KILLED DIFFUSIONS AND ITS CONDITIONING

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KILLED DIFFUSIONS AND ITS CONDITIONING

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Abstract

Let $X = (X_t)_{t \geq 0}$ be a diffusion determined by an elliptic differential operator L in R^n $(n \geq 1)$. For any bounded $C^{1,1}$ domain D, we define the conditional killed diffusion X^ϕ on D by the semigroup:

$$T_t^\phi f(x) = \phi_0(x)^{-1} E_x[f(X_t)\phi_0(X_t), \tau_D > t] e^{\lambda_0 t} \qquad (t > 0)$$

where λ_0 and ϕ_0 is the principle eigenvalue and eigenfunction of L on D. In this paper, we prove that X^ϕ is a strong Feller process on D and $\{T_t^\phi\}$ has the strong continuity on $C(\overline{D})$. For any $T > 0$ we consider the conditioned process X^T, i.e. the process X in D conditioned on $\{\tau_D > T\}$, and prove that X^T weakly converges to X^ϕ as $T \to \infty$ without any additional hypotheses.

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1. INTRODUCTION

We consider the second order differential operator on R^n:

$$Lu = \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j} a_{ij}(x) \frac{\partial u}{\partial x_i} + \sum b_j(x) \frac{\partial u}{\partial x_j}$$

where $a_{ij}(x)$, $b_j(x) \in C^{1,\alpha}_{loc}(R^n)$, $i,j = 1,\ldots,n$, $\alpha > 0$, and for $\forall\ N > 0$, there exist $\beta_{i,N} > 0$ $(i = 1,2)$ such that $a = (a_{ij}(x))$ satisfies

$$0 < \beta_{1,N} I \leq a(x) \leq \beta_{2,N} I \quad (|x| < N) \tag{1}$$

L generates a diffusion process $X = (X_t)_{t \geq 0}$ on R^n with L as the infinitesimal generator of the corresponding Markov semigroup. Let us deal with L and X locally, i.e. we confine $\{X_s: 0 \leq s \leq T\}$ on the pre-exit event $\{\tau_D > T\}$ (where $T > 0$, $\tau_D = \inf\{t > 0: X_t \overline{\in} D\}$), and then obtain the killed diffusion X^D. The corresponding killed semigroup is

$$T_t f(x) = E^x[f(X_t): t < \tau_D].$$

In this paper, it is given that for a bounded open $C^{1,1}$ domain D and $\forall\ t_0 > 0$, there exists a constant $C = C(L,D,t_0) > 0$, such that

$$\frac{1}{C} d(x) e^{-\lambda_0 t} \leq P_x(\tau_D > t) \leq C d(x) e^{-\lambda_0 t} \tag{3}$$

for all $(t,x) \in [t_0,\infty) \times D$, where λ_0 is the principle eigenvalue of L,

$$d(x) \triangleq \mathrm{dis}(x, \partial D),$$

And a uniform estimate is given

$$\sup_{x \in D} \left| \frac{e^{\lambda_0 t} Px(\tau_D > t)}{\phi_0(x)} - \int_D \psi_0(x) dx \right| \to 0 \tag{4}$$

for time-space parameters, where ϕ_0, ψ_0 are principle eigenfunctions of L and its formal adjoint $\mathcal{L}$ respectively.

However, the killed diffusion disappears gradually as $t \to \infty$. It is natural to consider the diffusion $\{X_t: 0 \le t \le T\}$ under the conditional probability measure $P^T_{sx} = P_x(\cdot|\tau_D > T - s)$ (for each fixed $T > 0$) and its weak limit as $T \to \infty$. In the case of Markov chain, this was done by [2], [14]. For diffusions Ross Pinsky proved [12] that the conditioned diffusion X^T up to time T is inhomogeneous with the generator L^T depending on T, and under three hypotheses (which seem hard to check) the coefficients of L^T converge *locally* uniformly to those of the operator $L + a(\nabla \ln \phi_0)\nabla$ (therefore the weak convergence of the processes follows). But even in the case of L being symmetric those three hypotheses have not been justified rigorously there. In this paper we approach this conditioning problem in a different way. We define directly a Markov process X^ϕ in D given by the semigroup

$$T^\phi_t f(x) = \phi_0(x)^{-1}E^x[f(X_t)\phi_0(X_t): \tau_D > t]e^{\lambda_0 t} \tag{5}$$

Since ϕ_0 is a positive "harmonic" function on D with respect to $-(L + \lambda_0)$, this definition is a natural generalization of the concept of conditional Brownian motion given by Doob [3]. We call X^ϕ the conditional killed diffusion, which has been considered by Zhao [18] in a different situation. Moreover, we prove that T^ϕ_t has the strong Feller property. A more delicate property is the strong continuity of T^ϕ_t on $C_U(D)$ (the Banach space of uniformly continuous functions on D), or equivalently, for any given $\delta > 0$

$$\sup_{x\in D} |\phi_0(x)^{-1}E_x(\phi(X_t), |X_t - x| \geq \delta, \tau_D > t)| \to 0 \quad (t \to 0) \tag{6}$$

The strong continuity of T^ϕ_t is not only an analytical property of the semigroup, but also is the key step to get the tightness of the measures $P^T_{s,x}$. On the other hand, with the help of (5) and (4) we have the convergence of the transition functions of X^T. Thus, it turns out that X^T weakly converges to X^ϕ without any additional assumptions.

