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BILINEAR OPTIMAL CONTROL OF A KIRCHHOFF PLATE

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Abstract. We consider the problem of optimal control of a Kirchhoff plate. A bilinear control is used as a force to make the plate close to a desired profile taking into the account, a quadratic cost of control. We prove the existence of an optimal control and characterize it uniquely through the solution of an optimality system.

Key words. Kirchhoff plate, optimal control, bilinear control

AMS(MOS) subject classifications. 49A22, 35K35

1. Introduction. We consider the problem of controlling the solution of a Kirchhoff plate equation. The equation with appropriate boundary conditions describes the motion of a thin plate which is clamped along one portion of its boundary and has free vibrations on the other portion of its boundary. Our control acts as a bilinear force.

Given control

$$h \in U_M = \{h \in L^\infty(Q) | -M \leq h \leq M\},$$

the "displacement" solution $w = w(h)$ of our state equation, satisfies

$$(1.1) \qquad \begin{cases} w_{tt} + \Delta^2 w = h(x,y,t)w \qquad \text{on } \; Q = \Omega \times (0,T) \\ w(x,y,0) = w_0(x,y), w_t(x,y,0) = w_1(x,y) \quad \text{when } t = 0 \\ w = \frac{\partial w}{\partial \nu} = 0 \text{ on } \Sigma_0 = \Gamma_0 \times (0,T) \\ \left.\begin{array}{l} \Delta w + (1-\mu)B_1 w = 0 \\ \frac{\partial \Delta w}{\partial \nu} + (1-\mu)B_2 w = 0 \end{array}\right\} \text{on } \Sigma_1 = \Gamma_1 \times (0,T) \end{cases}$$

where $\Omega \subset \mathbf{R}^2$ with C^2 boundary, $\partial\Omega = \Gamma_0 \cup \Gamma_1, \Gamma_0 \neq \emptyset$, $\vec{\nu} = \langle n_1, n_2 \rangle$ is the outward unit normal vector on $\partial\Omega$, and

$$\begin{aligned} B_1 w &= 2n_1 n_2 w_{xy} - n_1^2 w_{yy} - n_2^2 w_{xx} \\ B_2 w &= \frac{\partial}{\partial \tau}[(n_1^2 - n_2^2)w_{xy} + n_1 n_2 (w_{yy} - w_{xx})]. \end{aligned}$$

The direction τ in $B_2 w$ is the tangential direction along Γ_1. The plate is clamped along Γ_0 and has free vibrations along Γ_1. The constant $\mu, 0 < \mu < \frac{1}{2}$, represents Poisson's ratio.

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We take as our cost functional

$$J(h) = \frac{1}{2}\left(\alpha \int_Q (w-z)^2 dQ + \beta \int_Q h^2 dQ\right), \tag{1.2}$$

where z is the desired profile for the plate and the quadratic term in h represents the cost of implementing the control. We seek to minimize the cost functional, i.e., find optimal control $h^* \in U_M$ such that

$$J(h^*) = \min_{h \in U_M} J(h).$$

The goal of this paper is to characterize the unique optimal control in terms of the unique solution of the optimality system. The optimality system consists of the state equation coupled with an adjoint equation.

In section 2, we show the well-posedness of our state problem in an appropriate solution space. Then we show the existence of an optimal control by a minimizing sequence argument. In section 3, the optimality system is derived by differentiating the cost functional with respect to the control. We also show how the solution depends in a differentiable way on the control. Then for sufficiently small time T, we prove uniqueness of the optimality system, which characterizes the unique optimal control. The uniqueness proof is unusual due to the opposite time orientations of the two equations.

For a general introduction to optimal control and partial differential equations, associated adjoint equations, and representation of the optimal control in terms of the solution of the optimality system, see the fundamental book by Lions [17]. For detailed background on plate models, see the books by Lagnese and Lions [10] and Lagnese [9]. Much work has been done in the stabilization and exact controllability of thin plates. The results predominantly fall into three classes: controllability results for linear plates [6,12,13,14] and nonlinear plates [8], uniform stability for linear plates using boundary feedback control [7,13,14,19], and exponential stability of nonlinear plates (such as von Kármán plate models) using boundary controls [4,5,20].

The problem here is of a different nature than those results. We seek a bilinear optimal control which acts as a coefficient of the state variable. Optimal control problems for plate equations in the form

$$y' = Ay + Bu$$

with quadratic cost functionals have been treated by use of Riccati equations. See the book by Lasiecka and Triggiani [11] for a complete treatment. Our work concerning bilinear control is new for plate equations and does not fit into the Riccati framework.

In a series of three papers, Ball, Marsden and Slemrod [1,2,3], treated feedback stabilization and controllability using bilinear control for beam and wave equations. Stojanovic [21, 22] treated optimal control of elliptic and parabolic equations with a bilinear "damping" control. See Lenhart and Bhat [15] and Lenhart and Wilson [16] for other types of bilinear control on parabolic equations. We plan in the future to investigate further generalizations in nonlinear control.

