1} \cdots e_n^{a_n}$ a *term* of f, and $e_1^{a_1} \cdots e_n^{a_n}$ a *monomial* of f. Define $\deg(f) := \max\{\sum_i a_i : c_a \neq 0\}$ and then the *degree-initial form* of f is

$$\mathrm{in}_{\deg}(f) := \sum_{\substack{a \\ \sum_i a_i = \deg(f)}} c_a \, e_1^{a_1} \cdots e_n^{a_n}.$$

A *monomial ordering* is a total ($=$ linear) order $\prec$ *well-ordering* on the set of all of monomials m in $\mathbb{Z}[e_1, \ldots, e_n]$ which respects multiplication in the sense that $m \prec m'$ implies $m \cdot m'' \prec m' \cdot m''$. Define the $\prec$-leading monomial $\mathrm{in}_\prec(f)$ to be the $\prec$-highest monomial of f. Say that $\prec$ is a *degree order* if it is compatible with $\mathrm{in}_{\deg}$ in the sense that $\mathrm{in}_\prec(f) = \mathrm{in}_\prec(\mathrm{in}_{\deg}(f))$ for all f; see Sturmfels [60, Chapter 1] for more on these notions. Given a collection $\mathcal{G} = \{g_i\}_{i \in I}$ of polynomials, say that a monomial m is a $\mathrm{in}_\prec(\mathcal{G})$*-standard* if it is divisible by none of $\{\mathrm{in}_\prec(g_i)\}_{i \in I}$.

4.2.2 Filtrations and Associated Graded Rings

Let R be a commutative ring with unit. An *(ascending) filtration* of R is a sequence $F_0 \subseteq F_1 \subseteq F_2 \subseteq \ldots$ of nested $\mathbb{Z}$-submodules of R with the property that if $f \in F_c$ and $g \in F_d$, then $f \cdot g \in F_{c+d}$. In this paper, we consider the *degree filtration*

$\{F_d\}_{d\geq 0}$ for quotient rings $R = \mathbb{Z}[e_1,\ldots,e_n]/I$, where I is an ideal of $\mathbb{Z}[e_1,\ldots,e_n]$: define F_i to be the image within R of the polynomials in $\mathbb{Z}[e_1,\ldots,e_n]$ having degree at most d. Define the associated graded ring

$$\mathfrak{gr}(R) := \mathfrak{gr}_{\mathcal{F}}(R) := \bigoplus_{d\geq 0} F_d/F_{d-1}$$

where we define $F_{-1} := 0$.

Recall that the *rank* of a $\mathbb{Z}$-module M is $\mathrm{rank}_{\mathbb{Z}}(M) := \dim_{\mathrm{Int}_\ell}(\mathrm{Int}_\ell \otimes_{\mathbb{Z}} M)$, see [23, Section 11.6]. In the setting of a degree filtration, each F_d is a finitely generated $\mathbb{Z}$-module, allowing us to define the Hilbert series of the associated graded ring:

$$\mathrm{Hilb}(\mathfrak{gr}(R), t) := \sum_{d\geq 0} \mathrm{rank}_{\mathbb{Z}}(F_d/F_{d-1})\ t^d.$$

For example, we will wish to consider the associated graded ring of the Varchenko-Gel'fand ring with its degree filtration $\mathcal{V}(\mathcal{K}) := \mathfrak{gr}(VG(\mathcal{K}))$, with Hilbert series $\mathrm{Hilb}(\mathcal{V}(\mathcal{K}), t)$.

The proof of Theorem 4.1 in Section 4.3 uses a certain general lemma, which we state and prove now. Experts may recognize this lemma as a standard fact from Gröbner basis theory when the polynomial rings are defined over a field, but the modification here relates to polynomial rings over $\mathbb{Z}$; see Remark 4.8 below.

Lemma 4.7. *Assume one has a $\mathbb{Z}$-algebra surjection $S := \mathbb{Z}[e_1,\ldots,e_n] \overset{\varphi}{\twoheadrightarrow} R$ in which R is a free $\mathbb{Z}$-module of rank r, and $\mathcal{G} = \{g_i\}_{i\in I} \subset S$ has these properties:*

(i) $\mathcal{G} \subset \ker\varphi$.

(ii) Each g_i is $\preceq$-monic, meaning $\mathrm{in}_\prec(g_i)$ has coefficient ± 1 in g_i.

(iii) The set of $\mathrm{in}_\prec\mathcal{G}$-standard monomials $\mathcal{N} = \{m_1,\ldots,m_t\}$ has cardinality $t \leq r$.

Then one has these implications:

(a) $\ker(\varphi) = (\mathcal{G})$*, so that* φ *induces a* $\mathbb{Z}$*-algebra isomorphism* $S/(\mathcal{G}) \cong R$*.*

(b) The cardinality $\#\mathcal{N} = t = r$*, and* R *has* $\varphi(\mathcal{N}) = \{\varphi(m_1), \dots, \varphi(m_r)\}$ *as a* $\mathbb{Z}$*-basis.*

If $\preceq$ *is also a degree-ordering, then one has two further implications:*

(c) The map $S \xrightarrow{\psi} \mathfrak{gr}(R)$ *sending* $e_i \mapsto \bar{x}_i$ *in* F_1/F_0 *is surjective, with* $\ker(\psi) = (\mathrm{in}_{\deg}(\mathcal{G}))$*, so that it induces a* $\mathbb{Z}$*-algebra isomorphism*

$$S/(\mathrm{in}_{\deg}(\mathcal{G})) \xrightarrow{\psi} \mathfrak{gr}(R)$$

(d) For $d \geq 0$*, each* F_d/F_{d-1} *is a free* $\mathbb{Z}$*-module on the basis* $\{m_i \in \mathcal{N} : \deg(m_i) = d\}$*, so that*

$$\mathrm{Hilb}(\mathfrak{gr}(R), t) = \sum_{i=1}^{r} t^{\deg(m_i)}.$$

Proof. Note since each g_i in $\mathcal{G}$ is $\prec$-monic, the usual *multivariate division algorithm* with respect to $\mathcal{G}$ using the order $\prec$ (see Cox, Little and O'Shea [20, §2.3, Theorem 3]) shows that every f in S lies in $\mathbb{Z}m_1 + \cdots + \mathbb{Z}m_t + (\mathcal{G})$. Therefore, if one defines a $\mathbb{Z}$-module map $\mathbb{Z}^t \to S$ that sends the i^{th} standard basis element of $\mathbb{Z}^t$ to the $\mathrm{in}_\prec(\mathcal{G})$-standard monomial m_i, then the composite $\mathbb{Z}$-module map $\mathbb{Z}^t \to S \twoheadrightarrow S/(\mathcal{G})$ is surjective. This composite is the map α in this sequence of $\mathbb{Z}$-module surjections/isomorphisms

$$\mathbb{Z}^t \overset{\alpha}{\twoheadrightarrow} S/(\mathcal{G}) \overset{\beta}{\twoheadrightarrow} S/\ker(\varphi) \cong R \cong \mathbb{Z}^r, \tag{4.1}$$

where β comes from our assumption (i) above. It is well-known (see [6, Chapter 2, Exercise 12], for example) that if M is a $\mathbb{Z}$-module with $\mathrm{rank}_{\mathbb{Z}}(M) = r \geq t$, then any surjection $\mathbb{Z}^t \to M$ must in fact be an isomorphism, with $t = r$. It follows that the composite of all maps in (4.1) is an isomorphism. Thus β and $\beta \circ \alpha$ are isomorphisms, proving assertions (a) and (b), respectively.