2. SOME RESULTS IN ANALYSIS

Let $a = (a_{ij}(x))$, $b = (b_1(x),\dots,b_n(x))$ be $C^{1,\alpha}$ functions on D, where $i,j = 1,2,\dots,n$, and

$$0 < \gamma_1 I \leq a \leq \gamma_2 I \qquad (x \in D) \tag{7}$$

Denote

$$\begin{aligned} L &= \frac{1}{2}\nabla a \nabla + b\nabla - c(x) \qquad (c(x) \geq 0,\ c(x) \in C^{0,\alpha}(D)) \\ L_a &= \frac{1}{2}\nabla a \nabla \end{aligned} \tag{8}$$

and G_L, G_{L_a}, G_Δ are Green functions of L, L_a, $\frac{1}{2}\Delta$ respectively. Some analytic facts which we need are listed below

Theorem (A) (Widman [17])

$$G_{L_a}(x,y) \leq C \frac{1}{|x - y|^{n-1}} (|x - y| \wedge d(x))$$

$$|\nabla_x G_{L_a}(x,y)| \leq K \frac{1}{|x - y|^{n-1}} . \qquad (n \geq 3)$$

Theorem (B) (Hueber and Sieveking [7])

$$\frac{1}{c} G_\Delta \leq G_L \leq c G_\Delta \qquad (c > 0)$$

Now we have

<u>Lemma 1</u>

$$\sup_{x \in D} \int_D \frac{dy}{|x - y|^{n-1}} < \infty$$

and measures $\int_A \frac{dy}{|x - y|^{n-1}}$ are absolutely continuous with respect to dy and uniformly for $x \in D$.

Proof: Taking the n-dimensional spherical polar coordinate centered at x, we get

$$\sup_{x\varepsilon D} \int_A \frac{dy}{|x-y|^{n-1}} < \sup_{x\varepsilon D} \Big(\int_{\{|x-y|<\delta\}\cap A} + \int_{\{|x-y|>\delta\}\cap A} \Big) \frac{dy}{|x-y|^{n-1}}$$

$$< \int_0^\delta \int \frac{r^{n-1}drd\sigma}{r^{n-1}} + \frac{1}{\delta^{n-1}}\, m(A) = \delta\sigma + \frac{1}{\delta^{n-1}}\, m(A)$$

where $m(\cdot)$ is the Lebesque measure and σ is the area of the unit sphere in R^n. Then the proof is complete.

Lemma 2 If $f \varepsilon M_b(D)$ (bounded measurable function on D) $f \geq 0$ and $f > 0$ at least on a small ball, then there are $c_1 > 0$, c_2 such that

$$c_1 < \frac{\int_D G_L(x,y)f(y)dy}{d(x)} < c_2 \qquad (x \varepsilon D).$$

Proof. Since Theorem (B), (A) and Lemma 1, the upper bound follows and for the lower bound we only need to investigate when $L = \Delta$ and x satisfies $d(X) < \delta$ with small $\delta > 0$. If, on the contrary, this lemma fails, then there are $x_m \varepsilon D$ such that

$$\frac{\int_D G_\Delta(x_m,y)f(y)dy}{d(x_m)} \to 0 \qquad (m \to \infty)$$

Without loss of generality, we can assume $x_m \to x_0 \varepsilon \partial D$. Denote the inward normal of ∂D at x_m to be n_m and pick up $x_m^* \varepsilon \partial D$ such that $|x_m - x_m^*| = d(x_m)$. Because $\nabla_x G_\Delta(x,y)$ can be extended continuously to $\overline{D}\setminus\{y\}$ [5],[6], with the help of the mean value theorem we obtain

$$\begin{aligned} G_\Delta(x_m,y) &= G_\Delta(x_m,y) - G_\Delta(x_m^*,y) \\ &= \frac{\partial G_\Delta}{\partial n_m}(\xi_m^*(y),y)d(x_m) \quad (\xi_m^*(y) \varepsilon \overline{x_m x_m^*}). \end{aligned}$$

Thus

$$\int_D \frac{G_\Delta(x_m,y)}{d(x_m)} f(y)dy = \int_D \frac{\partial G_\Delta}{\partial n_m} (\xi_m^*(y),y)f(y)dy$$

Now the estimate in Theorem (A) and Lemma 1 allows us to pass the limit under the integral and then get

$$\int_D \frac{G_\Delta(x_m,y)}{d(x_m)} f(y)dy \to \int_D \frac{\partial G_\Delta}{\partial n} (x_0,y)f(y)dy \qquad (m \to \infty)$$

with $\frac{\partial G_\Delta}{\partial n}\Big|_{\partial D} \geq \text{const} > 0$ [5], [6]. Hence it should be positive. This contradiction fulfills the validity of our lemma.

From now on, we assume a, b satisfying conditions in Section 1 and $c(x) \in C^{0,\alpha}_{loc}(R^n)$.

<u>Proposition 1.</u> Let

$$L^{(\lambda)} \triangleq (L - \lambda), \quad G_\lambda \triangleq G_{L(\lambda)} \qquad (\lambda > 0)$$

Then there are $\alpha > 0$ and $C > 0$ such that

$$G_\lambda(x,y) \leq C \frac{e^{-\alpha\sqrt{\lambda}|x-y|}}{|x - y|^{n-2}} \qquad (n \geq 3) \tag{9}$$

where

$$G_\lambda(x,y) = \int_0^\infty e^{-\lambda t} p^D(t,x,y)dt \tag{10}$$

and $p^D(t,x,y)$ is the transition density of the killed diffusion X^D of X at ∂D.