2. Well-posedness of the state equation and existence of the optimal control. We begin by proving existence, uniqueness, and regularity results for the state equation (1.1). These results will provide the *a priori* estimates needed to prove the existence of an optimal control.

To define our notion of weak solution, we first define the following Hilbert spaces:

$$H^2_{\Gamma_0}(\Omega) = \{w \in H^2(\Omega) : \quad w = \frac{\partial w}{\partial \nu} = 0 \text{ on } \Gamma_0\}$$

and

$$\mathcal{H} = H^2_{\Gamma_0}(\Omega) \times L^2(\Omega).$$

The bilinear form

$$a(w,v) = \int_\Omega \{\Delta w \Delta v + (1-\mu)[2w_{xy}v_{xy} - w_{xx}v_{yy} - w_{yy}v_{xx}]\}\, d\Omega$$

induces a norm on $H^2_{\Gamma_0}(\Omega)$ given by $a(w,w)$, which is equivalent to the usual norm on $H^2_{\Gamma_0}(\Omega)$.

DEFINITION. Given $h \in U_M$, $\tilde{w} = \tilde{w}(h) = (w, w_t)$ is a weak solution of (1.1) if $\tilde{w} \in C([0,T];\mathcal{H})$, $\tilde{w}(0) = (w_0, w_1)$, and $\tilde{w}$ satisfies

$$\frac{d}{dt}\int_\Omega w_t\phi \;d\Omega + a(w,\phi) = \int_\Omega hw\phi \;d\Omega \quad \text{for all} \quad \phi \in H^2_{\Gamma_0}(\Omega).$$

LEMMA 2.1. (Well-posedness and Regularity)

(i) Let $\tilde{w}(0) = (w_0, w_1) \in \mathcal{H}$ and $h \in U_M$, then the state equation (1.1) has a unique weak solution $\tilde{w} = \tilde{w}(h) = (w, w_t)$ with $(w, w_t) \in C([0,T];\mathcal{H})$.

(ii) If $(w_0, w_1) \in D_0$ where

$$D_0 = \{(w_0, w_1) \in (H^4(\Omega) \cap H^2_{\Gamma_0}(\Omega)) \times H^2_{\Gamma_0}(\Omega) :$$

$$\Delta w_0 + (1-\mu)B_1 w_0 = 0 \quad \textit{on} \quad \Gamma_1$$

$$\textit{and} \quad \frac{\partial \Delta w_0}{\partial \nu} + (1-\mu)B_2 w_0 = 0 \quad \textit{on} \quad \Gamma_1\}$$

and $h \in C^1(Q) \cap U_M$, then the weak solution $\tilde{w} = \tilde{w}(h)$ satisfies

$$\begin{aligned} \tilde{w} &\in C([0,T]; (H^4(\Omega) \cap H^2_{\Gamma_0}(\Omega)) \times H^2_{\Gamma_0}(\Omega)) \\ w_{tt} &\in C([0,T]; L^2(\Omega)) \end{aligned}$$

and $\tilde{w}(0) = (w_0, w_1)$.

Also $\tilde{w}$ satisfies equation (1.1) in the L^2 sense.

Proof. (i) We begin by defining the operator

$$\begin{aligned} \mathcal{A} : H^4(\Omega) \quad &\to L^2(\Omega) \text{ by} \\ &\mathcal{A}w = \Delta^2 w \text{ with} \\ D(\mathcal{A}) \quad &= \{w \in H^4(\Omega) \cap H^2_{\Gamma_0}(\Omega) : \Delta w + (1-\mu)B_1 w = 0 \text{ on } \Gamma_1 \\ &\qquad \frac{\partial \Delta w}{\partial \nu} + (1-\mu)B_2 w = 0 \text{ on } \Gamma_1\}. \end{aligned}$$

We write (1.1) in semigroup formulation,

$$\text{(2.2)} \qquad \begin{aligned} \frac{d}{dt}\tilde{w}(t) &= \begin{bmatrix} 0 & I \\ -\mathcal{A} & 0 \end{bmatrix} \tilde{w}(t) + \begin{bmatrix} 0 \\ h(t)\tilde{w}_1(t) \end{bmatrix} \\ \tilde{w}(0) = \tilde{w}_0 &= \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \end{aligned}$$

where $\tilde{w}_1$ denotes the first component of $\tilde{w}$. Define the operator $A : H^4(\Omega) \times H^2_{\Gamma_0}(\Omega) \to \mathcal{H}$ by

$$A\tilde{w} = \begin{bmatrix} 0 & I \\ -\mathcal{A} & 0 \end{bmatrix} \tilde{w}$$

with $D(A) = \{(w_1, w_2) : w_2 \in H^2_{\Gamma_0}(\Omega), w_1 \in D(\mathcal{A})\}$. Formulation (2.2) may be written as

$$\text{(2.3)} \qquad \frac{d}{dt}\tilde{w}(t) = A\tilde{w}(t) + B_h(\tilde{w})(t)$$

where $B_h(\tilde{w}) = \begin{bmatrix} 0 \\ h\tilde{w}_1 \end{bmatrix}$. The operator A is skew-adjoint, so that $D(A) = D(A^*)$, and A generates a unitary group on $\mathcal{H}$.