Now assume further that $\prec$ is a degree ordering. Since $\mathrm{in}_\prec(\mathrm{in}_{\deg}(g_i)) = \mathrm{in}_\prec(g_i)$, replacing $\mathcal{G}$ with $\mathrm{in}_{\deg}(\mathcal{G})$, we conclude as in the above proof that the composite map

$$\mathbb{Z}^r \to S \twoheadrightarrow S/(\mathrm{in}_{\deg}(\mathcal{G}))$$

is surjective. The fact that $S \overset{\varphi}{\twoheadrightarrow} R$ is surjective implies that $S \overset{\psi}{\twoheadrightarrow} \mathfrak{gr}(R)$ is also surjective. Furthermore, the definitions of $\mathrm{in}_{\deg}(-)$ and $\mathfrak{gr}(R)$, together with $\mathcal{G} \subset \ker(S \overset{\varphi}{\to} R)$, imply

$$\mathrm{in}_{\deg}(\mathcal{G}) \subset \ker(S \overset{\psi}{\twoheadrightarrow} \mathfrak{gr}(R)).$$

Hence we again have a sequence of surjections and isomorphisms:

$$\mathbb{Z}^r \overset{\gamma}{\twoheadrightarrow} S/(\mathrm{in}_{\deg}(\mathcal{G})) \overset{\delta}{\twoheadrightarrow} S/\ker(\psi) \cong \mathfrak{gr}(R). \tag{4.2}$$

Note that $\mathsf{rank}(\mathfrak{gr}(R)) = \mathsf{rank} R = r$, since $\mathsf{rank}(-)$ is additive along short exact sequences and direct sums. Thus we can again conclude that the composite of the surjections in (4.2) is an isomorphism. Hence δ is an isomorphism, proving (c). Then (d) follows from $\delta \circ \gamma$ being an isomorphism, upon noting that a monomial m in S has $\psi(m)$ lying in F_d/F_{d-1} where $d = \deg(m)$. $\square$

Remark 4.8. Replacing $\mathbb{Z}$ by a field $\mathbb{F}$, and replacing $\mathsf{rank}(-)$ with $\dim_{\mathbb{F}}(-)$, the proof of Lemma 4.7 shows $\mathcal{G}$ and $\mathrm{in}_{\deg}(\mathcal{G})$ give Gröbner bases for the ideals presenting the rings R and $\mathfrak{gr}(R)$.

4.3 Proof of Theorem 4.1

Given a cone $\mathcal{K}$ in an arrangement $\mathcal{A}$, let $\mathcal{G}$ be the elements shown in the second column of the table in Table 4.1. In this section, we give two propositions regarding $\mathcal{G}$, which together prove Theorem 4.1. First we show that the polynomials $\mathcal{G}$ lie in the kernel of the map $\varphi : \mathbb{Z}[e_1, \dots, e_n] \longrightarrow VG(\mathcal{K})$ which sends the variable e_i to the Heaviside function x_i for each $i \in [n]$. After that, we fix a monomial order $\prec$ on $\mathbb{Z}[e_1, \dots, e_n]$ whose restriction to the variables is $e_1 \prec \cdots \prec e_n$. We will show that the $\mathcal{K}$-NBC monomials are exactly the $\mathrm{in}_\prec(\mathcal{G})$-standard monomials. In fact, since Equation (2.1) implies $VG(\mathcal{K}) \cong \mathbb{Z}^{\#\mathcal{C}(\mathcal{K})} = \mathbb{Z}^{\#NBC(\mathcal{K})}$, using Lemma 4.7 it suffices to show that the $\mathrm{in}_\prec(\mathcal{G})$-standard monomials are a subset of the $\mathcal{K}$-NBC monomials.

Proposition 4.9. *Every polynomial in $\mathcal{G}$ lies in* $\ker \varphi$.

Proof. This holds for $\mathsf{Idempotent}$ relations $e_i^2 - e_i$ since Heaviside functions x_i have $x_i(C) \in \{0, 1\}$. It holds for Unit relations $e_i - 1$ with $i \in W$, since then $H_i^+ \supseteq \mathcal{K}$, so $x_i(C) \equiv 1$ for all C in $\mathcal{C}(\mathcal{K})$.

To understand the $\mathsf{Circuit}$ and $\mathsf{Cone\ Circuit}$ relations, note that the existence of a signed circuit $C = (C^+, C^-)$ implies that these two intersections are empty, and hence contain no chambers:

$$\bigcap_{i \in C^+} H_i^+ \cap \bigcap_{j \in C^-} H_j^- = \emptyset = \bigcap_{i \in C^-} H_i^+ \cap \bigcap_{j \in C^+} H_j^-$$

Consequently, if one writes the $\mathsf{Circuit}$ relation as the difference $f_+ - f_-$ of these two products

$$f_+ := e_{C^+} \cdot \prod_{j \in C^-} (e_j - 1) \quad \text{and} \quad f_- := e_{C^-} \prod_{j \in C^+} (e_j - 1), \tag{4.3}$$

one finds that both f_+, f_- lie in $\ker \varphi$, and hence so does the $\mathsf{Circuit}$ relation $f_+ - f_-$.

For a Cone Circuit relation, assume without loss of generality that the signed circuit $C = (C^+, C^-)$ has $\emptyset \neq W \cap C_+ = W \cap C$. Then since $\ker \varphi$ contains the product f_+ defined in (4.3) along with the Unit relations $e_i - 1$ for $i \in W \cap C^+$, it also contains the Cone Circuit relation $e_{C^+ \setminus W} \cdot \prod_{j \in C^-}(e_j - 1)$. $\square$

Remark 4.10. Note that if the signed circuit $C = (C^+, C^-)$ has $W \cap C^+ \neq \emptyset$, the element f_- is divisible by Unit relations $e_i - 1$ for $i \in W \cap C^+$, and hence is superfluous in generating $\ker \varphi$. Similarly, if $W \cap C^- \neq \emptyset$, then f_+ is a redundant generator. Combining these: if both $W \cap C^- \neq \emptyset$ and $W \cap C^+ \neq \emptyset$, then both f_+, f_- are redundant and so is the corresponding Circuit relation.

Also note, when $H_i \in \mathcal{A}$ is neither a wall hyperplane nor cuts through the cones, the union $W \cup \{i\}$ contains a signed circuit $C = (C^+, C^-)$. Furthermore, one can show that $i \in \underline{C}$ and that the Cone-Circuit relation does not vanish, i.e. one of $e_i - 1$ or e_i is in $\mathcal{G}$.