Proof. It is well known that $p^D(t,x,y)$ is the Green function of $\frac{\partial u}{\partial t} = Lu$ on D and $p^D(t,x,y)|_{x\in\partial D} = 0$. Meanwhile estimation

$$p(t,x,y) \leq C_{\varepsilon,T} \frac{1}{t^{n/2}} e^{-\frac{(\gamma_1 - \varepsilon)|x - y|^2}{4t}} \tag{11}$$

$$(0 < t < T, \; x,y \in R^n, \; 0 < \varepsilon < \gamma_1)$$

holds for the transition density $p(t,x,y)$ of X with a constant $C_{\varepsilon,T}$ (γ_1 is defined in (7)). [10] And the existence of $p^D(t,x,y)$ and its property can be established in a purely probabilistic way without the help of PDE just like what is done in the killed brownian motion case [13]. Now the estimate (11) reduces our case to the classical brownian one. Combinind (11) and

$$p^D(t,x,y) < p(t,x,y),$$

we have for small $\eta > 0$ that

$$\int_0^\delta e^{-\lambda t} p^D(t,x,y)dy <$$

$$< C_{\varepsilon,\delta} \frac{e^{-\sqrt{(\gamma_1-\varepsilon)/(1+\eta)}\sqrt{\lambda}|x-y|}}{|x-y|^{n-2}} \int_0^{\delta\cdot d(D)^2} \frac{1}{u^{n/2}} e^{-\frac{(\gamma_1-\varepsilon)\eta}{1+\eta}\frac{1}{u}} du$$

($d(D)$ = diamter of D)

$$\triangleq C_{\varepsilon,\delta,\eta} \frac{e^{-\sqrt{\frac{\gamma_1-\varepsilon}{1+\eta}}\sqrt{\lambda}\,|x-y|}}{|x-y|^{n-2}}$$

On the other hand

$$\int_\delta^\infty e^{-\lambda t}\, p^D(t,x,y)dt$$

$$< e^{\frac{\gamma_1-\varepsilon}{1+\eta}\frac{d(D)}{4\delta}} e^{-\sqrt{\frac{\gamma_1-\varepsilon}{1+\eta}}\sqrt{\lambda}|x-y|} \int_\delta^\infty p^D(y,x,y)dt,$$

And by Theorem (B) and (A)

$$\int_\delta^\infty p^D(t,x,y)dy < \int_0^\infty p^D(t,x,y)dy = G_L(x,y)$$

$$< cG_\Delta(x,y) < \frac{\tilde{C}}{|x-y|^{n-2}}$$

Thus (9) holds for $\alpha = \sqrt{\frac{\gamma_1 - \varepsilon}{1 + n}}$, $C = C_{\varepsilon,\delta,n} V(\tilde{C}e^{\frac{\gamma_1-\varepsilon}{1+n} \frac{d(D)}{4\delta}})$.

Remark Let

$$\hat{L} = \frac{1}{2} \nabla a \nabla - \nabla(b\cdot)$$

to be the formal adjoint of L, and $\lambda > -\inf\{(\mathrm{div}\, b(x))\}$, then proposition 1 keeps valid for $G_\lambda = G_{\hat{L}(\lambda)}$ where $\hat{L}^{(\lambda)} = \hat{L} - \lambda$.

Proposition 2 There exists $\beta > 0$ such that

$$G_\lambda(x,y) < \frac{Cd(x)e^{-\beta\sqrt{\lambda}|x-y|}}{|x - y|^{n-1}} \qquad (n > 3) \tag{12}$$

Proof Inequality (12) follows by modifying Widman's arguments in [17]. For the specified annuloid domain A (which is denoted as D in [17]), $G_\lambda(z,y)$ is majorized by $\frac{Ce^{-\beta\sqrt{\lambda}|x-y|}}{|x - y|^{n-2}}$ and zero on $\partial A \cap D$ and ∂D respectively by Proposition 1. Then by the maximum principle and Theorem (B) $G_\lambda(z,y)$ (λ sufficiently large) in A can be majorized by the harmonic function with the boundary value $\frac{ce^{-\beta\sqrt{\lambda}|x-y|}}{|x - y|^{n-2}}$ and zero on $\partial A \cap D$ and ∂D respectively. Thus the same arguments as [17] lead to the inequality (12).

From now on we always assume $c(x) \equiv 0$.

We need a version of Frobenius theorem of the elliptic PDE. A familiar fact is that this kind of counterpart can be proven by Krein-Rutman's theorem [9] and the strong maximum principle [5]. We state it as follows

Theorem (C)

$-L$ and $-\hat{L}$ with the Dirichlet boundary condition on ∂D have a common simple positive eigenvalue λ_0 at the left of their spectrum. The corresponding eigenfunctions, ϕ_0 and ψ_0, can be chosen positive on D and vanish on ∂D.

For this we have to do a little bit explanation both for the sketch of its proof and convenience of its application.

Let us introduce two Banach spaces:

$$B(D) \triangleq \{f: f(x) = d(x)\tilde{f}(x),\ \tilde{f} \in C_b(D),\ \|f\|_{B(D)} \triangleq \|\tilde{f}\|_{C_b(D)}\}$$

$$B_0(D) \triangleq \{f: f(x) = d(x)\tilde{f}(x),\ \tilde{f} \in C_U(D),\ \|f\|_{B_0(D)} \triangleq \|\tilde{f}\|_{C_U(D)}\}$$

where $C_b(D)$ = space of bounded continuous functions and $C_U(D)$ is equivalent to $c(\overline{D})$. $B_0(D)$ is a closed subspace of $B(D)$.

$B_0(D)$ is included in $C_0(D)$ ($C_b(D)$ functions vanishing at ∂D) but equipped with a stronger topology.

<u>Lemma 3</u>

1° Let G_λ be the operator with the kernel $G_\lambda(x,y)$. Then we have

$$G_\lambda M_b(D) \subset B_0(D).$$

And G_λ is compact on $B_0(D)$.

The semigroup T_t^D $(t > 0)$ in (2) maps $M_b(D)$ into $B_0(D)$. Besides, T_t^D is compact on $B_0(D)$ and on $C_0(D)$ and strongly continuous on $C_0(D)$.