Motivated by the semigroup formulation of (2.3), we seek a solution of the form

$$\tilde{w}(t) = e^{At}\tilde{w}_0 + \int_0^t e^{A(t-\tau)} B_h(\tilde{w})(\cdot, \tau) d\tau.$$

We will prove that the map T_h,

$$T_h\tilde{w}(t) = e^{At}\tilde{w}_0 + \int_0^t e^{A(t-\tau)} B_h(\tilde{w})(\cdot, \tau) d\tau,$$

has a unique fixed point in $C([0,T];\mathcal{H})$. Our first step is to prove existence of a unique fixed point on $C([0,T_0];\mathcal{H})$ for T_0 sufficiently small. To use the contraction mapping theorem, we show

$$T_h : C([0,T_0];\mathcal{H}) \to C([0,T_0];\mathcal{H})$$

is bounded and contractive. We observe that

$$\begin{aligned} \|T_h\tilde{w}\|_{C([0,T_0];\mathcal{H})} &\leq \|e^{At}\tilde{w}_0\|_{C([0,T_0];\mathcal{H})} \\ &+ \sup_{0\leq t\leq T_0} \int_0^t \|e^{A(t-\tau)} B_h(\tilde{w})(\cdot, \tau)\|_{\mathcal{H}} d\tau \\ &\leq \|w_0\|_{\mathcal{H}} + \sup_{0\leq t\leq T_0} \int_0^t \|h(\cdot,\tau)\tilde{w}_1(\cdot,\tau)\|_{L^2(\Omega)} d\tau, \end{aligned}$$

where we have used that A generates a unitary group, and thus $\|e^{At}\| = 1$.

For spatial dimension 2, the Sobolev imbeddings give $H^2_{\Gamma_0}(\Omega) \hookrightarrow C(\Omega)$. Using $\tilde{w}_1 \in C([0,T]; H^2_{\Gamma_0}(\Omega))$ and $\|h\|_\infty \leq M$, we obtain

$$\|T_h\tilde{w}\|_{C([0,T_0];\mathcal{H})} \leq \|\tilde{w}_0\|_{\mathcal{H}} + MT_0\|\tilde{w}_1\|_{C([0,T_0];L^2(\Omega))}.$$

Hence T_h is bounded.

Similarly,

$$\begin{aligned}\|T_h\tilde{v} - T_h\tilde{w}\|_{C([0,T_0];\mathcal{H})} &\leq MT_0\|\tilde{v}_1 - \tilde{w}_1\|_{C([0,T_0];L^2(\Omega))} \\ &\leq MT_0\|\tilde{v} - \tilde{w}\|_{C([0,T_0];\mathcal{H})}.\end{aligned}$$

Choosing $T_0 < \frac{1}{M}$, we have T_h is contractive for $t < T_0$. Thus we have the existence of a unique fixed point on $C([0,T_0];\mathcal{H})$.

This result extends to a solution on $[0,T]$ by selecting a "new initial data" as $\tilde{w}_{T_0} = \tilde{w}(T_0) \in \mathcal{H}$. By a second contraction argument, we have a unique solution on $C([0,2T_0],\mathcal{H})$. Repeating this argument, a finite number of times, we obtain the existence of a weak solution to (1.1) with $\tilde{w} \in C([0,T];\mathcal{H})$.

(ii) Assume $\tilde{w}_0 \in D_0$ and $h \in C^1(Q) \cap U_M$. We seek to prove stronger regularity in this case. Taking a time derivative of our solution map, we obtain

$$\begin{aligned}F\tilde{w} &= \frac{d}{dt}(T_h\tilde{w})(t) \\ &= Ae^{At}\tilde{w}_0 + \int_0^t Ae^{A(t-\tau)}B_h(\tilde{w})(\cdot,\tau)d\tau \\ &\quad + \begin{bmatrix} 0 \\ h(\cdot,t)\tilde{w}_1(\cdot,t) \end{bmatrix}.\end{aligned} \tag{2.4}$$