Proposition 4.11. *The* $\mathrm{in}_\prec(\mathcal{G})$*-standard monomials are (a subset of the)* $\mathcal{K}$*-NBC monomials.*

Proof. Let $m \in \mathbb{Z}[e_1, \ldots, e_n]$ be any $\mathrm{in}_\prec(\mathcal{G})$-standard monomial. We show that m is a $\mathcal{K}$-NBC monomial in several reduction steps.

Reduction 1. Since $e_i^2 \in \mathrm{in}_\prec(G)$ for $1 \leq i \leq n$, we may assume that $m = e_N$ for some $N \subseteq \{1, \ldots, n\}$.

Reduction 2. Since $e_i \in \mathrm{in}_\prec(G)$ for $i \in W$, we may assume that $m = e_N$ with $W \cap N = \emptyset$.

Reduction 3. We can assume that $m = e_N$ where N contains no broken circuits, i.e. $N \in NBC(\mathcal{A})$. To see this, suppose N contains a signed circuit C with $i_0 = \min(\underline{C})$ such that the corresponding broken circuit $\underline{C} \backslash \{i_0\}$ is contained in N.

Since $W \cap N = \emptyset$ (from Reduction 2) and $\underline{C} \backslash \{i_0\} \subseteq N$, either $i_0 \in W$ or $W \cap \underline{C}$ is empty. We obtain a contradiction in both cases. First, if $W \cap \underline{C} = \{i_0\}$, then

$$N \supseteq \underline{C} \setminus \{i_0\} = \underline{C} \setminus W,$$

forcing e_N to be divisible by $e_{\underline{C} \backslash W}$, and contradicting that e_N is $\text{in}_{\prec}(\mathcal{G})$-standard. On the other hand, if $\# W \cap \underline{C} = \emptyset$, then e_N is divisible by $e_{\underline{C} \backslash \{i_0\}}$ which contradicts the assumption that e_N is $\text{in}_{\prec}(\mathcal{G})$-standard.

Reduction 4. Assuming that $m = e_N$ where N is in $NBC(\mathcal{A})$, we will show that it also lies in $NBC(\mathcal{K})$, that is, $X := \cap_{j \in N} H_j^0$ has $\mathcal{K} \cap X \neq \emptyset$. For the sake of contradiction, assume

$$\mathcal{K} \cap X = \bigcap_{i \in W} H_i^+ \cap \bigcap_{j \in N} H_j^0 = \emptyset.$$

It is easy to see that there is a choice of signs $\varepsilon \in \{+, -\}^N$ for which

$$\bigcap_{i \in W} H_i^+ \cap \bigcap_{j \in N} H_j^{\varepsilon_j} = \emptyset \tag{4.4}$$

(see Observation 4.12, below, for details). Translating Equation (4.4) into the language of oriented matroids, we have that there is no covector $F = (F^+, F^-)$ of the matroid on $E = W \cup N$ with

$$F_j = \begin{cases} + & \text{if } j \in W \\ \varepsilon_j & \text{if } j \in N. \end{cases}$$

From Observation 4.13 (below) or, equivalently, Gordan's Theorem [33], the fact that no such F exists, means that there does exist a (nonzero) signed dependence $D = (D^+, D^-) \in \mathfrak{D}$ with $\underline{D} \subseteq W \cup N$ and $(W \cap \underline{D}) \subseteq D^+$.

Recall from Section 2.2 that every signed dependence D is a composition of circuits and that at least one of these circuits [4] must conform to D. Let C be such a circuit conforming to D. Then C has has $\underline{C} \subseteq W \cup N$ and $(W \cap \underline{C}) \subseteq C^+$. Since N is an NBC set, the corresponding collection of vectors $\{v_i\}_{i \in N}$ is independent and we can assume that $W \cap \underline{C}$ is nonempty. Thus its Cone-Circuit relation has initial form $e_{\underline{C} \setminus W}$ dividing e_N, contradicting e_N being $\mathrm{in}_\prec(\mathcal{G})$-standard. $\square$

Combining the preceding proposition with Lemma 4.7 gives a proof of Theorem 4.1. For completeness, we now state two observations about oriented matroids, which were used in the preceding proof. The first concerns the geometric interpretation of covectors as faces of hyperplane arrangements and the second observation connects the non-existence of a covectors to the existence of a vector.

Observation 4.12. If $\mathcal{K} \cap X = \bigcap_{i \in W} H_i^+ \cap \bigcap_{j \in N} H_j^0 = \emptyset$ then there some choice of signs $(\varepsilon_j)_{j \in N}$ in $\{+,-\}^N$ such that

$$\mathcal{K} \cap X = \bigcap_{i \in W} H_i^+ \cap \bigcap_{j \in N} H_j^{\varepsilon_j} = \emptyset.$$

In particular, the signed set $F^\varepsilon = ((F^\varepsilon)^+, (F^\varepsilon)^-)$ with

$$F_i^\varepsilon = \begin{cases} + & \text{if } i \in W \\ \varepsilon_i & \text{if } i \in N. \end{cases}$$

is not a covector of the oriented matroid on ground set $E = W \cup N$.

Another way to phrase $X \cap \mathcal{K} = \emptyset$ is: there are no covectors F having $F_i = +$ for all $i \in W$ and $F_j = 0$ for all $j \in N$. With that in mind, the observation holds because if no such choice of signs $(\varepsilon_j)_{j \in N}$ existed, one would obtain a family of

[4] As mentioned in Section 2.2, there is always a choice of composition for which *every* circuit conforms to D but we only need one conformal circuit, see [9, Proposition 3.7.2].

covectors $\{F^\varepsilon\}_{\varepsilon\in\{+,-\}^N}$ to which one could repeatedly apply the elimination axiom V3 and reach such a covector F having $F_j = 0$ for $j \in N$.

Observation 4.13 ([66, Section 6.3]). From the definition of an oriented matroid dual, we know that every $F \in \mathfrak{D}^*$ is orthogonal to *every* vector $D \in \mathfrak{D}$. In particular if there is a signed set F on $[n]$ that is *not* in $\mathfrak{D}^*$, then there is some $D \in \mathfrak{D}$ such that $\{F_e \cdot D_e \mid e \in E\}$ contains exactly one of $+$ or $-$.

Remark 4.14. The crux of the preceding proof only uses statements about vectors and covectors valid for oriented matroids. Hence our results remain valid in that setting, as in the generalization by Gel'fand and Rybnikov [29] of the work by Gel'fand and Varchenko [61] to oriented matroids.

4.4 Proof of Theorem 4.4

In this section we prove Theorem 4.4, asserting that when one works over a field $\mathbb{F}$, the associated graded ring $\mathcal{V}_\mathbb{F}(\mathcal{A})$ is Koszul whenever $\mathcal{A}$ is a supersolvable arrangement. Some of the most well-studied hyperplane arrangements are supersolvable, and supersolvable arrangements are interesting, for example, because their Poincaré polynomial $\mathrm{Poin}(\mathcal{A}, t)$ factors into linear factors[5] [56, Corollary 4.9]. Koszulity, on the other hand, is also interesting from an algebraic perspective. Koszul algebras come equipped with a natural Koszul dual quadratic algebra $A^!$, and the relationship between $A, A^!$ has implications for the coefficients of the Hilbert series of A.