Proof:

1°. We have $G_\lambda f \in C_b(D)$ and

$$\left| \frac{G_\lambda f(x')}{d(x')} - \frac{G_\lambda f(x)}{d(x)} \right| < \int_D \left| \frac{G_\lambda(x,y)}{d(x)} - \frac{G_\lambda(x',y)}{d(x')} \right| |f(y)|dy$$

for any $f \in M_b(D)$. The left hand side goes to zero when $x' \to x \in D$. In the case of $x' \to x \in \partial D$, $\nabla_x G(x,y)$ can be extended continuously to $\overline{D}$ $\{y\}$ [5], [6] which ensures that the left hand side turns also to zero. That means $\frac{G_\lambda f}{d(x)}$ can be taken as a function in $C(\overline{D})$, i.e. $G_\lambda f \in B_0(D)$. A very similar argument shows that $\frac{G_\lambda f}{d(x)}$ is equivalently continuous on $\overline{D}$ for all $f = \tilde{f}d$ with $\|\tilde{f}\|_C < 1$ which leads to the compactness of G_λ on $B_0(D)$.

2°. First we point out that for any $f \in M_b(D)$

$$\frac{1}{d(x)} \int_D p^D(t,x,y)f(y)dy \in C_b(D)$$

holds, which says $T_t^D M_b(D) \subset B(D)$. In fact, $\int p^D(t,x,y)f(y)dy$ is continuous [10]. Combining Theorem (B), (A) and Lemma 1, we get

$$\begin{aligned}
&\sup_{x\in D} \frac{1}{d(x)} \left| \int_D p^D(t,x,y)f(y)dy \right| \\
&\quad < \sup_{x\in D} \frac{\|f\|_{C_b}}{d(x)} P_x(\tau_D > t) \\
&\quad < \|f\|_{C_b} \frac{1}{t} \sup_{x\in D} \frac{E_x\tau_D}{d(x)} \\
&\quad < \frac{\|f\|_{C_b}}{t} \sup_{x\in D} \int_D \frac{G_L(x,y)}{d(x)} dy < \frac{C\|f\|_{C_b}}{t} \sup_{x\in D} \int_D \frac{G_\Delta(x,y)}{d(x)} dy \\
&\quad < C \frac{\|f\|_{C_b}}{t} \sup_{x\in D} \int_D \frac{dy}{|x-y|^{n-1}} < \infty.
\end{aligned}$$

Next we show that $\frac{1}{d(x)} \int_D p^D(t,x,y)f(y)dy$ is uniformly continuous on D. This can be done as Lemma 2 with the well-known fact that $\nabla_x p^D(t,x,y)$ can be extended continuously to $\overline{D}$ with respect to (x,y) ([10], Th 16.3). From these two steps we get $T_t^D f \in B_0(D)$. The compactness of T_t^D $(t > 0)$ on $B_0(D)$ or $C_0(D)$ can be proven just as we did in 1° with G_λ. Finally, let us prove that T_t^D is strongly continuous on $C_0(D)$. For this we pick up $\overline{a}_{ij}(x), \overline{b}_i(x) \in C_b(R^n) \cap C^{1,\alpha}_{loc}(R^n)$ such that

$$\overline{a}_{ij}(x) = a_{ij}(x), \quad \overline{b}_i(x) = b_i(x) \qquad (x \in D,\ i,j = 1,\ldots,n)$$

The Markov process $\overline{X}$ generated by $\overline{L}$ with coefficients $\overline{a}_{ij}$, $\overline{b}_i$ is a strong Feller process with strongly continuous semigroup on $C_0(D)$. However $\overline{X}^D$ and X^D have the same (killed) semigroup on D. By a fact in [1], T_t^D is strongly continuous on $c_0(D)$. That accomplishes the lemma.

Late on, let G_λ be the operator with the kernel $\hat{G}_\lambda(x,y)$: the Green function of $-\hat{L} - \lambda$.

We have $\int_D \frac{G_\lambda(x,y)}{d(x)} f(y)dy \geq \varepsilon_\lambda > 0$ (for some ε_λ and $\forall f \in B_0^+(D)$ i.e. $f \in B_0(D)$ and $f \geq 0$) as in Lemma 2 (Similar inequality holds for $\hat{G}_\lambda$ with λ large enough). It implies that G_λ is strictly positive on $B_0^+(D)$ in the sense of Krein-Rutman [9]. Now their famous theorem works: The first eigenvalue $\gamma(\lambda)$ of G_λ is positive and simple with a positive eigenfunction $\phi_0^\lambda(x)$. And the resolvent identity $G_\mu - G_\lambda = (\lambda - u)G_\mu G_\lambda$ assures that $\phi_0(x)$ is independent of λ, i.e. $\phi_0^\lambda(x) = \phi_0(x)$, and $\gamma(\lambda) = \frac{1}{\lambda + \lambda_0}$. Similarly $\hat{G}_\lambda$ has a positive eigenfunction $\psi_0(x)$ with positive simple first eigenvalue $\frac{1}{\lambda + \hat{\lambda}_0}$ when λ is large enough. Since we have $\langle G_\lambda f,g\rangle_{L^2(D)} = \langle f,\hat{G}_\lambda g\rangle_{L^2(D)}$ for $f,g \in L^2(D)$ and G_λ, $\hat{G}_\lambda$ are compact both on $B_0(D)$ and $L^2(D)$, we should have $\hat{\lambda}_0 = \lambda_0$, and the other generalized eigenspace M_j, $\hat{M}_j$ of G_λ, $\hat{G}_\lambda$ are finite dimensional and can be chosen that the corresponding point spectrum are $\frac{1}{\lambda + \lambda_j}$ and $\frac{1}{\lambda + \overline{\lambda}_j}$, and $M_j \perp \hat{M}_k$ in $L^2(D)$ when $j \neq k$. Meanwhile ϕ_0, ψ_0 are determined uniquely by

$$
\begin{aligned}
-L\phi &= \lambda_0\phi & \qquad -\hat{L}\psi &= \lambda_0\psi \\
\phi|_{\partial D} &= 0 \qquad \text{and} & \psi|_{\partial D} &= 0 \\
\phi &> 0 \text{ on } D & \psi &> 0 \text{ on } D.
\end{aligned}
$$