We will prove that F has a unique fixed point in $C([0,T];\mathcal{H})$. Notice that

$$\begin{aligned}\int_0^t Ae^{A(t-\tau)}B_h(\tilde{w})(\cdot,\tau)d\tau &= -\int_0^t \frac{d}{d\tau}(e^{A(t-\tau)}B_h(\tilde{w})(\cdot,\tau))d\tau \\ &\quad + \int_0^t e^{A(t-\tau)}\begin{bmatrix} 0 \\ h_\tau(\cdot,\tau)w(\cdot,\tau) + h(\cdot,\tau)w_\tau(\cdot,\tau) \end{bmatrix} d\tau \\ &= -B_h(\tilde{w})(\cdot,t) + e^{At}\begin{bmatrix} 0 \\ h(\cdot,0)w_0 \end{bmatrix} \\ &\quad + \int_0^t e^{A(t-\tau)}\begin{bmatrix} 0 \\ h_\tau(\cdot,\tau)w(\cdot,\tau) + h(\cdot,\tau)w_\tau(\cdot,\tau) \end{bmatrix} d\tau\end{aligned}$$

We rewrite (2.4) as

$$\begin{aligned}F\tilde{w}(t) &= Ae^{At}\tilde{w}_0 + e^{At}\begin{bmatrix} 0 \\ h(\cdot,0)w_0 \end{bmatrix} \\ &\quad + \int_0^t e^{A(t-\tau)}\begin{bmatrix} 0 \\ h_\tau(\cdot,\tau)w(\cdot,\tau) + h(\cdot,\tau)w_\tau(\cdot,\tau) \end{bmatrix} d\tau.\end{aligned}$$

By our assumptions on $\tilde{w}_0$ and h, we see that

$$F : C([0,T_0];\mathcal{H}) \to C([0,T_0];\mathcal{H})$$

is bounded. To show F is contractive, consider

$$\|F\tilde{v} - F\tilde{w}\|_{C([0,T];\mathcal{H})}$$

$$= \| \int_0^t e^{A(t-\tau)} \begin{bmatrix} 0 \\ h_\tau(\cdot,\tau)(\tilde{v}_1 - \tilde{w}_1)(\cdot,\tau) + h(\cdot,\tau)(\tilde{v}_2 - \tilde{w}_2)(\cdot,\tau) \end{bmatrix} d\tau \|_{C([0,T_0],\mathcal{H})}$$

$$\leq \sup_{0\leq t\leq T_0} \int_0^t \|h_\tau(\cdot,\tau)(\tilde{v}_1 - \tilde{w}_1)(\cdot,\tau) + h(\cdot,\tau)(\tilde{v}_2 - \tilde{w}_2)(\cdot,\tau)\|_{L^2(\Omega)} d\tau$$

$$\leq CT_0 \|\tilde{v} - \tilde{w}\|_{C([0,T_0];\mathcal{H})},$$

since $h \in C^1(Q)$. For $T_0 < \frac{1}{C}$, we obtain a unique fixed point on $C([0,T_0],\mathcal{H})$. Using the same idea as above, we obtain our solution in $C([0,T],\mathcal{H})$.

We now observe the regularity that this solution provides. We have

$$(w_t, w_{tt}) \in C([0,T];\mathcal{H}).$$

Since $hw \in L^2(Q)$, from the equation (1.1), we see

$$\Delta^2 w \in C([0,T];L^2(\Omega)).$$

This combined with homogeneous boundary conditions gives

$$w \in C([0,T];H^4(\Omega) \cap H^2_{\Gamma_0}(\Omega)),$$

by standard elliptic theory [18]. We have obtained the desired regularity. □

To prove the existence of an optimal control, we need the following *a priori* estimate.

LEMMA 2.2. *Given $\tilde{w}_0 = (w_0, w_1) \in \mathcal{H}$ and $h \in U_M$, the weak solution $\tilde{w} = \tilde{w}(h) = (w, w_t)$ of (1.1) satisfies*

$$\|\tilde{w}\|_{C([0,T];\mathcal{H})} \leq C_1 e^{C_2 MT} \tag{2.5}$$

where $C_1 = \|\tilde{w}_0\|_{\mathcal{H}}$.

Proof. There exist sequences $(w_{0n}, w_{1n}) \in D_0$ and $h_n \in U_M \cap C^1(Q)$ such that

$$(w_{0n}, w_{1n}) \to (w_0, w_1) \quad \text{strongly in} \quad \mathcal{H}$$
$$h_n \to h \quad \text{strongly in } L^2(Q).$$

Denote the weak solution of (1.1) corresponding to initial data (w_{0n}, w_{1n}) with control h_n by (w_n, w_n') (where $'$ denotes $\frac{d}{dt}$). Then (w_n, w_n') satisfy the additional regularity of