Let $\mathbb{F}$ be a field. Recall that $VG_\mathbb{F}(\mathcal{K})$ is the collection of maps $\{f : \mathcal{C}(\mathcal{K}) \to \mathbb{F}\}$ with pointwise addition and multiplication. Theorem 4.1 extends without modification to $VG_\mathbb{F}(\mathcal{K})$, and in fact some of the proofs are easier since $\mathcal{G}$ forms an honest Gröbner basis (see Remark 4.8).

[5] This does not hold for cones of supersolvable arrangements, see [22, Remark 5.6].

Before beginning the proof of Theorem 4.4, we remind the reader of some standard results relating to supersolvable lattices and Koszul algebras. For a more detailed reference on Koszul algebras, we point the reader toward [26]. Let R be a *commutative standard graded* $\mathbb{F}$*-algebra*, i.e. $R \cong \mathbb{F}[e_1, \ldots, e_n]/I$ where I is a homogeneous ideal and each e_i has degree exactly 1. Suppose

$$F_\bullet : \cdots \xrightarrow{\varphi_3} R^{\beta_2} \xrightarrow{\varphi_2} R^{\beta_1} \xrightarrow{\varphi_1} R \longrightarrow \mathbb{F}$$

is a *minimal free resolution* of the R-module $\mathbb{F} = R/R_+$ where R_+ is the maximal homogeneous ideal, consisting of all elements of positive degree. For details on free resolutions, see [23]. We say R is *Koszul* if the nonzero entries of each φ_i matrix are homogeneous of degree 1. Say that an ideal is *monomial* if it is generated by monomials. A monomial ideal is $\mathcal{G}$*-quadratic* if it has a Gröbner basis of monomials of degree two. It is well-known (see [18], [23, Chapter 15], [26, Section 4], for example) that:

Proposition 4.15. *If I a homogeneous ideal in $\mathbb{F}[e_1, \ldots, e_n]$ is generated by a Gröbner basis $\mathcal{G}$ consisting of quadratic elements for some monomial order $\prec$, then $R = \mathbb{F}[e_1, \ldots, e_n]/I$ is Koszul.*

We are now prepared to define a supersolvable arrangement.

Definition 4.16 ([54, Definition 1.1], [56, Definition 4.13])**.** A lattice L is *supersolvable* if there exists a maximal chain Δ satisfying: for every chain K of L, the sublattice generated by Δ and K is distributive. An arrangement $\mathcal{A}$ is *supersolvable* if $\mathcal{L}(\mathcal{A})$ is supersolvable.

The following result and its proof are analogous to a result of Peeva [41, Thm. 4.3].

Theorem 4.4. If $\mathcal{A}$ is a supersolvable arrangement, then $\mathcal{V}_{\mathbb{F}}(\mathcal{A})$ is Koszul.

Proof. A theorem of Björner and Ziegler [14, Theorem 2.8] tells us that when $\mathcal{A}$ is a supersolvable arrangement, one can choose a linear ordering of the hyperplanes $H_1, \ldots, H_n$ such that every broken circuit $\underline{C} \setminus \{i_0\}$ contains some broken circuit of size two. Choose $\prec$ a degree monomial order on $\mathbb{Z}[e_1, \ldots, e_n]$ which restricts to the same linear order on the variables $e_1 \prec \cdots \prec e_n$.

We wish to use the presentation $\mathcal{V}_{\mathbb{F}}(\mathcal{A}) = \mathbb{F}[e_1, \ldots, e_n]/I$ where $I = (\mathrm{in}_{\deg}(\mathcal{G}))$ that comes from Theorem 4.1. From Remark 4.8, the generators $\mathrm{in}_{\deg}(\mathcal{G})$ form a Gröbner basis for $I \subseteq \mathbb{F}[e_1, \ldots, e_n]$. Because $\mathcal{A}$ is a full arrangement, not a cone, $\mathrm{in}_{\deg}(\mathcal{G})$ will contain only $\mathrm{in}_{\deg}(g)$ for $\mathsf{Idempotent}$ and $\mathsf{Circuit}$ relations g. The $\mathsf{Idempotent}$ relations correspond to generators $\mathrm{in}_{\deg}(g) = e_i^2$ that are all quadratic.

Each $\mathsf{Circuit}$ relation corresponds to a generator $\mathrm{in}_{\deg}(g)$ which may not be quadratic: its degree is the size of the broken circuit $C \setminus \{i_0\}$, with $\mathrm{in}_{\prec}(g) = e_{C\setminus\{i_0\}}$. Since each is a squarefree monomial, it suffices to consider the monomials who indexing set is minimal under inclusion. The Björner-Ziegler result [14, Theorem 2.8] implies that the minimal (under inclusion) broken circuits all have cardinality 2. From Proposition 4.15, it follows that $\mathcal{V}_{\mathbb{F}}(\mathcal{A})$ is Koszul. $\square$

One might ask if Theorem 4.4 has a cone analogue. Sadly there are cones $\mathcal{K}$ of supersolvable arrangements whose $\mathcal{V}_{\mathbb{F}}(\mathcal{K})$ are not Koszul as we demonstrate by example. A particularly well-studied family of supersolvable arrangements are the *Type A reflection arrangements* or *braid arrangements*; see [56, Cor. 4.10, Example 4.11(c)]. The braid arrangement A_{n-1}, consists of the $\binom{n}{2}$ hyperplanes

$$H_{ij} = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid x_i - x_j = 0\}$$

for each pair $\{i, j\}$.

We wish to exhibit a cone $\mathcal{K}$ inside a braid arrangement for which $\mathcal{V}(\mathcal{K})$ is not Koszul. One way to prove something is *not* Koszul uses the following.

Theorem 4.17 ([27, Section 4]). *Let A be a Koszul algebra. Then there is another algebra $A^!$, the quadratic dual of A, whose Hilbert series is* $\mathrm{Hilb}(A^!, t) = 1/\mathrm{Hilb}(A, -t)$. *In particular, if A is Koszul, then* $1/\mathrm{Hilb}(A, -t)$ *has positive coefficients when considered as a power series in* $\mathbb{Z}[[t]]$.