Surely, we can assume that $\int \phi_0(x)\psi_0(x)dx = 1$. Moreover $\{\phi_0,\phi_{j\ell}: j,\ell \text{ varies}\}$ ($\{\psi_0,\psi_{j\ell} : j,\ell \text{ varies}\}$) generates $L^2(D)$, where $\phi_{j\ell} \in M_j$, $\psi_{j\ell} \in \hat{M}_j$ are given by

$$(\lambda_j + L)^{m_{j\ell}} \phi_{j\ell} = 0 \qquad \text{and} \qquad (\lambda_j + \hat{L})^{\hat{m}_{j\ell}} \psi_{j\ell} = 0$$
$$\phi_{j\ell}|_{\partial D} = 0 \qquad\qquad \psi_{j\ell}|_{\partial D} = 0$$

with $m_{j\ell} < \dim M_j$, $\hat{m}_{j\ell} < \dim \hat{M}_j$ respectively.

Proposition 3

$$\phi_0,\ \psi_0 \in B_0(D).$$

And there are $\beta_2 > \beta_1 > 0$ such that

$$\beta_1 d(x) < \phi_0(x) < \beta_2 d(x),$$
$$\beta_1 d(x) < \psi_0(x) < \beta_2 d(x).$$

Proof: Since we can take a sufficiently large λ such that

$$\frac{\psi_0(x)}{d(x)} = \frac{(\lambda + \lambda_0)\int_D \hat{G}_\lambda(x,y)\psi_0(y)dy}{d(x)},$$

the proposition follows immediately from Theorem (B) and Lemma 2.

Corollary $\phi_0(x)$ or $\psi_0(x)$ can be used instead of $d(x)$ in the definition of $B_0(D)$.

Let us denote the Green function of $\frac{\partial u}{\partial t} = \hat{L}u$ on D by $\hat{p}^D(t,x,y)$ $(= p^D(t,y,x))$ and set

$$\hat{T}_t^D f(x) = \int_D \hat{p}^D(t,x,y)f(y)dy.$$

Proposition 4

Semigroups T_t^D and $\hat{T}_t^D$ on $B_0(D)$ have a common first eigenvalue $e^{-\lambda_0 t}$ which is simple with eigenfunctions ϕ_0 and ψ_0 respectively.

Proof: By Lemma 3, T_t^D only has non-zero pure point spectrum. Then it follows from the spectral mapping theorem ([8] Th. 16.7.2) that T_t^D and $\hat{T}_t^D$ have the first eigenvalue $e^{-\lambda_0 t}$. Since G_λ and $\hat{G}_\lambda$ have no generalized eigenfucntions with the eigenvalue $e^{-\lambda_0 t}$ either. That shows $e^{-\lambda_0 t}$ is a simple eigenvalue of T_t^D and $\hat{T}_t^D$.

Theorem 1

T_t^D, $\hat{T}_t^D$ as semigroups on $B_0(D)$ are strongly continuous.

Proof: By Hille-Yosida Theorem, it suffices to prove that

$$\|\lambda G_\lambda - f\|_{B_0(D)} \to 0,\ \|\lambda \hat{G}_\lambda f - f\|_{B_0(D)} \to 0 \quad (\lambda \to \infty) \tag{13}$$

It follows from Proposition 2 that

$$\frac{1}{d(x)} \int_D \lambda G_\lambda(x,y) I_{\{|x-y|>\delta\}} dy$$

$$< \frac{c}{\delta^{n-1}} \lambda e^{-\beta\sqrt{\lambda}\delta} m(D) \to 0 \quad (\lambda \to \infty)$$

(uniformly with respect to x).

Thus Proposition 3 gives us

$$\sup_{x \in D} \int_{D \ \{|y-x|>\delta\}} \frac{\lambda G_\lambda(x,y)}{\phi_0(x)} dy \to 0 \quad (\lambda \to \infty) \tag{14}$$

Now for any $f \in B_0(D)$, by the corollary of Proposition 3 there is a $\tilde{f} \in C_U(D)$ such that $f = \tilde{f}\phi_0$. Then we have

$$\|\lambda G_\lambda f - f\|_{B_0(D)} = \left\| \frac{\lambda G_\lambda f}{\phi_0} - \tilde{f}\right\|_{C(\bar{D})}$$

$$< \sup_{x\varepsilon D} \int_D \frac{\lambda G_\lambda(x,y)\phi_0(y)}{\phi_0(x)} |\tilde{f}(y) - \tilde{f}(x)|dy +$$

$$+ \sup_{x\varepsilon D} |\tilde{f}(x)| \left|\int_D \frac{\lambda G_\lambda(x,y)\phi_0(y)}{\phi_0(x)} dy - 1\right|$$

$$< C_1 \sup_{x\varepsilon D} \int_{|x-y|>\delta} \frac{\lambda G_\lambda(x,y)}{\phi_0(x)} dy + \sup_{\substack{|x-y|<\delta \\ x,y\ \varepsilon\ D}} |\tilde{f}(y) - \tilde{f}(x)| \frac{\lambda}{\lambda + \lambda_0} +$$

$$+ \|\tilde{f}\|_{C(\bar{D})} \left(\frac{\lambda}{\lambda + \lambda_0} - 1\right).$$

Hence (13) follows from (14).

3. KILLED DIFFUSIONS

Let us assume that $X = (\Omega, F, F_t, P_x, \theta_t, X)$ is the Markov process generated by L on R^n.