Lemma 2.1 (ii). Multiplying the PDE (1.1) by w_n' and integrating over $Q_t = \Omega \times (0,t)$, we obtain

$$\begin{aligned}
0 &= \int_{Q_t} (w_n'' w_n' + \Delta^2 w_n w_n' - h_n w_n w_n') dQ_t \\
&= \int_{Q_t} \left(\frac{1}{2}\frac{d}{dt}\left[(w_n')^2 + (\Delta w_n)^2 \right] - h_n w_n w_n' \right) dQ_t \\
&\quad + \int_{\Gamma_1 \times (0,t)} \left[\frac{\partial}{\partial \nu} \Delta w_n w_n' - \Delta w_n \frac{\partial}{\partial \nu} w_n' \right] d\Gamma_1 dt \\
&= \int_{Q_t} \left(\frac{1}{2}\frac{d}{dt}[(w_n')^2 + (\Delta w)^2] - h_n w_n w_n' \right) dQ_t \\
&\quad + \int_{\Gamma_1 \times (0,t)} (1-\mu) \left((B_1 w_n) \frac{\partial w_n'}{\partial \nu} - (B_2 w_n) w_n' \right) d\Gamma_1 dt \\
&= \int_{Q_t} \frac{1}{2}\frac{d}{dt}(w_n')^2 dQ_t + \frac{1}{2}\int_0^t \frac{d}{dt} a(w_n, w_n) dt - \int_{Q_t} h_n w_n w_n' dQ_t,
\end{aligned}$$

where we used the identity (see [9] p. 68)

$$\int_\Omega (2 w_{xy} v_{xy} - w_{xx} v_{yy} - w_{yy} v_{xx}) d\Omega = \int_{\partial\Omega} \left((B_1 w) \frac{\partial v}{\partial \nu} - (B_2 w) v \right) d\Omega$$

with w_n, w_n'. We obtain

$$\begin{aligned}
\int_\Omega (w_n')^2(x,t) d\Omega \quad + \quad a(w_n, w_n)(t) &= \int_{Q_t} h_n w_n w_n' dQ_t + \|(w_{0n}, w_{1n})\|_{\mathcal{H}}^2 \\
&\leq \frac{M}{2} \int_{Q_t} ((w_n)^2 + (w_n')^2) dQ_t + \|(w_{0n}, w_{1n})\|_{\mathcal{H}}^2 \\
&\leq CM(\int_{Q_t} (w_n')^2 dQ_t + \int_0^t a(w_n, w_n) dt) + \|(w_{0n}, w_{1n})\|_{\mathcal{H}}^2
\end{aligned}$$

where we have used that $a(w,w)$ is equivalent to the norm on $H^2_{\Gamma_0}(\Omega)$. Applying Gronwall's Inequality, we obtain

$$\sup_{0 \leq t \leq T} \left(\int_\Omega (w_n')^2(x,t) d\Omega + a(w_n, w_n)(t) \right) \leq \|(w_{0n}, w_{1n})\|_{\mathcal{H}} e^{CMT} \tag{2.6}$$

Now letting $(w_{0n}, w_{1n}) \to (w_0, w_1)$ in $H^2_{\Gamma_0}(\Omega) \times L^2(\Omega)$ and $h_n \to h$ in $L^2(\Omega)$, we see that

$$(w_n, w_n') \to (w, w') \text{ in } C([0,T]; \mathcal{H})$$

and $(w, w') = \tilde{w}(h)$ is a weak solution of (1.1). Estimate (2.5) holds by passing to the limit as $n \to \infty$ and noticing (2.6) does not depend on C^1 regularity of the control. □

Remark. If we consider the plate equation with a source term f,

$$w_{tt} + \Delta^2 w = hw + f,$$

with $f \in C([0,T];H^2(\Omega)), f_t \in C([0,T];L^2(\Omega))$, the same well-posedness and regularity results as in Lemma 2.1 hold. However in Lemma 2.2,

$$C_1 = \|\tilde{w}_0\|^2_{\mathcal{H}} + \|f\|^2_{L^2(Q)}, \quad C_2 = C(M+1).$$

We now prove the main result of this section. □

THEOREM 2.1. *There exists an optimal control $h^* \in U_M$ which minimizes the cost functional $J(h)$ for $h \in U_M$.*

Proof. Let $\{h^n\} \in U_M$ be a minimizing sequence such that

$$\lim_{n\to\infty} J(h^n) = \inf_{h \in U_M} J(h).$$

We denote the corresponding solution to (1.1) by $\tilde{w}^n = \tilde{w}(h^n)$. By Lemma 2.2,

$$\|\tilde{w}^n\|_{C([0,T],\mathcal{H})} \le C_1 e^{C_2 M T}.$$

On a subsequence, we have

$$\begin{aligned} w^n &\rightharpoonup w^* \text{ weakly in } L^2([0,T];H^2_{\Gamma_0}(\Omega)) \\ w^n &\to w^* \text{ strongly in } L^2(Q) \\ w^n_t &\rightharpoonup w^*_t \text{ weakly in } L^2(Q) \end{aligned}$$