Example 4.18. The cone of A_5 given by

$$\mathcal{K} = \{\mathbf{x} \in \mathbb{R}^6 \mid x_1 \leq x_2,\ x_3 \leq x_4,\ x_5 \leq x_6\}$$

does not yield a Koszul $\mathcal{V}_{\mathbb{F}}(\mathcal{K})$. The Hilbert series of $\mathcal{V}_{\mathbb{F}}(\mathcal{K})$ is

$$\mathrm{Hilb}(\mathcal{V}_{\mathbb{F}}(\mathcal{K}), t) = 1 + 12t + 43t^3 + 30t^3 + 4t^4.$$

The first few terms of $1/\mathrm{Hilb}(\mathcal{V}_{\mathbb{F}}(\mathcal{K}), -t)$ are

$$\begin{aligned}\frac{1}{1 - 12t + 43t^3 - 30t^3 + 4t^4} &= 1 + 12t + 101t^2 + 725t^3 + 4725t^4 + 28464t^5 + 159769t^6 \\ &+ 832122t^7 + 3950417t^8 + 16302972t^9 + 50092317t^{10} \\ &+ 15264030t^{11} - 1497513779t^{12} + \cdots\end{aligned}$$

The coefficient of t^{12} is negative, meaning that $1/\mathrm{Hilb}(\mathcal{V}_{\mathbb{F}}(\mathcal{K}), -t)$ is not the Hilbert series of a ring.

Given this counterexample, one might ask if there are certain families of cones $\mathcal{K}$ for which $\mathcal{V}_{\mathbb{F}}(\mathcal{K})$ is Koszul.

Question 4.19. Is there some simple, combinatorial condition on $\mathcal{L}^{\mathrm{int}}(\mathcal{K})$ (for a cone $\mathcal{K}$) in the spirit of supersolvability for the full lattice $\mathcal{L}(\mathcal{A})$ of the arrangement, that implies Koszulity of $\mathcal{V}_{\mathbb{F}}(\mathcal{K})$?

Even in the arrangement case, the connection between supersolvable arrangements and Koszul $\mathcal{V}_{\mathbb{F}}(\mathcal{A})$ remains opaque. Fröberg tells us that the converse to Proposition 4.15 is false in general [27, Note on p.39], but one might ask: if $\mathcal{V}_{\mathbb{F}}(\mathcal{A})$ is Koszul, is $\mathcal{A}$ supersolvable? The analogous question is famously open for Orlik-Solomon algebras [41, Example 4.5], [63, Example 6.22].

5 Remarks and Questions

There are a number of other interesting questions regarding hyperplane arrangements, which I have not yet investigated. Some of these questions were discussed in earlier chapters of this thesis. In this chapter, we collect some of those questions. This chapter is divided into three sections. The first section consists of some questions from previous chapters, the second section concerns a simplicial complex called the *nbc-complex of a cone* (defined in that section), and the third section concerns a family of affine arrangements called *Shi arrangements*.

5.1 Some Questions from Previous Chapters

In Chapter 3, we saw that the Poincaré polynomial $\mathrm{Poin}(P,t)$ for the antichain poset $P = P_{(1^n)} = \mathsf{Antichain}_n$ has *only real roots* (see equation (3.6)). One might wonder whether this holds for some more general class of posets. It does not hold for all posets, and not even for all disjoint unions of chains $P_{\overline{a}}$, since

$$\mathrm{Poin}(P_{(2,2,2)}, t) = 1 + 12t + 43t^2 + 30t^3 + 4t^4$$

has a pair of non-real complex roots. Incidentally, this poset's cone defines a Varchneko-Gel'fand ring whose associated graded is not Koszul, see Example 4.18.

Real-rootedness can even fail even for rectangular Ferrers posets, e.g., $\lambda = (3, 3, 3)$

has $\mathrm{Poin}(P(\lambda), t) = \mathrm{Poin}(3 \times 3, t) = 1 + 9t + 19t^2 + 11t^3 + 2t^4$ in Table 3.1, with two non-real complex roots.

On the other hand, computations show that $\mathrm{Poin}(P, t)$ *is* real-rooted for all posets P of width two having at most 9 elements, so we ask the following question.

Question 3.61. Is $\mathrm{Poin}(P, t)$ real-rooted when the poset P has width two?

On a more algebraic note, in Chapter 4, we use Gröbner bases to show that when a hyperplane arrangement is *supersolvable*, a certain associated graded ring of the Varchenko-Gel'fand ring, denoted $\mathcal{V}_{\mathbb{F}}(\mathcal{A})$, is supersolvable. One might hope that this result extends to cones of supersolvable arrangements. Unfortunately it does not, and in Section 4.4, we provide a cone $\mathcal{K}$ of the Type A reflection arrangement (a supersolvable arrangement) whose associated graded ring $\mathcal{V}_{\mathbb{F}}(\mathcal{K})$ is not Koszul. This is somewhat disappointing but leads to an interesting open question.

Question 4.19. Is there some simple, combinatorial condition on $\mathcal{L}^{\mathrm{int}}(\mathcal{K})$ (for a cone $\mathcal{K}$) in the spirit of supersolvability for the full lattice $\mathcal{L}(\mathcal{A})$ of the arrangement, that implies Koszulity of $\mathcal{V}_{\mathbb{F}}(\mathcal{K})$?

5.2 The (Sometimes) Non-Shellability of the Broken Circuit Complex of a Cone

An *(abstract) simplicial complex* Δ on ground set E is a finite collection of subsets of E satisfying the following two conditions

- $\emptyset \in \Delta$, and
- If $F \in \Delta$ and $G \subseteq F$, then $G \in \Delta$.

We call the elements of Δ the *faces* of the simplicial complex. The dimension of a face F of Δ is $\dim F = \#F - 1$. A complex is *pure* if the facets all have the same dimension. For each face F of Δ, we define the *simplex* of F by

$$\overline{F} = \{G \mid G \subseteq F\}$$

Definition 5.1 ([13, Definition 2.1])**.** Let Δ be a simplicial complex. We say that Δ is *(nonpure) shellable* if its facets can be arranged in a linear order $F_1, F_2, \ldots, F_t$ in such a way that the subcomplex

$$\left(\bigcup_{i=1}^{k-1} \overline{F}_i \right) \cap \overline{F}_k$$

is pure and has dimension $\dim(F_k) - 1$ for each $k = 1, \ldots, t$.

Intuitively, a (possibly nonpure) complex Δ is shellable if we can build Δ by successively gluing on facets in a well-behaved way. In the case when Δ is a pure simplicial complex, the following definition agrees with the definition of a shellable simplicial complex. In particular, if a complex is pure and not shellable, then it is not nonpure shellable.

5.2.1 The nbc-complex of a cone

Recall that a no broken circuit set of a matroid M on ground set E, is a subset $N \subseteq E$ such that N does not contain any broken circuit sets of M. The collection of all such sets, $NBC(M)$, naturally forms a simplicial complex, as no subset N' of a no broken circuit set N can contain a broken circuit; see [62, Chapter 7] for details. We call this complex the *nbc-complex*[1] of a matroid M on E and denote it by $BC(M)$, where E is implicitly imbued with some linear order $e_1 < \cdots < e_n$. The following classical result will be of particular interest for us:

[1] In the case of the full arrangement, this sometimes also called the bc-complex.