Denote

$$T_t^\phi f(x) \triangleq e^{\lambda_0 t} \frac{T_t^D(\phi_0 f)(x)}{\phi_0(x)} \triangleq \int_D p^\phi(t,x,y)f(y)dy$$

and

$$\hat{T}_t^\psi f(x) = e^{\lambda_0 t} \frac{\hat{T}_t^D(\psi_0 f)(x)}{\psi_0(x)} = \int_D \hat{p}^\psi(t,x,y)f(y)dy$$

where $f \ \varepsilon\ C_U(D)$ and

$$p^{\phi}(t,x,y) \triangleq \frac{e^{\lambda_0 t} p^D(t,x,y)\phi_0(y)}{\phi_0(x)}$$

$$\hat{p}^{\psi}(t,x,y) \triangleq \frac{e^{\lambda_0 t} \hat{p}^D(t,x,y)\psi_0(y)}{\psi_0(x)} = \frac{e^{\lambda_0 t} p^D(t,y,x)\psi_0(y)}{\psi_0(x)}$$

It is easy to see that $T_t^{\phi}1 = 1$, $\hat{T}_t^{\psi}1 = 1$.

Theorem 2 There exist C_2 (depending on t_0) $> C_1 > 0$, such that for $t > t_0$ and $x \varepsilon D$

$$C_1 e^{-t\lambda_0} d(x) < P_x(\tau_D > t) < C_2 e^{-t\lambda_0} d(x) \tag{15}$$

and

$$C_1 e^{-t\lambda_0} \phi_0(x) < P_x(\tau_D > t) < C_2 e^{-t\lambda_0} \phi_0(x) \tag{16}$$

hold.

Proof: Since Proposition 3 we only need to check (16). For the lower bound, we have

$$\frac{P_x(\tau_D > t)}{e^{-\lambda_0 t}\phi_0(x)} = \int_D \frac{p^D(t,x,y)\phi_0(y)e^{\lambda_0 t}}{\phi_0(x)} \cdot \frac{1}{\phi_0(y)}\, dy$$

$$> \frac{1}{\sup\limits_{x\varepsilon D} \phi_0(y)} \triangleq C_1 > 0.$$

For the upper bound, we recall from Lemma 3 and Proposition 3 that there exists a constant C_2^t such that $g_t(x) \triangleq \int_D \frac{p^D(t,x,y)}{\phi_0(x)}\, dy < C_2^t$. Hence for $t > t_0$

$$\frac{P_x(\tau_D > t)}{e^{-\lambda_0 t}\phi_0(x)} = T_{t-t_0}^{\phi}\, g_{t_0}(x) < C_2^{t_0}\, T_{t-t_0}^{\phi}1 < C_2^{t_0} \triangleq C_2,$$

since $T_t^{\phi}1 = 1$.

Theorem 3

There exists a constant c only depending on D, such that

$$\sup_{x\varepsilon D} \left| \frac{e^{\lambda_0 t} P_x(\tau_D > t)}{\phi_0(x)} - c \right| \to 0 \qquad (t \to \infty) \tag{17}$$

Moreover

$$\sup_{x\varepsilon D} \left| \frac{e^{\lambda_0 t} \int_D p^D(t,x,y)f(y)dy}{\phi_0(x)} - \int_D f(y)\psi_0(y)dy \right| \to 0 \qquad (t \to \infty) \tag{18}$$

$$\sup_{x\varepsilon D} \left| \frac{e^{\lambda_0 t} \int \hat{p}^D(t,x,y)f(y)dy}{\psi_0(x)} - \int_D f(y)\phi_0(y)dy \right| \to 0 \tag{18'}$$

where $f \varepsilon M_b(D)$.

Proof:

First we assume $f \varepsilon B_0(D)$.

The compactness of T_t^D on $B_0(D)$ implies that there exists a decomposition of $B_0(D)$ into the direct sum of its subspaces M_{λ_0} and N_{λ_0} which are invariant under $e^{-\lambda_0 t} - T_t^D$, and

$$B_0(D) = M_{\lambda_0} + N_{\lambda_0}, \quad M_{\lambda_0} = \{\alpha\phi_0 : \alpha \varepsilon R'\}$$

$$T_t^D N_{\lambda_0} \subset N_{\lambda_0}$$

and T_t^D is compact on N_{λ_0}, while $e^{-\lambda_0 t}$ is no more in the spectrum of T_t on N_{λ_0} ([11] Section 6.2, Th. 6).

Now f can be written into

$$f = c_f\phi_0 + g \qquad g \varepsilon N_{\lambda_0}.$$

And then

$$T_t^D f = c_f e^{-\lambda_0 t}\phi_0 + T_t^D g,$$

Let λ_1 be in the point spectrum of T_t^D with the smallest real part bigger than λ_0. The spectrum radius theorem tells us that

$$\lim_{t\to\infty} \|T_t\|_{N_{\lambda_0}}^{1/t} = e^{-Re\lambda_1} < e^{-\mu} \quad (\lambda_0 < \mu < Re\lambda_1).$$

Thus we have

$$\|T_t^D\|^{1/t} < e^{-\mu} \qquad (t \text{ large enough}).$$

Hence

$$\sup_{x\varepsilon D} \left| \frac{e^{\lambda_0 t} T_t^D f}{\phi_0(x)} - C_f \right| = \sup_{x\varepsilon D} \left| \frac{e^{\lambda_0 t} \; T_t^D g}{\phi_0(x)} \right|$$

$$\leq \text{const} \, \| e^{\lambda_0 t} \; T_t^D g \|_{B_0(D)} = \text{const} \; e^{\lambda_0 t} \, \| T_t^D g \|_{N_{\lambda_0}}$$

$$\leq \text{const} \; e^{\lambda_0 t} \; e^{-\mu t} \, \|g\| \to 0 \qquad (t \to \infty).$$