and

$$h^n \rightharpoonup h^* \text{ weakly in } L^2(Q).$$

We may now pass to the limit on (1.1) as $n \to \infty$, to obtain that $\tilde{w} = \tilde{w}(h) = (w^*, w^*_t)$ solves the state equation (1.1) with control h^*. Since the cost functional is lower semicontinuous with respect to weak convergence (basically Fatou's Lemma), we obtain

$$J(h^*) \le \underline{\lim}_{n\to\infty} J(h^n) = \inf_{h\in U_M} J(h).$$

Hence h^* is an optimal control. □

3. Characterization of the Optimal Control. We now derive the optimality system by differentiating the cost functional $J(h)$ with respect to the control h. Since $\tilde{w} = \tilde{w}(h)$ is involved in $J(h)$, we first must prove appropriate differentiability of the mapping

$$h \to \tilde{w}(h).$$

LEMMA 3.1. *The mapping*

$$h \in U_M \to \tilde{w}(h) \in \mathcal{H}$$

is differentiable in the following sense:

$$\frac{\tilde{w}(h+\varepsilon\ell) - \tilde{w}(h)}{\varepsilon} \rightharpoonup \tilde{\psi} \text{ weakly in } \mathcal{H}$$

as $\varepsilon \to 0$, *for any* $h, h+\varepsilon\ell \in U_M$. *Moreover* $\tilde{\psi} = (\psi, \psi_t)$ *is a weak solution of the following problem:*

$$\psi_{tt} + \Delta^2\psi - h\psi = \ell w \text{ in } Q$$
$$\psi(x,0) = \psi_t(x,0) = 0 \text{ in } \Omega$$
$$\psi = \frac{\partial \psi}{\partial \nu} = 0 \text{ on } \Sigma_0 \tag{3.1}$$
$$\left.\begin{array}{r} \Delta\psi + (1-\mu)B_1\psi = 0 \\ \frac{\partial}{\partial\nu}\Delta\psi + (1-\mu)B_2\psi = 0 \end{array}\right\} \text{ on } \Sigma_1$$

where $\tilde{w} = \tilde{w}(h) = (w, w_t)$.

Proof. Denote $\tilde{w}^\varepsilon = \tilde{w}(h+\varepsilon\ell) = (w^\varepsilon, w_t^\varepsilon)$ and $\tilde{w} = \tilde{w}(h)$. Then $\frac{\tilde{w}^\varepsilon - \tilde{w}}{\varepsilon}$ is a weak solution of

$$\left(\frac{w^\varepsilon - w}{\varepsilon}\right)_{tt} + \Delta^2\left(\frac{w^\varepsilon - w}{\varepsilon}\right) = h\left(\frac{w^\varepsilon - w}{\varepsilon}\right) + \ell w^\varepsilon \text{ in } Q$$

$$\text{with } \left(\frac{w^\varepsilon - w}{\varepsilon}\right)(x,y,0) = \left(\frac{w^\varepsilon - w}{\varepsilon}\right)_t(x,y,0) = 0 \text{ in } \Omega$$

and satisfies zero boundary conditions on $\partial\Omega \times (0,T)$. Using the result of Lemma 2.2 and Remark 2.1, we obtain

$$\|\frac{\tilde{w}^\varepsilon - \tilde{w}}{\varepsilon}\|_{C([0,T],\mathcal{H})} \le \|\ell w^\varepsilon\|_{L^2(Q)} e^{CMT} \le C_3$$

where C_3 depends on the L^∞ bound on ℓ, but is independent of ε, due to a bound on $\|\tilde{w}^\varepsilon\|_{L^2(Q)}$, independent of ε. Hence on a subsequence,

$$\frac{\tilde{w}^\varepsilon - \tilde{w}}{\varepsilon} \rightharpoonup \tilde{\psi} \text{ weakly in } \mathcal{H}.$$

This convergence and the above *a priori* estimates are sufficient to guarantee that $\tilde{\psi}$ is a weak solution of (3.1). □

Finally, we derive our optimality system.

THEOREM 3.1. *Given an optimal control h and corresponding solution* $\tilde{w} = \tilde{w}(h) = (w, w_t)$, *there exists a weak solution* $\tilde{p} = (p, p_t)$ *in* $\mathcal{H}$ *to the adjoint problem,*

$$p_{tt} + \Delta^2 p = hp + w - z \text{ in } Q$$

$$p = \frac{\partial p}{\partial \nu} = 0 \text{ on } \Sigma_0 \tag{3.2}$$

$$\left.\begin{array}{r} \Delta p + (1-\mu)B_1 p = 0 \\ \frac{\partial}{\partial\nu}\Delta p + (1-\mu)B_2 p = 0 \end{array}\right\} \text{ on } \Sigma_1$$

and tranversality conditions $p(x,y,T) = p_t(x,y,T) = 0$ *when* $t = T$.