Theorem 5.2 ([62, Theorem 7.4.3]). *If M is a loopless matroid, then $BC(M)$ is pure and shellable.*

Using the definition of $\mathcal{K}$-nbc set introduced earlier, we can naturally define the *broken circuit complex of a cone* $\mathcal{K}$ to be the complex of $\mathcal{K}$-nbc. We denote the broken circuit complex of $\mathcal{K}$ by $BC(\mathcal{K})$. One might hope that Theorem 5.2 extends to cones, but it does not. Recall, for example, that the $(4-1)$st braid arrangement A_3 has hyperplanes $H_{ij} = \{x \in \mathbf{R}^4 \mid x_i - x_j = 0\}$ for all pairs $i \neq j$ in $\{1, 2, 3, 4\}$. Consider the cone $\mathcal{K} = H_{34}^+ \cap H_{12}^+$ of this arrangement. Under the linear order

$$H_{34} < H_{23} < H_{24} < H_{12} < H_{13} < H_{14},$$

the nbc-complex of $\mathcal{K}$ has two facets $\{23, 14\}$ and $\{24, 13\}$. It is a pure complex, but not shellable. There are also cones $\mathcal{K}$ for which $BC(\mathcal{K})$ is not pure, but is shellable. In A_5, for example, the cone given by

$$\mathcal{K} = H_{12} \cap H_{24} \cap H_{56}$$

has a nonpure nbc-complex with facets $\{1\}, \{5\}, \{2, 4\}$. This complex is shellable with shelling order $F_1 = \{2, 4\}$, $F_2 = \{1\}$, $F_3 = \{5\}$. Given these examples, one might ask the following question:

Question 5.3. Is there a simple property on $\mathcal{K}$ guaranteeing that $BC(\mathcal{K})$ is (nonpure) shellable?

5.3 Dominant Cones of Shi Arrangements

This section concerns an *irreducible crystallographic root system* Φ with choice of *positive roots* Φ^+ and *Weyl group* W. We will use the *Cartan type* classification of irreducible crystallographic root systems and have included some relevant data

W	Roots Φ	Degrees
A_{n-1}	$\pm(e_i - e_j)$ for $1 \leq i < j \leq n$	$2, 3, 4, \ldots n+1$
B_n	$\pm e_i \pm e_j$ for $1 \leq i < j \leq n$ and $\pm e_i$ for $1 \leq i \leq n$	$2, 4, 6, \ldots 2n$
C_n	$\pm e_i \pm e_j$ for $1 \leq i < j \leq n$, and $\pm 2e_i$ for $1 \leq i \leq n$	$2, 4, 6, \ldots 2n$
D_n	$\pm e_i \pm e_j$ for $1 \leq i < j \leq n$	$n, 2, 4, 6, \ldots 2n-2$
E_6	$\pm e_i \pm e_j$ for $1 \leq i < j \leq 5$ and $\pm\frac{1}{2}(e_8 - e_7 - e_6 + \sum_{i=1}^{5} \pm e_i)$ (odd number of - signs)	$2, 5, 6, 8, 9, 12$
E_7	$\pm e_i \pm e_j$ for $1 \leq i < j \leq 6$ $\pm(e_7 - e_8)$ $\pm\frac{1}{2}\left(e_7 - e_8 + \sum_{i=1}^{6} \pm e_i\right)$ (odd number of - signs)	$2, 6, 8, 12, 14, 18$
E_8	$\pm e_i \pm e_j$ for $1 \leq i < j \leq 8$ and $\frac{1}{2}\left(\pm e_1 \pm \cdots \pm e_8\right)$ (even number of + signs)	$2, 6, 8, 12, 14, 18, 20, 24, 30$
F_4	$\pm e_i \pm e_j$ for $1 \leq i < j \leq 4$, and $\pm e_i$ for $1 \leq i \leq 4$, $\frac{1}{2}\left(\pm e_1 \pm e_2 \pm \cdots \pm e_4\right)$.	$2, 6, 8, 12$
G_2	$\pm(e_i - e_j)$ for $1 \leq i < j \leq 3$, and $\pm(2e_i - e_j - e_k)$ for $\{i, j, k\} \in \{1, 2, 3\}$	$2, 6$

Table 5.1: Root systems for Weyl groups by their Cartan type. For more detailed descriptions of these root systems, see [34, Section 2.10].

for this classification in Table 5.1. Whenever possible, we will identify specific references, but we recommend Humphreys [34] as a general resource concerning reflection groups and Coxeter groups.

For a root system Φ and choice of positive roots Φ^+, the *Shi arrangement* of Φ is

$$\mathsf{Shi}_\Phi := \{H_{\alpha,t} \mid \alpha \in \Phi^+, t = 0, 1\}$$

where $H_{\alpha,t} := \{\mathbf{x} \in V \mid \mathbf{x} \cdot \alpha = t\}$ [53]. The choice of positive roots gives an orientation of Shi_Φ by

$$H^+_{\alpha,t} := \{\mathbf{x} \in V \mid \mathbf{x} \cdot \alpha > 0\} \qquad \text{for } \alpha \in \Phi^+,\ t = 0, 1.$$

The *dominant cone* $\mathcal{K}^+$ of Shi_Φ is the intersection of positive haflspaces

$$\mathcal{K}^+ := \bigcap_{\alpha \in \Phi^+} H^+_{\alpha,0}.$$

The *Coxeter-Catalan number* of W is

$$\mathsf{Cat}(W) := \prod_{i=1}^{n} \frac{d_i + h}{d_i}$$

where $d_1, \ldots, d_n$ are the degrees of W and $h := d_n$ is the *Coxeter number* of W [4, Equation 1.1]. In Type A, this number reduces to the usual definition of a Catalan number

$$\mathsf{Cat}^{(A_{n-1})} = \prod_{k=1}^{n} \frac{k + n}{k}.$$

More generally, when we specialize W, we obtain more specialized formulas, see Petersen [42, Table 12.2] for a comprehensive list.

The following theorem was conjectured by Postnikov [46, Remark 2] and proved (independently) by Athanasiadis [5, Corollary 1.3] and Cellini-Papi [17, Theorem 1 & Section 4.2].

Theorem 5.4. *Let Φ be an irreducible crystallographic root system and W its Weyl group. The number of chambers of the dominant cone of Shi_Φ is $\mathsf{Cat}(W)$.*

Applying Zaslavsky's theorem gives the following corollary.

Corollary 5.5. *Let Φ be an irreducible crystallographic root system and W its Weyl group. Then the Whitney numbers $c_k(\mathcal{K}^+)$ of the dominant cone of Shi_Φ sum to $\mathsf{Cat}(W)$, i.e.*

$$\mathsf{Cat}(W) = \sum_{k \geq 0} c_k(\mathcal{K}^+)\,.$$

Given the topic of this thesis, we ask: what are these Whitney numbers? Preliminary data, collected using SageMath, suggests that the Whitney numbers agree with the *W-Narayana numbers*, which we define now.