Let us specify c_f:

$$0 = \lim_{t\to\infty} \int_D \left(\frac{e^{\lambda_0 t} \; T_t f}{\phi_0(x)} - C_f \right) \phi_0(x)\psi_0 x) dx$$

$$= \lim_{t\to\infty} \iint_{DD} e^{\lambda_0 t} \; \hat{p}^D(t,y,x)\psi_0(x)f(y)dxdy - c_f$$

$$= \int \psi_0(y)f(y)dy - c_f.$$

Secondly, for $f \varepsilon M_b(D)$, we have $T_{t_0} f \varepsilon B_0(D)$ from Lemma 3. Applying the formula just got, we obtain

$$\sup_{x\varepsilon D} \left| \frac{e^{\lambda_0 (t - t_0)} \; T_{t-t_0} T_{t_0} f}{\phi_0(x)} - \int_D \psi_0(y) \; T_{t_0} f(y)dy \right| \to 0 \qquad (t \to \infty)$$

Then (18) follows.

4. CONDITIONAL KILLED DIFFUSIONS

Theorem 4

T_t^{ϕ} and $\hat{T}_t^{\psi}$ are strongly continuous semigroups on $C_U(D)$. If $f \in C_K^{\infty}(D)$ (infinitely differential functions with compact support), then

$$A^{\phi}f = \left(\frac{1}{2}\nabla a\nabla + b\nabla + a(\nabla \log \phi_0)\nabla\right)f, \tag{19}$$

$$\hat{A}^{\psi}f = \left(\frac{1}{2}\nabla a\nabla - b\nabla + a(\nabla \log \psi_0)\nabla\right)f, \tag{19'}$$

where A^{ϕ}, $\hat{A}^{\psi}$ are the generator of T_t^{ϕ} and $\hat{T}_t^{\psi}$ on $C_U(D)$ respectively.

Proof: The first assertion is just a restatement of Theorem 1. For the second one, we have $f \in D(A^{\phi})$ iff $\phi f \in D(A)$ (A: generator of T_t) and under this condition $A^{\phi}f = A(\phi_0 f)/\phi_0$. Then (19) follows from the direct calculation.

As usual $\{p^{\phi}(t,x,y)\}$ and $\{\hat{p}^{\psi}(t,x,y)\}$ generate families of Markov measures $\{P_x^{\phi}\}$ and $\{\hat{P}_x^{\psi}\}$ with state space D respectively.

Theorem 5

X is a continuous homogeneous conservative Markov process with strong Feller (and $C_u(D)$ strongly continuous semigroup) under both $\{P_x^{\phi}\}$ and $\{\hat{P}_x^{\psi}\}$ with invariant measure $\mu_0(\Gamma) \overset{\Delta}{=} \int_{\Gamma} \phi_0(x)\psi_0(x)dx$ $(\Gamma \in B(D))$.

$\{\hat{P}_x^{\psi}\}$ is the time reversal of $\{P_x^{\phi}\}$ with respect to the invariant measure μ_0 which is mixing.

And we have for any initial measure $\mu(\Gamma)$ $(\Gamma \in B(D))$:

$$\iint_{D\times D} p^{\phi}(t,x,y)f(y)\mu(dx)dy \to \int f(y)\mu_0(dy) \quad (f \in M_b(D)) \tag{20}$$

$$\sup_{x\in D} \int_D |p^{\phi}(t,x,y) - \phi_0(y)\psi_0(y)|dy \to 0 \qquad (t \to \infty) \tag{21}$$

(the same for $\hat{p}^{\psi}$!)

Proof:

First, we show the path continuity of X under $\{P_x^\phi\}$ (or $\{\hat{P}_x^\psi\}$). For any $\zeta \in \sigma\{X_s: s < t\}$, we have

$$E_x^\phi \xi = \frac{e^{\lambda_0 t}}{\phi_0(x)} E_x(\xi \phi_0(X_t)).$$

It follows that

$$P_x^\phi\{w: X_s(w) \text{ discontinuous on } [0,n]\}$$
$$= \frac{e^{\lambda_0 t}}{\phi_0(x)} E_x\ (I_{\{X_s \text{ discontinuous on } [0,n]\}}\phi_0(X_n)) = 0$$

Next, we set

$$q_{t_0}(y) = \int_D \frac{p^D(t_0,x,y)}{\phi_0(x)} \mu(dx).$$

Then Theorem 2 implies

$$\int_D p^\phi(t,x,y)f(y)\mu(dx) = c^{\lambda_0 t} \iint_{D\times D} p^D(t - t_0,y,z)\phi_0(z)f(z)q_{t_0}(y)dydz$$

$$= \int_D e^{\lambda_0(t - t_0)} T^D_{t - t_0}(\phi_0 f)(y)e^{\lambda_0 t_0} q_{t_0}(y)dy$$

$$\to \int \langle\phi_0 f,\psi_0\rangle_{L^2(D)}\ \phi_0(y)e^{\lambda_0 t_0} q_{t_0}(y)dy \quad (t \to \infty)$$

$$= \langle\phi_0 f,\psi_0\rangle_{L^2(D)} = \int_D f(y)\mu_0(dy).$$

To see the mixing of μ_0, we calculate, for instance, for bounded measurable functions f and g

$$E_\mu^\phi[f(C_t,X_{t+s_1},X_{t+s_2})g(X_{u_1},X_{u_2})]$$

$$= \int \phi_0(x)\psi_0(x)p^\phi(u_1,x,y)p^\phi(u_2-u_1,y,z)g(y,z)p^\phi(t - u_2,z,a)$$

$$p^\phi(s_1,a,b)p^\phi(s_2 - s_1,b,c)f(a,b,c)dxdydzdadbdc$$

The right hand side goes to $E_{\mu}^{\phi}f(X_0,X_{s_1},X_{s_2})E_{\mu}^{\phi}g(X_{u_1},X_{u_2})$ as $t \to \infty$, because we have