Furthermore, h satisfies

$$h = \max(-M, \min(-\frac{\alpha w p}{\beta}, M)). \tag{3.3}$$

Proof. Let $h \in U_M$ be an optimal control and $\tilde{w} = \tilde{w}(h)$ be the corresponding optimal solution. Let $h + \varepsilon\ell \in U_M$ for $\varepsilon > 0$ and $\tilde{w}^\varepsilon = \tilde{w}(h + \varepsilon\ell)$ be the corresponding weak solution of the state equation (1.1). We compute the directional derivative of the cost functional $J(h)$ with respect to h in the direction of ℓ. Since $J(h)$ is a minimum value,

$$
\begin{aligned}
0 &\leq \lim_{\varepsilon\to 0^+} \frac{J(h+\varepsilon\ell) - J(h)}{\varepsilon} \\
&= \lim_{\varepsilon\to 0^+} \frac{\alpha}{2\varepsilon} \int_Q ((w^\varepsilon - z)^2 - (w - z)^2) dQ + \frac{\beta}{2\varepsilon} \int_Q ((h + \varepsilon\ell)^2 - h^2) dQ \\
&= \lim_{\varepsilon\to 0^+} \alpha \int_Q \left(\frac{w^\varepsilon - w}{\varepsilon}\right)\left(\frac{w^\varepsilon + w - 2z}{2}\right) dQ + \frac{\beta}{2} \int_Q (2h\ell + \varepsilon\ell^2) dQ \qquad (3.4)\\
&= \alpha \int_Q \psi(w - z) dQ + \beta \int_Q h\ell \ dQ,
\end{aligned}
$$

where ψ is defined in Lemma 3.1.

Let $\tilde{p} = (p, p_t)$ be the weak solution of the adjoint problem (3.2). Existence and uniqueness of $\tilde{p}$ is proved by arguments similar to those in Section 2. Substituting the adjoint solution into (3.4) for $(w - z)$, we obtain

$$
0 \leq \alpha \int_0^T -\frac{d}{dt} \int_\Omega p_t \psi \ d\Omega dt + \alpha \int_0^T a(p, \psi) dt - \alpha \int_Q \psi h p \ dQ + \int_Q \beta h\ell \ dQ.
$$

Using the weak form of (3.1),

$$
0 \leq \int_Q \ell(\alpha w p + \beta h) dQ.
$$

By a standard control argument concerning the sign of the variation ℓ depending on the size of h, we obtain the desired characterization of h,

$$
h = \max(-M, \min(-\frac{\alpha w p}{\beta}, M)). \qquad \square
$$

Substituting (3.3) for h into the state equation (1.1) and the adjoint equation (3.2), we obtain the optimality system (OS):

$$(OS) \qquad w_{tt} + \Delta^2 w = \max(-M, \min(-\tfrac{\alpha w p}{\beta}, M))w \quad \text{in} \quad Q$$

$$p_{tt} + \Delta^2 p = \max(-M, \min(-\tfrac{\alpha w p}{\beta}, M))p + w - z \text{ in } Q$$

$$w = p = \tfrac{\partial w}{\partial \nu} = \tfrac{\partial p}{\partial \nu} = 0 \quad \text{on} \quad \Sigma_0$$

$$\left.\begin{array}{r} \Delta w + (1-\mu)B_1 w = \Delta p + (1-\mu)B_1 p = 0 \\ \tfrac{\partial}{\partial \nu}\Delta w + (1-\mu)B_2 w = \tfrac{\partial}{\partial \nu}\Delta p + (1-\mu)B_2 p = 0 \end{array}\right\} \text{ on } \Sigma_1$$

$$w(x,y,0) = w_0(x,y), \quad w_t(x,y,0) = w_1(x,y) \quad \text{on } \Omega$$

$$p(x,y,T) = p_t(x,y,T) = 0.$$

Weak solutions of the optimality system exist by Theorems 2.1 and 3.1. For small time T, we now prove uniqueness of weak solutions of (OS), which gives the characterization (3.3) of the unique optimal control in terms of the solution of (OS).

THEOREM 3.2. *For T sufficiently small, weak solutions of the optimality system (OS) are unique.*

Proof. Suppose we have two weak solutions,

$$\tilde{w} = (w, w_t), \tilde{p} = (p, p_t), \hat{w} = (\overline{w}, \overline{w}_t), \hat{p} = (\overline{p}, \overline{p}_t).$$

Since $w, \overline{w}, p, \overline{p} \in C(0,T; H^2(\Omega)), w, \overline{w}, p, \overline{p}$ are bounded on $\overline{Q}$.