Let Φ be a crystallographic root system and Φ^+ a choice of positive roots. Then the *root poset* of Φ is a poset on ground set Φ^+ with the order relations $\alpha \prec \beta$ if $\beta - \alpha$ is a linear combination of positive roots with nonnegative coefficients. Let W be the Weyl group of Φ. Then the kth W-Narayana number is

$$\mathsf{Nar}^{(W)}(k) := \# \left\{ A \subseteq \Phi^+ \,\middle|\, \begin{array}{l} A \text{ is an antichain of the} \\ \text{root poset and } \#A = k \end{array} \right\}.$$

Just as in the Catalan case, the W-Narayana coincide with the usual Narayana numbers in Type A and there are more specialized formulas when we specialize to specific W, see Armstrong [4, Figure 3.4] for a comprehensive list.

Conjecture 5.6. *The Whitney numbers of the dominant cone of Shi_W are the W-Narayana numbers, i.e. $c_k(\mathcal{K}^+) = \mathsf{Nar}^{(W)}(k)$.*

We can see by direct computation that this conjecture is true in rank 2 (see Section 5.3.1 for a stronger statement in rank 2). Using SageMath, we verified for Type A up to rank $n = 6$, Type B up to $n = 4$, D_4, and F_4.

5.3.1 Extended Shi arrangements and Fuss-Narayana Numbers

Given the previous discussion, it is natural to consider the m-extended Shi arrangements. For $t \in \mathbb{Z}$, define $H_{\alpha,t} := \{\mathbf{x} \in V \mid \mathbf{x} \cdot \alpha = 1\}$. The m-extended Shi arrangement associated to a crystallographic root system Φ is

$$\mathsf{Shi}_{\Phi}^{(m)} := \{H_{\alpha,t} \mid \alpha \in \Phi^+, t = 0, \ldots, m\}.$$

In [5, Corollary 1.3], Athanasiadas proved that the number of chambers in the dominant cone of the m-extended Shi arrangements is the *Fuss-Catalan number* (= Fuß-Catalan)

$$\mathsf{Cat}^{(m)}(W) := \prod_{i=1}^{n} \frac{d_i + mh}{d_i}.$$

We ask: are the Whitney numbers of the dominant cone of the m-extended Shi arrangement given by the *Fuss-Narayana numbers* (given in Armstrong [4, Figure 3.4])? In rank 2, the only Weyl groups are A_2, B_2, and G_2. It is easy to compute the Whitney numbers of the dominant cones of the m-extended Shi arrangements for A_2, B_2, and G_2 for all m. The Whitney numbers are

$$\begin{aligned} c_0(\mathcal{K}^+) &= 1 \\ c_1(\mathcal{K}^+) &= \#(\Phi^+) \cdot m \\ c_2(\mathcal{K}^+) &= \mathsf{Cat}^{(m)}(W) - c_0(\mathcal{K}^+) - c_1(\mathcal{K}^+) . \end{aligned}$$

This pattern is precisely the Fuss-Narayana distribution. Unfortunately, this observation does not continue in higher ranks for $m \geq 2$. In B_3 for example, the dominant cone of the 3-extended Shi arrangement has Whitney sequence $(1, 18, 47, 18)$ but the Fuss-Narayana numbers are $(1, 18, 45, 20)$. Thus we propose the following open problem:

Open Problem 5.7. *Give a combinatorial description for the Whitney numbers of the m-extended Shi arrangement for ranks ≥ 3 and $m \geq 2$.*

Bibliography

[1] William W. Adams and Philippe Loustaunau. *An introduction to Gröbner bases*, volume 3 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1994.

[2] Marcelo Aguiar and Swapneel Mahajan. *Topics in Hyperplane Arrangements*, volume 226 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2017.

[3] G. L. Alexanderson and John E. Wetzel. Dissections of a tetrahedron. *J. Combinatorial Theory Ser. B*, 11:58–66, 1971.

[4] Drew Armstrong. Generalized noncrossing partitions and combinatorics of Coxeter groups. *Mem. Amer. Math. Soc.*, 202(949):x+159, 2009.

[5] Christos A. Athanasiadis. Generalized Catalan numbers, Weyl groups and arrangements of hyperplanes. *Bull. London Math. Soc.*, 36(3):294–302, 2004.

[6] M. F. Atiyah and I. G. Macdonald. *Introduction to Commutative Algebra*. Addison-Wesley Series in Mathematics. Westview Press, Boulder, CO, 2016.

[7] Hans-Jürgen Bandelt, Victor Chepoi, and Kolja Knauer. COMs: Complexes of Oriented matroids. *J. Combin. Theory Ser. A*, 156:195–237, 2018.

[8] Matthias Beck and Raman Sanyal. *Combinatorial Reciprocity Theorems*, volume 195 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2018. An invitation to enumerative geometric combinatorics.

[9] Anders Björner, Michel Las Vergnas, Bernd Sturmfels, Neil White, and Günter M. Ziegler. *Oriented Matroids*, volume 46 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, second edition, 1999.

[10] Anders Björner and Michelle L. Wachs. Generalized Quotients in Coxeter Groups. *Trans. Amer. Math. Soc.*, 308(1):1–37, 1988.

[11] Anders Björner and Michelle L. Wachs. q-hook Length Formulas for Forests. *J. Combin. Theory Ser. A*, 52(2):165–187, 1989.

[12] Anders Björner and Michelle L. Wachs. Permutation Statistics and Linear Extensions of Posets. *J. Combin. Theory Ser. A*, 58(1):85–114, 1991.

[13] Anders Björner and Michelle L. Wachs. Shellable Nonpure Complexes and Posets. I. *Trans. Amer. Math. Soc.*, 348(4):1299–1327, 1996.

[14] Anders Björner and Günter M. Ziegler. Broken Circuit Complexes: Factorizations and Generalizations. *J. Combin. Theory Ser. B*, 51(1):96–126, 1991.

[15] Petter Brändén. q-Narayana Numbers and the Flag h-vector of $J(2 \times \mathbf{n})$. *Discrete Math.*, 281(1-3):67–81, 2004.

[16] Kenneth S. Brown. Semigroups, Rings, and Markov Chains. *J. Theoret. Probab.*, 13(3):871–938, 2000.

[17] Paola Cellini and Paolo Papi. Ad-nilpotent ideals of a Borel subalgebra. II. volume 258, pages 112–121. 2002. Special issue in celebration of Claudio Procesi's 60th birthday.

[18] Aldo Conca. Gröbner Bases for Spaces of Quadrics of Low Codimension. *Adv. in Appl. Math.*, 24(2):111–124, 2000.

[19] R. Cordovil. A Commutative Algebra for Oriented Matroids. *Discrete Comput. Geom.*, 27:73–84, 2002.

[20] David A. Cox, John Little, and Donal O'Shea. *Ideals, Varieties, and Algorithms*. Undergraduate Texts in Mathematics. Springer, fourth edition, 2015. An introduction to computational algebraic geometry and commutative algebra.

[21] Galen Dorpalen-Barry. The Varchenko-Gel'fand Ring of a Cone. *arXiv 2104.02740*, pages 1–16, 2021.

[22] Galen Dorpalen-Barry, Jang Soo Kim, and Victor Reiner. Whitney Numbers for Poset Cones. *Order*, 2021.