$$\int\phi_0(x)\psi_0(x)p^{\phi}(u_1,x,y)dx = \phi_0(y)\psi_0(y)$$

$$\int\phi_0(y)\psi_0(y)p^{\phi}(u_2 - u_1,y,z)g(y,z)dy = $$
$$= e^{\lambda_0(u_2-u_1)}\phi_0(z)\int p^D(u_2-u_1,y,z)\psi_0(y)g(y,z)dz$$
$$\triangleq \phi_0(z)\overline{g}(z)$$

and

$$\int \phi_0(z)\overline{g}(z)p^{\phi}(t - u_2,z,a)dz = \phi_0(a)\int e^{\lambda_0(t - u_2)}\hat{p}^D(t - u_2,a,z)\overline{g}(z)dz$$

$$\to \phi_0(a)\psi_0(a)\int \overline{g}(z)\phi_0(z)dz \quad (t \to \infty, \text{ uniformly}):$$

What we get above deduces that for any $\sigma(X_t,0 \le t < \infty)$ cylinder sets A and B, it is valid that

$$P_{\mu_0}^{\phi}[\theta_t A \cap B] \to P_{\mu_0}^{\phi}(A)P_{\mu_0}^{\phi}(B) \qquad (t \to \infty).$$

Thus the standard way of approximating $\sigma(X_t,0 \le t < \infty)$ measurable sets A, B by cylinder sets A_n, B_n ensures that

$$|P_{\mu_0}^{\phi}(\theta_t A\cap B) - P_{\mu_0}^{\phi}(\theta_t A_n\cap B_n)| \le P_{\mu_0}^{\phi}(\theta_t A \,\Delta\, \theta_t A_n) + P_{\mu_0}^{\phi}(B \,\Delta\, B_n)$$

$$= P_{\mu_0}^{\phi}(A \,\Delta\, A_n) + P_{\mu_0}^{\phi}(B \,\Delta\, B_n) \to 0 \quad (n \to \infty, \text{ uniformly for } t)$$

which implies

$$P_{\mu_0}^{\phi}(\theta_t A \cap B) \to P_{\mu_0}^{\phi}(A)P_{\mu_0}^{\phi}(B) \quad (t \to \infty)$$

Finally, we estimate

$$\sup_{x\in D}\int_D |p^\phi(t,x,y) - \phi_0(y)\psi_0(y)|dy$$

$$= \sup_{x\in D}\int_D \left|\int_D (p^\phi T,x,y) - p^\phi(t,x',y)\phi_0(x')\psi_0(x')dx'\right|dy$$

$$< \sup_{x\in D}\int_D\int_D |p^\phi(t,x,y) - p^\phi(t,x',y)dy \ \phi_0(x')\psi_0(x')dx'$$

$$= \sup_{x\in D}\int_D \sup_{\|f\|_{M_b(D)}<1} |T_t f(x) - T_t f(x')|\phi_0(x')\psi_0(x')dx'$$

$$\to 0 \qquad (t \to \infty)$$

since (18) holds uniformly for $(f: \|f\| < 1)$.

The other conclusions of the Theorem are easy to obtain.

Corollary If L is symmetric, then $\psi_0(x) = \text{const.}\ \phi_0(x)e^{2\int^x(a^{-1}b)}$.

The next theorem says that X under $\{P_x^\phi\}$ (we denote it as X^ϕ) can be regarded as original process X (under $\{P_x\}$) conditioned on D in the following sense: $\{P_x^\phi\}$ is the weak limit (when $T \to \infty$) of X^T the killed diffusion conditioned up to time T. Actually X^T is determined by the nonhomogeneous Markov measures $\{P_{s,x}^T\}$ [12] with the density:

$$p^T(s,t,x,y) \triangleq p^D(t - s,x,y)\frac{P_y(\tau_D > T - t)}{P_x(\tau_D > T - s)}\,. \qquad (22)$$

Hence we call $\{P_x^\phi\}$ conditional killed diffusion on D.

Theorem 6 For any $T_0 > 0$, X^T converges weakly to X^ϕ in $D[0,T_0]$ as $T \to \infty$.

Proof.

Two things have to be done for the proof. First, the convergence of the finite distributions of $P^T_{s,x}$ on $[0,T_0]$. Second, the uniform tightness of the family $P^T_{s,x}$ with $T_0 < T$. In the light of Theorem 1 and 2, we have

$$p^T(s,t,x,y) < \text{Const. } p^\phi(t - s,x,y) \tag{23}$$

and

$$p^T(s,t,x,y) \to p^\phi(t - s,x,y) \quad (T \to \infty).$$

It implies that the finite distributions of $P^T_{s,x}$ converge to those of P^ϕ_x. On the other hand, since Thoerem 3 T^ϕ_t is strongly continuous on $C_U(D)$, then $p^\phi(t,x,y)$ satisfies that

$$\sup_{x\varepsilon D} \int_{|x-y|>\delta} p^\phi(t,x,y)dy \to 0 \quad (t \to 0)$$

by an argument similar to that in [4].

It follows from (23) that

$$\sup_{x\varepsilon D} \int_{|x-y|>\delta} p^T(s,t,x,y)dy \to 0 \quad (t \downarrow s)$$

uniformly with respect to s and T. Now a theorem [16] about the tightness of the Markov families of $D[0,T_0]$ provides us that $P^T_{s,x}$ is compact in $D[0,T_0]$. Thus it implies that $P^T_{s,x} \overset{w}{\to} P^\phi_x$ in $D[0,T_0]$. □

We see that this Theorem gives us a comprehensive understanding of the Markov process X^ϕ.

All these Propositions and Theorems above remain true with a usual modification in the case of $n = 2$ or $n = 1$.

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