We change variables

$$w = e^{\lambda t}u, \quad p = e^{-\lambda t}q, \quad \overline{w} = e^{\lambda t}\overline{u}, \quad \overline{p} = e^{-\lambda t}\overline{q}.$$

Then u, q satisfy in a weak sense

$$\begin{aligned} u_{tt} + 2\lambda u_t + \lambda^2 u + \Delta^2 u &= \max(-M, \min(-\tfrac{\alpha u q}{\beta}, M))u \\ -q_{tt} + 2\lambda q_t - \lambda^2 q - \Delta^2 q &= \max(-M, \min(-\tfrac{\alpha u q}{\beta}, M))(-q) \\ &\quad - e^{2\lambda t}u + e^{\lambda t}z. \end{aligned}$$

One can check that u, q satisfy the same boundary and initial/terminal conditions as before. Also $\overline{u}, \overline{q}$ satisfy a similar equation.

Using multiplier $(u - \overline{u})_t$ on the $u - \overline{u}$ equation and multiplier $(q - \overline{q})_t$ on the $q - \overline{q}$ equation, and combining, we have the following estimate (similar to the proof of Lemma 2.1)

$$
\begin{aligned}
&\tfrac{1}{2}\int_\Omega((u-\overline{u})_t)^2(x,T)d\Omega + \frac{1}{2}\int_\Omega((q-\overline{q})_t)^2(x,0)d\Omega \\
&+\tfrac{\lambda^2}{2}\int_\Omega((u-\overline{u})^2(x,T)+(q-\overline{q})^2(x,0))d\Omega \\
&+a(u-\overline{u},u-\overline{u})(T)+a(q-\overline{q},q-\overline{q})(0) \\
&+2\lambda\int_Q[((q-\overline{q})_t)^2+((u-\overline{u})_t)^2]dQ \\
&=\int_Q\left[(hu-\overline{h}\overline{u})(u-\overline{u})_t-(hq-\overline{h}\overline{q})(q-\overline{q})_t-e^{2\lambda t}(u-\overline{u})(q-\overline{q})_t\right]dQ
\end{aligned} \tag{3.5}
$$

where $h=\max(-M,\min(-\frac{\alpha uq}{\beta},M))$ and $\overline{h}=\max(-M,\min(-\frac{\alpha\overline{u}\overline{q}}{\beta},M))$.

Rewriting,

$$
hu-\overline{h}\overline{u}=h(u-\overline{u})+\overline{u}(h-\overline{h})
$$

and

$$
hq-\overline{h}\overline{q}=h(q-\overline{q})+\overline{q}(h-\overline{h}),
$$

and noting

$$
|h-\overline{h}|\le\frac{\alpha}{\beta}|\overline{u}\overline{q}-uq|\le\frac{\alpha}{\beta}\left(|\overline{u}-u||\overline{q}|+|\overline{q}-q||u|\right),
$$

we can estimate the RHS of (3.5). Now from (3.5), we obtain

$$
\begin{aligned}
&2\lambda\int_Q[((q-\overline{q})_t)^2+((u-\overline{u})_t)^2]dQ \\
&\le\int_Q[((q-\overline{q})_t)^2+((u-\overline{u})_t)^2]dQ \\
&\quad+(Ce^{C(M+\lambda)T})\int_Q[(u-\overline{u})^2+(q-\overline{q})^2]dQ,
\end{aligned} \tag{3.6}
$$

where C is independent of λ, T but does depend on the L^∞ bounds on $w,\overline{p}$. But

$$
\begin{aligned}
\int_Q(u-\overline{u})^2dQ &= \int_\Omega\int_0^T\left(\int_0^t(u-\overline{u})_t(x,y,s)ds\right)^2dtd\Omega \\
&\le \int_\Omega\int_0^T t\left(\int_0^t((u-\overline{u})_t)^2ds\right)dtd\Omega \\
&\le \int_0^T tdt\int_Q((u-\overline{u})_t)^2dsd\Omega \\
&\le \frac{T^2}{2}\int_Q((u-\overline{u})_t)^2dQ.
\end{aligned}
$$

Similarly, we have

$$\int_Q (q-\overline{q})^2 dQ = \int_0^T \int_\Omega \left(-\int_t^T (q-\overline{q})_t(x,y,s)ds \right)^2 d\Omega dt$$
$$\leq \frac{T^2}{2} \int_Q ((q-\overline{q})_t)^2 dQ.$$

Thus further estimation on the RHS of (3.6) yields

$$(2\lambda - 1) \int_Q [((q-\overline{q})_t)^2 + ((u-\overline{u})_t)^2] dQ \leq$$
$$T^2 (Ce^{C(M+\lambda)T}) \int_Q [((q-\overline{q})_t)^2 + ((u-\overline{u})_t)^2] dQ.$$

Now fix λ such that $2\lambda - 1 > 0$. Choose T small so that

$$2\lambda - 1 > T^2 (Ce^{C(M+\lambda)T}),$$

and thus $(q-\overline{q})_t = (u-\overline{u})_t \equiv 0$ in Q.

Due to agreement of $q, \overline{q}$ and $u, \overline{u}$ at top and bottom of the cylinder Q respectively,

$$q = \overline{q}, \quad u = \overline{u}. \qquad \square$$

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