[23] David Eisenbud. *Commutative Algebra*, volume 150 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995. With a view toward algebraic geometry.

[24] Dominique Foata. *Etude Algébrique de Certains Problèmes d'analyse Combinatoire et du Calcul des Probabilités*. PhD thesis, Publ. Inst. Statist. Univ. Paris, 1965.

[25] Dominique Foata and Pierre Cartier. *Problèmes Combinatoires de Commutation et Réarrangements.* Springer Verlag, Lecture Notes in Mathematics, 1969.

[26] R. Fröberg. *Koszul Algebras*, volume 205 of *Lecture Notes in Pure and Appl. Math.* Dekker, New York, 1999.

[27] Ralph Fröberg. Determination of a Class of Poincaré Series. *Math. Scand.*, 37(1):29–39, 1975.

[28] Stavros Garoufalidis, Thang T.Q. Lê, and Doron Zeilberger. The Quantum MacMahon Master Theorem. *J. Algebra*, March 2003.

[29] I. M. Gel'fand and G. L. Rybnikov. Algebraic and Topological Invariants of Oriented Matroids. *Dokl. Akad. Nauk SSSR*, 307(4):791–795, 1989.

[30] Regina Gente. *The Varchenko Matrix for Cones.* PhD thesis, Universität Marburg, 2013.

[31] Ira M. Gessel. Multipartite P-partitions and Inner Products of Skew Schur functions. 34:289–317, 1984.

[32] Jacob E. Goodman, Joseph O'Rourke, and Csaba D. Tóth, editors. *Handbook of Discrete and Computational Geometry.* Discrete Mathematics and its Applications (Boca Raton). CRC Press, Boca Raton, FL, 2018.

[33] P. Gordan. Ueber die Auflösung Linearer Gleichungen mit reellen Coefficienten. *Math. Ann.*, 6(1):23–28, 1873.

[34] James E. Humphreys. *Reflection Groups and Coxeter Groups*, volume 29 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, 1990.

[35] Donald Ervin Knuth. *The Art of Computer Programming*. Addison-Wesley, 2015.

[36] Matjaz Konvalinka and Igor Pak. Non-commutative Extensions of MacMahon's Master Theorem. *Advances in Mathematics*, 2007.

[37] G. I. Lehrer and Louis Solomon. On the Action of the Symmetric Group on the Cohomology of the Complement of its Reflecting Hyperplanes. *J. Algebra*, 104(2):410–424, 1986.

[38] Daniel Moseley. Equivariant Cohomology and the Varchenko-Gelfand Filtration. *J. Algebra*, 472:95–114, 2017.

[39] Peter Orlik and Louis Solomon. Unitary Reflection Groups and Cohomology. *Invent. Math.*, 59(1):77–94, 1980.

[40] Peter Orlik and Hiroaki Terao. *Arrangements of Hyperplanes*, volume 300 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1992.

[41] Irena Peeva. Hyperplane Arrangements and Linear Strands in Resolutions. *Trans. Amer. Math. Soc.*, 355(2):609–618, 2003.

[42] T. Kyle Petersen. *Eulerian numbers*. Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks]. Birkhäuser/Springer, New York, 2015. With a foreword by Richard Stanley.

[43] Alexander Postnikov, Victor Reiner, and Lauren Williams. Faces of Generalized Permutohedra. *Documenta Mathematica*, 13:207–273, 2008.

[44] Nicholas Proudfoot. The Equivariant Orlik-Solomon Algebra. *J. Algebra*, 305(2):1186–1196, 2006.

[45] Victor Reiner. Quotients of Coxeter Complexes and P-partitions. *Mem. Amer. Math. Soc.*, 95(460):vi+134, 1992.

[46] Victor Reiner. Non-crossing partitions for classical reflection groups. *Discrete Math.*, 177(1-3):195–222, 1997.

[47] Victor Reiner, Franco Saliola, and Volkmar Welker. Spectra of Symmetrized Shuffling Operators. *Mem. Amer. Math. Soc.*, 228(1072):vi+109, 2014.

[48] Victor Reiner and Volkmar Welker. On the Charney-Davis and Neggers-Stanley conjectures. *Journal of Combinatorial Theory, Series A*, 109(2):247–280, February 2005.

[49] Alfréd Rényi. Théorie des éléments saillants d'une suite d'observations. *Ann. Fac. Sci. Univ. Clermont-Ferrand*, 8:7–13, 1962.

[50] Gian-Carlo Rota. On the Foundations of Combinatorial Theory. I. Theory of Möbius functions. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 2:340–368 (1964), 1964.

[51] Bruce E. Sagan. A Generalization of Rota's NBC theorem. *Adv. Math.*, 111(2):195–207, 1995.

[52] Kyoji Saito. Principal Γ-cone for a Tree. *Adv. Math.*, 212(2):645–668, 2007.

[53] Jian Yi Shi. Alcoves corresponding to an affine Weyl group. *J. London Math. Soc. (2)*, 35(1):42–55, 1987.

[54] Richard Stanley. Supersolvable Lattices. *Algebra Universalis*, 2:197–217, 1972.

[55] Richard Stanley. *Enumerative Combinatorics, Volume 2.* Cambridge University Press, Cambridge, UK, 1999.

[56] Richard Stanley. An Introduction to Hyperplane Arrangements. 13:389–496, 2007.

[57] Richard Stanley. *Enumerative Combinatorics.* Cambridge University Press, New York, NY, USA, 2 edition, 2012.

[58] Robert Steinberg. Differential Equations Invariant under Finite Reflection Groups. *Trans. Amer. Math. Soc.*, 112:392–400, 1964.

[59] John R. Stembridge. Coxeter Cones and their h-vectors. *Adv. Math.*, 217(5):1935–1961, 2008.

[60] Bernd Sturmfels. *Gröbner Bases and Convex Polytopes.* University Lecture Series, Volume 8. American Mathematical Society, Providence, RI, 1996.

[61] A. N. Varchenko and I. M. Gel'fand. Heaviside Functions of a Configuration of Hyperplanes. *Funktsional. Anal. i Prilozhen.*, 21(4):1–18, 96, 1987.

[62] Neil White, editor. *Matroid Applications.* Encyclopedia of Mathematics and its Applications, Volume 40. Cambridge University Press, Cambridge, 1992.

[63] S. Yuzvinsky. Orlik-solomon algebras in algebra and topology. *Russian Mathematical Surveys*, 56(2):293–364, 2001.

[64] Thomas Zaslavsky. Facing up to Arrangements: Face-Count Formulas for Partitions of Space by Hyperplanes. *Mem. Amer. Math. Soc.*, 1(1):vii+102, 1975.

[65] Thomas Zaslavsky. A Combinatorial Analysis of Topological Dissections. *Advances in Math.*, 25(3):267–285, 1977.

[66] Günter M. Ziegler. *Lectures on Polytopes.* Graduate Texts in Mathematics, Volume 152. Springer-Verlag, New York, 1995.